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WIENER'S THEOREM FOR PERIODIC AT INFINITY FUNCTIONS WITH SUMMABLE WEIGHTED FOURIER SERIES

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Abstract. In the article we define a Banach algebra of periodic at infinity functions. For this class of functions we introduce the notions of a Fourier series, its absolute convergence, and invertibility. We obtain an analogue of Wiener's theorem on absolutely convergent Fourier series for periodic at infinity functions whose Fourier coefficients are summable with a weight.

Keywords: Banach space, slowly varying at infinity functions, periodic at infinity functions, Wiener's theorem, absolutely convergent Fourier series, invertibility.

Mathematics Subject Classification: 46J10.

1. Introduction

Let $l^1(\mathbb{Z})$ be the Banach space of two-sided summable sequences $a : \mathbb{Z} \to \mathbb{C}$ with the norm $\|a\|_1 = \sum_{k\in\mathbb{Z}} |a(k)| < \infty$.

By the symbol $C_\omega(\mathbb{R})$ we shall indicate the Banach space of all continuous ω-periodic functions $f : \mathbb{R} \to \mathbb{C}$.

We say that a function $f \in C_\omega(\mathbb{R})$ has an *absolutely convergent Fourier series* if it can be represented as the series $f(t) = \sum_{k\in\mathbb{Z}} a(k)e^{i\frac{2\pi k}{\omega}t}$, $t \in \mathbb{R}$, where $a \in l^1(\mathbb{Z})$. We denote the set of all such functions by $AC_\omega(\mathbb{R})$. We observe that $AC_\omega(\mathbb{R})$ is a Banach algebra (cf. [1]) with the pointwise multiplication and the norm

$$\|f\|_{AC} = \|a\|_1 = \sum_{k\in\mathbb{Z}} |a(k)|.$$

In terms of the introduced notations Wiener's theorem reads as follows.

Theorem 1. *If a function f belongs to $AC_\omega(\mathbb{R})$ and $f(t) \neq 0$ for each $t \in \mathbb{R}$, then $1/f \in AC_\omega(\mathbb{R})$, i.e. $1/f(t) = \sum_{k\in\mathbb{Z}} b(k)e^{i\frac{2\pi k}{\omega}t}$, where $b \in l^1(\mathbb{Z})$.*

The proof of Theorem 1 is given in [2].

Wiener's theorem was generalized in several directions. We mention Bochner-Fillips theorem [3] for the functions with values in a Banach algebra, as well as papers [4], [5], where Wiener's theorem was proved for the operators whose matrices have absolutely summable diagonals. The references to the studies related with applications of the results are given in [6].

In the present paper we extend Wiener's theorem for the class of periodic at infinity functions.

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We introduce the set of periodic at infinite functions. Let X be a complex Banach space, $End\, X$ be the Banach algebra of linear bounded operators acting in X.

By the symbol $C_{b,u} = C_{b,u}(\mathbb{R}, X)$ we denote the Banach space of continuous and bounded on $\mathbb{R}$ functions with values in X, the norm in this space is $\|x\|_\infty = \sup\limits_{t\in\mathbb{R}} \|x(t)\|_X$. The symbol $C_0 = C_0(\mathbb{R}, X)$ will be employed to indicate the closed subspace of $C_{b,u}$ consisting of the functions decaying at infinity.

In the Banach space $C_{b,u}$ we consider an isometric group of operators (or representation) $S : \mathbb{R} \to End\, C_{b,u}$ acting by the rule

$$(S(\alpha)x)(t) = x(t + \alpha),\ \alpha \in \mathbb{R}. \tag{1}$$

Definition 1. A function $x \in C_{b,u}(\mathbb{R}, X)$ is called *slowly varying or stationary at infinity* if

$$S(\alpha)x - x \in C_0(\mathbb{R}, X) \text{ for each } \alpha \in \mathbb{R}.$$

For instance, a function $f \in C_{b,u}(\mathbb{R}, \mathbb{C})$ being $f(t) = \sin \ln(1 + t^2)$ is slowly varying at infinity.

Definition 2. A function $x \in C_{b,u}(\mathbb{R}, X)$ is called *periodic at infinity of period* $\omega > 0$ if

$$S(\omega)x - x \in C_0(\mathbb{R}, X).$$

The definition of periodic at infinity function was suggested by A.G. Baskakov and was employed in paper [7].

