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ON SOME SPECIAL SOLUTIONS OF EISENHART EQUATION

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Abstract. In this note we study a 6-dimensional pseudo-Riemannian space $V^6(g_{ij})$ with the signature $[++----]$, which admits projective motions, i.e., continuous transformation groups preserving geodesics. A general method of determining pseudo-Riemannian spaces admitting some nonhomothetic projective group G_r was developed by A.V. Aminova. A.V. Aminova classified all Lorentzian manifolds of dimension greater than three admitting nonhomothetic projective or affine infinitesimal transformations. The problem of classification is not solved for pseudo-Riemannian spaces with arbitrary signature.

In order to find a pseudo-Riemannian space admitting a nonhomothetic infinitesimal projective transformation, one has to integrate Eisenhart equation

$$h_{ij,k} = 2g_{ij}\varphi_{,k} + g_{ik}\varphi_{,j} + g_{jk}\varphi_{,i}.$$

Pseudo-Riemannian manifolds for which there exist nontrivial solutions $h_{ij} \neq cg_{ij}$ to the Eisenhart equation are called *h-spaces*. It is known that the problem of describing such spaces depends on the type of the h-space, i.e., on the type of the bilinear form $L_X g_{ij}$ determined by the characteristic of the λ-matrix $(h_{ij} - \lambda g_{ij})$. The number of possible types depends on the dimension and the signature of an h-space

In this work we find the metric and determine quadratic first integrals of the corresponding geodesic lines equations for 6-dimensional h-spaces of the type $[(21\ldots1)(21\ldots1)\ldots(1\ldots1)]$.

Keywords: differential geometry, pseudo-Riemannian manifolds, systems of partial differential equations.

Mathematics Subject Classification: 53C50, 53B30.

1. Introduction

A curve $x^i(t)$ is called *geodesic*, if its speed vector $T^i = dx^i/dt$ is parallel along the curve (cf. [1]): $\nabla_t T = 0$. In local coordinates, the geodesic equation reads as

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \tag{1}$$

where Γ^i_{jk} are the components of connection of a pseudo-Riemannian manifold (M, g). Hereinafter, the summation is made over repeating indices.

A transformation f of a pseudo-Riemannian manifold M onto itself is called *projective transformation* if it maps geodesics into geodesics.

A vector field X is called *infinitesimal projective transformation* or *projective motion* if the local one-parametric transformation group generated by this field in a neighborhood of each point $p \in M$ consists of locally projective transformations.

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A vector field X is called infinitesimal projective transformation on a manifold M with an affine connection ∇ if and only if [2] (see also [8])

$$\nabla_Y(L_X Z - \nabla_X Z) - (L_X - \nabla_X)\nabla_Y Z = R(X,Y)Z - \varphi(Y)Z - Y\varphi(Z) \tag{2}$$

for the field of 1-form φ and all vector fields Y, Z on M, where R is the Ricci tensor.

In local coordinates, we have

$$L_X\Gamma^i_{jk} = \delta^i_j\varphi_k + \delta^i_k\varphi_j \tag{3}$$

that is equivalent to

$$L_X\Gamma^i_{jk} \equiv \partial_{jk}\xi^i + \xi^l\partial_l\Gamma^i_{jk} - \Gamma^l_{jk}\partial_l\xi^i + \Gamma^i_{lk}\partial_j\xi^l + \Gamma^i_{jl}\partial_k\xi^l \equiv$$
$$\equiv \xi^i_{,jk} + \xi^l R^i_{jlk} = \delta^i_j\varphi_k + \delta^i_k\varphi_j.$$

If M a pseudo-Riemmanian manifold with the metric g and Riemannian connection ∇, then condition (2) is equivalent to the equations (cf. [2], [8])

$$L_X g = h, \tag{4}$$

$$\nabla h(Y,Z,W) = 2g(Y,Z)W\varphi + g(Y,W)Z\varphi + g(Z,W)Y\varphi, \tag{5}$$

where $(Y,Z,W) \in T(M)$, $\varphi = \frac{1}{n+1}\mathrm{div}X$. Equation (4) is called *generalized Killing equation*, the second equation in (5) is called *Eisenhart equation*.

First the problem on determining $2D$ Riemannian manifolds admitting projective motions or infinitesimal projective transformations, i.e., continuous groups of transformation preserving geodesics, was considered by S. Lie and G. Koenigs (cf. [3]). Other important results were obtained by A.Z. Petrov in work [4]. He classified geodesically equivalent pseudo-Riemannian spaces V^3. Later A.V. Aminova solved completely this problem in [5]. For a Riemannian manifold of dimension greater than two, a similar problem was solved by G. Fubini in [6] and by A.S. Solodovnikov in [7][1]. In their works, they provide a complete classification of pseudo-Riemannian spaces of dimension greater than two over local groups of projective transformations which are wider than the homotheties groups. We note that their conclusion based on the assumption of positive definiteness for the considered metric. Once we renounce the positive definiteness condition, the problem becomes much more complicated and requires a completely new method of solving.

In work [8], A.V. Aminova classified all Lorentzian manifolds of dimension greater than three admitting nonhomothetical infinitesimal projective and affine transformations. In each case, the corresponding maximal and affine Lie algebras were determined. This problem is not solved for a pseudo-Riemannian space with an arbitrary signature.

In order to find a pseudo-Riemannian space admitting nonhomothetical infinitesimal projective transformation, we need to integrate Eisenhart equation (5). The problem on determining such spaces depends on the type of h-space, i.e., on the type of the bilinear form $L_X g$ determined by Segre characteristics of the λ-matrix $(h - \lambda g)$ (see [8]). If the characteristics of a tensor $L_X g$ is $[abc\ldots]$, we call the corresponding space as h-space of type $[abc\ldots]$. These ideas were first suggested by P.A. Shirokov (see [10]). Thus, a pseudo-Riemannian space for which a nontrivial solution $h \neq cg$ to Eisenhart equation exists is called *h-space*.

The number of possible types depends on the dimension and signature of the pseudo-Riemannian space. In particular, for a 6-dimensional pseudo-Riemannian space $V^6(g_{ij})$ with the signature $[++----]$, the following types are possible:

1) $[(1...1)...(1...1)]$, i.e., $[111111]$, $[(11)1111]$, $[(111)111]$ and so forth;
2) $[1\bar{1}(1...1)...(1...1)]$, i.e., $[1\bar{1}1111]$, $[1\bar{1}(11)11]$, $[1\bar{1}(111)1]$ and so forth;
3) $[11\overline{11}(1...1)...(1...1)]$, i.e., $[(11)\overline{11}11]$, $[(11)(\overline{11})11]$, $[11\overline{11}(11)]$ and so forth;
4) $[(21...1)...(1...1)]$, i.e., $[21111]$, $[(21)111]$, $[(211)11]$ and so forth;

[1]It should be noted that the complete survey on this subject was given in work [8] as well as in the PhD thesis of the author [9].

5) [(21)11$\bar{1}$], [(211)1$\bar{1}$], [2(11)1$\bar{1}$];
6) [(21...1)(21...1)(1...1)(1...1)], i.e., [2211], [(22)11], [2(21)1], [(21)(21)] and so forth;
7) [2$\bar{2}$11], [2$\bar{2}$(11)];
8) [(31...1)...(1...1)], i.e., [3111], [(31)11], [(311)1] and so forth;
9) [311$\bar{1}$], [(31)1$\bar{1}$];
10) [321], [3(21)], [(32)1], [(321)];
11) [33], [(33)];
12) [411], (41)1], [4(11)], [(411)];
13) [51], [(51)].

