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INTERPOLATION WITH MULTIPLICITY BY SERIES OF EXPONENTIALS IN $H(\mathbb{C})$ WITH NODES ON REAL AXIS

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Abstract. In the space of entire functions, we study the problem on interpolation with multiplicity by the functions from a closed subspace which is invariant with respect to the operator of differentiation. The discrete set of the nodes for the interpolation with multiplicity is located on the real axis in the complex plane. The proof is based on the passage from the subspace to its subspace consisting of all series of exponentials converging in the topology of uniform convergence on compact sets. We obtain a solvability criterion for the problem of interpolation with multiplicity by series of exponentials for real nodes in the terms of location of exponents of exponentials.

Keywords: entire function, interpolation with multiplicity, series of exponents, ideal, Fischer representation.

Mathematics Subject Classification: 30E05, 46N20.

1. Preliminaries

Denote by $H(\mathbb{C})$ the space of entire functions with the topology of uniform convergence on compact sets. As a corollary of classical Weierstrass and Mittag-Leffler theorems, we obtain that the following problem of the interpolation with multiplicities by the functions in $H(\mathbb{C})$ (see [1, p. 32]) is solvable.

For each locally analytic function ω on a given discrete set of points $\{\mu_k\}$ in the complex plane and natural numbers m_k, $k \in \mathbb{N}$, there exists an entire function g such that $|\omega(z) - g(z)| = O(|z-\mu_k|^{m_k})$ as $z \to \mu_k$, $k \in \mathbb{N}$. We shall call points μ_k interpolation nodes and numbers m_k will be called multiplicities of nodes μ_k.

This statement is equivalent to the solvability of the problem on interpolation with multiplicity by entire functions in the traditional formulation.

For an arbitrary discrete set of interpolation nodes $\mu_k \in \mathbb{C}$ with multiplicities m_k and for each interpolation data

$$b_k^j \in \mathbb{C}, \quad j = 0, 1, \cdots, m_k - 1, \quad k \in \mathbb{N},$$

there exists an entire function g such that

$$g^{(j)}(\mu_k) = b_k^j, \quad j = 0, 1, \cdots, m_k - 1, \quad k \in \mathbb{N}.$$

In the case of a finite number of interpolation nodes μ_k, $k = 1, 2, \ldots, r$, with multiplicities m_k, there was studied the problem of interpolation not only by polynomials, but also by the functions in the kernels of linear and nonlinear differential operators in various functional spaces, for instance, the multi-points Vallée Poussin problem ([2]). In the category of the spaces of holomorphic functions, for a finite-dimensional kernel of a differential operators with constant coefficients of finite order, this problem is equivalent to the algebraic problem on the

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existence and uniqueness of solutions to a non-homogeneous system of linear equations with constant coefficients. It leads one to the known results ([3]) on the unique solvability of a global holomorphic Cauchy problem (or holomorphic multi-points Vallée Poussin problem) and there are also some advances in the case of partial differential operators of special form in space $H(\mathbb{C}^n)$ ([4]-[6]).

In the monograph [7], there were considered computational aspects of the problem on interpolation by finite sums of exponents in a finite number of nodes.

We also mention that in surveys [8], [9], there was formulated an unsolved problem on interpolation by the functions in the infinite-dimensional kernel of so-called algebraic differential operator in space $H(\mathbb{C})$ and some results were mentioned.

We denote by $P_{\mathbb{C}}$ the space of entire functions of exponential type with the traditional (LN^*)-topology ensuring a topological isomorphism between the strong dual space $H^*(\mathbb{C})$ and the space $P_{\mathbb{C}}$, and this isomorphism is implemented by the Laplace transform $\mathcal{L}$ of functionals $F \in H^*(\mathbb{C})$. More precisely, a linear continuous one-to-one Laplace transform $\mathcal{L}$ of functionals $F \in H^*(\mathbb{C})$ is defined as $\mathcal{L} : F \longmapsto \mathcal{L}F(z) = \langle F_\lambda, e^{\lambda z}\rangle$, $\mathcal{L}F \in P_{\mathbb{C}}$, and then the usual duality is employed.

