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ONE DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS: PATHWISE APPROACH

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Abstract. We study path-wise analogues of one dimensional stochastic differential equations with symmetric integrals. We find existence and uniqueness conditions for solutions, the conditions of continuity and differentiability w.r.t. a parameter, as well as the conditions of linearization for such equations. We also study the structure of the solutions.

Keywords: symmetric integral, differential equations with symmetric integral.

Mathematics Subject Classification: 60H10, 60H05

1. Introduction. Symmetric integral and its properties.

In the stochastic analysis (see, for instance, [5]) two kinds of integrals are mainly employed, these are Itō integral and Stratonovic integral. The latter happened to be more susceptible to a generalization and its deterministic analogue is a symmetric integral. Symmetric integrals were introduced by the third author, and the systematic exposition of the theory of symmetric integrals and some results on the theory of equations with symmetric integrals were provided in monograph [8].

The main aim of these studies is to move to the language of the theory of functions the part of the stochastic analysis which can constructed with involving the notion of symmetric integral. Under such approach the integration can be made along arbitrary continuous function (realization of a stochastic process), and the integrands are not necessary to be predictible.

In what follows we provide the definition and some properties of symmetric integral (for more detail cf. [6], [8]).

Let $X(s)$, $s \in [0, +\infty)$, be an an arbitrary continuous function, then as a *symmetric integral* we call

$$\int_0^t f(s, X(s)) * dX(s) = \lim_{n\to\infty} \int_0^t f(s, X^{(n)}(s))(X^{(n)})'(s)ds, \tag{1}$$

where $X^{(n)}(s)$ is the polyline constructed by the partition $\{t_k^{(n)}\}$ of the segment $[0, t]$ and the function $X(s)$, at that, $\max\limits_k(t_k^{(n)} - t_{k-1}^{(n)}) \to 0$ as $n \to \infty$.

We shall say that condition (S) holds true for a pair of functions $(X(s), f(s, u))$ if the following assumptions are satisfied:

(a) $X(s)$, $s \in [0, t]$, is a continuous function;

(b) For a.e. u the function $f(s, u)$, $s \in [0, t]$, is right continuous and has a bounded variation;

(c) The total variation $|f|(t, u)$ w.r.t. variable s of the function $f(s, u)$ on $[0, t]$ is locally summable w.r.t. variable u;

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(d) For a.e. u we have $\int_0^t \mathbf{1}(s : X(s) = u)|f|(ds, u) = 0$.

It was shown in [6], [8] that if functions $(X(s), f(s,u))$ satisfy condition (S), then symmetric integral $\int_0^t f(s, X(s) * dX(s)$ exists.

Important properties of symmetric integral.

1. Suppose that $(X(s), f(s,u))$ satisfy condition (S), then

$$\int_0^t f(s, X(s)) * dX(s) = \int_{X(0)}^{X(t)} f(t,u)du - \int_R \int_0^t \kappa(u, X(0), X(s)) f(ds, u)du, \tag{2}$$

where $\kappa(u,a,b) = sign(b-a)\mathbf{1}(a \wedge b < u < a \vee b)$.

2. For symmetric integral, the differential corresponding to the stochastic differential with Stratonovich integral holds true.
Let a function $F(t,u)$ has continuous partial derivatives F'_t and F'_u, then there exists the symmetric integral $\int_0^t F'_u(s, X(s)) * dX(s)$ and the formula

$$F(t, X(t)) - F(0, X(0)) = \int_0^t F'_u(s, X(s)) * dX(s) + \int_0^t F'_s(s, X(s))ds \tag{3}$$

holds true.

3. Let $Y(s)$, $s \in R^+$, be a continuous function and let $X(s) = g(s, Y(s))$, where function $g(s,y)$, $s \in R^+$, $y \in R$ and its partial derivatives $g'_s(s,y)$ and $g'_y(s,y)$ are jointly continuous. Assume the conditions:

- Functions $f(s,u)$, $f'_s(s,u)$, $\phi(s,y)$, $\phi'_s(s,y)$, where $\phi(s,y)=f(s,g(s,y))g'_y(s,y)$, are jointly continuous.
- Pairs of functions $(X(s), f(s,u))$ and $(Y(s), \phi(s,y))$ satisfy condition (S) on $[0,t]$.

Then the formula for the change of variables in the symmetric integral

$$\int_0^t f(s, X(s)) * dX(s) = \int_0^t f(s, g(s, Y(s)))g'_y(s, Y(s)) * dY(s) +$$
$$+ \int_0^t f(s, g(s, Y(s)))g'_s(s, Y(s))ds.$$

holds true.

Let $W(s) = W(s, \omega)$ be a standard Wiener process, then in the framework of Itō formula for almost all trajectories of the Wiener process the pathwise symmetric integral $\int_0^t f(s, W(s)) * dW(s)$ coincides with the stochastic Stratonovich integral $\int_0^t f(s, W(s)) * dW(s)$. This is why we shall employ the same notation for both kinds of integrals.

2. On existence and uniqueness of solutions to ordinary differential equations with symmetric integrals

A new type of integral posed the problem on moving a part of the results in the theory of stochastic differential equations (SDE) to the deterministic language. Although the class of integrands for the symmetric integral is rather narrow, it happened to be sufficient for constructing a meaningful theory of deterministic analogues of SDE.

We consider the Cauchy problem for the equation with the symmetric integral

$$d\xi_t = \sigma(t, X(t), \xi_t) * dX(t) + b(t, X(t), \xi_t)dt, \quad \xi_0 = \xi(0), \quad t \in [0, t_0], \tag{4}$$

where $X(t)$, $t \in [0, t_0]$, is a continuous function.

A *solution* to Cauchy problem (4) is a function $\xi_t = \varphi(t, X(t))$ possessing continuous derivatives $\varphi'_t(t,v)$, $\varphi'_v(t,v)$ such that the differential with the symmetric integral of this function coincide with the right hand side of equation in (4).

Theorem 1. *Let $X(t)$, $t \in [0, t_0]$, be a continuous function and the coefficients of the equation $\sigma(t, v, \phi)$ and $b(t, v, \phi)$ are jointly continuous. Suppose that a function $\varphi(t, v)$, $\varphi(0, X(0)) = \xi(0)$ possesses continuous derivatives $\varphi'_t(t, v)$, $\varphi'_v(t, v)$, and for a.e. $t \in [0, t_0]$ satisfies the system of equations*

$$\varphi'_t(t, X(t)) = b(t, X(t), \varphi(t, X(t))), \tag{5}$$

$$\varphi'_v(t, X(t)) = \sigma(t, X(t), \varphi(t, X(t))). \tag{6}$$

Then for each $t \in [0, t_0]$ there exists the symmetric integral

$$\int_0^t \sigma(s, X(s), \varphi(s, X(s))) * dX(s)$$

and $\xi_t = \varphi(t, X(t))$ is a solution to the equation in (4).