We note that h-spaces in Items 1), 2), 3) were studied by G. Fubini in [6] and by A.S. Solodovnikov in [7], h-spaces in Items 4), 5), 8), 9) were studied by A.V. Aminova in [8], h-spaces in Items 6), 7), 10), 11), 12), 13) were studied by the author in PhD thesis [9]. Some results were published in [11]-[15].

The aim of the present work is to study 6-dimensional pseudo-Riemannian spaces $V^6(g_{ij})$ with signature $[++----]$. In particular, we find the metric of 6-dimensional h-spaces of types [22(11)], [2(21)1],[2(211)], [(22)11], [(221)1], [(2211)], [(22)(11)], [(21)(21)] and determine the first quadratic integrals for the geodesic equations in these h-spaces. The metric in the h-space of type [2211] was obtained by the author in [11].

The main method of determining pseudo-Riemannian manifolds admitting nonhomothetical projective group G_r was developed by A.V. Aminova (see [8])[1]. Employing the technique of integration in a moving skew-normal frame in the present work, we find the metric in the considered h-spaces.

In a skew-normal frame, Eisenhart equation

$$h_{ij,k} = 2g_{ij}\varphi_{,k} + g_{ik}\varphi_{,j} + g_{jk}\varphi_{,i} \tag{6}$$

casts into the form (see [8])[2]

$$X_r\overline{a}_{pq} + \sum_{h=1}^{n} e_h(\overline{a}_{hq}\gamma_{\tilde{h}pr} + \overline{a}_{ph}\gamma_{\tilde{h}qr}) = \overline{g}_{pr}X_q\varphi + \overline{g}_{qr}X_p\varphi \quad (p,q,r = 1,\dots,n), \tag{7}$$

where

$$X_r\varphi \equiv \underset{r}{\xi}^i \frac{\partial\varphi}{\partial x^i}, \quad \gamma_{pqr} = -\gamma_{qpr} = \underset{p}{\xi}_{i,j}\underset{q}{\xi}^i\underset{r}{\xi}^j, \quad a_{ij} = h_{ij} - 2\varphi g_{ij},$$

$\underset{i}{\xi}^j$ are the components of the skew-normal frame, $\overline{g}_{pr} = e_p\delta_{\tilde{p}}^{\,r}$ and $\overline{a}_{pq}$ are canonical forms of tensors g_{pr}, a_{pq}, respectively, $\gamma^p_{lk} = e_p\gamma_{l\tilde{p}k}$ are the components of connection in skew-normal frame X. The commutators of vector field X_k and X_h are determined by the formula (cf. [8])

$$[X_k, X_h] = \sum_{l=1}^{n} e_l(\gamma_{lkh} - \gamma_{lhk})X_{\tilde{l}}, \tag{8}$$

[1]The technique of integration in skew-normal frame was employed first in works [16], [17].

[2]The mapping $\sim$ which maps indices into the others was first introduced by A.V. Aminova in works [16], [17] (cf. also [8]) in the definition of a moving skew-normal frame. It should be noted that these papers by A.V. Aminova can be found in the Internet by the link http://www.mathnet.ru/php/person.phtml?option_lang=rus&personid=8394. Omitting a cumbersome definition of a moving skew-normal frame, it is sufficient to provide several examples in order to understand the action of mapping $\sim$. For instance, for h-space of type [2211], $\tilde{1}=2$, $\tilde{2}=1$, $\tilde{3}=4$, $\tilde{4}=3$, $\tilde{5}=5$, $\tilde{6}=6$; for h-space of type [321], $\tilde{1}=3$, $\tilde{2}=2$, $\tilde{3}=1$, $\tilde{4}=5$, $\tilde{5}=4$, $\tilde{6}=6$; for h-space of type [411], $\tilde{1}=4$, $\tilde{2}=3$, $\tilde{3}=2$, $\tilde{4}=1$, $\tilde{5}=5$, $\tilde{6}=6$. The same relations remain true also in the case of multiple primitive divisors, i.e., under the existence of brackets in the types of h-spaces, for instance, in the case [22(11)], [2(211)] and so forth.

that is equivalent to

$$[X_k, X_h] = \sum_{l=1}^{n}\sum_{m=1}^{n}(\gamma_{mkh} - \gamma_{mhk})\overline{g}^{ml}X_l.$$

We note that for 6-dimensional h-spaces of type $[(21\dots1)(21\dots1)\dots(1\dots1)]$, $\tilde{1} = 2$, $\tilde{2} = 1$, $\tilde{3} = 4$, $\tilde{4} = 3$, $\tilde{5} = 5$, $\tilde{6} = 6$ (see [8]).

For 6-dimensional h-spaces of type $[(21\dots1)(21\dots1)\dots(1\dots1)]$, canonical forms $\overline{g}_{pr}$ and $\overline{a}_{pq}$ read as (cf. [4])

$$\overline{g}_{pr} = \begin{pmatrix} 0 & e_2 & 0 & 0 & 0 & 0 \\ e_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_4 & 0 & 0 \\ 0 & 0 & e_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_6 \end{pmatrix}, \tag{9}$$

$$\overline{a}_{pq} = \begin{pmatrix} 0 & e_2\lambda_2 & 0 & 0 & 0 & 0 \\ e_2\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_4\lambda_4 & 0 & 0 \\ 0 & 0 & e_4\lambda_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_5\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_6\lambda_6 \end{pmatrix},$$

where $e_1 = e_2, e_3 = e_4, e_i = \pm1, (i = 1, 2, ..., 6)$, $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, λ_5, λ_6 are real functions which can coincide. These functions are the roots of the characteristic equation $\det(h_{ij} - \lambda g_{ij}) = 0$.

2. Metric of h-space of type [22(11)]

We substitute canonical forms $\overline{g}_{pr}$ and $\overline{a}_{pq}$ from (9) into (7) and take into consideration that for h-space of type [22(11)] we have $\lambda_5 = \lambda_6$, $\tilde{1} = 2$, $\tilde{2} = 1$, $\tilde{3} = 4$, $\tilde{4} = 3$, $\tilde{5} = 5$, $\tilde{6} = 6$. Then we obtain the system of equations

$$\begin{gathered} X_r\lambda_2 = 0 \ (r \neq 2), \quad X_r\lambda_4 = 0 \ (r \neq 4), \quad X_r\lambda_6 = 0, \\ X_2(\lambda_2 - \varphi) = X_4(\lambda_4 - \varphi) = 0, \quad \gamma_{121} = e_2X_2\varphi, \quad \gamma_{343} = e_4X_4\varphi, \\ \gamma_{142} = \gamma_{241} = \frac{e_2X_4\varphi}{\lambda_2 - \lambda_4}, \quad \gamma_{242} = -\frac{e_2X_4\varphi}{(\lambda_2 - \lambda_4)^2}, \quad \gamma_{324} = \gamma_{423} = \frac{e_4X_2\varphi}{\lambda_4 - \lambda_2}, \\ \gamma_{424} = -\frac{e_4X_2\varphi}{(\lambda_4 - \lambda_2)^2}, \quad \gamma_{244} = \frac{e_4X_2\varphi}{(\lambda_2 - \lambda_4)^2}, \quad \gamma_{s\sigma\sigma} = \frac{e_\sigma X_s\varphi}{(\lambda_s - \lambda_\sigma)}, \end{gathered} \tag{10}$$

where $r = 1, 2, ..., 6$, $\sigma = 5, 6$, $s = 2, 4$, γ_{56r} are arbitrary. Other γ_{pqr} are zero.

