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SYMMETRIES AND GOURSAT PROBLEM FOR SYSTEM OF EQUATIONS $u_{xy} = e^{u+v} u_y$, $v_{xy} = -e^{u+v} v_y$

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Abstract. We describe the higher symmetries and construct the general solution for a hyperbolic system of equations. We also obtain the explicit formula for the solution of Goursat problem.

Keywords: symmetries, Goursat problem, integrals.

Mathematics Subject Classification: 35L53, 76M60, 58J70

1. Introduction

In work [1], the authors considered Goursat problem for exponential system of equations

$$\frac{\partial^2 u^i}{\partial x \partial y} + \sum_{k=1}^{r} a_{ik} e^{u^k} = 0, \quad i = 1, \dots, r, \tag{1.1}$$

$$u^i(x, y) - \ln\left(\tau_i \phi^i(x) \bar{\phi}^i(y)\right) = 0 \quad \text{as} \quad xy = 0, \tag{1.2}$$

a_{ik} are the entries of Cartan matrix of a simple Lie algebra, and studied the dependence of a solution on parameters $\tau_1, \dots, \tau_r$ involved in boundary conditions (1.2). A scheme was suggested for constructing a solution to this problem with employing higher symmetries admitted by system of equations (1.1). Examples of reduction to a closed system of ordinary differential equations were provided.

In works [2]–[4], for linear hyperbolic system of equations with zero generalized Laplace invariants, the general solution was constructed and an algorithm for solving boundary values problems was given. In work [5], basing on the symmetry approach, there was constructed an exact solution to Goursat problem for a nonlinear scalar hyperbolic equation of Liouville type.

In the present paper we consider the system of equations

$$\begin{cases} u_{xy} = e^{u+v} u_y, \\ v_{xy} = -e^{u+v} v_y, \end{cases} \tag{1.3}$$

obeying $\det(H_1 \cdot K_1) = 0$, $\operatorname{ord}(H_1, K_1) = 1$, and its chain of the generalized Laplace invariants breaks at the second step (see [6]), where H_1, K_1 are the main invariants of linearized system (1.3). We describe the higher symmetries and construct the general solution to system (1.3), which allows us to obtain an exact solution to Goursat problem.

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2. Symmetries

For the sake of convenience of presentation, we introduce the notations

$$u_1 = u_x,\ u_2 = u_{xx},\ \ldots, v_1 = v_x,\ v_2 = v_{xx}, \ldots,$$

$$\bar{u}_1 = u_y,\ \bar{u}_2 = u_{yy},\ \ldots, \bar{v}_1 = v_y,\ \bar{v}_2 = v_{yy}, \ldots.$$

It was shown in work [6] that system of equations (1.3) possesses the integrals of first and second order

$$\begin{array}{cc} w = u_1 - v_1 - e^{u+v} \quad \text{and} \quad \bar{w} = \bar{u}_1\bar{v}_1, \\ W = u_2 - u_1 v_1 - e^{u+v}u_1 \quad \text{and} \quad \bar{W} = \frac{\bar{u}_2}{\bar{u}_1} + \bar{v}_1 - \bar{u}_1, \end{array} \tag{2.1}$$

such that $\bar{D}w = 0$, $\bar{D}W = 0$, $D\bar{w} = 0$, $D\bar{W} = 0$, where D,$\bar{D}$ are the operators of total differentiation w.r.t. x, y, respectively.

