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MINIMUM OF MODULUS OF THE SUM OF DIRICHLET SERIES CONVERGING IN A HALF-PLANE

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Abstract. The estimate of the sum of Dirichlet series near the convergence line and outside some exceptional set of disks is obtained in terms of minimum of modulus on continuums close to vertical line segments. This result generalizes the known theorem on minimum of modulus on vertical segments lying in the convergence half-plane.

Keywords: Dirichlet series, convergence half-plane, minimum modulus theorem.

Mathematics Subject Classification: 30D10

1. Introduction

The methods of obtaining asymptotic estimates for the sum of an entire Dirichlet series on vertical segments in terms of maximum or minimum of modulus as well as on curves passing to infinity in a certain way are well-known at the present (on this subject see, for instance, [1] – [3]). The behavior of the sum of Dirichlet series on curves happens to be exactly its global behavior outside certain set of exceptional circles. This is the matter of the present paper in the case when the convergence domain of the Dirichlet series is a half-plane.

Let $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ be a sequence with a finite upper density, a $D_c(\Lambda)$ be a class of functions F which can be represented in the half-plane $\Pi_c = \{s : \operatorname{Re} s < c\}$, $-\infty < c \leqslant +\infty$, by the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \quad (s = \sigma + it) \tag{1}$$

converging only in this half-plane. In what follows, for the sake of convenience as “maximum of modulus” we call the quantity

$$M_F(\sigma) = \sup_{|t|<\infty} |F(\sigma + it)|, \quad \sigma < c.$$

We describe briefly the general scheme of our arguments allowing us to obtain the estimates for the maximum of modulus $M_F(\sigma)$ in terms of the minimum of modulus F on vertical segments.

The asymptotic estimate of quantity $M_F(\sigma)$ for the sum F of entire Dirichlet series (1) in terms of maximum $|F|$ on the vertical segment $I = \{s = \sigma + it : |t - t_o| \leqslant H\}$ of a certain length was obtained in [1]. At that, the length $|I|$ of segment I should not be less than a certain characteristics close to similar characteristics allowing one to determine the completeness radius for the system of exponents $\left\{e^{i\lambda_k x}\right\}$ in space $C[a, b]$ (or $L^2[a, b]$) (on this subject see [1, 4, 5, 6]). It is natural to expect that while estimating $M_F(\sigma)$ by the minimum of modulus, in a general situation the length of segment I can not be too large. It is clear that the more $|I|$ the better the estimate. Under natural restrictions for the sequence of central indices for entire series (1), it was shown in [3] that as $\sigma \to +\infty$, the length of segment $|I|$ can grow as $O(\sigma^q)$ $(0 < q < 1)$.

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Let us clarify the main ideas employed in obtaining asymptotic estimates for each function $F \in D_\infty(\Lambda)$ (as we shall see, after an appropriate modification this scheme is applicable also for the case $D_o(\Lambda)$):

a) for each curve γ passing to infinity in a proper way there exists a sequence $\{\xi_n\}$, $\xi_n \in \gamma$ such that

$$\ln M_F(\sigma_n) = (1+o(1)) \ln |F(\xi_n)|, \quad \sigma_n = \operatorname{Re}\xi_n, \quad \text{as } \xi_n \to \infty;$$

b) as $\sigma \to \infty$, outside some set $E \subset \mathbb{R}_+$

$$\ln M_F(\sigma) = (1+o(1)) \ln m_F(\sigma),$$

where $m_F(\sigma) = \min\limits_{it\in I} |F(\sigma+it)|$, $I = I(\sigma)$ is the segment of, generally speaking, arbitrary length.

Let Λ be a sequence satisfying natural conditions [2]:

$$1)\ \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty; \qquad 2)\ \int_1^{\infty} \frac{c(t)}{t^2} dt < \infty, \tag{2}$$

where $c(t) = \max\limits_{\lambda_n \leqslant t} q_n, \quad q_n = -\ln |Q'(\lambda_n)|, \quad Q(z) = \prod\limits_{n=1}^{\infty} \left(1 - \dfrac{z^2}{\lambda_n^2}\right)$.