Each function $\varphi \in P_{\mathbb{C}}$, $\varphi \not\equiv 0$, generates a linear continuous surjective convolution operator $M_\varphi : H(\mathbb{C}) \longmapsto H(\mathbb{C})$ in the space of entire functions $H(\mathbb{C})$. Its action on functions $f \in H(\mathbb{C})$ is defined as

$$M_\varphi[f](z) = \langle F_\lambda, f(z+\lambda)\rangle \,, \; z \in \mathbb{C},$$

where $F = \mathcal{L}^{-1}\varphi \in H^*(\mathbb{C})$, see details in [10], [11], and below in the proof of Theorem 1. An entire function φ of exponential type is called characteristic function for the convolution operator M_φ.

By $\operatorname{Ker} M_\varphi = \{f \in H(\mathbb{C}) : M_\varphi[f] = 0\}$ we denote the kernel of convolution operator M_φ. This kernel is a closed subspace in $H(\mathbb{C})$ and it is invariant w.r.t. the differentiation operator.

In paper [12], the following theorem was proven (see also [13]).

Theorem A. *Fix a number $\alpha \in \big[0, \frac{\pi}{2}\big)$. Denote by φ an entire function of exponential type with simple zeroes λ_n so that there are infinitely many of λ_n in each of the angles $A_\alpha(0) = \{z \in \mathbb{C} : |\arg z| \leqslant \alpha\}$ and $A_\alpha(\pi) = \{z \in \mathbb{C} : |\arg z - \pi| \leqslant \alpha\}$. Suppose that an infinite discrete set of interpolation nodes $\{\mu_{\pm k}\}_{k=1}^{+\infty}$ lies on the real axis and $\mu_{\pm k} \to \pm\infty$, $k \to +\infty$. Denote by ψ an arbitrary entire function having simple zero at each of the nodes $\mu_{\pm k}$ and no zeroes at any other point.*

Then for each entire function g, there exists an entire function $f \in \operatorname{Ker} M_\varphi$ such that the function $r = (g-f)/\psi$ is entire.

In the proof in [12], there was considered the case of simple interpolation (i.e., the multiplicity for each $\mu_{\pm k}$ was equal to one) and it was noticed that the method of the proof can be extended for the case of the interpolation with multiplicity. Such extension is indeed possible but serious technical problems appear.

We observe that the divisibility condition in Theorem A is equivalent to the relation $|f(z) - g(z)| = O(|z - \mu_{\pm k}|)$ as $z \to \mu_{\pm k}$ for each $k \in \mathbb{N}$. Hence, in contrast to the classical problem of interpolation by functions in space $H(\mathbb{C})$, it is stated in Theorem A that in space of entire function $H(\mathbb{C})$, we have the solvability for the following problem on simple interpolation by the functions in closed subspace $\operatorname{Ker} M_\varphi$ with an infinite sequence of nodes $\mu_{\pm k}$ of multiplicity 1 on the real axis.

For each interpolation data $b_{\pm k} \in \mathbb{C}$, $k = 1, 2, \ldots$, there exists an entire function $f \in \operatorname{Ker} M_\varphi$, such that $f(\mu_{\pm k}) = b_{\pm k}$, $k = 1, 2, \cdots$

In other words, in Theorem A, the solvability of the Vallée Poussin multi-points problem for convolution operators in space $H(\mathbb{C})$ was proven (see also [14]).

Theorem A has such a formulation since Fischer representations are used in the proof (see, for instance, [3]–[6], [12], [13] and the references therein). If $\psi \in H(\mathbb{C})$, by (ψ) we denote the closed ideal in $H(\mathbb{C})$ generated by function ψ,

$$\big(\psi\big) = \{h \in H(\mathbb{C}) : h = \psi \cdot r,\, r \in H(\mathbb{C})\}.$$

Theorem A on the solvability of the interpolation problem is equivalent to the Fischer representation

$$H(\mathbb{C}) = \operatorname{Ker} M_\varphi + \big(\psi\big),$$

where ψ is an arbitrary entire function having simple zero at each of nodes $\mu_{\pm k}$ and no zero at any other point.