The system for higher symmetries of system of equations (1.3) reads as

$$\begin{cases} D\bar{D}p = e^{u+v}\bar{D}p + e^{u+v}\bar{u}_1(p+q), \\ D\bar{D}q = -e^{u+v}\bar{D}q - e^{u+v}\bar{v}_1(p+q). \end{cases} \tag{2.2}$$

By formulae (2.1), the symmetries for system of equations (1.3) depending on variables u, v, u_1, v_1, ... can be sought as

$$p = p(u, v, v_1, w, W, w_1, W_1, \ldots), \quad q = q(u, v, v_1, w, W, w_1, W_1, \ldots).$$

We calculate $\bar{D}p$, $\bar{D}q$, $D\bar{D}p$, $D\bar{D}q$ and substitute them into system (2.2). Then, we equate the expressions at $\bar{u}_1$, $\bar{v}_1$ and obtain the following system of equations

$$\begin{cases} Dp_u = e^{u+v}(p+q), \\ D\left(p_v - e^{u+v}p_{v_1}\right) = 2e^{u+v}\left(p_v - e^{u+v}p_{v_1}\right), \\ Dq_u = -2e^{u+v}q_u, \\ D\left(q_v - e^{u+v}q_{v_1}\right) = -e^{u+v}(p+q). \end{cases} \tag{2.3}$$

The second and third equations of system (2.3) imply that

$$p_v - e^{u+v}p_{v_1} = 0, \quad q_u = 0.$$

Summing the first and the fourth equation of system (2.3) and integrating the obtained identity w.r.t. u, we obtain the following expression for p,

$$p = -q_v u + e^{u+v}q_{v_1} + Cu + h(v, v_1, w, W, \ldots), \tag{2.4}$$

where $h(v, v_1, w, W, \ldots)$ is an arbitrary function, and C is an arbitrary constant. It remains to substitute the found function p into the second and first equation of system (2.3) that leads us to functions p and q, namely,

$$p = (D + u_1)a - b,\ q = v_1 a + b. \tag{2.5}$$

Here $a(w, W, w_1, W_1, \ldots)$, $b(w, W, w_1, W_1, \ldots)$ are arbitrary functions.

Then the symmetries depending on variables u, v, $\bar{u}_1$, $\bar{v}_1$, ... can be sought as

$$p = p(\bar{u}_1, \bar{w}, \bar{W}, \bar{w}_1, \bar{W}_1, \ldots), \quad q = q(\bar{u}_1, \bar{w}, \bar{W}, \bar{w}_1, \bar{W}_1, \ldots).$$

In system of equations (1.3), we make the change of variables,

$$u + v = U,\ \ u - v = V.$$

Then system of equations (1.3) is equivalent to the system

$$\begin{cases} u_{xy} = e^u v_y, \\ v_{xy} = e^u u_y. \end{cases} \tag{2.6}$$

For the sake of convenience, here we denote new variables U, V again by u, v. Then the linearized system (see (2.2)) becomes

$$\begin{cases} D\bar{D}p = e^u(\bar{D}q + \bar{v}_1 p), \\ D\bar{D}q = e^u(\bar{D}p + \bar{u}_1 p). \end{cases} \tag{2.7}$$

Integrating the second equation of system (2.7) w.r.t. y, we obtain the following system of equations equivalent to the previous one

$$\begin{cases} D\bar{D}p = e^u(\bar{D}q + \bar{v}_1 p), \\ \qquad Dq = e^u p. \end{cases} \tag{2.8}$$

We shall seek the solution to system of equations (2.8) as

$$p = \sum_{k=0}^{n} p_k f^{(k)}(y), \quad q = \sum_{k=0}^{n} q_k f^{(k)}(y), \tag{2.9}$$

where $f^{(k)}(y) = \bar{D}^{(k)} f(\bar{w}, \bar{W}, \bar{w}_1, \bar{W}_1, \ldots)$.

Then we substitute functions (2.9) into system of equations (2.8) and equation coefficients at like derivatives. We obtain a system of equations equivalent to system (2.8), namely,

$$\begin{cases} Dq_k = e^u p_k, \ k = 0, 1, \ldots, n, \\ D\bar{D}p_0 = e^u(\bar{D}q_0 + \bar{v}_1 p_0), \\ D\bar{D}p_k + D(p_{k-1}) = e^u(\bar{D}q_k + q_{k-1} + \bar{v}_1 p_k), \ k = 1, 2, \ldots, n, \\ Dp_n = e^u q_n. \end{cases} \tag{2.10}$$