The general scheme of the approach allowing us to obtain estimates like a) and b) is as follows. We first find a partial sum of the series $F_v(s) = \sum\limits_{\lambda_n \leqslant v} a_n e^{\lambda_n s}$, $v = v(\sigma)$, obeying

$$|F(s) - F_v(s)| < 1 \tag{3}$$

for each $\sigma \geqslant \sigma_0$ outside E, $\operatorname{mes} E < \infty$. At the next step we show that outside E

$$M_F(\sigma) \leqslant e^{w(v)} \max_{|z-\alpha|\leqslant\delta} |F_v(z)|, \tag{4}$$

where $\alpha \in \mathbb{C}$ $(\operatorname{Re}\alpha = \sigma)$ is arbitrary, $\delta = \dfrac{w^*(v)}{v}$, w, w^* are some continuous monotonically increasing functions for the convergence class W, i.e., $x^{-2}w(x)$, $x^{-2}w^*(x)$ belongs to $L^1[1,\infty)$, $w(x) = o(w^*(x))$ as $x \to \infty$, $w(v) \geqslant N(v) + c(v)$, $w(v) = o(\ln M_F(\sigma)$ as $\sigma \to \infty$ [2]. Here c is a function in (2), $N(x) = \int\limits_0^x \frac{n(t)}{t} dt$, $n(t) = \sum\limits_{\lambda_n \leqslant t} 1$. In view of (3) we see that estimate like (4) is valid also for F.

Statement a) follows from (4) by applying two constants theorem (it is assumed that $\alpha \in \gamma$) and a lemma like Borel-Nevalinna theorem (see [2]).

To obtain Statement b) from (4), we need to pass to a similar estimate for the circle $\{z\colon |z-\alpha| \leqslant \delta^2\}$. This is why we act as follows (see, for instance, [7]). We apply two constants theorem and take into consideration the asymptotic estimate (being a corollary of a lemma of Borel-Nevalinna kind [2])

$$\ln M_F(\sigma+\delta) < (1+o(1)) \ln M_F(\sigma), \quad \sigma \to \infty, \quad \sigma \notin E.$$

Then we can assume that estimate (4) holds true for a vertical segment I of length 2δ centered at a point α [7]. Now the problem is reduced to replacing segment I by a small segment $J \subset I$ of length $2\delta^2$. The latter problem is usually solved by means of the following P. Turan's lemma [8]: if $\mu_1 < \mu_2 < \cdots < \mu_n$ and

$$p(t) = \sum_{j=1}^{n} b_j e^{it\mu_j},$$

then

$$\|p\|_I \leqslant \left(2e\frac{|I|}{|J|}\right)^n \|p\|_J. \tag{5}$$

Here I, J are segments on the imaginary axis, $J \subset I$, $\|p\|_I = \max_{it\in I} |p(t)|$.

If I, J are the above segments, bearing in mind estimate (5), it remains to pass from segment J to the circle $\{z\colon |z-\alpha| \leqslant \delta^2\}$.

Sometimes Turan's lemma is replaced by another statement based on the properties of Fourier transform (see [1], [3]).

There is another approach based on applying well-known A.F. Leont'ev formulae for the coefficients of quasipolynomial F_v [9, Ch. I, Sec. 2, Subsec. 1]. In this case outside some set $E \subset \mathbb{R}_+$ of finite measure

$$M_F(\sigma) \leqslant e^{w(v)} H_v(\delta^2) \max_{|z-\alpha|\leqslant\delta^2} |F_v(z)|, \tag{6}$$

where $H_v(\delta) = \int\limits_0^\infty M(r, q_v) e^{-r\delta}\, dr$, $q_v(z) = \prod\limits_{\lambda_n\leqslant v} \Big(1 - \dfrac{z^2}{\lambda_n^2}\Big)$. In view of estimate $N(v) \leqslant w(v)$ it follows from (6) that

$$M_F(\sigma) \leqslant 2e^{3w(v)} \exp\Big(\max_{r\geqslant 0} \varphi(r)\Big) \max_{|z-\alpha|\leqslant\delta^2} |F_v(z)|,$$

where $\varphi(r) = n(v) \ln\Big(1 + \dfrac{r^2}{v^2}\Big) - r\delta^2$. But the maximum of function φ is attained at the point $r_0 \leqslant \dfrac{2n(v)}{\delta^2}$ and this is why

$$\varphi(r_0) \leqslant n(v) \ln\Big(1 + 4\frac{v^2}{n^2(v)}\Big) = O\Big(n(v) \ln \frac{v}{n(v)}\Big)$$

as $\sigma \to \infty$. Thus,

$$M_F(\sigma) \leqslant e^{3w(v) + An(v)\ln\frac{v}{n(v)}} \max_{|z-\alpha|\leqslant\delta^2} |F_v(z)|. \tag{7}$$