Proof. Let a function $\varphi(t, v)$, $\varphi(0, X(0)) = \xi(0)$, possessing continuous derivatives $\varphi'_t(t, v)$, $\varphi'_v(t, v)$ satisfies conditions (5) and (6) for a.e. $t \in [0, t_0]$. We note that due to the continuity of the expressions in the both sides of identities (5) and (6), the validity of identities (5) and (6) for a.e. $t \in [0, t_0]$ is equivalent to its validity for each $t \in [0, t_0]$. Then in view of the formulae for the differential with a symmetric integral we have

$$\varphi(t, X(t)) - \xi(0) = \int_0^t \varphi'_v(s, X(s)) * dX(s) + \int_0^t \varphi'_s(s, X(s))\, ds =$$

$$= \int_0^t \sigma(s, X(s), \xi_s) * dX(s) + \int_0^t b(s, X(s), \xi_s)\, ds.$$

Therefore, function $\xi(t) = \varphi(t, X(t))$ solves Cauchy problem (4). $\square$

Theorem 2. *Let $X(t)$, $t \in [0, t_0]$, be a continuous almost nowhere differentiable function and the coefficients of equations $\sigma(t, v, \phi)$ and $b(t, v, \phi)$ satisfy the conditions*

(a) Function $\sigma(t, v, \phi)$ is jointly continuous and has continuous partial derivatives $\sigma'_t(t, v, \phi)$ and $\sigma'_\phi(t, v, \phi)$;

(b) Function $b(t, v, \phi)$ is jointly continuous.

Then the following conditions are equivalent

1. Cauchy problem (4) has the solution $\xi_t = \varphi(t, X(t))$ with function $\varphi(t, v)$;

2. Function $\varphi(t, v)$ possessing continuous derivatives $\varphi'_t(t, v)$, $\varphi'_v(t, v)$, $\varphi''_{tv}(t, v)$ and being such that $\varphi(0, X(0)) = \xi(0)$ for a.e. $t \in [0, t_0]$ satisfies conditions (5) and (6).

Proof. Let $\xi_t = \varphi(t, X(t))$ be a solution to the equation in (4), then due to the formulae for the differential and for calculating the symmetric integral for each $t \in [0, t_0]$ we have

$$\varphi(t, X(t)) - \varphi(0, X(0)) = \int_{X(0)}^{X(t)} \varphi'_v(t, v)dv + \int_0^t \varphi'_s(s, X(0))\, ds =$$

$$= \int_{X(0)}^{X(t)} \sigma(t, v, \varphi(t, v))dv + \int_0^t \left[b(s, X(s), \varphi(s, X(s))) - \int_{X(0)}^{X(s)} (\sigma(s, v, \varphi(s, v))'_s dv \right] ds.$$

We denote

$$\Phi(t, v) = \varphi'_v(t, v) - \sigma(t, v, \varphi(t, v)),$$

$$g(t, X(t)) = b(t, X(t), \varphi(t, X(t))) - \varphi'_t(t, X(0)) - \int_{X(0)}^{X(t)} (\sigma(t, v, \varphi(t, v))'_t dv,$$

then the identity obtained above can be written as

$$\int_{X(0)}^{X(t)} \Phi(t, v)dv = \int_0^t g(s, X(s))ds. \tag{7}$$

We note that the right hand side in identity (7) is continuously differentiable w.r.t. t as the integral with the variable upper limit, hence, the same true for the left hand side. Differentiating then both sides of identity (7) w.r.t. variable t, we get

$$g(t, X(t)) - \int_{X(0)}^{X(t)} \Phi_t'(t, v)dv = \frac{d}{dt}\left(\int_{X(0)}^{X(t)} \Phi(p, v)dv\right)\Bigg|_{p=t}, \quad t \in [0, t_0]. \tag{8}$$

We let

$$a_t(h) = \frac{1}{h}\int_{X(t)}^{X(t+h)} \Phi(t, v)dv, \quad r_t(h) = \frac{1}{X(t+h) - X(t)}\int_{X(t)}^{X(t+h)} \Phi(t, v)dv,$$

if $X(t+h) \neq X(t)$, $r_t(h) = \Phi(t, X(t))$ in the case $X(t+h) = X(t)$. By formula (8), for each t there exists a finite limit $a(t) = \lim\limits_{h\to 0} a_t(h)$, and the limit $r(t) = \lim\limits_{h\to 0} r_t(h)$ exists and is finite due to the continuity of function $\Phi(t, v)$ w.r.t. variable v, and at that, for each $h \neq 0$ the identity

$$\frac{a_t(h)}{r_t(h)} = \frac{X(t+h) - X(t)}{h} \tag{9}$$

is valid. But function $X(t)$ is almost nowhere differentiable and hence, for a.e. t the limit in the left hand side of (9) as $h \to 0$ is either infinite or does not exist. The latter is possible only in the case when either $a(t) \neq 0$, $r(t) = 0$ or $a(t) = r(t) = 0$, i.e., in any case $r(t) = 0$. Thus, for each $t \in [0, t_0]$ relation (6) holds true. Then, in view of (6), formula

$$\varphi_v'(t, X(t)) * dX(t) + \varphi_t'(t, X(t))dt = \sigma(t, X(t), \xi_t) * dX(t) + b(t, X(t), \xi_t)dt$$

implies the validity of condition (5). □

It seems that there are no well-developed methods for solving system of equations (5), (6), which is an unusual system of equations along a trajectory of function $X(s)$. This is why it is reasonable to solve the following chain of equations instead of system (5), (6):

$$\varphi_v'(t, v) = \sigma(t, v, \varphi(t, v)), \ \ \varphi_t'(t, X(t)) = b(t, X(t), \varphi(t, X(t))), \ \ \varphi(0, X(0)) = \xi(0). \tag{10}$$

In this case we can employ the methods of the theory of ordinary differential equations (ODE). Indeed, if it exists, the general solution to the first equation in (10) depends on an arbitrary function $C(t)$: $\varphi(t, v) = \varphi^*(t, v, C(t))$. Here $\varphi^*(t, v, C)$ is a known function and by the theorem on differentiability w.r.t. the parameter of solution to ODE function $C(t)$ is smooth. Substituting the found function $\varphi^*(t, v, C(t))$ into the second equation in (10), we arrive at the Cauchy problem for unknown function $C(t)$

$$(\varphi^*)_t'(t, X(t), C(t)) + (\varphi^*)_C'(t, X(t), C(t))C'(t) = b(t, X(t), \varphi^*(t, X(t), C(t))),$$

$\varphi^*(0, X(0), C(0)) = \xi(0)$. It is clear that if it exists, a solution to chain (10) gives a solution to system (5), (6). It means that the solvability of equation (1) can be reduced to the issue on solvability conditions for ODE in chain (10).