It is known that a system of partial differential equations

$$X_q\theta = \underset{q}{\xi^i}\partial_i\theta = 0, \quad (q = 1, \dots, m, i = 1, \dots, 6, m < 6), \tag{11}$$

where $\underset{q}{\xi^i}$ are the components of a skew-normal frame, is completely integrable, i.e., it admits $6 - m$ independent solutions, if and only if all the commutators of the operators of system ([2], see also [8])

$$[X_q, X_r] = X_qX_r - X_rX_q = \sum_{p=1}^{6} e_p(\gamma_{pqr} - \gamma_{prq})X_{\tilde{p}} \tag{12}$$

are linearly expressed in terms of operators X_q.

Employing formulae (10) and (12), we write down the commutators of operators X_i ($i = 1, \dots, 6$) in the considered h-space,

$$
\begin{gathered}
[X_1, X_2] = -e_1\gamma_{121}X_2, \quad [X_1, X_3] = 0, \quad [X_2, X_3] = e_4\gamma_{423}X_3, \\
[X_1, X_4] = -e_2\gamma_{241}X_1, \quad [X_3, X_4] = -e_3\gamma_{343}X_4, \\
[X_2, X_4] = -e_2\gamma_{242}X_1 - e_1\gamma_{142}X_2 + e_4\gamma_{424}X_3 + e_3\gamma_{324}X_4, \\
[X_p, X_\sigma] = -e_\tau\gamma_{\tau\sigma p}X_\tau, \quad [X_q, X_\sigma] = e_\sigma\gamma_{\sigma q\sigma}X_\sigma - e_\tau\gamma_{\tau\sigma q}X_\tau, \\
[X_5, X_6] = -e_5\gamma_{565}X_5 + e_6\gamma_{656}X_6,
\end{gathered}
\tag{13}
$$

where $p = 1, 3$, $q = 2, 4$, $\sigma, \tau = 5, 6$ ($\sigma \neq \tau$).

Then, forming completely integrable system (11) by (13), we define independent solutions admitted by these systems and indicate these solutions by θ^i. After that, by the coordinate transformation $x^{i'} = \theta^i(x)$, we can vanish some of the components $\underset{q}{\xi}^i$ of the skew-normal frame introduced above. In particular, completely integrable systems from (13) are $X_1\theta = X_3\theta = X_4\theta = X_5\theta = X_6\theta = 0$, $X_1\theta = X_2\theta = X_3\theta = X_5\theta = X_6\theta = 0$, $X_1\theta = X_2\theta = X_3\theta = X_4\theta$, $X_3\theta = X_4\theta = X_5\theta = X_6\theta = 0$, $X_1\theta = X_2\theta = X_5\theta = X_6\theta = 0$. We denote the solution to the first system by θ^2, the solution to the second system is θ^4, and the solutions to the third system are θ^5 and θ^6. The forth system has two independent solutions. One of them is indicated by θ^1, while the other is chosen to coincide with θ^2. The latter system has two independent solutions as well. One of them is denoted by θ^3, while the other coincides with θ^4. Making the coordinate transformation $x^{i'} = \theta^i(x)$ in the new coordinate system, omitting the primes, we determine

$$
\underset{p}{\xi}^i = P_p(x)\delta_p{}^i, \quad \underset{2}{\xi}^3 = \underset{2}{\xi}^4 = \underset{2}{\xi}^\sigma = \underset{4}{\xi}^1 = \underset{4}{\xi}^2 = \underset{4}{\xi}^\sigma = \underset{\sigma}{\xi}^\alpha = 0,
\tag{14}
$$

where $p = 1, 3$, $\sigma = 5, 6$, $\alpha = 1, 2, 3, 4$, $P_p(x)$ are arbitrary functions.

By means of identities (14), employing the part of equations (10) not involving γ_{pqr}, we find

$$
2\varphi = \sum_{i=1}^{6} f_i + c, \quad \lambda_i = f_i,
\tag{15}
$$

where $f_1 = f_2(x^2)$, $f_3 = f_4(x^4)$ are arbitrary functions, $f_5 = f_6 = \lambda$, c are constants.

Equating the coefficients at like derivatives $\partial/\partial x^i$ in the left hand side and the right hand side of identities (13), by formulae (10) and (14) we obtain the system of equations for components $\underset{i}{\xi}^j$ of the skew-normal frame,

1° $\underset{1}{\xi}^1 \partial_1 \underset{2}{\xi}^1 - \underset{2}{\xi}^1 \partial_1 \underset{1}{\xi}^1 - \underset{2}{\xi}^2 \partial_2 \underset{1}{\xi}^1 = -f'{}_2 \underset{2}{\xi}^2 \underset{2}{\xi}^1$,

2° $\underset{1}{\xi}^1 \partial_1 \underset{2}{\xi}^2 = -f'{}_2 (\underset{2}{\xi}^2)^2$,

3° $\underset{3}{\xi}^3 \partial_3 \underset{1}{\xi}^1 = \underset{3}{\xi}^3 \partial_3 \underset{2}{\xi}^1 = \underset{3}{\xi}^3 \partial_3 \underset{2}{\xi}^2 = 0$,

4° $\partial_4 \underset{1}{\xi}^1 = \frac{f'_4}{f_2 - f_4} \underset{1}{\xi}^1$,

5° $\partial_4 \underset{2}{\xi}^1 = \frac{f'_4}{f_2 - f_4} \underset{2}{\xi}^1 - \frac{f'_4}{(f_2 - f_4)^2} \underset{1}{\xi}^1$,

6° $\partial_4 \underset{2}{\xi}^2 = \frac{f'_4}{f_2 - f_4} \underset{2}{\xi}^2$,

7° $\underset{3}{\xi}^3 \partial_3 \underset{4}{\xi}^3 - \underset{4}{\xi}^3 \partial_3 \underset{3}{\xi}^3 - \underset{4}{\xi}^4 \partial_4 \underset{3}{\xi}^3 = -f'{}_4 \underset{4}{\xi}^4 \underset{4}{\xi}^3$,

8° $\underset{3}{\xi}^3 \partial_3 \underset{4}{\xi}^4 = -f'{}_4 (\underset{4}{\xi}^4)^2$,

9° $\underset{1}{\xi}^1 \partial_1 \underset{3}{\xi}^3 = \underset{1}{\xi}^1 \partial_1 \underset{4}{\xi}^3 = \underset{1}{\xi}^1 \partial_1 \underset{4}{\xi}^4 = 0$,

10° $\partial_2 \underset{3}{\xi}^3 = \frac{f'_2}{f_4 - f_2} \underset{3}{\xi}^3$, 11° $\partial_2 \underset{4}{\xi}^3 = \frac{f_2'}{f_4-f_2}\underset{4}{\xi}^3 - \frac{f_2'}{(f_4-f_2)^2}\underset{3}{\xi}^3,$

12° $\partial_2 \underset{4}{\xi}^4 = \frac{f_2'}{f_4-f_2}\underset{4}{\xi}^4,$

13° $\underset{1}{\xi}^1\partial_1\underset{\sigma}{\xi}^\sigma = -\gamma_{\tau\sigma 1}\underset{\tau}{\xi}^\sigma, \quad (\tau\neq\sigma),$

14° $\underset{1}{\xi}^1\partial_1\underset{\sigma}{\xi}^\tau = -\gamma_{\tau\sigma 1}\underset{\tau}{\xi}^\tau, \quad (\tau\neq\sigma),$

15° $\underset{3}{\xi}^3\partial_3\underset{\sigma}{\xi}^\sigma = -\gamma_{\tau\sigma 3}\underset{\tau}{\xi}^\sigma, \quad (\tau\neq\sigma),$

16° $\underset{3}{\xi}^3\partial_3\underset{\sigma}{\xi}^\tau = -\gamma_{\tau\sigma 3}\underset{\tau}{\xi}^\tau, \quad (\tau\neq\sigma),$