Consider the case $n = 0$. Here system (2.10) casts into the form

$$\begin{cases} D\bar{D}p_0 = e^u(\bar{D}q_0 + \bar{v}_1 p_0), \\ \qquad Dq_0 = e^u p_0, \\ \qquad Dp_0 = e^u q_0. \end{cases} \tag{2.11}$$

We make the replacement $\bar{v}_1 = \sqrt{\bar{u}_1^2 - 4\bar{w}}$ in the second equation of system (2.11) to obtain the equation

$$(q_0)_{\bar{u}_1} \sqrt{\bar{u}_1^2 - 4\bar{w}} = p_0.$$

We differentiate this equation w.r.t. $\bar{u}_1$ and expressing $(p_0)_{\bar{u}_1}$ by the third equation of system (2.11), we obtain

$$\left((q_0)_{\bar{u}_1}(\bar{u}_1^2 - 4\bar{w}) - \bar{u}_1 q_0\right)'_{\bar{u}_1} = 0$$

or

$$(q_0)_{\bar{u}_1}(\bar{u}_1^2 - 4\bar{w}) - \bar{u}_1 q_0 = A(\bar{w}, \bar{W}, \ldots), \tag{2.12}$$

where $A(\bar{w}, \bar{W}, \ldots)$ is arbitrary function. Equation (2.12) is a first order linear differential equation whose solution can be represented as

$$q_0 = -\frac{A}{4\bar{w}}\bar{u}_1 + B\sqrt{\bar{u}_1^2 - 4\bar{w}}, \tag{2.13}$$

where $B(\bar{w}, \bar{W}, \ldots)$ is an arbitrary function.

Then we substitute expression for q_0 (2.13) into the second and first equations of system (2.11) to find that p_0, q_0 read as

$$p_0 = \bar{u}_1 B(\bar{w}, \bar{W}, \ldots), \ \ q_0 = \bar{v}_1 B(\bar{w}, \bar{W}, \ldots). \tag{2.14}$$

Now we consider the case $n = 1$ in system (2.10). Here system (2.10) can be rewritten as

$$Dq_0 = e^u p_0, \tag{2.15}$$

$$Dq_1 = e^u p_1, \tag{2.16}$$

$$D\bar{D}p_0 = e^u(\bar{D}q_0 + \bar{v}_1 p_0), \tag{2.17}$$

$$D\bar{D}p_1 + Dp_0 = e^u(\bar{D}q_1 + q_0 + \bar{v}_1 p_1), \tag{2.18}$$

$$Dp_1 = e^u q_1. \tag{2.19}$$

Equations (2.16) and (2.19) coincide with equations of system (2.11), therefore, p_1 and q_1 are located above and read as

$$p_1 = \bar{u}_1 B - \frac{A}{4\bar{w}}\sqrt{\bar{u}_1^2 - 4\bar{w}}, \;\; q_1 = -\frac{A}{4\bar{w}}\bar{u}_1 + B\sqrt{\bar{u}_1^2 - 4\bar{w}}. \tag{2.20}$$

Differentiating equation (2.19) w.r.t. y, we obtain

$$D\bar{D}p_1 = e^u(\bar{D}q_1 + \bar{u}_1 q_1). \tag{2.21}$$

We subtract equation (2.21) from equation (2.18) and after simple transformation we obtain

$$Dp_0 = e^u(q_0 + A(\bar{w}, \bar{W}, \ldots)). \tag{2.22}$$

We differentiate equation (2.22) w.r.t. y and subtract then equation (2.17), we find the expression for p_0,

$$p_0 = \frac{1}{\bar{v}_1}(\bar{D}A + \bar{u}_1 q_0 + \bar{u}_1 A). \tag{2.23}$$