Let us consider in more details the Cauchy problem for a simpler equation with symmetric integral

$$d\xi_t = \sigma(t, \xi_t) * dX(t) + b(t, \xi_t)dt, \tag{11}$$

where $X(t)$ is a continuous function.

Suppose that in the domain $G = \{(s, \phi)\} \subseteq R^2$ the following conditions hold true:

- There exists a constant σ_0 such that $|\sigma(t, \phi)| \geq \sigma_0 > 0$.
- Function $\sigma(t, \phi)$ is continuously differentiable w.r.t. both variables.
- Function $b(t, \phi)$ is continuous and satisfies local Lipshitz condition w.r.t. ϕ, i.e., for each point $(t_0, \phi_0) \in G$ there exists a neighborhood U of this point such that $|b(t, \phi_1) - b(t, \phi_2)| \leq L|\phi_1 - \phi_2|$ for all $(t, \phi_k) \in U$, $k = 1, 2$, where L is independent of the points in U.

We note that the first relation in (10) leads us immediately to the identity

$$\Phi(t,\phi,v)\equiv\int\frac{d\phi}{\sigma(t,\phi)}-v=0.$$

Since by our assumptions for $(t,\phi)\in G$, $v\in R$ there exist continuous derivatives $\Phi'_t(t,\phi,v)$, $\Phi'_v(t,\phi,v)$ and the non-zero derivative $\Phi'_\phi(t,\phi,v)$, by the implicit function theorem (see [4]) there exists (locally) a function $\phi=\phi^*(t,v)$ being continuously differentiable w.r.t. t and v. For each point $(t_0,\phi_0)\in G$, by $J_0(t_0,\phi_0)$ we denote the maximal interval where there exists the implicit function $\phi=\phi(t,v,t_0,\phi_0)$ with initial conditions $\phi(t,X(t_0))=\phi_0$.

The second equation in (10) can be written as

$$C'(t)=\frac{b(t,\phi(t,X(t)+C(t)))-(\phi)'_t(t,v)|_{v=X(t)+C(t)}}{\sigma(t,\phi(t,X(t)+C(t)))}. \tag{12}$$

Let $C(t_0)$ be determined by the identity $\phi(t_0,X(t_0)+C(t_0))=\phi_0$. We observe that the right hand side of equation (11) is continuous in G and in view of the fact that $\sigma(t,\phi)$ is continuously differentiable on G and $\phi(t,v)$ is continuously differentiable w.r.t. both variables, formula $(\phi)'_v(t,v)$=$\sigma(t,v,\phi(t,v))$ implies the continuous differentiability of the function $(\phi)'_t(t,v)|_{v=X(t)+C}$ w.r.t. variable C. Therefore, by Theorem 2.3.2 in [4], $G^{(1)}=\{(t,C):(t,\phi(t,X(t)+C))\in G\}$ is the uniqueness domain for equation (11) and $C(t)=C(t,t_0,C_0)$ is a solution to equation (11) with initial condition (t_0,C_0) defined on the set $D=\{(t,t_0,C_0):(t_0,C_0)\in G^{(1)},t\in J_1(t_0,C_0)\}$, where $J_1(t_0,C_0)$ is the maximal interval of existence for the solution to equation (11). This is why for each initial condition $(t_0,\phi_0)\in G$ there exists the solution $\xi_t=\phi^*(t,X(t)+C(t))$ on the interval $J(t_0,\phi_0)=J_0(t_0,\phi_0)\cap J_1(t_0,C_0)$.

Example 1. Consider the linear Itō equation with constant coefficients

$$\xi_t-\xi_0=\int_0^t[a\xi_s+b]dW(s)+\int_0^t[e\xi_s+f]ds.$$

Here the first term in the right hand side is the stochastic Itō integral. Passing to the corresponding equation with stochastic Stratonovich integral, we obtain

$$\xi_t-\xi_0=\int_0^t[a\xi_s+b]*dW(s)+\int_0^t[h\xi_s+g]ds, \tag{13}$$

where $h=e-a^2/2$, $g=f-ab/2$. We seek a solution to (13) as $\xi_t=\phi(t,W(t))$. We compose two equations

$$\phi'_u(t,u)=a\phi(t,u)+b, \tag{14}$$

$$\phi'_t(t,u)|_{u=W(t)}=h\phi(t,W(t))+g,\quad \phi(0,W(0))=\xi_0. \tag{15}$$

Solution to equation (14) reads as $\ln(a\phi+b)=u+C(t)$ or

$$\phi(t,u)=\frac{1}{a}\left(e^{u+C(t)}-b\right). \tag{16}$$

Substituting the obtained expression into equation (12), we arrive at the Cauchy problem for unknown function $C(t)$:

$$\frac{1}{a}e^{W(t)+C(t)}C'(t)=\frac{h}{a}\left(e^{W(t)+C(t)}-b\right)+g,\ \ \frac{1}{a}\left(e^{W(0)+C(0))}-b\right)=\xi_0.$$

By the change $z(t)=e^{C(t)}$ the obtained ODE is reduced to the linear non-homogeneous ODE with constant coefficients

$$z'(t)-hz(t)=e^{-W(t)}(ag-bh),$$

whose solution is

$$z(t)=\left((ag-bh)\int_0^t e^{-W(s)}e^{-hs}ds+C^*\right)e^{hs}.$$

In order to construct solution to equation (13), it remains to find constant C^* by means of initial condition and substitute it into the latter formula.

Let us prove a generalization of Grönwall's lemma.