17° $\underset{2}{\xi}^1\partial_1\underset{\sigma}{\xi}^\sigma + \underset{2}{\xi}^2\partial_2\underset{\sigma}{\xi}^\sigma = -\frac{f_2'}{f_2-\lambda}\underset{2}{\xi}^2\underset{\sigma}{\xi}^\sigma - \gamma_{\tau\sigma 2}\underset{\tau}{\xi}^\sigma, \quad (\tau\neq\sigma),$

18° $\underset{2}{\xi}^1\partial_1\underset{\sigma}{\xi}^\tau + \underset{2}{\xi}^2\partial_2\underset{\sigma}{\xi}^\tau = -\frac{f_2'}{f_2-\lambda}\underset{2}{\xi}^2\underset{\sigma}{\xi}^\tau - \gamma_{\tau\sigma 2}\underset{\tau}{\xi}^\tau, \quad (\tau\neq\sigma),$

19° $\underset{4}{\xi}^3\partial_3\underset{\sigma}{\xi}^\sigma + \underset{4}{\xi}^4\partial_4\underset{\sigma}{\xi}^\sigma = -\frac{f_4'}{f_4-\lambda}\underset{4}{\xi}^4\underset{\sigma}{\xi}^\sigma - \gamma_{\tau\sigma 4}\underset{\tau}{\xi}^\sigma, \quad (\tau\neq\sigma),$

20° $\underset{4}{\xi}^3\partial_3\underset{\sigma}{\xi}^\tau + \underset{4}{\xi}^4\partial_4\underset{\sigma}{\xi}^\tau = -\frac{f_4'}{f_4-\lambda}\underset{4}{\xi}^4\underset{\sigma}{\xi}^\tau - \gamma_{\tau\sigma 4}\underset{\tau}{\xi}^\tau, \quad (\tau\neq\sigma),$

21° $\underset{5}{\xi}^5\partial_5\underset{6}{\xi}^5 + \underset{5}{\xi}^6\partial_6\underset{6}{\xi}^5 - \underset{6}{\xi}^5\partial_5\underset{5}{\xi}^5 - \underset{6}{\xi}^6\partial_6\underset{5}{\xi}^5 = -\gamma_{565}\underset{5}{\xi}^5 + \gamma_{656}\underset{6}{\xi}^5,$

22° $\underset{5}{\xi}^5\partial_5\underset{6}{\xi}^6 + \underset{5}{\xi}^6\partial_6\underset{6}{\xi}^6 - \underset{6}{\xi}^5\partial_5\underset{5}{\xi}^6 - \underset{6}{\xi}^6\partial_6\underset{5}{\xi}^6 = -\gamma_{565}\underset{5}{\xi}^6 + \gamma_{656}\underset{6}{\xi}^6,$

23° $(\underset{\sigma}{\xi}^\sigma\partial_\sigma + \underset{\sigma}{\xi}^\tau\partial_\tau)\underset{\alpha}{\xi}^\beta = 0, \quad (\tau\neq\sigma),$

where $\alpha,\beta = 1,2,3,4$, $\sigma,\tau = 5,6$, $f_2' = \frac{df_2}{dx^2}$, $f_4' = \frac{df_4}{dx^4}$.

It follows from equation 23° that $\underset{\alpha}{\xi}^\beta$ do not depend on variables x^5, x^6. Integrating equations 3°, 4°, 9°, 10°, we find

$$\underset{1}{\xi}^1 = (f_4-f_2)^{-1}F_1(x^1,x^2),$$

$$\underset{3}{\xi}^3 = (f_2-f_4)^{-1}F_3(x^3,x^4),$$

where F_1, F_3 are functions of the mentioned variables and these functions are not zero due to the linear dependence of the vectors in the frame and due to formulae (14). Equation 3° also implies that $\underset{2}{\xi}^1$ is independent of variable x^3.

The expressions for the found components of the frame can be simplified by the coordinate transformation

$$\overline{x}^1 = \int\frac{dx^1}{F_1}, \quad \overline{x}^2 = x^2, \quad \overline{x}^3 = \int\frac{dx^3}{F_3}, \quad \overline{x}^4 = x^4, \quad \overline{x}^\sigma = x^\sigma,$$

which does not change the identities (14). In the new coordinate system we obtain

$$\underset{1}{\xi}^1 = (f_4-f_2)^{-1}, \quad \underset{3}{\xi}^3 = (f_2-f_4)^{-1}. \tag{16}$$

After that, integrating equation 2° and 6° and taking into consideration 3°, we get

$$\underset{2}{\xi}^2 = (f_4-f_2)^{-1}(f_2'x^1+\theta(x^2))^{-1},$$

where $\theta(x^2)$ is an arbitrary function of variable x^2.

There are two possible cases: 1) $f_2' = 0$, 2) $f_2' \neq 0$. In the second case we make the coordinate transformation $\overline{x}^2 = f_2(x^2)$, $\overline{x}^p = x^p$ $(p\neq 2)$ and we let $\overline{\theta} = (f_2')^{-1}\theta$. Omitting the overline, we can unify both the case by one formula

$$\underset{2}{\xi}^2 = (f_4-f_2)^{-1}A^{-1}, \tag{17}$$

where

$$A = \epsilon x^1 + \theta, \quad f_1 = f_2 = \epsilon x^2,$$

ϵ is 0 or 1, θ is a function of variable x^2 being non-zero as $\epsilon = 0$.

Following similar lines and integrating equations 8°, 12° with 9° taken into consideration, we obtain

$$\underset{4}{\xi}^4 = (f_2 - f_4)^{-1}\tilde{A}^{-1}. \tag{18}$$

Here

$$\tilde{A} = \tilde{\epsilon}x^3 + \omega, \quad f_3 = f_4 = \tilde{\epsilon}x^4 + a,$$

$\tilde{\epsilon}$ is 0 or 1, a is a constant being non-zero as $\tilde{\epsilon} = 0$, ω is a function of variable x^4 being non-zero as $\tilde{\epsilon} = 0$.

Integrating equations 1°, 5°, 7° and 11°, due to 3°, 9° we find

$$\underset{2}{\xi}^1 = (f_4 - f_2)^{-1}((f_4 - f_2)^{-1} + Q(x^2)),$$
$$\underset{4}{\xi}^3 = (f_2 - f_4)^{-1}((f_2 - f_4)^{-1} + R(x^4)),$$

where $Q(x^2)$, $R(x^4)$ are the functions of the indicated variables.

By the coordinate transformation

$$\overline{x}^1 = x^1 - \int Q dx^2, \quad \overline{x}^3 = x^3 - \int R dx^2, \quad \overline{x}^p = x^p \ (p \neq 1, 3),$$

we can vanish functions Q and R keeping the formulae obtained before. Then components $\underset{2}{\xi}^1$ and $\underset{4}{\xi}^3$ of the skew-normal frame cast into the form

$$\underset{2}{\xi}^1 = (f_4 - f_2)^{-2}, \quad \underset{4}{\xi}^3 = (f_2 - f_4)^{-2}. \tag{19}$$

Employing the obtained results and the formula (see [8])

$$g^{ij} = \sum_{h=1}^{6} e_h \underset{h}{\xi}^i \underset{\tilde{h}}{\xi}^j, \tag{20}$$

one can calculate the following contravariant components of the metric tensor for the considered h-space,

$$g^{11} = 2e_2(f_4 - f_2)^{-3}, \quad g^{12} = e_2(f_4 - f_2)^{-2}A^{-1},$$
$$g^{33} = 2e_4(f_2 - f_4)^{-3}, \quad g^{34} = e_4(f_2 - f_4)^{-2}\tilde{A}^{-1}.$$