We denote the set of slowly varying at infinity functions by the symbol $C_{sl} = C_{sl}(\mathbb{R}, X)$, while the symbol $C_{\omega,\infty} = C_{\omega,\infty}(\mathbb{R}, X)$ stands for the functions periodic at infinity of period ω.

In case $X = \mathbb{C}$, the considered spaces will be indicated as $C_{b,u}(\mathbb{R})$, $C_0(\mathbb{R})$, $C_{sl}(\mathbb{R})$, $C_{\omega,\infty}(\mathbb{R})$.

We note that $C_{\omega,\infty}(\mathbb{R}, X)$ is a Banach space with the norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|_X.$$

Moreover, $C_{sl}(\mathbb{R}, X)$ and $C_{\omega,\infty}(\mathbb{R}, X)$ form Banach algebras with pointwise multiplication if X is a Banach algebra.

Definition 3. Given a function $x \in C_{\omega,\infty}(\mathbb{R}, X)$, we call the series

$$x(t) \sim \sum_{n\in\mathbb{Z}} x_n(t) e^{i\frac{2\pi n}{\omega}t}, \quad t \in \mathbb{R},$$

its *generalized Fourier series*, where the functions x_n, $n \in \mathbb{Z}$, are defined by the formulae

$$x_n(t) = \frac{e^{-i\frac{2\pi n}{\omega}t}}{\omega} \int_0^\omega x(t+\tau) e^{-i\frac{2\pi n}{\omega}\tau} d\tau, \quad t \in \mathbb{R},\ n \in \mathbb{Z}, \tag{2}$$

and are called *Fourier coefficients* for function x. We shall say that the generalized Fourier series of function x converges absolutely if there exist functions $y_n \in C_{sl}(\mathbb{R}, X)$, $n \in \mathbb{Z}$, such that $y_n - x_n \in C_0(\mathbb{R}, X)$ and $\sum_{n\in\mathbb{Z}} \|y_n\|_\infty < \infty$.

In what follows we shall omit the word "generalized". It is also possible that a considered Fourier series does not converge to function x. In this case it is regarded as a formal series.

Example 1. As an example of a function in $C_{\omega,\infty}(\mathbb{R})$ with an absolutely convergent Fourier series we consider the function $f : \mathbb{R} \to \mathbb{C}$ defined as

$$f(t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{n^2} \sin(\alpha_n \ln(1 + t^2)) \right) e^{i\frac{2\pi n}{\omega}t}, \quad t \in \mathbb{R}, \quad \alpha_n \in \mathbb{R}. \tag{3}$$

We note that the functions f_n, $n \in \mathbb{Z}$, constructed for function f by formula (2) do not coincide with the functions $y_n : t \mapsto \frac{1}{n^2} \sin \alpha_n \ln(1 + t^2)$, $t \in \mathbb{R}$, $n \in \mathbb{Z}$, however, $f_n - y_n \in C_0(\mathbb{R})$.

Remark 1. If $x \in C_\omega(\mathbb{R})$, then the Fourier series in Definition 3 coincides with the usual Fourier series of function x.

In what follows we shall make use of the notation

$$e_n(t) = e^{i\frac{2\pi n}{\omega}t}, \quad t \in \mathbb{R},\ n \in \mathbb{Z}.$$

We observe that the mapping $x \mapsto P_n x = x_n e_n : C_{\omega,\infty}(\mathbb{R}, X) \to C_{\omega,\infty}(\mathbb{R}, X)$, $n \in \mathbb{Z}$, is a bounded operator obeying $\|P_n\| \le 1$. Moreover, $\mathrm{Im}\,(P_n^2 - P_n) \subset C_0(\mathbb{R}, X)$ for the image $\mathrm{Im}\,(P_n^2 - P_n)$ of the operator $P_n^2 - P_n$ (the proof is given in the end of Section 3), however, the operators P_n, $n \in \mathbb{Z}$, are not projectors.

Till the end of this section the symbol X will indicate a Banach algebra.