Lemma 1. *Let $\delta(s,v)$, $B(s,v)$, $s\in[a,t_0]$, $v\in R$, and $X(s)$, $s\in[a,t_0]$, be continuous functions, $C(s)$, $s\in[a,t_0]$, be a continuously differentiable function. Suppose that the pair of functions $(\delta(s,v)B(s,v),X(s))$ satisfies condition (S) on $[a,t_0]$ and the identity*

$$\delta(t,X(t))=C(t)+\int_a^t\delta(s,X(s))B(s,X(s))*dX(s),\quad t\in[a,t_0],\tag{17}$$

holds true. If $\delta(s,X(s))\neq 0$ as $s\in(a,t_0]$ and the integrals

$$\int_a^t(\delta(s,X(s)))^{-1}C'(s)ds,\qquad \int_a^t B(s,X(s))*dX(s),\quad t\in(a,t_0),$$

are finite, then for the same t

$$|\delta(t,X(t))|=|\delta(a,X(a))|\ \exp\left\{\int_a^t(\delta(s,X(s)))^{-1}C'(s)ds+\int_a^t B(s,X(s))*dX(s)\right\}.$$

Proof. Let $a<\varepsilon<t\le t_0$, then by the formula for the differential and relation (17) we have

$$d\left(\ln|\delta(s,X(s))|\right)=\frac{d\delta(s,X(s))}{\delta(s,X(s))}=(\delta(s,X(s)))^{-1}C'(s)ds+B(s,X(s))*dX(s).$$

Therefore,

$$|\delta(t,X(t))|=|\delta(\varepsilon,X(\varepsilon))|\ \exp\left\{\int_\varepsilon^t(\delta(s,X(s)))^{-1}C'(s)ds+\int_\varepsilon^t B(s,X(s))*dX(s)\right\}.$$

Passing in the latter expression to the limit as $\varepsilon\to a$, we obtain formula (17). □

Basing on Lemma 1, we can prove the uniqueness theorem for solutions to ordinary differential equations with symmetric integral.

Theorem 3. *Suppose that function $\sigma(t,v,\phi)$ is jointly continuous and has continuous partial derivatives $\sigma'_t(t,v,\phi)$ and $\sigma'_\phi(t,v,\phi)$, and a function $b(t,v,\phi)$ and its derivative $b'_\phi(t,v,\phi)$ are jointly continuous. If there exists a solution to Cauchy problem (4), then it is unique.*

Proof. Let $\xi_t^{(1)}=\varphi^{(1)}(t,X(t))$, $\xi_t^{(2)}=\varphi^{(2)}(t,X(t))$ be two solutions to Cauchy problem (4). Let us show that then $\xi_t^{(1)}=\xi_t^{(2)}$ for all $t\in[0,t_0]$. We let $\delta(t,X(t))=\varphi^{(1)}(t,X(t))-\varphi^{(2)}(t,X(t))$, and by the initial condition $\delta(t,X(t))=0$ as $t=0$. By the continuity of functions $\xi_t^{(1)}$, $\xi_t^{(2)}$ the set $\{t\in[0,t_0]:\xi_t^{(1)}=\xi_t^{(2)}\}$ is closed. Suppose that the function $\delta(t,X(t))$ is non-zero on some non-empty set L_0, then L_0 is open and thus it can be represented as the union of at most finitely many intervals $L_0=\cup_k(a_k,b_k)$. We fix an interval (a_k,b_k), and in what follows we shall denote it as (a,b). For $s\in(a,b)$ we let

$$g_1(s,X(s))=\frac{\sigma(s,X(s),\xi_s^{(1)})-\sigma(s,X(s),\xi_s^{(2)})}{\xi_s^{(1)}-\xi_s^{(2)}},$$

$$g_2(s,X(s))=\frac{b(s,X(s),\xi_s^{(1)})-b(s,X(s),\xi_s^{(2)})}{\xi_s^{(1)}-\xi_s^{(2)}}.$$

We adopt the notations of Lemma 1:

$$C(t)=\int_a^t\left[b(s,\varphi^{(1)}(s,X(s)))-b(s,\varphi^{(2)}(s,X(s)))\right]ds,$$

$$B(t,u)=\sigma(t,\varphi^{(1)}(t,u))-\sigma(t,\varphi^{(2)}(t,u)).$$

Since the pair of functions $(g_1(s,v), X(s))$ possesses property (S), and $g_2(s, X(s))$ is summable on each segment $[a, a_1] \subset [a, b)$, then the hypothesis of Lemma 1 is satisfied and therefore relation (17) holds true on set $t \in [a, b)$. But $\delta(a, X(a)) = 0$, and thus $\xi_t^{(1)} = \xi_t^{(2)}$ for each $t \in [a, b)$ and $L_0 = \emptyset$. Therefore, Theorem 3 holds true. $\square$

Remark. It is well-known that in the stochastic analysis there is a notion of weak solution and a strong solution is a weak one. Let us show that in the pathwise analysis function $X(t)$ can be found by means of solution ξ_t to equation (1), namely, the following identity holds true:

$$X(t) - X(0) = \int_0^t \frac{1}{\sigma(s, X(s), \xi_s)} * d\xi_s - \int_0^t \frac{b(s, X(s), \xi_s)}{\sigma(s, X(s), \xi_s)} ds. \tag{18}$$

Indeed, in view of the formula of change the variables in symmetric integrals and equation in (4), the right hand side of relation (18) is equal to

$$\int_0^t \frac{1}{\sigma(s, X(s), \xi_s)} [\sigma(s, X(s), \xi_s) * dX(s) + b(s, X(s), \xi_s) ds] - \int_0^t \frac{b(s, X(s), \xi_s)}{\sigma(s, X(s), \xi_s)} ds.$$

3. On structure of solution to equations with symmetric integral

It was shown in the previous section that solution to equation (11) reads as $\xi(t) = \phi(t, X(t) + C(t))$. This fact can be rather useful while studying SDE. Suppose, for instance, that function $\sigma(t, \phi)$ and $b(t, \phi)$ in equation (11) are deterministic and $X(t) \equiv W(t)$ is a trajectory of a Wiener process. Then all the probabilistic information on solution to SDE is contained in $W(t) + C(t)$ since the diffusion process determined as a solution to equation (11) is a deterministic function of the sum of the Wiener process and a random smooth drift. The structure of solution to equation (11) was first found in works [8], [7] for the case $\sigma(s, \phi, u) = \sigma(s, \phi) \neq 0$.

The aim of the present section is to find the structure of a solution to equation (4) in more general situation since in many cases the knowledge of the structure allows one to simplify essentially the studies of both the equations with symmetric integrals and SDE. The methods of group analysis happened to be effective for solving this problem.

The group analysis is one of the methods by means of which it is possible to know a lot of studied differential equation. The techniques of group analysis are well-developed and are used for studying both ordinary and partial differential equations. It was expounded rather in detail in works [1], [2].