It also follows from formula (20) that in the considered h-space we have $g^{\sigma\tau} = e_\sigma \underset{\sigma}{\xi}^\sigma \underset{\sigma}{\xi}^\tau + e_\tau \underset{\tau}{\xi}^\sigma \underset{\tau}{\xi}^\tau$, $\sigma, \tau = 5, 6$. By means of equations 13°, 14°, 15°, 16°, one can prove easily that $\underset{1}{\xi}^1 \partial_1 g^{\sigma\tau} = \underset{3}{\xi}^3 \partial_3 g^{\sigma\tau} = 0$. It implies $\partial_1 g^{\sigma\tau} = \partial_3 g^{\sigma\tau} = 0$. Then it follows from equations 17°, 18°, 19°, 20° that

$$\partial_2 g^{\sigma\tau} = -2\frac{f_2'}{f_2 - \lambda} g^{\sigma\tau}, \quad \partial_4 g^{\sigma\tau} = -2\frac{f_4'}{f_4 - \lambda} g^{\sigma\tau}.$$

Integrating these equations and bearing in mind equations 21°, 22°, we find

$$g^{\sigma\tau} = (f_2 - \lambda)^{-2}(f_4 - \lambda)^{-2}F^{\sigma\tau}(x^5, x^6), \tag{21}$$

where $F^{\sigma\tau}$ are arbitrary functions of variables x^5, x^6.

Then, calculating covariant components g_{ij} of the metric tensor, by formulae (see [8])

$$\underset{h}{\xi}_i = g_{ij} \underset{h}{\xi}^j, \quad a_{ij} = \sum_{h,l=1}^{n} e_h e_l \overline{a}_{hl} \underset{\tilde{h}}{\xi}_i \underset{\tilde{l}}{\xi}_j, \tag{22}$$

we find the components of tensor a_{ij}.

We write the final results as the following theorem.

Theorem 1. *If a symmetric tensor h_{ij} of type [22(11)] and a scalar φ satisfy equations (1) in $V^6(g_{ij})$, then there exists a holonomic coordinate system, in which φ, g_{ij}, and h_{ij} are determined by the formulae*

$$\begin{gathered} g_{ij}dx^i dx^j = e_2 A(f_4 - f_2)\{(f_4 - f_2)dx^1 dx^2 - A(dx^2)^2)\} + \\ + e_4 \tilde{A}(f_2 - f_4)\{(f_2 - f_4)dx^3 dx^4 - \tilde{A}(dx^4)^2\} + \\ + F_{\sigma\tau}(f_2 - \lambda)^2 (f_4 - \lambda)^2 dx^\sigma dx^\tau, \end{gathered} \tag{23}$$

$$\begin{gathered} a_{ij}dx^i dx^j = f_2 g_{i_1 j_1} dx^{i_1} dx^{j_1} + \\ + A g_{12}(dx^2)^2 + f_4 g_{i_2 j_2} dx^{i_2} dx^{j_2} + \tilde{A} g_{34}(dx^4)^2 + \lambda g_{\sigma\tau} dx^\sigma dx^\tau, \end{gathered} \tag{24}$$

$$h_{ij} = a_{ij} + 2\varphi g_{ij}, \quad 2\varphi = 2f_2 + 2f_4 + c, \tag{25}$$

$$A = \epsilon x^1 + \theta(x^2), \quad \tilde{A} = \tilde{\epsilon} x^3 + \omega(x^4), \tag{26}$$

where $\epsilon, \tilde{\epsilon} = 0, 1$, $f_2 = \epsilon x^2$, $f_4 = \tilde{\epsilon} x^4 + a$, λ, c and a are constants, $a \neq 0$ as $\tilde{\epsilon} = 0$, $F_{\sigma\tau}(x^5, x^6)$, $\theta(x^2)$, $\omega(x^4)$ are arbitrary functions, $\theta \neq 0$ as $\epsilon = 0$, $\omega \neq 0$ as $\tilde{\epsilon} = 0$, $i_1, j_1 = 1, 2$, $i_2, j_2 = 3, 4$, $\sigma, \tau = 5, 6$, $e_2, e_4 = \pm 1$.

3. Metric of h-spaces of types [2(21)1], [2(211)]

In this case and in the cases considered below we make calculations similar with ones made for h-space of type [22(11)]. And this is why we omit some arguments.

We substitute canonical forms $\overline{g}_{pr}$ and $\overline{a}_{pq}$ from (9) into (7). Since $\lambda_4 = \lambda_5$ for h-space of type [2(21)1], we obtain

$$\begin{gathered} X_r \lambda_2 = 0 \ (r \neq 2), \quad X_r \lambda_5 = 0, \quad X_r \lambda_6 = 0 \ (r \neq 6), \\ X_2(\lambda_2 - \varphi) = X_6(\lambda_6 - \varphi) = 0, \quad \gamma_{121} = e_2 X_2 \varphi, \\ \gamma_{162} = \gamma_{261} = \frac{e_2 X_6 \varphi}{\lambda_2 - \lambda_6}, \quad \gamma_{262} = -\frac{e_2 X_6 \varphi}{(\lambda_2 - \lambda_6)^2}, \quad \gamma_{3s4} = \gamma_{4s3} = \frac{e_4 X_s \varphi}{\lambda_5 - \lambda_s}, \\ \gamma_{4s4} = -\frac{e_4 X_s \varphi}{(\lambda_5 - \lambda_s)^2}, \quad \gamma_{2\sigma\sigma} = \frac{e_\sigma X_2 \varphi}{(\lambda_2 - \lambda_\sigma)}, \quad \gamma_{565} = \frac{e_5 X_6 \varphi}{\lambda_5 - \lambda_6}, \end{gathered} \tag{27}$$

where $r = 1, 2, ..., 6$, $\sigma = 5, 6$, $s = 2, 6$, γ_{45r} are arbitrary, while other γ_{pqr} are zero.

The commutators of the operators of h-space of type [2(21)1] read as

$$\begin{gathered} [X_1, X_2] = -e_1 \gamma_{121} X_2, \quad [X_1, X_3] = 0, \\ [X_2, X_3] = e_4 \gamma_{423} X_3, \quad [X_1, X_4] = -e_5 \gamma_{541} X_5, \\ [X_1, X_5] = -e_4 \gamma_{451} X_3, \quad [X_1, X_6] = -e_2 \gamma_{261} X_1, \\ [X_2, X_4] = e_3 \gamma_{324} X_4 + e_4 \gamma_{424} X_3 - e_5 \gamma_{542} X_5, \\ [X_2, X_5] = e_5 \gamma_{525} X_5 - e_4 \gamma_{425} X_3, \\ [X_2, X_6] = -e_2 \gamma_{262} X_1 - e_1 \gamma_{162} X_2 + e_6 \gamma_{626} X_6, \\ [X_3, X_4] = -e_5 \gamma_{543} X_5, \quad [X_3, X_5] = -e_4 \gamma_{453} X_3, \\ [X_3, X_6] = -e_4 \gamma_{463} X_3, \quad [X_4, X_5] = -e_4 \gamma_{454} X_3 + e_5 \gamma_{545} X_5, \\ [X_4, X_6] = -e_3 \gamma_{364} X_4 - e_4 \gamma_{464} X_3 + e_5 \gamma_{546} X_5, \\ [X_5, X_6] = e_4 \gamma_{456} X_3 - e_5 \gamma_{565} X_5. \end{gathered} \tag{28}$$

It follows that systems $X_i\theta = 0$ $(i \neq 2)$, $X_j\theta = 0$ $(j \neq 4)$, $X_k\theta = 0$ $(k \neq 6)$ are completely integrable and have, respectively, the solutions: θ^2, θ^4, θ^6. Systems $X_3\theta = X_4\theta = X_5\theta = X_6\theta = 0$, $X_1\theta = X_2\theta = X_3\theta = X_6\theta = 0$ and $X_1\theta = X_2\theta = X_6\theta = 0$ are also completely integrable.