We substitute expression (2.23) for p_0 into equation (2.15) and replacing $\bar{v}_1 = \sqrt{\bar{u}_1^2 - 4\bar{w}}$, we obtain first order linear differential equation for function q_0, namely,

$$(q_0)_{\bar{u}_1} = \frac{\bar{u}_1}{\bar{u}_1^2 - 4\bar{w}} q_0 + \frac{\bar{D}A}{\bar{u}_1^2 - 4\bar{w}} + \frac{\bar{u}_1 A}{\bar{u}_1^2 - 4\bar{w}}.$$

The solutions to this equation can be represented as

$$q_0 = -\bar{u}_1 \frac{\bar{D}A}{4\bar{w}} - A + R\sqrt{\bar{u}_1^2 - 4\bar{w}}, \tag{2.24}$$

where $R = R(\bar{w}, \bar{W}, \ldots)$ is an arbitrary function. We substitute expression (2.24) into equation (2.23) that allows us to find p_0,

$$p_0 = -\bar{v}_1 \frac{\bar{D}A}{4\bar{w}} + \bar{u}_1 R. \tag{2.25}$$

As a result, we obtain that system of equations (2.15)–(2.19) has solutions (2.20), (2.24), (2.25). It follows from (2.9) that the symmetries for system of equations (2.6) read as

$$p = \left(-\bar{v}_1 \frac{\bar{D}A}{4\bar{w}} + \bar{u}_1 R\right) f + \left(\bar{u}_1 B - \frac{A\bar{v}_1}{4\bar{w}}\right) \bar{D}f, \tag{2.26}$$

$$q = \left(-\bar{u}_1 \frac{\bar{D}A}{4\bar{w}} - A + \bar{v}_1 R\right) f + \left(-\frac{A}{4\bar{w}}\bar{u}_1 + \bar{v}_1 B\right) \bar{D}f. \tag{2.27}$$

In view of formulae (2.14), symmetries (2.26), (2.27) can be represented as

$$p = \frac{\bar{v}_1}{\bar{w}}\bar{D}G, \quad q = 4G + \frac{\bar{u}_1}{\bar{w}}\bar{D}G, \tag{2.28}$$

where $G = -\frac{1}{4}Af$. We remind that found symmetries (2.14), (2.28) are defined in terms of new variables U, V. Returning to variables $u = \frac{U+V}{2}$, $v = \frac{U-V}{2}$, we obtain the following representation for symmetries of system of equations (1.3)

$$p = \bar{u}_1 \frac{1}{\bar{w}} \bar{D}G + \bar{u}_1 B + 2G, \quad q = -\bar{v}_1 \frac{1}{\bar{w}} \bar{D}G + \bar{v}_1 B - 2G. \tag{2.29}$$

3. Construction of general solution

By employing higher symmetries (2.5), (2.29), the problem on integrating system of equations (1.3) is reduced to the following dynamical system (see [1]),

$$\begin{cases} \tau\frac{\partial u}{\partial \tau} = (D+u_1)\psi^1 - \psi^2 = \bar{u}_1 \frac{1}{\bar{w}}\bar{D}\bar{\psi}^1 + \bar{u}_1\bar{\psi}^2 + 2\bar{\psi}^1, \\ \tau\frac{\partial v}{\partial \tau} = v_1\psi^1 + \psi^2 = -\bar{v}_1 \frac{1}{\bar{w}}\bar{D}\bar{\psi}^1 + \bar{v}_1\bar{\psi}^2 - 2\bar{\psi}^1, \\ \tau\frac{\partial \bar{u}_1}{\partial \tau} = e^{u+v}\bar{u}_1\psi^1, \\ \tau\frac{\partial \bar{v}_1}{\partial \tau} = -e^{u+v}\bar{v}_1\psi^1, \\ \tau\frac{\partial u_1}{\partial \tau} = e^{u+v}\bar{u}_1\frac{1}{\bar{w}}\bar{D}\bar{\psi}^1 + e^{u+v}\bar{u}_1\bar{\psi}^2, \\ \tau\frac{\partial v_1}{\partial \tau} = e^{u+v}\bar{v}_1\frac{1}{\bar{w}}\bar{D}\bar{\psi}^1 - e^{u+v}\bar{v}_1\bar{\psi}^2. \end{cases} \tag{3.1}$$

We shall assume that $\psi^1 = \psi^1(x)$, $\psi^2 = \psi^2(x)$, $\bar{\psi}^1 = \bar{\psi}^1(y)$, $\bar{\psi}^2 = \bar{\psi}^2(y)$.