Let G be an one-parametric group $\bar{u} = f(u, \phi, a)$, $\bar{\phi} = g(u, \phi, a)$ with the infinitesimal operator $X = \xi(u, \phi)\frac{\partial}{\partial u} + \eta(u, \phi)\frac{\partial}{\partial \phi}$. We shall say the the first order differential equation $\frac{d\phi}{du} = \sigma(\phi, u)$ admits group G, if

$$\frac{d\bar{\phi}}{d\bar{u}} = \sigma(\bar{\phi}, \bar{u}).$$

As a rule, it is impossible to find the group admitted by a first order differential equation, but there exist equations whose admitted group is known. Each group G with the operator $X = \xi(u, \phi)\frac{\partial}{\partial u} + \eta(u, \phi)\frac{\partial}{\partial \phi}$ can be reduced to the group of shifts along one of the axis by an appropriate change of variables. Canonical variables $\hat{\phi} = \hat{\phi}(\phi, u)$, $\hat{u} = \hat{u}(\phi, u)$ are determined by the relations (cf. [1])

$$\xi(u, \phi)\frac{\partial \hat{\phi}}{\partial u} + \eta(u, \phi)\frac{\partial \hat{\phi}}{\partial \phi} = 0, \quad \xi(u, \phi)\frac{\partial \hat{u}}{\partial u} + \eta(u, \phi)\frac{\partial \hat{u}}{\partial \phi} = 1. \tag{19}$$

At that, if group G is admissible, by passing to the canonical variables one reduces this equation to the form where one of these variables is missing and the equation can be solved by quadrature.

Suppose the conditions of the unique solvability for equation in (4) hold true and the coefficients of the equation are smooth enough, say, possess third continuous derivatives. We note that in accordance with Theorem 1 solving of equation in (4) can be reduced to solving of a chain of two ordinary differential equations (10).

Consider the first equation in the chain

$$\phi'_u(s,u) = \sigma(s,u,\phi(s,u)). \tag{20}$$

This is a first order ordinary differential equation where s is a parameter. In a general case, if a solution to (20) exists, then the structure of the solution to (20) is $\Psi(\phi(s,u),u,C(s)) = 0$, where $C(s)$ is an arbitrary function. In order to find function $C(s)$, one should express $\phi(s,u) = \phi^*(s,u,C(s))$ from this identity and substitute the result into the second equation in chain (10). Finally we get an ordinary differential equation for unknown function $C(s)$. Initial conditions $\phi(0,X(0)) = \eta(0)$ pass to the initial condition $\phi^*(0,X(0),C(0)) = \xi(0)$ for $C(s)$.

We note that we aim to determine the structure of equation in (4), and the latter, as we see from the above arguments, is determined completely by equation (20). This is why we shall consider particular equations (20) with known admissible groups, we shall integrate them finding in this way the structure of solutions to the corresponding equations with symmetric integral.

In what follows we provide some examples of constructing the structure of solutions. A more detailed set of possible structures of solutions to equation in (4) is given in Table 1.

. Consider equation

$$\xi(t) - \xi(0) = \int_0^t \sigma(s,\xi(s)) * dX(s) + \int_0^t b(s,\xi(s))ds,\ t \in [0,T]. \tag{21}$$

The first equation in chain (10) can be integrated in this case

$$\phi'_u(s,u) = \sigma(s,\phi(s,u)), \qquad \phi(s,u) = \Psi(s,u+C(s)), \tag{22}$$

where Ψ is found by (22), and $C(t)$ can be determined as a solution to a differential equation if we substitute function $\Psi(s,X(s)+C(s))$ into the second equation in chain (10). We observe that the infinitesimal operator of the group admitted by equation (22) reads as $X = \frac{\partial}{\partial u}$. Thus, $\xi(s) = \Psi(s,X(s)+C(t))$ is a solution to SDE.

B. Let

$$\sigma(s,u,\phi) = F(s,ku+l\phi), \tag{23}$$

where k and l are known constants. The infinitesimal operator of the transformations group admitted by equation (23) is $X{=}l\frac{\partial}{\partial u} - k\frac{\partial}{\partial\phi}$. Let us find the canonical variables

$$l\frac{\partial\hat\phi}{\partial u} - k\frac{\partial\hat\phi}{\partial\phi} = 0,\ \ l\frac{\partial\hat u}{\partial u} - k\frac{\partial\hat u}{\partial\phi} = 1.$$

Solving these equation, we shall get, for instance, $\hat\phi = \frac{u}{l} + \frac{\phi}{k}, \quad \hat u = \frac{u}{l}$. The inverse change $\phi = k(\hat\phi - \hat u), \quad u = l\hat u$. Passing to new variables, we have

$$\hat\phi'_{\hat u} = \frac{lF(lk\hat\phi)}{k} + 1,\ \hat\phi = \Psi(s,\hat u + C(s)).$$

$$\text{Structure of solution: } \eta(s){=}k\left(\Psi\left(s,\frac{X(s)}{l}{+}C(s)\right) - \frac{X(s)}{l}\right).$$

C. Consider the case

$$\sigma(s,u,\phi) = F\left(s,\frac{\phi}{u}\right),\ s \in [t_1,t_2],\ t_1 > 0. \tag{24}$$

Since in this case equation (20) is a homogeneous type, it admits the group with the infinitesimal operator $X = u\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial \phi}$. Equations (19) for finding canonical variables are

$$\phi\frac{\partial\hat{\phi}}{\partial u} + u\frac{\partial\hat{\phi}}{\partial\phi} = 0, \quad \phi\frac{\partial\hat{u}}{\partial u} + u\frac{\partial\hat{u}}{\partial\phi} = 1.$$

Solutions to these equations can be represented as $\hat{\phi} = \frac{\phi}{u}$, $\hat{u} = \ln u$. The inverse change is $\phi = \hat{\phi}e^{\hat{u}}$, $u = e^{\hat{u}}$. Passing to new variables, we have

$$\hat{\phi}'_{\hat{u}} = (F(s,\hat{\phi}) - \hat{\phi}), \qquad \hat{\phi} = \Psi(s, \hat{u} + C(s)).$$

The structure of solution read as $\xi(s) = X(s)\Psi\left(s, \ln|X(s)| + C(s)\right)$.

D. Consider the following example

$$\sigma(s,u,\phi) = \frac{\phi}{u + F(s,\phi)}. \tag{25}$$

The infinitesimal operator of the group admitted by equation (25) is known: $X = \phi\frac{\partial}{\partial u}$. Equations (19) determining canonical variables are $\phi\frac{\partial\hat{\phi}}{\partial u} = 0$, $\phi\frac{\partial\hat{u}}{\partial u} = 1$. Solutions to these equations can be represented as $\hat{\phi} = \phi$, $\hat{u} = \frac{u}{\phi}$. The inverse change is $\phi = \hat{\phi}$, $u = \hat{\phi}\hat{u}$. Passing to new variables, we have $\hat{u}'_{\hat{\phi}} = \frac{F(s,\hat{\phi})}{\hat{\phi}^2}$, $\hat{\phi} = \Psi(s, \hat{u} + C(s))$. We obtain the structure of solutions in the form $\phi - \Psi\left(s, \frac{X(s)}{\phi} + C(s)\right) = 0$.