Definition 4. We call a function $x \in C_{b,u}(\mathbb{R}, X)$ *invertible w.r.t. subspace* $C_0(\mathbb{R}, X)$ if there exists a function $y \in C_{b,u}(\mathbb{R}, X)$ such that $xy - 1 \in C_0(\mathbb{R}, X)$. We call function y *inverse for x w.r.t. subspace* $C_0(\mathbb{R}, X)$.

Remark 2. From Definition 4 it immediately follows that a function $x \in C_{\omega,\infty}(\mathbb{R}, X)$ is invertible w.r.t. subspace $C_0(\mathbb{R}, X)$ if and only if it can be represented as $x = y + x_0$, where $x_0 \in C_0(\mathbb{R}, X)$, and function $y \in C_{\omega,\infty}(\mathbb{R}, X)$ is so that $\inf\limits_{t\in\mathbb{R}} \|y(t)\|_X > 0$. Definition 4 also implies that a function $x \in C_{\omega,\infty}(\mathbb{R}, X)$ is invertible w.r.t. subspace $C_0(\mathbb{R}, X)$ if and only if there exists a number $A > 0$ such that $\inf\limits_{|t|>A} \|x(t)\|_X > 0$.

It is easy to see that if y_1, y_2 are inverse for $x \in C_{b,u}(\mathbb{R}, X)$ w.r.t. subspace $C_0(\mathbb{R}, X)$, then $y_1 - y_2 \in C_0(\mathbb{R}, X)$.

Consider a function $a \in C_{\omega,\infty}(\mathbb{R}, X)$ and introduce the notation $d_a(k) = \|a_k\|_\infty$, $k \in \mathbb{Z}$, where a_k is the k-th Fourier coefficient for function a defined by formula (2).

The considered function is supposed to satisfy one of the conditions in the following assumption.

Assumption 1. *Function $a \in C_{\omega,\infty}(\mathbb{R}, X)$ satisfies one of the conditions:*
1) $\sum\limits_{k\in\mathbb{Z}} d_a(k)\alpha(k) < \infty$, *where* $\alpha : \mathbb{Z} \to \mathbb{R}_+$ *is a weight obeying the relation* $\lim\limits_{|k|\to\infty} \frac{\ln \alpha(k)}{|k|} = 0$;
2) $\lim\limits_{|k|\to\infty} d_a(k)|k|^\gamma = 0, \quad k \in \mathbb{Z},\ \gamma > 1$;
3) $d_a(k) \le Const \exp(-\varepsilon|k|), \quad k \in \mathbb{Z},\ \varepsilon > 0$.
In particular, the assumption holds true, if the Fourier series of function a has a finite number of non-zero Fourier coefficients, that is, there exists $M \in \mathbb{N}$ such that $d_a(k) = 0$, $|k| \ge M + 1$.

The main result of the present work is

Theorem 2. *If an invertible w.r.t. subspace $C_0(\mathbb{R}, X)$ function $a \in C_{\omega,\infty}(\mathbb{R}, X)$ satisfies one of the conditions 1)–3) of Assumption 1, then its inverse b w.r.t. $C_0(\mathbb{R}, X)$ obeys the corresponding condition among the following ones:*
1') $\sum\limits_{k\in\mathbb{Z}} d_b(k)\alpha(k) < \infty$;
2') $\lim\limits_{|k|\to\infty} d_b(k)|k|^\gamma = 0$;
3') $d_b(k) \le Const \exp(-\varepsilon_0|k|)$, $k \in \mathbb{Z}$ *for some* $\varepsilon_0 > 0$.

We note that quantities $Const$ and ε_0 depend on quantities $Const$ and ε in the conditions of Assumption 1.

2. Periodic vectors and their Fourier series

Let $\mathcal{B}$ be a Banach algebra with the unit and ω is a positive number. Consider a $\omega-$periodic isometric strongly continuous group of operators (representation) $T : \mathbb{R} \to End\,\mathcal{B}$ acting in B

and having the properties

$$\begin{aligned} &T(t)(ab) = T(t)a \cdot T(t)b, \\ &T(t)e = e, \quad t \in \mathbb{R}, \end{aligned} \tag{4}$$

where a, b are arbitrary elements in $\mathcal{B}$, and e is the unit in algebra $\mathcal{B}$.

Thus, each of the operators $T(t)$, $t \in \mathbb{R}$, is a homomorphism of algebra $\mathcal{B}$, and each function $t \mapsto T(t)a : \mathbb{R} \to \mathcal{B}$, $a \in \mathcal{B}$ is a continuous ω-periodic function.