The first and second equation of system (3.1) are first order partial differential equation for functions u, v, respectively. The solutions to these equations can be represented as

$$u = -\ln\psi^1 + \int \frac{\psi^2}{\psi^1}dx + F(a,y), \quad v = -\int \frac{\psi^2}{\psi^1}dx + G(a,y), \tag{3.2}$$

where $F(a,y)$, $G(a,y)$ are arbitrary functions, and by a we denote the expression

$$a = \ln\tau + \int \frac{dx}{\psi^1}.$$

Then we substitute the found functions (3.2) into system (3.1), we obtain the system of equations for functions F and G

$$F_a = \bar{D}\bar{\psi}^1 \frac{1}{G_y} + \bar{\psi}^2 F_y + 2\bar{\psi}^1, \tag{3.3}$$

$$G_a = -\bar{D}\bar{\psi}^1 \frac{1}{F_y} + \bar{\psi}^2 G_y - 2\bar{\psi}^1, \tag{3.4}$$

$$F_{ya} = e^{F+G}F_y, \tag{3.5}$$

$$G_{ya} = -e^{F+G}G_y, \tag{3.6}$$

$$F_{aa} = e^{F+G}\left(\bar{D}\bar{\psi}^1 \frac{1}{G_y} + \bar{\psi}^2 F_y\right), \tag{3.7}$$

$$G_{aa} = e^{F+G}\left(\bar{D}\bar{\psi}^1 \frac{1}{F_y} - \bar{\psi}^2 G_y\right). \tag{3.8}$$

In view of (3.3), (3.4), equations (3.7), (3.8) can be rewritten as

$$\begin{cases} f_{aa} = e^{f+g}f_a, \\ g_{aa} = -e^{f+g}g_a, \end{cases} \tag{3.9}$$

where $f = F - 2a\bar{\psi}^1$, $g = G + 2a\bar{\psi}^1$. We subtract first equation of system (3.9) from the second one and integrate the obtained identity w.r.t. a,

$$f_a - g_a = e^{f+q} + C_1(y), \tag{3.10}$$

where $C_1(y)$ is an arbitrary function. Then we multiply the first equation of system (3.9) by g_a, the second equation by f_a and sum the obtained expressions to find

$$g_a = \frac{C_2(y)}{f_a}, \tag{3.11}$$

where $C_2(y)$ is an arbitrary function. We substitute the found formula for g_a (3.11) into equation (3.10) and we get

$$g = -f + \ln\left(f_a^2 - C_1 f_a - C_2\right) - \ln f_a. \tag{3.12}$$

We return back to system of equations (3.9) and in view of formula (3.12), the first equation can be rewritten as

$$f_{aa} = f_a^2 - C_1 f_a - C_2.$$

The right hand side of this expression is a polynomial of second degree and we factorize it

$$f_{aa} = (f_a - \alpha)(f_a - \beta),$$

α, β are arbitrary functions of y. Integrating this equation, we find f, namely,

$$f = \frac{\alpha}{\alpha - \beta}\left[(\alpha - \beta)a + \gamma\right] - \ln\left[1 - \exp\{(\alpha - \beta)a + \gamma\}\right] + \delta(y),\ \ \alpha \neq \beta, \tag{3.13}$$

$$f = \alpha a - \ln(\varepsilon - a) + \kappa(y),\ \ \alpha = \beta, \tag{3.14}$$

where $\alpha(y)$, $\beta(y)$, $\gamma(y)$, $\delta(y)$, $\varepsilon(y)$, $\kappa(y)$ are arbitrary functions.