There is no uniqueness of the interpolation under the assumptions of the considered problem, i.e. $\operatorname{Ker} M_\varphi \cap \big(\psi\big) \neq \{0\}$ (see Proposition below). In view of this, it seems natural to pass to a closed subspace of kernel $\operatorname{Ker} M_\varphi$ in order to solve the problem on interpolation by the functions in a smaller subspace of $H(\mathbb{C})$. In particular, it is also reasonable that Theorem A treats the case when all zeroes of the characteristic function φ are simple.

As it is well-known, the kernel of each convolution operator M_φ in $H(\mathbb{C})$ admits the spectral synthesis, i.e., $\operatorname{Ker} M_\varphi$ coincides with the closure of the linear span of all polynomial-exponential monomials $z^\nu e^{\lambda_n z}$ lying in this kernel; the closure is understood in the sense of the topology in $H(\mathbb{C})$.

In particular, in the present paper we prove the solvability of the problem on the interpolation with multiplicity by the functions in the closed subspace of kernel $\operatorname{Ker} M_\varphi$ consisting of all entire functions f represented by the series of exponentials converging in space $H(\mathbb{C})$,

$$f(z) = \sum_{n=1}^{\infty} c_n e^{\lambda_n z},\; z \in \mathbb{C},$$

that gives a new and simpler proof of Theorem A.

2. Auxiliary results

In what follows we shall need some properties of the polynomials of exponentials with real exponents. Such polynomials were studied in monograph [18].

Consider an arbitrary polynomial of exponentials

$$p(z) = \sum_{k=0}^{s} a_k(z) e^{\omega_k z},\; \omega_0 < \omega_1 < \cdots < \omega_s, \tag{1}$$

where $a_k(z)$ are some polynomials and let $a_0 \cdot a_s \not\equiv 0$.

By Theorem 12.9 of monograph [18], one can get easily

Lemma 1. *There exists $c_1 > 0$ such that in the exterior of the circle $\{z \in \mathbb{C} : |z| \geqslant c_1\}$ the following statement holds true: there exist positive constants c_2, c_3 and two real numbers m_0, m_s obeying either $m_0 > m_s$ or $m_0 = m_s = 0$ such that*

$$|p(z)| \geqslant c_2 e^{\omega_0 \operatorname{Re} z}, \tag{2}$$

for each z in the domain $U_0 = \{z \in \mathbb{C} : \operatorname{Re}(z + m_0 \ln z)\} < -c_3$, and

$$|p(z)| \geqslant c_2 e^{\omega_s \operatorname{Re} z}, \tag{3}$$

for each z in the domain $U_s = \{z \in \mathbb{C} : \operatorname{Re}(z + m_s \ln z)\} > c_3$.

For each fixed $c \in \mathbb{R}$, we consider the curve $\operatorname{Re}(z + m \ln z) = c$, $m \neq 0$. It is symmetric w.r.t. the real axis. As $m > 0$, this curve lies in some half-plane $\operatorname{Re} z < A$, $A > 0$, while as $m < 0$, it lies in some half-plane $\operatorname{Re} z > -A$, $A > 0$. If a point $z = x + iy$ lies on this curve, then $\left|\dfrac{y}{x}\right| \to \infty$, $\arg z \to \dfrac{\pi}{2}$, $|z| = |y|(1 + o(1))$ as $|z| \to \infty$. The considered curve tends asymptotically to the exponential curve $x + m \ln |y| = c$.

Given $\alpha \in (0, \frac{\pi}{2})$, we denote

$$A_\alpha(\pi) = \{z \in \mathbb{C} : |\arg z - \pi| \leqslant \alpha\}, \quad A_\alpha(0) = \{z \in \mathbb{C} : |\arg z| \leqslant \alpha\}.$$

Lemma 2. *For an arbitrary polynomial of exponentials p of form (1), there exists such $r = r(p) > 0$ that for each z, $|z| > r$, the following estimates hold true.*