The above properties immediately yield that if an element $a \in \mathcal{B}$ is invertible, then

$$e = T(t)e = T(t)(aa^{-1}) = (T(t)a)T(t)a^{-1} = (T(t)a^{-1})T(t)a, \quad a \in \mathcal{B},$$

and hence, $(T(t)a)^{-1} = T(t)a^{-1}$.

Consider the Fourier series (see [8])

$$T(t)a \sim \sum_{n\in\mathbb{Z}} a_n e^{i\frac{2\pi n}{\omega}t}, \quad t \in \mathbb{R},$$

of the function $t \mapsto T(t)a : \mathbb{R} \to \mathcal{B}$, $a \in \mathcal{B}$, where the Fourier coefficients are defined by the formulae

$$a_n = \frac{1}{\omega}\int_0^\omega T(t)ae^{-i\frac{2\pi n}{\omega}t}dt, \quad n \in \mathbb{Z}. \tag{5}$$

We call the series $a \sim \sum_{n\in\mathbb{Z}} a_n$ *Fourier series* of an element $a \in \mathcal{B}$ and a_n, $n \in \mathbb{Z}$, are called *Fourier coefficients* of this element.

If the Fourier series of an element $a \in \mathcal{B}$ *converges absolutely*, i.e. the condition $\sum_{n\in\mathbb{Z}} \|a_n\| < \infty$ holds true, then the identity $a = \sum_{n\in\mathbb{Z}} a_n$ is valid.

Lemma 1. *Let $a \in \mathcal{B}$. Then $T(\alpha)a_n = e^{i\frac{2\pi n}{\omega}\alpha}a_n$, $n \in \mathbb{Z}$, for each $\alpha \in \mathbb{R}$, where a_n, $n \in \mathbb{Z}$, are the Fourier coefficients of an element a. At that, the operators P_n defined by the formula $P_n a = a_n = \frac{1}{\omega}\int_0^\omega T(t)ae^{-i\frac{2\pi n}{\omega}t}dt$, $n \in \mathbb{Z}$, are the projectors with $\|P_n\| \le 1$, $n \in \mathbb{Z}$.*

Proof. We take an arbitrary element $a \in \mathcal{B}$ and fix an arbitrary number $\alpha \in \mathbb{R}$. Let a_n, $n \in \mathbb{Z}$, be the Fourier coefficient of element a defined by formula (5). Then they satisfy the following chain of identities

$$\begin{aligned} T(\alpha)a_n =&T(\alpha)\left(\frac{1}{\omega}\int_0^\omega T(t)ae^{-i\frac{2\pi n}{\omega}t}dt\right) = \frac{1}{\omega}\int_0^\omega T(\alpha)T(t)ae^{-i\frac{2\pi n}{\omega}t}dt \\ =&\frac{1}{\omega}\int_0^\omega T(\alpha+t)ae^{-i\frac{2\pi n}{\omega}t}dt = \frac{e^{i\frac{2\pi n}{\omega}\alpha}}{\omega}\int_\alpha^{\omega+\alpha} T(\tau)ae^{-i\frac{2\pi n}{\omega}\tau}d\tau \\ =&\frac{e^{i\frac{2\pi n}{\omega}\alpha}}{\omega}\int_0^\omega T(\tau)ae^{-i\frac{2\pi n}{\omega}t}d\tau = e^{i\frac{2\pi n}{\omega}\alpha}a_n, \quad n \in \mathbb{Z}. \end{aligned}$$

That is, we have shown that $T(\alpha)a_n = e^{i\frac{2\pi n}{\omega}\alpha}a_n$, $n \in \mathbb{Z}$, for each $\alpha \in \mathbb{R}$.

Now let us show that the operators P_n, $n \in \mathbb{Z}$, defined by the formula $P_n a = a_n$ are projectors, i.e. $P_n^2 = P_n$, $n \in \mathbb{Z}$.