On the other hand, for $p = F_v$ the first factor in (5) as $\sigma \to \infty$ is the quantity

$$\exp\Big(O\Big(n(v) \ln \frac{1}{\delta}\Big)\Big) \leqslant \exp\Big(O\Big(n(v) \ln \frac{v}{n(v)}\Big)\Big),$$

since $\ln \frac{1}{\delta} \leqslant \ln \frac{v}{n(v)}$. To obtain from (7) the desired estimate for the minimum of modulus $m_F(\sigma)$ it is important [7] that for the function $n(t) \ln \frac{t}{n(t)}$ there exists a majorant w from class W. It is equivalent [3] to

$$\int\limits_{\lambda_1}^{\infty} \frac{n(t) \ln \frac{t}{n(t)}}{t^2}\, dt < \infty. \tag{8}$$

But then as $\sigma \to \infty$ in all cases outside E, $\operatorname{mes} E < \infty$,

$$M_F^{1+o(1)}(\sigma) \leqslant \max_{|z-\alpha|\leqslant\delta^2} |F(z)| = |F(\xi)|.$$

It is the main estimate for the maximum of modulus. In the same way as in [7] one can deduce the desired estimate if for the circle $D(\xi, 2\delta)$ one applies to F the following lemma on a lower estimate of an analytic bounded function in the unit circle.

Lemma 1. [10] *Suppose that a function g is analytic and bounded in the circle $\{z\colon |z| < R\}$, $|g(0)| \geqslant 1$. If $0 < r < 1 - N^{-1}$ $(N > 1)$, then there exists at most countably many circles*

$$V_n = \big\{z\colon |z - z_n| \leqslant \rho_n\big\}, \quad \sum_n \rho_n \leqslant R r^N (1-r),$$

such that for each z in the circke $\{z\colon |z| \leqslant Rr\}$ but outside $\bigcup\limits_n V_n$ the estimate

$$\ln|g(z)| \geqslant \frac{R-|z|}{R+|z|}\ln|g(0)| - 5NL \tag{9}$$

holds true, where

$$L = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln^+|g(Re^{i\theta})|\,d\theta - \ln|g(0)|.$$

Condition (8) is natural and it is likely optimal in consideration of problems like b). In some sense it is confirmed by the fact that omitting this condition leads one to a deformation of segment I, see below and [3].

For functions $F \in D_\infty(\Lambda)$ the issue on the minimum of modulus was studied rather in detail in work [3], where the complete results were obtained. It is important to note that in the present work function F can have arbitrarily fast growth. Some important theorems on minimum of modulus of Dirichlet series converging only in the half-plane Π_o were established in [11]. Similar results on less regular behavior of functions $F \in D_o(\Lambda)$, namely, on curves adjacent to the imaginary axis were proven in paper [12].

The aim of this paper is to move the results of work [3] on minimum of modulus of a function in $D_\infty(\Lambda)$ for the case of a function in class $D_o(\Lambda)$ and to strengthen and generalized the corresponding statements in [11], [12]. Thereupon we note that in the case $F \in D_o(\Lambda)$ there appear specific difficulties related with estimating the sizes of exceptional sets $e \subset [-1, 0)$. This is why in the case of half-plane Π_o (or the unit circle for lacunar power series $f(z) = \sum\limits_{n=1}^{\infty} a_n z^{p_n}$, $p_n \in \mathbb{N}$) the usual way is as follows. We fix some monotonically increasing continuous function Φ and choose certain subclass of functions $F \in D_o(\Lambda)$ satisfying, say, the condition

$$\varliminf_{\sigma\to 0-} \frac{\ln\mu(\sigma)}{\Phi\left(\frac{1}{|\sigma|}\right)} > 0,$$

where $\mu(\sigma)$ is the maximal term of series (1). Then the variable relative density