TABLE 1. Structure of solutions to some equations of the form $\xi(t) - \xi(0) = \int\limits_0^t \sigma(s,\xi(s),X(s)) * dX(s) + \int\limits_0^t b(s,\xi(s),X(s))ds$.

Equation $\phi_u = \sigma(s,u,\phi)$	Operator of the group	Structure of solution $\eta(t)$
1. $\sigma = F(s,\phi)$	$\frac{\partial}{\partial u}$	$\Psi(s, X(s) + C(s))$
2. $\sigma = F(s,u)$	$\frac{\partial}{\partial \phi}$	$\Psi(s, X(s)) + C(t)$
3. $\sigma = F(s, ku + l\phi)$, k, l – const	$l\frac{\partial}{\partial u} - k\frac{\partial}{\partial\phi}$	$k(\Psi(s, \frac{X(s)}{l} + C(s)) - \frac{X(s)}{l})$
4. $\sigma = F(s, \frac{\phi}{u})$, $u \neq 0$	$u\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial\phi}$	$X(s) \cdot \Psi(s, \ln|X(s)| + C(s))$
5. $\sigma = u^k F(s, \frac{\phi}{u^k})$, $u \neq 0$	$u\frac{\partial}{\partial u} + k\phi\frac{\partial}{\partial\phi}$	$(X(s))^k\Psi(s, \ln|X(s)| + C(s))$
6. $u\sigma = F(s, ue^{-\phi})$, $u \neq 0$	$u\frac{\partial}{\partial u} + \frac{\partial}{\partial\phi}$	$\ln|X(s)| - \Psi(s, \ln|X(s)| + C(s))$
7. $\sigma = \phi F(s, \phi e^{-u})$	$\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial\phi}$	$e^{X(s)}\Psi(s, X(s) + C(s))$
8. $\sigma = \frac{\phi}{u} + uF(s, \frac{\phi}{u})$, $u \neq 0$	$\frac{\partial}{\partial u} + \frac{\phi}{u}\frac{\partial}{\partial\phi}$	$X(s)\Psi(s, X(s) + C(s))$
9. $u\sigma = \phi + uF(s, \frac{\phi}{u})$, $u \neq 0$	$u^2\frac{\partial}{\partial u} + u\phi\frac{\partial}{\partial\phi}$	$X(s)\Psi(s, -\frac{1}{X(s)} + C(s))$
10. $\sigma = \frac{\phi}{u + F(s,\phi)}$	$\phi\frac{\partial}{\partial u}$	$\phi - \Psi(s, \frac{X(s)}{\phi} + C(s)) = 0$
11. $u\sigma = \phi + F(s,u)$, $u \neq 0$	$u\frac{\partial}{\partial\phi}$	$X(s)(\Psi(s, X(s)) + C(s))$
12. $u\sigma = \frac{\phi}{\ln|u| + F(s,\phi)}$, $u \neq 0$	$\phi u\frac{\partial}{\partial u}$	$\phi - \Psi(s, \frac{\ln|X(s)|}{\phi} + C(s)) = 0$

* function Ψ has a certain form and is determined by equation (20)
** in cases 10 and 12 structure of solution Ψ is determined implicitly
*** function $C(s)$ can be found by the second equation in chain (12).

4. On continuous and differentiability w.r.t. parameter of equations with symmetric integrals

Consider the equation with symmetric integral

$$d\xi_t = \sigma(t,\mu,\xi_t) * dX(t) + b(t,\mu,\xi_t)dt. \tag{26}$$

Suppose that in the domain $G_\mu = \{(t,\mu,\phi)\}$ functions $\sigma(t,\mu,\phi)$ and $b(t,\mu,\phi)$ are defined and they satisfy the conditions

1. Functions $\sigma(t,\mu,\phi)$, $b(t,\mu,\phi)$, $\sigma'_t(t,\mu,\phi)$, $\sigma'_\phi(t,\mu,\phi)$ and $b'_\phi(t,\mu,\phi)$ are continuous.
2. There exists a positive number σ_0 such that $|\sigma(t,\mu,\phi)| \geq \sigma_0$.

Then by the arguments from Section 2, the domain $G = \{(t,\xi) : (t,\mu,\xi) \in G_\mu\}$ is the uniqueness domain for the solution to equation (26) with $\xi_t = \xi(t,t_0,\xi(0),\mu)$ and with the initial condition $(t_0,\xi(0))$ which is defined on the set D_μ=$\{(t,t_0,\xi(0),\mu)$:$(t_0,\xi(0),\mu) \in G_\mu\}$, $t \in J(t_0,\xi(0),\mu)$, where $J(t_0,\xi(0),\mu)$ is the maximal existence interval for solution to (26).

Theorem 4. *Under the above assumptions the function $\xi(t,t_0,\xi(0),\mu)$ is continuous in D_μ.*

Proof. Solution to equation (26) reads as

$$\xi_t = \varphi(t,\mu,X(t)+C(t,\mu)), \tag{27}$$

where the function $\varphi(t,\mu,v)$ is determined by the relation

$$\Phi(t,\mu,\phi,v) \equiv \int \frac{d\phi}{\sigma(t,\mu,\phi)} - v = 0. \tag{28}$$

By the hypothesis of the theorem, in the domain $G_\mu \times R$ the function $\Phi(t,\mu,\phi,v)$ is jointly continuous and has continuous derivatives $\Phi'_t(t,\mu,\phi,v)$, $\Phi'_v(t,\mu,\phi,v)$ and a non-zero derivative $\Phi'_\phi(t,\mu,\phi,v)$. This is why by the implicit function theorem there exists a function $\phi(t,\mu,v)$ with continuous partial derivatives $\phi'_t(t,\mu,v)$ and $\phi'_v(t,\mu,v)$. For each point $(t_0,\mu,\phi_0) \in G_\mu$ by $J^{(0)}(t_0,\mu,\phi_0)$ we denote the maximal interval in which the implicit function $\phi = \phi^*(t,\mu,v,t_0,\phi_0)$ with initial data $\phi^*(t_0) = \phi_0$ is defined.