Let $a \in \mathcal{B}$. Then

$$P_n a = \frac{1}{\omega}\int_0^{\omega} T(t) a e^{-i\frac{2\pi n}{\omega}t} dt, \quad n \in \mathbb{Z},$$

$$P_n^2 a = P_n(P_n a) = \frac{1}{\omega}\int_0^{\omega} T(t) a_n e^{-i\frac{2\pi n}{\omega}t} dt = \frac{1}{\omega}\int_0^{\omega} a_n dt = a_n = P_n a, \quad n \in \mathbb{Z}.$$

Let us show that $\|P_n\| \leq 1$, $n \in \mathbb{Z}$. Employing the property $\|T(t)\| = 1$, $t \in \mathbb{R}$, we obtain

$$\|P_n\| = \sup_{\|a\|\leq 1} \|P_n a\| = \sup_{\|a\|\leq 1} \|\frac{1}{\omega}\int_0^{\omega} T(t) a e^{-i\frac{2\pi n}{\omega}t} dt\| \leq$$

$$\leq \sup_{\|a\|\leq 1} \frac{1}{\omega}\int_0^{\omega} \|T(t)a\| dt \leq \sup_{\|a\|\leq 1} \frac{1}{\omega}\int_0^{\omega} \|T(t)\|\|a\| dt \leq 1.$$

The proof is complete. □

Given an element $a \in \mathcal{B}$, we consider the operator $A \in End\,\mathcal{B}$ of the form

$$Ax = ax, \quad x \in \mathcal{B}.$$

We associate with this operator a ω-periodic operator-valued function $\Phi_A : \mathbb{R} \to End\,\mathcal{B}$ defined by the formula

$$\Phi_A(t) = T(t)AT(-t), \quad t \in \mathbb{R}.$$

We associate with function Φ_A its Fourier series

$$\Phi_A(t) \sim \sum_{n\in\mathbb{Z}} A_n e^{i\frac{2\pi n}{\omega}t}, \quad t \in \mathbb{R},$$

where the Fourier coefficients are defined by the formulae

$$A_n = \frac{1}{\omega}\int_0^{\omega} T(t)AT(-t) e^{-i\frac{2\pi n}{\omega}t} dt, \quad n \in \mathbb{Z}. \tag{6}$$

We call a series $\sum_{n\in\mathbb{Z}} A_n$ *Fourier series of operator* A, and the operators A_n are called *Fourier coefficients of this operator*. We define a two-sided number sequence $(d_A(n))$ by letting $d_A(n) = \|A_n\|$, $n \in \mathbb{Z}$.

Lemma 2. *The Fourier coefficients A_n, $n \in \mathbb{Z}$, of an operator A satisfy the representations $A_n x = a_n x$, $n \in \mathbb{Z}$, $x \in \mathcal{B}$. At that, $\|A_n\| = \|a_n\|$, $n \in \mathbb{Z}$.*

Proof. Let us show that $A_n x = a_n x$ for each $x \in \mathcal{B}$.

Employing formulae (5) and (6) as well as the fact that the operators $T(t)$, $t \in \mathbb{R}$, form a homomorphism of the algebra, we obtain

$$\begin{aligned} A_n x =& \frac{1}{\omega}\int_0^{\omega} T(t)AT(-t)x e^{-i\frac{2\pi n}{\omega}t} dt = \frac{1}{\omega}\int_0^{\omega} T(t)(aT(-t)x) e^{-i\frac{2\pi n}{\omega}t} dt \\ =& \frac{1}{\omega}\int_0^{\omega} (T(t)a)T(t)(T(-t)x) e^{-i\frac{2\pi n}{\omega}t} dt = \left(\frac{1}{\omega}\int_0^{\omega} T(t) a e^{-i\frac{2\pi n}{\omega}t} dt\right) x = a_n x. \end{aligned}$$

The inequality $\|A_n x\| \leq \|a_n\|\|x\|$ holds true for each $x \in \mathcal{B}$.

Since $a_n = A_n e$ and $\|e\| = 1$, then $\|A_n\| = \|a_n\|, n \in \mathbb{Z}$. The proof is complete. □

We observe that if the Fourier series of an operator A converges absolutely, i.e.

$$\sum_{n\in\mathbb{Z}} d_A(n) = \sum_{n\in\mathbb{Z}} \|a_n\| < \infty,$$

then function Φ_A is continuous in the uniform operator topology.

We suppose that for the considered operator one of the conditions in the following assumption is fulfilled.