$$\Delta(\sigma) = \frac{\operatorname{mes}(e\cap[\sigma, 0))}{|\sigma|}$$

of the exceptional set $e \subset [-1, 0)$ outside with function F satisfies desired estimates depends usually only on the behavior of the quantity

$$\varphi(t)\int\limits_t^\infty \frac{w(x)}{x^2}dx,$$

where φ is the inverse function for Φ, while $w = w(x)$ is a distribution function of sequence Λ [11]. If, for instance, $w \in \underline{W}_\varphi$ (see definition below) and $\varphi(x)w(x) = o(x)$ as $x \to \infty$, then the lower density de of set e happens to vanish. In other respects the ways of proof are the same as in the case $D_\infty(\Lambda)$. This is why the main aim of the present work is to specify the location and sizes of exceptional circles outside which the desired estimate holds for function $\ln|F(s)|$ in terms of the minimum of modulus in half-plane Π_0.

2. Definitions and preliminaries

Let L be the class of continuous unbounded and increasing on $[0,\infty)$ functions,

$$W=\{w\in L:\int\limits_1^\infty \frac{w(x)}{x^2}dx<\infty\}$$

be the convergence class, and

$$\underline{W}_\varphi=\{w\in W:\ \varliminf_{t\to\infty}\varphi(t)J(t;w)=\ 0\},$$

where $\varphi\in L,\ \ J(t;w)=\int\limits_t^\infty\frac{w(x)}{x^2}dx$. We also introduce the set $W_\varphi\subset\underline{W}_\varphi$:

$$W_\varphi=\{w\in W:\ \lim_{t\to\infty}\varphi(t)J(t;w)=\ 0\}.$$

We shall say that two functions φ and w from class L are consistent if $\varphi(x)w(x)=o(x)$ as $x\to\infty$.

Let $e\subset[-1,0)$ be a Lebesgue measurable set. The upper De and the lower de densities are the quantities [11]

$$De=\varlimsup_{\sigma\to0-}\frac{\mathrm{mes}(e\cap[\sigma,0))}{|\sigma|},\qquad de=\varliminf_{\sigma\to0-}\frac{\mathrm{mes}(e\cap[\sigma,0))}{|\sigma|}.$$

If $De=de$, then set e is said to have a density.

Theorems on minimum of modulus are based on the statements related with a lower estimate for the logarithm of modulus of an analytic and bounded from below function outside some set of circles. As it was mentioned above, Lemma 1 is useful in obtaining similar estimates. Let us make a remark on exceptional circles from this lemma. As $L=0$, estimate (9) follows from Harnack's inequality and it is valid everywhere in the circle $\{z\colon |z|<R\}$ (see [10]). Suppose now $L>0$. Estimate (9) is valid in each so-called light point of the circle $\overline{D}=\{z\colon |z|\leqslant Rr\}$ [10]. Other points of circle $\overline{D}$ are called heavy. To each heavy point z, the circle (see [10], [13]) $K_z=\{\xi\colon |\xi-z|\leqslant\rho_z\}$ is associated. As it is known, the covering of the set of heavy points by the circles K_z of a bounded radius ρ_z contains at most countable subcovering such that each heavy point is covered by at most six circles [14]. In circle $\overline{D}$ function g has just a finite number of zeroes $a_1,\ a_2,\ \ldots,\ a_n$. It is obvious that all of them are heavy points.

Slightly increasing the radii of exceptional points, we can assume that estimate (9) is valid outside the union of open circles $V_n=\{z\colon |z-z_n|<\rho_n\}$ with the total sum of radii

$$\sum_n\rho_n\leqslant Rr^N,\qquad r<1-\tfrac{1}{N},\quad N>1.$$

Then for each $z\in\overline{D}$ outside $V=\bigcup\limits_n V_n$ the estimate

$$G(z)>-6NL,\quad G(z)=\ln|g(z)|-\frac{R-|z|}{R+|z|}\ln|g(0)| \tag{10}$$

holds true. Discarding from $\overline{D}$ all open circles in V containing $a_1,\ a_2,\ \ldots,\ a_n$ (their total number is at most $6n$), we obtain a closed set which we indicate by C. Let

$$B=\big\{z\in C\colon G(z)\leqslant-6NL\big\}.$$

Set B is closed and $B\subset V$. Therefore, by Heine-Borel lemma there exists a finite number of circles in V covering B. Thus, for each z in $C\setminus B$ estimate (10) holds true outside mentioned circles. Thus, we arrive at