Now we substitute the found formulae (3.13), (3.14), (3.12) into equation (3.5). We obtain the following relations:

1. as $\alpha \neq \beta$,

$$\beta' + 2\bar{D}\bar{\psi}^1 = 0,\ \ \alpha = \beta + c,\ \ \delta' + \frac{\beta'\gamma}{\alpha - \beta} = 0, \tag{3.15}$$

where c is an arbitrary constant;

2. as $\alpha = \beta$,

$$\alpha' + 2\bar{D}\bar{\psi}^1 = 0,\ \ \alpha\varepsilon' + \kappa' = 0. \tag{3.16}$$

Then we substitute functions (3.13), (3.14), (3.12) into equations (3.6), (3.3), (3.4), and in view of conditions (3.15), (3.16), we obtain the true identities. Thus, the solution to systems of equations (1.3) can be represented as (3.2), where functions $F = f + 2a\bar{\psi}^1$, $G = g - 2a\bar{\psi}^1$ are determined by relations (3.13), (3.14), (3.12), namely, as $\alpha \neq \beta$,

$$u = \ln\phi_1'(x) + \phi_2(x) - \frac{\alpha\delta'}{\alpha'} - \ln\left(1 - exp\{a - \frac{\delta'}{\alpha'}\}\right) + \delta,$$

$$v = -\phi_2(x) + (a - \frac{\delta'}{\alpha'}) + \frac{\alpha\delta'}{\alpha'} - \ln\left(\alpha - (\alpha - 1)\exp\{a - \frac{\delta'}{\alpha'}\}\right) - \delta,$$

and as $\alpha = \beta$,

$$u = \ln\phi_1'(x) + \phi_2(x) - \ln(\varepsilon(y) - a) + \kappa(y), \tag{3.17}$$

$$v = -\phi_2(x) - \ln\left[\frac{\kappa'}{\varepsilon'}(a - \varepsilon(y)) + 1\right] - \kappa(y). \tag{3.18}$$

Here $\phi_1'(x) = \frac{1}{\psi^1}$, $\phi_2'(x) = \frac{\psi^2}{\psi^1}$, $\varepsilon(y)$, $\kappa(y)$, $\alpha(y)$, $\delta(y)$ are arbitrary functions.

4. Exact solution to Goursat problem

Consider Goursat problem for system of equations (1.3),

$$u\mid_{y=0}=\ln p(x),\quad v\mid_{y=0}=\ln q(x),\quad u\mid_{x=0}=\ln \bar{p}(y),\quad v\mid_{x=0}=\ln \bar{q}(y). \tag{4.1}$$

We let $y=0$, $\tau=1$ in the solution to (3.17), (3.18) to obtain

$$u\mid_{y=0}=\ln p(x)=\ln\phi_1'+\phi_2-\ln(\varepsilon(0)-\phi_1)+\kappa(0), \tag{4.2}$$

$$v\mid_{y=0}=\ln q(x)=-\phi_2-\ln\left(\frac{\kappa'(0)}{\varepsilon'(0)}(\phi_1-\varepsilon(0))+1\right)-\kappa(0). \tag{4.3}$$

We sum expressions (4.2) and (4.3) and integrate the obtained identity w.r.t. x that allows us to find function $\phi_1(x)$,

$$\phi_1(x)=\varepsilon(0)-\left(C_2+\left\{\frac{1}{C_1}-C_2\right\}e^{\int\limits_0^x pqd\xi}\right)^{-1}, \tag{4.4}$$

where $C_1=\varepsilon(0)-\phi_1(0)$, $C_2=\frac{\kappa'(0)}{\varepsilon'(0)}$.