The first system has solutions θ^1 and θ^2, the second one has solutions θ^4 and θ^5, the third system has solutions θ^3, θ^4, and θ^5. Making coordinate transformation $x^{i'} = \theta^i(x)$ and omitting the primes, we obtain

$$\underset{p}{\xi}^i = P_p(x)\delta_p{}^i, \quad \underset{2}{\xi}^s = \underset{4}{\xi}^q = \underset{5}{\xi}^q = \underset{5}{\xi}^4 = 0, \tag{29}$$

where $p = 1, 2, 3,\ s = 3, 4, 5, 6,\ q = 1, 2, 6,\ P_p(x)$ are arbitrary functions.

Integrating system of equations (28) as in the previous case, taking into account (27) and (29), and calculating the components of tensors g_{ij} and a_{ij}, we arrive at the following result,

$$\begin{gathered} g_{ij}dx^i dx^j = e_2\{2(f_6 - f_2)Adx^1dx^2 - A^2(dx^2)^2\}+ \\ +(f_6 - \lambda)(f_2 - \lambda)^2\{2e_4dx^3dx^4 - e_4(\Sigma + \omega)(dx^4)^2 + e_5(dx^5)^2\}+ \\ +e_6(f_2 - f_6)^2(dx^6)^2, \end{gathered} \tag{30}$$

$$\begin{gathered} a_{ij}dx^i dx^j = f_2 g_{i_1j_1}dx^{i_1}dx^{j_1}+ \\ +g_{12}(dx^2)^2+\lambda g_{i_2j_2}dx^{i_2}dx^{j_2} + g_{34}(dx^4)^2 + f_6 g_{66}(dx^6)^2, \end{gathered} \tag{31}$$

$$h_{ij} = a_{ij} + (2f_2 + f_6 + c)g_{ij}, \quad \varphi = f_2 + \frac{1}{2}f_6 + c, \tag{32}$$

$$A = \epsilon x^1 + \theta(x^2), \quad \Sigma = 2(f_2 - \lambda)^{-1} + (f_6 - \lambda)^{-1}, \tag{33}$$

where $\epsilon = 0, 1,\ f_2 = \epsilon x^2$, λ and c are constants, $\theta(x^2)$, $\omega(x^4, x^5)$, $f_6(x^6)$ are arbitrary functions, $\theta \neq 0$ as $\epsilon = 0$, $i_1, j_1 = 1, 2,\ i_2, j_2 = 3, 4, 5,\ e_2, e_4, e_5, e_6 = \pm 1$.

Similar arguments for for h-space of type [2(211)] yield

$$\begin{gathered} g_{ij}dx^i dx^j = 2e_2Adx^1dx^2+ \\ +(f_2 - \lambda)^2\{2e_4dx^3dx^4 - e_4(\Sigma + \omega)(dx^4)^2 + g_{\sigma\tau}dx^\sigma dx^\tau\}, \end{gathered} \tag{34}$$

$$a_{ij}dx^i dx^j = 2f_2g_{12}dx^1dx^2 + g_{12}(dx^2)^2 + \lambda g_{pq}dx^p dx^q + g_{34}(dx^4)^2, \tag{35}$$

$$h_{ij} = a_{ij} + (2f_2 + c)g_{ij}, \quad \varphi = f_2 + c, \tag{36}$$

$$A = \epsilon x^1 + \theta(x^2), \quad \Sigma = 2(f_2 - \lambda)^{-1}, \tag{37}$$

where $\epsilon = 0, 1,\ f_2 = \epsilon x^2,\ \lambda, c$ are constants, $\theta(x^2), \omega(x^4, x^5, x^6), g_{\sigma\tau}(x^4, x^5, x^6)$ are arbitrary functions, $\theta \neq 0$ as $\epsilon = 0$, $p, q = 3, 4, 5, 6,\ \sigma, \tau = 5, 6,\ e_2, e_4 = \pm 1$.

We summarize the obtained results in

Theorem 2. *If a symmetric tensor h_{ij} of types [2(21)1], [2(211)] and a function φ satisfy Eisenhart equations in $V^6(g_{ij})$, then there exists a holonomic coordinate system in which function φ and tensors g_{ij}, h_{ij} are determined by formulae (30)–(37).*

4. Metric of h-spaces of types [(22)11], [(221)1]

For h-space of type [(22)11], from (7) it follows the system of equations

$$\begin{gathered} X_r\lambda_4 = 0, \quad X_r\lambda_\sigma = 0\ (r \neq \sigma), \quad X_\sigma(\lambda_\sigma - \varphi) = 0, \\ \gamma_{14r} = \gamma_{23r}, \quad \gamma_{1\sigma 2} = \gamma_{2\sigma 1} = \frac{e_2X_\sigma\varphi}{\lambda_4 - \lambda_\sigma}, \quad \gamma_{3\sigma 4} = \gamma_{4\sigma 3} = \frac{e_4X_\sigma\varphi}{\lambda_4 - \lambda_\sigma}, \\ \gamma_{s\sigma s} = -\frac{e_sX_\sigma\varphi}{(\lambda_4 - \lambda_\sigma)^2}, \quad \gamma_{\sigma\tau\sigma} = \frac{e_\sigma X_\tau\varphi}{\lambda_\sigma - \lambda_\tau}\ (\sigma \neq \tau), \end{gathered} \tag{38}$$

where $r = 1, 2, ..., 6,\ \sigma, \tau = 5, 6,\ s = 2, 4,\ \gamma_{24r}$ are arbitrary, and other γ_{pqr} are zero.

We write the commutators of operators,

$$
\begin{gathered}
[X_1, X_2] = e_4\gamma_{412}X_3 - e_4\gamma_{421}X_3 - e_3\gamma_{411}X_4,\\
[X_1, X_3] = e_4\gamma_{413}X_3 - e_2\gamma_{141}X_1,\\
[X_2, X_3] = -e_2\gamma_{142}X_1 + e_3\gamma_{413}X_4 + e_4\gamma_{423}X_3,\\
[X_1, X_4] = -e_1\gamma_{141}X_2 + e_4\gamma_{414}X_3 - e_2\gamma_{241}X_1,\\
[X_1, X_\sigma] = e_4\gamma_{41\sigma}X_3 - e_2\gamma_{2\sigma1}X_1,\\
[X_2, X_4] = -e_1\gamma_{142}X_2 + e_4\gamma_{424}X_3 + e_3\gamma_{414}X_4 - e_2\gamma_{242}X_1,\\
[X_2, X_\sigma] = e_3\gamma_{41\sigma}X_4 + e_4\gamma_{42\sigma}X_3 - e_2\gamma_{2\sigma2}X_1 - e_1\gamma_{1\sigma2}X_2,\\
[X_3, X_4] = -e_1\gamma_{143}X_2 - e_2\gamma_{243}X_1 + e_2\gamma_{144}X_1,\\
[X_3, X_\sigma] = e_2\gamma_{23\sigma}X_1 - e_1\gamma_{4\sigma3}X_3,\\
[X_4, X_\sigma] = e_1\gamma_{14\sigma}X_2 - e_3\gamma_{3\sigma4}X_4 + e_2\gamma_{24\sigma}X_1 - e_4\gamma_{4\sigma4}X_3,\\
[X_5, X_6] = -e_5\gamma_{565}X_5 + e_6\gamma_{656}X_6.
\end{gathered}
\tag{39}
$$

Forming completely integrable systems in (39), by coordinate transformations we find

$$\underset{\sigma}{\xi}^i = P_\sigma(x)\delta_\sigma{}^i, \underset{p}{\xi}^q = \underset{\alpha}{\xi}^\sigma = 0, \tag{40}$$

where $\alpha = 1,2,3,4,\ p = 1,3,\ q = 2,4,\ \sigma,\tau = 5,6,\ ,\ P_\sigma(x)$ are arbitrary functions.