Lemma 2. *Suppose the hypothesis of Lemma 1. Then there exists a finite number of circles* $V_n = \{z\colon |z - z_n| < \rho_n\}$ $(1 \leqslant n \leqslant m)$ *with the total sum of radii*

$$\sum_{n=1}^{m} \rho_n \leqslant Rr^N, \quad r < 1 - \tfrac{1}{N} \ (N \geqslant 1), \tag{11}$$

outside which in the circle $\{z\colon |z| \leqslant Rr\}$ *the estimate*

$$G(z) > -6NL$$

holds true, where G *is the function defined in formulae (10).*

3. Main result

Let $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$, $\overline{\lim\limits_{n\to\infty}} \frac{n}{\lambda_n} = D < \infty$,

$$Q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right). \tag{12}$$

It is clear that Q is an entire function of exponential type. Denote $M(r; Q) = \max\limits_{|z|=r} |Q(z)|$. Suppose that sequence Λ is distributed so that for some function $\psi \in W_\varphi$ the estimates

$$-\ln |Q'(\lambda_n)| \leqslant \psi(\lambda_n) \quad n \geq 1, \tag{13}$$

hold true. We note that in this case, if $F \in D_\infty(\Lambda)$, then estimate (13) should be satisfied for $\psi \in W$, and this condition is essential for validity of estimates a), b) (see Introduction) [2].

Let us formulate the main result. Under the above assumptions we have

Theorem 1. *Let* φ *be a fixed function in* L, $p \in \underline{W}_\varphi$, *where* $p(x) = \ln M(x; Q)$ *and* φ *and* p *are consistent. Suppose that the maximal term* $\mu(\sigma)$ *of series (1) satisfies the condition*

$$\underline{\lim_{\sigma\to 0-}} \frac{\ln \mu(\sigma)}{\Phi\left(\frac{1}{|\sigma|}\right)} > 0, \tag{14}$$

Φ *is inverse function for* φ. *Then for each function* $F \in D_o(\Lambda)$ *there exists a measurable set* $e \subset [-1, 0)$ *of zero lower density such that for each vertical segment*

$$I_H = I_H(\sigma) = \{s = \sigma + it : |t - t_o| \leqslant H,\ \sigma < 0\}, \quad H = \text{const},$$

for each σ, $-1 < \sigma_o \leqslant \sigma < 0$, *outside* e *there exists a deformed segment* $I_H^* = I_H^*(\sigma)$ *with the properties*

1) $\operatorname{mes}[I_H(\sigma) \cap I_H^*(\sigma)] \to |I_H| = 2H$ *as* $\sigma \to 0-$;
2) $\ln M_F(\sigma + d(\sigma)) < (1 + o(1)) \ln M_F(\sigma)$ *as* $\sigma \to 0-$ *outside* e, *where* $d(\sigma) = \max\limits_{\tau \in I_H^*} |\operatorname{Re}\tau - \sigma)|$;
3) $\ln M_F(\sigma) = (1 + o(1)) \ln m_F^*(\sigma)$ *as* $\sigma \to 0-$ *outside* e, *where* $m_F^*(\sigma) = \min\limits_{\tau \in I_H^*} |F(\tau)|$.

Доказательство. Let $w_1(x) = N(ex)$, $\psi \in W_\varphi$ be the function in condition (13). Since $p \in \underline{W}_\varphi$, then $w_1 \in \underline{W}_\varphi$. Therefore, function $w(x) = w_1(x) + \psi(x)$ belongs to class $\underline{W}_\varphi$. In view of the belonging $\psi \in W_\varphi$, we have $\varphi(x)\psi(x) = o(x)$ as $x \to \infty$. But then it follows obviously from the hypothesis of the theorem that $\varphi(x)w_1(x) = o(x)$ as $x \to \infty$. Hence, there exists a function such that $w^*(x) = \beta(x)w(x)$ $(0 < \beta(x) \uparrow \infty,\ \ x \to \infty)$, also belonging to $\underline{W}_\varphi$, $\varphi(x)w^*(x) = o(x)$ as $x \to \infty$. We denote $w_1^*(x) = \sqrt{\beta(x)}w(x)$.