Then by expression (4.2) we find function $\phi_2(x)$, namely,

$$\phi_2(x)=\ln\left[\left(q(0)e^{\phi_2(0)+\kappa(0)}-1\right)e^{-\int\limits_0^x pqd\xi}+1\right]-\ln q-\kappa(0). \tag{4.5}$$

Now we let $x=0$ in formulae (3.17), (3.18),

$$u\mid_{x=0}=\ln\bar{p}(y)=\ln\phi_1'(0)+\phi_2(0)-\ln(\varepsilon-\phi_1(0))+\kappa, \tag{4.6}$$

$$v\mid_{x=0}=\ln\bar{q}(y)=-\phi_2(0)-\ln\left(\frac{\kappa'}{\varepsilon'}(\phi_1(0)-\varepsilon)+1\right)-\kappa. \tag{4.7}$$

We sum identities (4.6), (4.7), then obtain a first order differential equations with separating variables for function $\varepsilon(y)$. Solving this equation, we find

$$\varepsilon(y)=\phi_1(0)+\phi_1'(0)\left(\int\limits_0^y \bar{q}\bar{p}'d\xi+\frac{\phi_1'(0)}{C_1}\right)^{-1}. \tag{4.8}$$

Then $\kappa(y)$ can be found by expression (4.6) and reads as

$$\kappa(y)=\ln\bar{p}-\phi_2(0)-\ln\left(\int\limits_0^y \bar{q}\bar{p}'d\xi+\bar{p}(0)e^{-\phi_2(0)-\kappa(0)}\right). \tag{4.9}$$

Now we employ the matching condition. We let $x=0$, $y=0$ in (3.17), (3.18) and obtain the relations

$$\phi_1'(0)=p(0)q(0)C_1(1-C_1C_2), \tag{4.10}$$

$$\phi_2(0)+\kappa(0)=-\ln\left(q(0)(1-C_1C_2)\right). \tag{4.11}$$

We substitute the found functions (4.4), (4.5), (4.8), (4.9) into solution (3.17), (3.18) and taking into consideration matching condition (4.10), (4.11), we finally obtain the representation for the solution to Goursat problem (1.3), (4.1),

$$u=\ln\left[\frac{p(x)\bar{p}(y)q(0)}{p(0)q(0)+\int\limits_0^y \bar{p}'\bar{q}d\xi(1-\exp\{\int\limits_0^x pqd\xi\})}\right],$$

$$v = \ln \left[\frac{q(x)\bar{q}(y)p(0)\exp\{\int\limits_0^x pqd\xi\}}{(\bar{p}\bar{q} - \int\limits_0^y \bar{p}'\bar{q}d\xi)(\exp\{\int\limits_0^x pqd\xi\} - 1) + p(0)q(0)} \right].$$

BIBLIOGRAPHY

1. A.N. Leznov, A.B. Shabat. *Conditions for gap series in perturbation theory* // Work collection "Integrable systems". Bashkir branch of AS USSR, Ufa. 1982. P. 34-44. (in Russian).
2. A.V. Zhiber, Yu.G. Mihaylova. *Algorithm of building the solution of hyperbolic system of equation* // Ufimskij matem. zhurn. 2009. V. 1, No. 3. P. 28-45. (in Russian).
3. Yu.G. Voronova. *On Cauchy problem for linear hyperbolic system of the equations with zero generalized Laplace invariants* // Ufimskij matem. zhurn. 2010. V. 2, No. 2. P. 20-26. (in Russian).
4. Yu.G. Mikhailova, A.V. Zhiber. *On hyperbolic systems of equations with zero generalized Laplace invariants* // Trudy inst. matem. mekhan. UrO RAN. 2007. V. 13, No. 4. P. 73-82. [Proc. Steklov Inst. Math. Suppl. 2008. V. 261, No. 1. P. 154-164.]
5. Yu.G. Voronova. *Construction of solution to Goursat problem for nonlinear hyperbolic equations with integrals of first and second order* // Proc. of International School-Conference for students, PhD students and young scientists. Ufa, BSU. 2012. V. 1. P. 51-58. (in Russian).
6. A.M. Gur'eva. *Laplace method of cascade integration and nonlinear hyperbolic systems of equations.* PhD thesis. 2005. (in Russian).

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