By (38), (39), (40) we obtain the system of equations for components $\underset{i}{\xi}^j$ of the skew-normal frame,

1° $\underset{1}{\xi}^\alpha\partial_\alpha\underset{2}{\xi}^\beta - \underset{2}{\xi}^\alpha\partial_\alpha\underset{1}{\xi}^\beta = \gamma_{412}\underset{3}{\xi}^\beta - \gamma_{421}\underset{3}{\xi}^\beta - \gamma_{411}\underset{4}{\xi}^\beta,$

2° $\underset{1}{\xi}^\alpha\partial_\alpha\underset{3}{\xi}^\beta - \underset{3}{\xi}^\alpha\partial_\alpha\underset{1}{\xi}^\beta = \gamma_{413}\underset{3}{\xi}^\beta - \gamma_{141}\underset{1}{\xi}^\beta,$

3° $\underset{1}{\xi}^\alpha\partial_\alpha\underset{4}{\xi}^\beta - \underset{4}{\xi}^\alpha\partial_\alpha\underset{1}{\xi}^\beta = -\gamma_{141}\underset{2}{\xi}^\beta - \gamma_{414}\underset{3}{\xi}^\beta - \gamma_{241}\underset{1}{\xi}^\beta,$

4° $\underset{\sigma}{\xi}^\sigma\partial_\sigma\underset{1}{\xi}^\beta = \frac{1}{2}\frac{f'_\sigma}{\lambda-f_\sigma}\underset{\sigma}{\xi}^\sigma\underset{1}{\xi}^\beta - \gamma_{41\sigma}\underset{3}{\xi}^\beta,$

5° $\underset{2}{\xi}^\alpha\partial_\alpha\underset{3}{\xi}^p - \underset{3}{\xi}^\alpha\partial_\alpha\underset{2}{\xi}^p = -\gamma_{142}\underset{1}{\xi}^\beta + \gamma_{413}\underset{4}{\xi}^p + \gamma_{423}\underset{3}{\xi}^p,$

6° $\underset{3}{\xi}^\alpha\partial_\alpha\underset{2}{\xi}^q = -\gamma_{413}\underset{4}{\xi}^q,$

7° $\underset{2}{\xi}^\alpha\partial_\alpha\underset{4}{\xi}^\beta - \underset{4}{\xi}^\alpha\partial_\alpha\underset{2}{\xi}^\beta = -\gamma_{142}\underset{2}{\xi}^\beta + \gamma_{414}\underset{4}{\xi}^\beta - \gamma_{242}\underset{1}{\xi}^\beta + \gamma_{424}\underset{3}{\xi}^\beta,$

8° $\underset{\sigma}{\xi}^\sigma\partial_\sigma\underset{2}{\xi}^\beta = \frac{1}{2}\frac{f'_\sigma}{\lambda-f_\sigma}\underset{\sigma}{\xi}^\sigma\underset{2}{\xi}^\beta - \frac{1}{2}\frac{f'_\sigma}{(\lambda-f_\sigma)^2}\underset{\sigma}{\xi}^\sigma\underset{1}{\xi}^\beta - \gamma_{41\sigma}\underset{4}{\xi}^\beta - \gamma_{42\sigma}\underset{3}{\xi}^\beta,$

9° $\underset{3}{\xi}^\alpha\partial_\alpha\underset{4}{\xi}^\beta - \underset{4}{\xi}^\alpha\partial_\alpha\underset{3}{\xi}^\beta = -\gamma_{143}\underset{2}{\xi}^\beta + \gamma_{144}\underset{1}{\xi}^\beta - \gamma_{243}\underset{1}{\xi}^\beta,$

10° $\underset{\sigma}{\xi}^\sigma\partial_\sigma\underset{3}{\xi}^\beta = \frac{1}{2}\frac{f'_\sigma}{\lambda-f_\sigma}\underset{\sigma}{\xi}^\sigma\underset{3}{\xi}^\beta - \gamma_{14\sigma}\underset{1}{\xi}^\beta,$

11° $\underset{\sigma}{\xi}^\sigma\partial_\sigma\underset{4}{\xi}^\beta = \frac{1}{2}\frac{f'_\sigma}{\lambda-f_\sigma}\underset{\sigma}{\xi}^\sigma\underset{4}{\xi}^\beta - \frac{1}{2}\frac{f'_\sigma}{(\lambda-f_\sigma)^2}\underset{\sigma}{\xi}^\sigma\underset{3}{\xi}^\beta - \gamma_{14\sigma}\underset{2}{\xi}^\beta - \gamma_{24\sigma}\underset{1}{\xi}^\beta,$

12° $\underset{\beta}{\xi}^\alpha\partial_\alpha\underset{\sigma}{\xi}^\sigma = 0,$

13° $\underset{\sigma}{\xi}^\sigma\partial_\sigma\underset{\tau}{\xi}^\tau = \frac{1}{2}\frac{f'_\sigma}{f_\tau-f_\sigma}\underset{\sigma}{\xi}^\sigma\underset{\tau}{\xi}^\tau, \quad (\tau \neq \sigma),$

where $\alpha,\beta = 1,2,3,4,\ p = 1,3,\ q = 2,4,\ \sigma,\tau = 5,6,\ f_\tau = f_\tau(x_\tau), f_\sigma = f_\sigma(x_\sigma)$ are arbitrary functions of the mentioned variables.

Integrating 12° and 13°, by a coordinate transformation of x^5, x^6 we find

$$\underset{5}{\xi}^5 = (f_6 - f_5)^{-1/2}, \quad \underset{6}{\xi}^6 = (f_5 - f_6)^{-1/2}.$$

Differentiating $g^{\alpha\beta} = \sum_{h=1}^{6} e_h \underset{h}{\xi^\alpha} \underset{\tilde{h}}{\xi^\beta}$ w.r.t. x^p $(p = 1, 3)$, in view of equations $1^\circ - 3^\circ$ and $5^\circ - 7^\circ$ we get $\partial_p g^{\alpha\beta} = 0$.

Differentiating $g^{\alpha\beta}$ w.r.t. x^σ $(\sigma = 5, 6)$, by equations 4°, 8°, 10°, 11° we obtain

$$\partial_\sigma g^{pr} = -\frac{f'_\sigma}{f_\sigma - \lambda} g^{pr} - \frac{f'_\sigma}{(f_\sigma - \lambda)^2} \Pi_\sigma (f_\sigma - \lambda)^{-1}, \quad \partial_\sigma g^{pq} = -\frac{f'_\sigma}{f_\sigma - \lambda} g^{pq},$$

where $p, r = 1, 3$, $q = 2, 4$. Integrating these equations, we find

$$g^{pr} = \Pi_\sigma (f_\sigma - \lambda)^{-1} (\Sigma + F^{pr}), \quad g^{pq} = \Pi_\sigma (f_\sigma - \lambda)^{-1} F^{pq},$$

where F^{pr}, F^{pq} are arbitrary functions of variables x^2, x^4, $\Sigma = \sum_\sigma (f_\sigma - \lambda)^{-1}$, $\sum_\sigma$ denotes the summation over σ, Π_σ denotes the product over σ.