Let $v = v(\sigma)$ be a solution to the equation

$$w^*(v) = 3 \ln \mu(\sigma). \tag{15}$$

It is clear that $v(\sigma)\uparrow\infty$ as $\sigma\uparrow 0$. Since $w^*\in\underline{W}_{\varphi}$, there exists a sequence $\{\tau_j\}$ $(\tau_j\uparrow 0)$ such that

$$\lim_{v_j\to\infty}\varphi(v_j)J(v_j;w^*)=0, \tag{16}$$

where $v_j=v(\tau_j)\to\infty$, $\tau_j\to 0-$,

$$J(v_j;w^*)=\int\limits_{v_j}^{\infty}\frac{w^*(x)}{x^2}dx.$$

It follows from conditions (14) — (16) the consistency of functions φ and w^* that [12]

$$\lim_{\tau_j\to 0-}\frac{1}{|\tau_j|}J(v_j;w^*)=0,\quad v_j=v(\tau_j), \tag{17}$$

$$\lim_{\sigma\to 0-}\frac{w^*(v(\sigma))}{|\sigma|v(\sigma)}=0. \tag{18}$$

But under conditions (15), (17), (18) Borel-Nevalinna kind lemma holds, in accordance with it as $\sigma\to 0-$, outside some set $e_1\subset[-1,0)$, $\mathrm{mes}(e_1\cap[\tau_j,0))=o(|\tau_j|)$, $\tau_j\to 0-$, the estimates [12] $\sigma+3\delta^*<0$, and

$$\mu(\sigma+3\delta^*)\le\mu(\sigma)^{1+o(1)},\quad \delta^*=\frac{w^*(v(\sigma))}{v(\sigma)}, \tag{19}$$

hold true. Applying estimates (10), (19) and other arguments mentioned in Section 1 (see, for instance,[12]), as $\sigma\to 0-$ outside exceptional set e_1 we obtain that

$$\begin{aligned} &1)\qquad \ln M_F(\sigma+\delta^*)<(1+o(1))\ln M_F(\sigma);\\ &2)\qquad M_F^{1+o(1)}(\sigma)\leqslant\max_{|\xi-\alpha|\leqslant\delta}|F(\xi)|,\end{aligned} \tag{20}$$

where α $(\operatorname{Re}\alpha=\sigma)$ is an arbitrary complex number in half-plane Π_o,

$$\delta=\delta(v)=\frac{w_1^*(v)}{v},\qquad \delta^*=\delta^*(v)=\frac{w^*(v)}{v},\qquad v=v(\sigma).$$

Let $D_a=[-1,0)\setminus e_1$. Then for $\sigma\in D_a$ and $\sigma\to 0-$ by (20) we obviously get

$$M_F^{1+o(1)}(\sigma)\leqslant\max_{\xi\in K}|F(\xi)|=|F(\xi^*)|, \tag{21}$$

where $\xi^*\in\partial K$, K is the square with the sides parallel to the axis and circumscribed around the circle $\bar D(\alpha,\delta)=\{\xi\colon|\xi-\alpha|\leqslant\delta\}\subset\Pi_o$.

We apply Lemma 2 to function $g(z)=F(z+\xi^*)$ assuming $N=4$, $R=\delta^*$, $r=\frac{2\sqrt{2}}{\sqrt{\beta(v)}}$. Since $Rr=2\sqrt{2}\,\delta$ is the length of the diagonal in square K, then $K\subset\bar D(\xi^*,Rr)$. If $r<1-\frac{1}{N}$, in accordance with Lemma 2 for each z in the circle $\bar D(\xi^*,Rr)$ outside a finite number of exceptional circles V_n with radii obeying condition (11), as $\sigma\in D_a$ and $\sigma\to 0-$, the estimate

$$F(\xi^*)|\leqslant|F(z)|^{1+o(1)}\leqslant M_F^{1+o(1)}(\sigma+\delta^*) \tag{22}$$

holds true. The number of exceptional circles depends on the square. Denoting this number by $m(K)$, we have

$$\sum_{n=1}^{m(K)}\rho_n\leqslant Rr^4\leqslant\frac{64\,\delta}{\beta^{3/2}(v)}. \tag{23}$$

Thus, by (21), (22) we obtain that as $\sigma\in D_a$ and $\sigma\to\infty$,

$$\ln M_F(\sigma)=(1+o(1))\ln|F(z)|, \tag{24}$$

if $z\in K\setminus\bigcup\limits_{n=1}^{m(K)}V_n$, where K is the above square centered at the point $\alpha=\sigma+it$.