Then function $C(t,\mu)$ in (27) is determined as the solution to the Cauchy problem

$$C'_t(t,\mu) = \frac{b(t,\mu,\phi(t,\mu,X(t)+C(t,\mu)))}{\sigma(t,\mu,\phi(t,\mu,X(t)+C(t,\mu)))} - \int_0^{\phi(t,\mu,X(t)+C(t,\mu))} \left(\frac{1}{\sigma(t,\mu,\psi)}\right)'_t d\psi, \tag{29}$$

$$\phi(t_0,\mu,X(t_0)+C(t_0,\mu)) = \xi_0.$$

Consider the set $G^{(1)}_\mu = \{(t,\mu,C) : (t,\mu,C,\phi(t,\mu,X(t)+C) \in G\mu)$, then by Theorem 2.3.2 in [4] the set $G^{(1)}$=$\{(t,C) : (t,C,\mu) \in G^{(1)}_\mu\}$ is the uniqueness domain for equation (29) for each fixed μ.

Let $C(t,t_0,C_0,\mu)$ be the solution to equation (29) with initial data $\phi(t_0,\mu,X(t_0)+C_0) = \xi_0$ defined on the set $D^{(1)}_\mu = \{(t,t_0,C_0,\mu) \in G^{(1)}_\mu,\ t \in J^{(1)}(t_0,C_0,\mu)\}$, where $J^{(1)}(t_0,C_0,\mu)$ is the maximal existence interval for solution. By Theorem 5.1.1 in [4], $D^{(1)}_\mu$ is the domain and $C(t,t_0,C_0,\mu)$ is continuous in $D^{(1)}_\mu$. Hence, solution (27) is continuous in D_μ, where $J(t_0,C_0,\mu) = J^{(0)}(t_0,\mu,\phi_0) \cap J^{(1)}(t_0,C_0,\mu)$. □

Theorem 5. *Assume the hypothesis of Theorem 4 and let the derivatives $\sigma'_\mu(t,\mu,\phi)$, $\sigma''_{t\mu}(t,\mu,\phi)$, $b'_\mu(t,\mu,\phi)$ be continuous in domain G_μ. Then solution $\xi_t = \phi(t,\mu,X(t)+C(t_0,\mu,\xi(0)))$ to Cauchy problem (26) has continuous derivatives $\eta^\mu_t = \frac{\partial}{\partial\mu}\xi_t$ and $\eta^{\xi(0)}_t = \frac{\partial}{\partial\xi(0)}\xi_t$ in the domain D_μ. At that the formula*

$$d\eta^\mu_t = \left[\frac{\partial}{\partial\mu}\sigma(t,\mu,\xi_t) + \frac{\partial}{\partial\phi}\sigma(t,\mu,\xi_t)\eta^\mu_t\right] * dX(t) + \left[\frac{\partial}{\partial\mu}b(t,\mu,\xi_t) + \frac{\partial}{\partial\phi}b(t,\mu,\xi_t)\eta^\mu_t\right] dt, \tag{30}$$

$$\frac{\partial}{\partial\mu}\eta_{t_0}^{\xi(0)}=\frac{\partial}{\partial\mu}\xi_{t_0},$$

hold true.

Proof. Here we follow the main lines of the proof of Theorem 4. The hypothesis of Theorem 5 implies that the solution to equation (26) reads as (27) and by the implicit function theorem $\phi(t,\mu,v)$ has continuous partial derivatives w.r.t. the variables (t,μ,v). For each point $(t_0,\mu,\phi_0)\in G_\mu$ by $J^{(0)}(t_0,\mu,\phi_0)$ we denote the maximal interval in which the implicit function $\phi=\phi^*(t,\mu,t_0,\phi_0)$ with initial data $\phi^*(t_0)=\phi_0$ exist.

Due to Theorem 5.2.1 in [4], the solution $C(t,t_0,C_0,\mu)$ in Theorem 4 has a continuous derivative w.r.t. variable μ. This is why the solution $\xi(t,t_0,\xi(0),\mu)$ is continuous and it has continuous derivatives $\eta_t^\mu=\frac{\partial}{\partial\mu}\xi_t$ and $\eta_t^{\xi(0)}=\frac{\partial}{\partial\xi(0)}\xi_t$ in D_μ. We employ the fact that by Theorem 2 the solution to equation (26) satisfies the relation like (5) and (6). Differentiating the obtained identities w.r.t. μ and applying then Theorem 2 once again for the resulting relation, we arrive at formula (30). □

5. Linearization of first order equations with symmetric integrals. Inverse transform.

1. Consider the equation with symmetric integral

$$d\xi_t=\sigma(t,X(t),\xi_t)*dX(t)+b(t,X(t),\xi_t)dt,\quad t\in[0,t_0],\tag{31}$$

where $X(t)$, $t\in[0,t_0]$, is a continuous almost nowhere differentiable function.

Our aim is to show how one can equation (31) into the linear equation

$$d\eta(t)=A(t)\eta(t)*dX(t)+B(t)\eta(t)dt\tag{32}$$

by an appropriate change of variables $\eta(t)=g(t,\xi_t)$ In what follow we assume that functions $\sigma(t,X,\xi)$, $b(t,X,\xi)$ satisfy the hypothesis of Theorem 1, $\sigma(t,X,\xi)\neq 0$ for each t,X,ξ, and functions $A(t)$, $B(t)$ are continuously differentiable.

By means of the formula for the differential of symmetric integral and relation (31) we have

$$dg(t,\xi_t)=[g_t'(t,\xi_t)+g_\xi'(t,\xi_t)b(t,X(t),\xi_t)]dt+[g_\xi'(t,\xi_t)\sigma(t,X(t),\xi_t)]*dX(t).$$

Comparing differentials $d\eta(t)$ calculated by the latter formula and formula (32), we arrive at the relation

$$0\equiv\left[g_t'(t,\xi_t)+g_\xi'(t,\xi_t)b(t,X(t),\xi_t)-B(t)g(t,\xi_t)\right]dt+$$
$$+\left[g_\xi'(t,\xi_t)\sigma(t,X(t),\xi_t)-A(t)g(t,\xi_t)\right]*dX(t).$$

By the arguments similar to those in Theorem 1, we obtain

$$g_\xi'(t,\xi_t)=\left[\frac{A(t)}{\sigma(t,X(t),\xi_t)}g(t,\xi_t)\right],$$

$$g_t'(t,\xi_t)=B(t)g(t,\xi_t)-g_\xi'(t,\xi_t)b(t,X(t),\xi_t)=\left[B(t)-\frac{A(t)b(t,X(t),\xi_t)}{\sigma(t,X(t),\xi_t)}\right]g(t,\xi_t).$$

Then we can employ formula (31), but in this case we obtain the solution to the equation in the form $\eta_t=\tilde{\eta}_t(t,X(t))$, while we need $\eta_t=\eta_t^*(t,\xi_t)$. This is why in the general case we need to express $X(t)$ in terms of ξ_t. It can be done by solving equation (31): $\xi_t=\varphi(s,X(s))$, and expressing then $X(t)$ in terms of ξ_t.