If $\omega_0 < 0$ and $z \in A_\alpha(\pi)$, then

$$|p(z)| \geqslant c_3 e^{(|\omega_0|\cos\alpha)|z|}. \tag{4}$$

If $\omega_s > 0$ and $z \in A_\alpha(0)$, then

$$|p(z)| \geqslant c_3 e^{(\omega_s \cos\alpha)|z|}. \tag{5}$$

Proof. It is easy to see that all the points z in angles $A_\alpha(\pi)$, $A_\alpha(0)$ located outside some circle lie in domains U_0, U_s, respectively. Therefore, estimates (2) and (3) for the polynomial of exponents p in domains U_0 and U_s as $|z| > c_1$ imply the estimates outside some circle in angles $A_\alpha(\pi)$ and $A_\alpha(0)$, respectively. Since

$$\omega_0 \operatorname{Re} z = |\omega_0| \cdot |\cos(\arg z)| \cdot |z| \geqslant |\omega_0| \cdot |\cos(\pi - \alpha)| \cdot |z|$$

in the angle $A_\alpha(\pi) = \{|\arg z - \pi| \leqslant \alpha < \pi/2\}$, estimate (2) implies estimate (4). Since

$$\omega_s \operatorname{Re} z = \big(\omega_s \cos(\arg z)\big)|z| \geqslant (\omega_s \cos\alpha)|z|$$

in the angle $A_\alpha(0) = \{|\arg z| \leqslant \alpha < \pi/2\}$, estimate (3) yields estimate (5). The proof is complete. □

Consider two arbitrary infinite discrete sequences of complex numbers $\mathcal{V}^- = \{v_{-j}\}$ and $\mathcal{V}^+ = \{v_j\}$ such that $\operatorname{Re} v_{-j} < 0$, $\operatorname{Re} v_j > 0$. Denote $\mathcal{V} = \mathcal{V}^- \cup \mathcal{V}^+$. We introduce the following conditions

$$\limsup_{j\to\infty} \frac{|\operatorname{Re} v_{-j}|}{\ln |v_{-j}|} = \infty. \tag{6}$$

$$\limsup_{j\to\infty} \frac{\operatorname{Re} v_j}{\ln |v_j|} = \infty. \tag{7}$$

If sequences $\mathcal{V}^-$, $\mathcal{V}^+$ lie in the angles $A_\alpha(\pi)$, $A_\alpha(0)$, respectively, then conditions (6) and (7) hold true.

We indicate by $I_{\mathcal{V}^\pm}$ the ideals in $P_{\mathbb{C}}$,

$$I_{\mathcal{V}^-} = \{f \in P_{\mathbb{C}} : f(v_{-j}) = 0,\ j \in \mathbb{N}\}.$$

$$I_{\mathcal{V}^+} = \{f \in P_{\mathbb{C}} : f(v_j) = 0,\ j \in \mathbb{N}\}.$$

They are closed subspaces in $P_{\mathbb{C}}$.

Lemma 3. *In the described situation, if at least one of conditions (6) or (7) holds true for $\mathcal{V}$, then none of polynomials of exponentials $p \not\equiv 0$ of form (2) can be an element of ideals $I_{\mathcal{V}^\pm}$.*

Proof. Suppose $p|_{\mathcal{V}^-} = 0$ or $p|_{\mathcal{V}^+} = 0$. It follows from estimates (2) and (3) that outside some circle, an arbitrary polynomial of exponentials p of form (1) has no zeroes in domains U_0, U_s, and in their definitions, constants c_1, c_2, m_0, m_s depend on p. It is easy to see that conditions (6) and (7) imply that for arbitrary domains U_0, U_s of such form there exists a circle whose radius depends on p, and outside this circle, there exist two infinite sequences of points in $\mathcal{V}^-$ and $\mathcal{V}^+$ lying in U_0 and U_s, respectively. We obtain the contradiction. The proof is complete. □

3. Main theorem

Suppose that we are given an infinite discrete set of real interpolation nodes $\mathcal{M} = \mathcal{M}^- \cup \mathcal{M}^+$, where $\mathcal{M}^- = \{\mu_{-k}\}_{k=1}^{\tau_2}$, $\mu_{-k} < 0$, or $\mathcal{M}^- = \varnothing$, and $\mathcal{M}^+ = \{\mu_k\}_{k=1}^{\tau_1}$, $\mu_k \geqslant 0$, or $\mathcal{M}^- = \varnothing$. Here $\tau_1 \leqslant +\infty$, $\tau_2 \leqslant +\infty$.

Let us assume that all the interpolation nodes are taken in the increasing order of index k, i.e., in such a way that $\mu_{-k-1} < \mu_{-k}$, $\mu_k < \mu_{k+1}$. Suppose that to each node $\mu_{\pm k} \in \mathcal{M}$, the multiplicity $m_{\pm k} \in \mathbb{N}$ is associated.