For each $\sigma \in D_a$ we consider the rectangular

$$P = \{z = x + iy\colon |\sigma - x| \leqslant \delta,\ |y - t_0| \leqslant H\} \quad (H = \text{const}).$$

It is clear that $P \subset \Pi_0$ as $\sigma' < \sigma < 0$.

Consider the minimal numbers of squares like K mutually having no common internal points and covering P. The exceptional set $e = \{e_i\}$ of rectangular P consists of exceptional circles of the covering squares and the number of these circles is finite. Circles e_i can intersect forming so-called clusters $d_K = \bigcup\limits_{i=1}^{m_K} e_i$ being connected components of e.

Let Π be the projection of sets d_K having non-empty intersection with segment $I_H(\sigma)$ on this segment. Then $\Pi = \bigcup\limits_{j=1}^{n} I_j$, where I_j are some mutual disjoint segments, $I_j \subset I_H(\sigma)$, and by (23)

$$\sum_{j=1}^{n} |I_j| \leqslant 2\sum_{j=1}^{n} \rho_j \leqslant \frac{\text{const}}{\beta^{3/2}(v)}$$

as $\sigma' < \sigma'' < \sigma < 0$.

The changed segment I_H^* is constructed as follows. For each $j = 1, 2, \dots, n$ we find the minimal rectangular P_j with the side I_j and covering the appropriate sets d_K. The part I_j of segment I_H is replaced by the polyline $\gamma_j = \partial P_j \setminus I_j$. If P_j is adjacent to a horizontal side of P, from γ_j we exclude the segment lying on this side of P. Making this procedure for each segment I_j, we obtain the required "segment" I_H^*.

The continuity of function F implies the validity of estimate (24) on the boundaries of clusters d_K. Therefore, estimate (24) holds on the whole "segment" I_H^*, and as $\sigma \in D_a$ and $\sigma \to 0-$

$$\ln M_F(\sigma) = (1 + o(1)) \ln m_F^*(\sigma),$$

where $m_F^*(\sigma) = \min\limits_{\tau \in I_H^*(\sigma)} |F(\tau)|$. The proof is complete. □

Remark 1. In [12] under the hypothesis of Theorem 1 a weaker asymptotic relation $d(F;\gamma) = 1$ was proven, where

$$d(F;\gamma) = \varlimsup_{s\in\gamma,\ \operatorname{Re} s \to 0-} \frac{\ln |F(s)|}{\ln M_F(\operatorname{Re} s)},$$

γ is an arbitrary curve in Π_o ending on the imaginary part.

Theorem 2. *Suppose hypothesis of Theorem 1, and function l,*

$$l(r) = N(r) \ln \frac{r}{N(r)}, \qquad N(t) = \int_0^t \frac{n(x)}{x} dx, \qquad n(t) = \sum_{\lambda_j \le t} 1,$$

belongs to class W_φ. Then as $\sigma \to 0-$, outside some set $e \subset [-1, 0)$ of zero density

$$\ln M_F(\sigma) = (1 + o(1)) \ln m_F(\sigma), \tag{25}$$

where $m_F(\sigma) = \min\limits_{\tau \in I_H} |F(\tau)|$, $I_H = I_H(\sigma)$, $\sigma < 0-$ is a vertical segment of length $2H$.

If $l \in \underline{W}_\varphi$, then asymptotic identity (25) is valid as $\sigma \to 0-$ outside a set $e \subset [-1, 0)$ of zero upper density.

Theorem 2 can be proven by the same approach as in [11], if one takes into consideration the way of estimating the measures of exceptional sets like e_1 in the proof of Theorem 1.

We note that in paper [11] function φ satisfies certain additional restrictions. In Theorems 1, 2 we require only $\varphi \in L$. Proof of Theorem 2 will be provided in another paper.

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