In any case we arrive at the case $\sigma(t,X(t),\xi_t)=\sigma(t,\xi_t)$, and by Theorem 1 the function $\eta_t=g(t,\xi_t)$ solves the linear equation with symmetric integral

$$d\eta_t=\left[\frac{A(t)}{\sigma(t,\xi_t)}\right]\eta_t*d\xi_t+\left[B(t)-\frac{A(t)b(t,X(t),\xi_t)}{\sigma(t,\xi_t)}\right]\eta_t dt.\tag{33}$$

We seek the solution to equation (33) as $\eta_t = g(t, \xi_t)$ and obtain the chain of two equations

$$g'_\xi(t,\xi) = \frac{A(t)}{\sigma(t,\xi)} g(t,\xi), \quad g'_t(t,\xi)|_{\xi=\xi_t} = \left[B(t) - \frac{A(t)b(t,X(t),\xi_t)}{\sigma(t,\xi_t)}\right] g(t,\xi_t).$$

The first equation determines $g(t,\xi)$ up to unknown function $C(t)$,

$$g(t,\xi) = C(t)\exp\left(A(t)\int \frac{d\xi}{\sigma(t,\xi)}\right). \tag{34}$$

By differentiating we get

$$g'_t(t,\xi) = \exp\left(A(t)\int \frac{d\xi}{\sigma(t,\xi)}\right)\left\{C'(t) - C(t)\left[A'(t)\int \frac{d\xi}{\sigma(t,\xi)} - A(t)\int \frac{\sigma'_t(t,\xi)}{\sigma^2(t,\xi)}d\xi\right]\right\}.$$

Substituting the determined expression (34) for $g(t,\xi)$ into the second equation, we obtain equation for unknown $C(t)$:

$$C'(t) = C(t)\left[B(t) - \frac{A(t)b(t,X(t),\xi_t)}{\sigma(t,\xi_t)} + A'(t)\int_{\xi_0}^{\xi_t} \frac{d\xi}{\sigma(t,\xi)} - A(t)\int_{\xi_0}^{\xi_t} \frac{\sigma'_t(t,\xi)}{\sigma^2(t,\xi)}d\xi\right].$$

Therefore,

$$\frac{C'(t)}{C(t)} = B(t) - \frac{A(t)b(t,X(t),\xi_t)}{\sigma(t,\xi_t)} - A'(t)\int_{\xi_0}^{\xi_t} \frac{d\xi}{\sigma(t,\xi)} - A(t)\int_{\xi_0}^{\xi_t} \frac{\sigma'_t(t,\xi)}{\sigma^2(t,\xi)}d\xi. \tag{35}$$

or

$$C(t) = C^* \exp\left(B(t) - \frac{A(t)b(t,X(t),\xi_t)}{\sigma(t,\xi_t)} - A'(t)\int_{\xi_0}^{\xi_t} \frac{d\xi}{\sigma(t,\xi)} - A(t)\int_{\xi_0}^{\xi_t} \frac{\sigma'_t(t,\xi)}{\sigma^2(t,\xi)}d\xi\right),$$

where C^* is an arbitrary constant. Substituting the found value of $C(t)$ into (34), we find the desired transformation $g(t,\xi_t)$:

$$g(t,\xi_t) = C^* \exp\left(A(t)\int_{\xi_0}^{\xi_t} \frac{d\xi}{\sigma(t,\xi)} + B(t) - \frac{A(t)b(t,X(t),\xi_t)}{\sigma(t,\xi_t)} - \right.$$
$$\left. -A'(t)\int_{\xi_0}^{\xi_t} \frac{d\xi}{\sigma(t,\xi)} - A(t)\int_{\xi_0}^{\xi_t} \frac{\sigma'_t(t,\xi)}{\sigma^2(t,\xi)}d\xi\right). \tag{36}$$

2. By simple transformations one can make sure that the function $\eta_t = g(t,\xi_t)$ is indeed a solution to the linear equation. In order to do it, we find the total derivative (formally since ξ_t is non-differentiable; one should calculate the differentials) of the expression

$$\left(A(t)\int_{\xi_0}^{\xi_t} \frac{d\xi}{\sigma(t,\xi)}\right)'_t =$$
$$= A'(t)\int_{\xi_0}^{\xi_t} \frac{d\xi}{\sigma(t,\xi)} + \frac{A(t)}{\sigma(t,\xi_t)}[\sigma(t,\xi_t)X'(t) + b(t,X(t),\xi_t)] + A(t)\int_{\xi_0}^{\xi_t}\left(\frac{1}{\sigma(t,\xi)}\right)'_t d\xi.$$

Employing this formula, by simple algebraic transformation we can rewrite the right hand side of relation (35) as

$$B(t) + A(t)X'(t) - \left(A(t)\int_{\xi_0}^{\xi_t} \frac{d\xi}{\sigma(t,\xi)}\right)'_t.$$

Thus, $\eta_t = g(t,\xi_t) = \tilde{C}\exp\left(\int_0^t B(s)ds + \int_0^t A(s) * dX(s)\right)$, and the function in the right hand side is the solution to linear equation (32).

3. Consider the inverse problem of passing from linear equation (32) to equation (31) by an appropriate change of variables.

Of course, the inverse transform can be found by formula (36), but it is possible to construct a direct method which is often more preferable. Consider the assumed change of variables $\xi_t = \Phi(t, \eta_t)$ and let us fund the differential of this function. We have

$$d\xi_t = \sigma(t, X(t), \xi_t) * dX(t) + b(t, X(t), \xi_t)dt = \left[\Phi'_t(t, \eta_t) + \Phi'_\eta(t, \eta_t)B(t)\eta_t\right] dt+$$
$$+\Phi'_\eta(t, \eta_t)A(t)\eta_t * dX(t).$$

Therefore, by (6) for a.e. t the identities

$$\sigma(t, X(t), \xi_t) = \Phi'_\eta(t, \eta_t)A(t)\eta_t, \quad b(t, X(t), \xi_t) = \Phi'_t(t, \eta_t) + \Phi'_\eta(t, \eta_t)B(t)\eta_t$$

hold true. Employing the former relation, we rewrite the latter as $b(t, X(t), \xi_t) = \Phi'_t(t, \eta_t) + \frac{B(t)}{A(t)}\sigma(t, X(t), \xi_t)$. Therefore, the function $\xi_t = \Phi(t, \eta_t)$ solves the equation with symmetric integral

$$d\xi_t = \frac{\sigma(t, X(t), \xi_t)}{A(t)\eta_t} * d\eta_t + \left[b(t, X(t), \xi_t) - \frac{B(t)}{A(t)}\sigma(t, X(t), \xi_t)\right] dt.$$

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