Assumption 2. *Operator $A \in End\,\mathcal{B}$ satisfies one of the following conditions:*
1) $\sum\limits_{k\in\mathbb{Z}} d_A(k)\alpha(k) < \infty$, where $\alpha : \mathbb{Z} \to \mathbb{R}_+$ is a weight satisfying the relation $\lim\limits_{|k|\to\infty} \frac{\ln \alpha(k)}{|k|} = 0$;
2) $\lim\limits_{|k|\to\infty} d_A(k)|k|^\gamma = 0$, $k \in \mathbb{Z}$, $\gamma > 1$;
3) $d_A(k) \le Const\exp(-\varepsilon|k|)$, $k \in \mathbb{Z}$, $\varepsilon > 0$.
In particular, the assumption holds true if the Fourier series of operator A comprises finitely many non-zero Fourier coefficients, i.e. there exists $M \in \mathbb{N}$ such that $d_A(k) = 0$, $|k| \ge M + 1$.

In what follows we shall make use of

Theorem 3. *Suppose that an operator $A \in End\,\mathcal{B}$ is invertible and satisfies one of Conditions 1)–3) of Assumption 2. Then the inverse operator $B = A^{-1} \in End\,\mathcal{B}$ satisfies the corresponding condition among the following ones:*
1') $\sum\limits_{k\in\mathbb{Z}} d_B(k)\alpha(k) < \infty$;
2') $\lim\limits_{|k|\to\infty} d_B(k)|k|^\gamma = 0$;
3') $d_B(k) \le Const\exp(-\varepsilon_0|k|)$, $k \in \mathbb{Z}$, for some $\varepsilon_0 > 0$.

This theorem follows from [9, Thm. 1].

3. Harmonic analysis of periodic at infinity functions

Throughout this section X stands for a Banach algebra with unit.

It is clear that the group of shifts S defined by formula (1) is not periodic in the space of periodic at infinity functions.

In what follows, by the symbol $\mathcal{B}$ we denote the factor-space $C_{\omega,\infty}(\mathbb{R}, X)/C_0(\mathbb{R}, X)$ which becomes an algebra if we define the multiplication as

$$\widetilde{x}\widetilde{y} = \widetilde{xy}, \quad \widetilde{x}, \widetilde{y} \in \mathcal{B}. \tag{7}$$

In this factor-space we construct as isometric group of operators $T : \mathbb{R} \to End\,\mathcal{B}$ acting by the rule

$$T(t)\widetilde{x} = \widetilde{S(t)x} = S(t)x + C_0(\mathbb{R}, X),\ t \in \mathbb{R}, \tag{8}$$

where x is an element of class $\widetilde{x} \in \mathcal{B}$.

Since

$$\begin{aligned} T(\omega)\widetilde{x} =& \widetilde{S(\omega)x} = S(\omega)x + C_0(\mathbb{R}, X) \\ =& (S(\omega)x - x) + x + C_0(\mathbb{R}, X) = x + C_0(\mathbb{R}, X) = \widetilde{x}, \end{aligned}$$

representation T is ω-periodic. Moreover, the strong continuity of presentation S implies the same for representation T.

In terms of group T, the belonging of a class $\widetilde{x}$ to algebra $\mathcal{B}$ means that $T(\omega)\widetilde{x} = \widetilde{x}$. The Fourier series of a function $x \in C_{\omega,\infty}(\mathbb{R}, X)$ being an element of a class $\widetilde{x}$ reads as $x(\tau) \sim$

$\sum_{n\in\mathbb{Z}} x_n(\tau)e^{i\frac{2\pi n}{\omega}\tau}$, where the Fourier coefficients x_n, $n \in \mathbb{Z}$, are determined by formula (2), while the mean x_0 is

$$x_0(t) = \frac{1}{\omega}\int_0^\omega x(t+\tau)d\tau, \quad t \in \mathbb{R}.$$

We have

Lemma 3. *The Fourier coefficients of a function $x \in C_{\omega,\infty}(\mathbb{R}, X)$ possess the property $x_n \in C_{sl}(\mathbb{R}, X)$, $n \in \mathbb{Z}$.*