Making similar calculations for h-space of type [(221)1], after appropriate coordinate transformations we finally get

$$g_{ij} dx^i dx^j = \Pi_\sigma (f_\sigma - \lambda) \{ 2g_{12} dx^1 dx^2 - e_2 (\Sigma + \theta_1)(dx^2)^2 + \\ +2g_{34} dx^3 dx^4 - e_4 (\Sigma + \theta_2)(dx^4)^2 + G \} + \sum_\sigma e_\sigma (f_\tau - f_\sigma)(dx^\sigma)^2, \tag{41}$$

$$a_{ij} dx^i dx^j = \lambda (g_{st} dx^s dx^t + G) + \\ g_{12} (dx^2)^2 + g_{34} (dx^4)^2 + \sum_\sigma f_\sigma g_{\sigma\sigma} (dx^\sigma)^2 + G, \tag{42}$$

$$h_{ij} = a_{ij} + (\sum_\sigma f_\sigma + c) g_{ij}, \quad \varphi = \frac{1}{2} \sum_\sigma f_\sigma + c. \tag{43}$$

$$\Sigma = \sum_\sigma (f_\sigma - \lambda)^{-1}, \tag{44}$$

$$G = 2e_5 \{ 1 + \theta_3 (f_6 - \lambda) \} dx^4 dx^5 + (f_6 - \lambda) g_{55} (dx^5)^2 + e_6 (dx^6)^2, \tag{45}$$

where λ, c are constants. Here for h-space of type [(22)11], we have $\tau, \sigma = 5, 6$ $(\tau \neq \sigma)$, $s, t = 1, 2, 3, 4$, $G = 0$, $g_{st}, \theta_1, \theta_2$ are arbitrary functions of variables x^2, x^4, f_σ is an arbitrary functions of variable x^σ. For h-space of type [(221)1], we have $\sigma = 6$, $s, t = 1, 2, 3, 4, 5$, $g_{st}, \theta_1, \theta_2, \theta_3$ are arbitrary functions of variables x^2, x^4, x^5, $f_\tau = \lambda$, f_6 is an arbitrary function of variable x^6.

We have

Theorem 3. *If a symmetric tensor h_{ij} of types* [(22)11], [(221)1] *and a function φ satisfy Eisenhart equation in $V^6(g_{ij})$, then there exists a holonomic coordinate system in which function φ and tensors g_{ij}, h_{ij} are determined by formulae (41)–(45).*

5. Metric of h-spaces of types [(2211)], [(22)(11)], [(21)(21)]

In all these cases we have $\varphi = const$, therefore, due to identity (6), tensor h_{ij} is covariantly constant. Omitting further calculations being integrating of equations w.r.t. $\underset{i}{\xi^j}$ together with appropriate coordinate transformations, we obtain:
for h-space of type [(2211)],

$$g_{ij} dx^i dx^j = 2g_{12} dx^1 dx^2 - e_2 \theta (dx^2)^2 + 2g_{34} dx^3 dx^4 + g_{rq} dx^r dx^q, \tag{46}$$

$$a_{ij} dx^i dx^j = \lambda g_{ij} dx^i dx^j + g_{12} (dx^2)^2, \tag{47}$$

$$h_{ij} = a_{ij} + c g_{ij}, \tag{48}$$

where $r, q = 5, 6$, λ, c are constant, θ, g_{12}, g_{34}, g_{rq} are arbitrary functions of variables x^2, x^4, x^5, x^6;

for h-space of type [(22)(11)],

$$g_{ij}dx^i dx^j = e_2\{2dx^1dx^2 - \theta(dx^2)^2\} + e_4\{2dx^3dx^4 - \omega(dx^4)^2\} + g_{\sigma\tau}dx^\sigma dx^\tau, \tag{49}$$

$$\begin{aligned} a_{ij}dx^i dx^j = \lambda_1\{g_{i_1j_1}dx^{i_1}dx^{j_1}+ \\ +e_2(dx^2)^2+g_{i_2j_2}dx^{i_2}dx^{j_2} + e_4(dx^4)^2\} + \lambda_2 g_{\sigma\tau}dx^\sigma dx^\tau, \end{aligned} \tag{50}$$

$$h_{ij} = a_{ij} + cg_{ij}, \tag{51}$$

where θ, ω are arbitrary functions of variables x^2, x^4, $g_{\sigma\tau}$ are arbitrary functions of variables x^5, x^6, λ_1, λ_2, c are constants, and $\lambda_1 \neq \lambda_2$, $i_1, j_1 = 1, 2$, $i_2, j_2 = 3, 4$, $\sigma, \tau = 5, 6$;
for h-space of type [(21)(21)],

$$\begin{aligned} g_{ij}dx^i dx^j = e_2\{2dx^1dx^2 - \theta(dx^2)^2\}+ \\ +e_3(dx^3)^2 + e_5\{2dx^4dx^5 - \omega(dx^5)^2\} + e_6(dx^6)^2, \end{aligned} \tag{52}$$

$$a_{ij}dx^i dx^j = \lambda_1 g_{i_1j_1}dx^{i_1}dx^{j_1} + e_2(dx^2)^2 + \lambda_2 g_{i_2j_2}dx^{i_2}dx^{j_2} + e_5(dx^5)^2, \tag{53}$$

$$h_{ij} = a_{ij} + cg_{ij}, \tag{54}$$

where θ is an arbitrary function of variables x^2, x^3, ω is an arbitrary function of variables x^5, x^6, λ_1, λ_2, c are constants, $\lambda_1 \neq \lambda_2$, $i_1, j_1 = 1, 2, 3$, $i_2, j_2 = 4, 5, 6$.

We summarize the results of this section in the following theorem.

Theorem 4. *If a tensor h_{ij} of types* [(2211)], [(22)(11)], [(21)(21)] *and function φ satisfy Eisenhart equation in $V^6(g_{ij})$, then there exists a holonomic coordinate system in which function φ and tensors g_{ij}, h_{ij} are determined by formulae (46)–(54).*

6. First quadratic integrals of geodesic equations in h-spaces of types $[(21\dots1)(21\dots1)\dots(1\dots1)]$

To each solution h_{ij} of equation (6), there corresponds the first quadratic integral of geodesic equations (see [8])

$$(h_{ij} - 4\varphi g_{ij})\dot{x}^i\dot{x}^j = \text{const}, \tag{55}$$

where $\dot{x}^i$ is the tangential vector to the geodesic.

Therefore, the first quadratic integrals of geodesic equations in h-space of types $[(21\dots1)(21\dots1)\dots(1\dots1)]$ are determined by formula (55), where tensors h_{ij}, g_{ij} and function φ are given in Theorems 1-4.

7. Conclusion

In the present work we have found a small class of 6-dimensional pseudo-Riemannian spaces with signature [++−−−−] admitting nonhomothetical infinitesimal projective transformations. In particular, we have found the metric of h-spaces of types [22(11)], [2(21)1],[2(211)], [(22)11], [(221)1], [(2211)], [(22)(11)], [(21)(21)] and have determined the first quadratic integrals of geodesic equations in these h-spaces. We note that the obtained results are easily generalized for the case of n-dimensional pseudo-Riemannian space with signature [+ + − − − − ... − −].

The determination of the metric of all h-spaces counted in Introduction solves completely the problem on finding 6-dimensional pseudo-Riemannian spaces with signature [+ + − − −−] admitting non-homothetical infinitesimal projective transformations or projective motions. This problem was completely solved by the author in PhD thesis [9].

The next problem is to study the projective-group properties of the considered spaces. Here we still have the open problem on recovering of the vector field determining infinitesimal projective transformation and the problem on the structure of projective Lie algebra. The solution of this problem is reduced to integrating Killing equation.

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