Proof. Let us show first that mean x_0 of function $x \in C_{\omega,\infty}(\mathbb{R}, X)$ belongs to space $C_{sl}(\mathbb{R}, X)$. We take an arbitrary number $\alpha \in \mathbb{R}$ and let us show that $(S(\alpha)x_0 - x_0) \in C_0(\mathbb{R}, X)$. From Lemma 1 it follows immediately that the class $\widetilde{x}_0$ comprising function x_0 obeys the identity $T(\alpha)\widetilde{x}_0 = \widetilde{x}_0$, i.e. x_0 satisfies $(S(\alpha)x_0 - x_0) \in C_0(\mathbb{R}, X)$. Since number $\alpha \in \mathbb{R}$ is arbitrary, the definition of slowly varying at infinity function yields $x_0 \in C_{sl}(\mathbb{R}, X)$.

Now let us prove this property for the Fourier coefficients x_n, $n \in \mathbb{Z}$, of function x. Introducing the notation $y(t) = x(t)e^{i\frac{2\pi n}{\omega}t}$, $t \in \mathbb{R}$, $n \in \mathbb{Z}$, we obtain that $S(\omega)y - y \in C_0(\mathbb{R}, X)$, i.e. $y \in C_{\omega,\infty}(\mathbb{R}, X)$. Then the mean of function y defined by the formula $y_0(t) = \frac{1}{\omega}\int_0^\omega x(t+\tau)e^{-i\frac{2\pi n}{\omega}(t+\tau)}d\tau$, $t \in \mathbb{R}$, belongs to space $C_{sl}(\mathbb{R}, X)$. Comparing the latter formula with formula (2), we obtain that $x_n \in C_{sl}(\mathbb{R}, X)$, $n \in \mathbb{Z}$. The proof is complete. □

Thus, we have the factor-algebra $\mathcal{B} = C_{\omega,\infty}(\mathbb{R}, X)/C_0(\mathbb{R}, X)$ and the ω-periodic strongly continuous isometric group of operators (representation) T acting in this factor-algebra and defined by formula (8).

With representation T we associate its Fourier series

$$T(t)\widetilde{x} \sim \sum_{n\in\mathbb{Z}} \widetilde{P_n}\widetilde{x}e^{i\frac{2\pi n}{\omega}t}, \quad t \in \mathbb{R}, \quad \widetilde{x} \in \mathcal{B}.$$

The Fourier coefficients of representation T read as

$$\widetilde{P_n}\widetilde{x} = \frac{1}{\omega}\int_0^\omega T(t)\widetilde{x}e^{-i\frac{2\pi n}{\omega}t}dt, \quad n \in \mathbb{Z}.$$

On the elements of the considered classes we have

$$(P_n x)(\tau) = \frac{1}{\omega}\int_0^\omega (S(t)x)(\tau)e^{-i\frac{2\pi n}{\omega}t}dt = \frac{1}{\omega}\int_0^\omega x(t+\tau)e^{-i\frac{2\pi n}{\omega}t}dt = x_n(\tau)e^{i\frac{2\pi n}{\omega}\tau},$$

where x_n, $n \in \mathbb{Z}$, are the Fourier coefficients of function x defined by formula (2).

Directly from formula (5) it follows that the Fourier coefficient of representation T satisfy the identity

$$\widetilde{P_n}\widetilde{x} = \widetilde{x_n}, \quad n \in \mathbb{Z}.$$

Let x be an element of class $\widetilde{x} \in \mathcal{B}$. Then the latter identity means that $\widetilde{P_n x} = \widetilde{x_n}$, i.e. $P_n x - x_n \in C_0(\mathbb{R}, X)$, $n \in \mathbb{Z}$. Since $\widetilde{P_n}$ are projectors, the identity $\widetilde{P_n}^2\widetilde{x} = \widetilde{P_n}\widetilde{x} = \widetilde{x_n}$, $n \in \mathbb{Z}$, holds true. This is why $\widetilde{P_n^2 x} = \widetilde{x_n}$, i.e. $P_n^2 x - x_n \in C_0(\mathbb{R}, X)$, $n \in \mathbb{Z}$. It follows that $P_n^2 x - P_n x \in C_0(\mathbb{R}, X)$, $n \in \mathbb{Z}$, i.e. $\mathrm{Im}\,(P_n^2 - P_n) \subset C_0(\mathbb{R}, X)$.

If the Fourier series of class $\widetilde{x} \in \mathcal{B}$ *converges absolutely*, i.e. the condition

$$\sum_{n\in\mathbb{Z}} \|\widetilde{x_n}\| < \infty$$

holds true, then from the properties of the norm in the factor-space it follows that in this case there exist elements y_n in classes $\widetilde{x_n}$ satisfying

$$\sum_{n\in\mathbb{Z}} \|y_n\|_\infty < \infty.$$

We note that function $x \in C_{\omega,\infty}(\mathbb{R}, X)$ is invertible w.r.t. $C_0(\mathbb{R}, X)$ if and only if the class $\widetilde{x} \in \mathcal{B}$, comprising it, is invertible. This statement is implied by Definition 4.

4. Proof of Theorem 2

In order to obtain the main results, as algebra B, we consider the factor-algebra $C_{\omega,\infty}(\mathbb{R}, X)/C_0(\mathbb{R}, X)$, and as the representation $T : \mathbb{R} \to End\,\mathcal{B}$, we consider the ω-periodic group of isometric operators $T : \mathbb{R} \to End\,\mathcal{B}$ defined by formula (8).

Let us show that group T possesses properties (4).

By employing formulae (7) and (8), we obtain that

$$\begin{aligned} T(t)(\widetilde{x}\widetilde{y}) =& T(t)(\widetilde{xy}) = \widetilde{S(t)(xy)} = S(t)xS(t)y + C_0(\mathbb{R}, X) \\ =& (T(t)\widetilde{x})\, T(t)\widetilde{y}, \quad x \in \widetilde{x}, y \in \widetilde{y}, \quad t \in \mathbb{R}, \end{aligned}$$

i.e. property (4) indeed holds for group T.

Consider the operator $A \in End\,\mathcal{B}$

$$A\widetilde{x} = \widetilde{a}\widetilde{x}, \quad \widetilde{a} \in \mathcal{B}. \tag{9}$$

With this operator we associate the ω-periodic operator-valued function $\Phi_A : \mathbb{R} \to End\,\mathcal{B}$ defined by the formula

$$\Phi_A(t) = T(t)AT(-t), \;\; t \in \mathbb{R}.$$

Theorem 3 holds true for the considered operator.

Proof of Theorem 2. Consider the Banach algebra $\mathcal{B} = C_{\omega,\infty}(\mathbb{R}, X)/C_0(\mathbb{R}, X)$ and the ω-periodic isometric group of operators T acting in this algebra and defined by formula (8).

Given the invertible function $a \in C_{\omega,\infty}(\mathbb{R}, X)$ introduced in the hypothesis of the theorem, we construct the class $\widetilde{a} \in \mathcal{B}$ which is invertible as well. Denoting the inverse class by the symbol $\widetilde{b}$, we obtain that $\widetilde{a}\widetilde{b} = \widetilde{1}$.

We introduce the operator $A \in End\,\mathcal{B}$ by formula (9). This is the operator of multiplication by element $\widetilde{a} \in \mathcal{B}$ and it is invertible. Then its inverse acts as

$$B\widetilde{x} = \widetilde{b}\widetilde{x}, \quad \widetilde{b} \in \mathcal{B}.$$

Theorem 3 also holds for operator A, and hence, there exists an element b of class $\widetilde{b}$ such that $ab - 1 \in C_0(\mathbb{R}, X)$ and it satisfies the appropriate condition in Theorem 2. The proof is complete. □

Corollary 1. *If a function $a \in C_{\omega,\infty}(\mathbb{R}, X)$ is invertible w.r.t. subspace $C_0(\mathbb{R}, X)$ and it has the absolutely convergent Fourier series, then the Fourier series of the inverse w.r.t. $C_0(\mathbb{R}, X)$ function converges absolutely as well.*

Corollary 2. *If a function $a \in C_{\omega,\infty}(\mathbb{R}, X)$ is invertible w.r.t. subspace $C_0(\mathbb{R}, X)$ and its Fourier series converges absolutely, then there exists a function $b \in C_{\omega,\infty}(\mathbb{R}, X)$ with an absolutely convergent Fourier series such that $ab - 1 \in C_0(\mathbb{R}, X)$.*

In conclusion we should mention that in recent paper [10] almost periodic at infinity functions were introduced. And there naturally appear the questions similar to ones studied in the present paper.

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