Consider an infinite discrete sequence of complex numbers $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$. Suppose that the condition

$$\limsup_{n\to\infty} \frac{\ln n}{|\lambda_n|} = d < \infty \tag{8}$$

holds true. We denote

$$\Sigma(\Lambda) = \{f \in H(\mathbb{C}) : f(z) = \sum_{n=1}^{\infty} c_n e^{\lambda_n z},\ z \in \mathbb{C}\}.$$

Under condition (8) for the exponents λ_n, the pointwise convergence of the series of exponentials for each $z \in \mathbb{C}$ implies that the series converges in the topology of space $H(\mathbb{C})$ ([15]).

Consider the following problem on interpolation with multiplicity by the series of exponentials with set of nodes $\mathcal{M}$.

Given an arbitrary entire function g, find a series of exponentials $f \in \Sigma(\Lambda)$ such that for each $\mu_{\pm k} \in \mathcal{M}$

$$|f(z) - g(z)| = O(|z - \mu_{\pm k}|^{m_{\pm k}}),\ z \to \mu_{\pm k}.$$

The solvability of this problem is equivalent to the following representation

$$H(\mathbb{C}) = \Sigma(\Lambda) + \big(\psi_{\mathcal{M}}\big).$$

Here $\psi_{\mathcal{M}}$ stands for an entire function with zeroes of multiplicity $m_{\pm k}$ at all the nodes $\mu_{\pm k} \in \mathcal{M}$ and with no zero at any other point. Moreover, we denote by

$$\big(\psi_{\mathcal{M}}\big) = \{h \in H(\mathbb{C}) : h = \psi_{\mathcal{M}} \cdot r,\ r \in H(\mathbb{C})\} \tag{9}$$

the closed ideal in $H(\mathbb{C})$ generated by function $\psi_{\mathcal{M}}$.

Theorem 1. *1. Suppose that set of nodes $\mathcal{M}^+$ is finite or infinite. The problem of interpolation with multiplicity by the series of exponentials in $\Sigma(\Lambda)$ with set of nodes $\mathcal{M}$ is solvable in space $H(\mathbb{C})$ if and only if the set $\Lambda \cap A_\alpha(\pi)$ is infinite for some $\alpha \in (0, \frac{\pi}{2})$.*

2. Suppose that both the sets of nodes $\mathcal{M}^-$ and $\mathcal{M}^+$ are infinite. The problem of interpolation with multiplicity by the series of exponentials in $\Sigma(\Lambda)$ with set of nodes $\mathcal{M}$ is solvable in space $H(\mathbb{C})$ if and only if both the sets $\Lambda \cap A_\alpha(\pi)$ and $\Lambda \cap A_\alpha(0)$ are infinite for some $\alpha \in (0, \frac{\pi}{2})$.

Proof. Let us prove the necessity of conditions in Assertions 1 and 2.

Suppose that the problem on interpolation with multiplicity by series of exponentials in $\Sigma(\Lambda)$ is solvable under the hypothesis of Assertion 1. Assume that for each $\alpha \in (0, \frac{\pi}{2})$, the set $\Lambda \cap A_\alpha(\pi)$ is finite or empty. Then

$$\lim_{n\to\infty} \frac{\operatorname{Re}\lambda_n}{|\lambda_n|} = 0.$$

For $x < 0$ consider the function

$$h(x) = \sup_n \{x \operatorname{Re}\lambda_n - |\lambda_n|\}, \quad h(x) < \infty.$$

Consider an arbitrary function $f \in \Sigma(\Lambda)$ and let us prove that for each $x < 0$ the estimate $|f(x)| \leqslant Ce^{h(x)}$, $C > 0$, holds true. Indeed,

$$|f(x)| \leqslant \sum_{n=1}^{\infty} |c_n|\, e^{x \operatorname{Re}\lambda_n} \leqslant e^{h(x)} \sum_{n=1}^{\infty} |c_n|\, e^{|\lambda_n|}.$$

For a fixed $\varepsilon > 0$ we denote $B = d + 2\varepsilon + 1$, where quantity d is defined in condition (8). It was shown in the proof of Theorem 3.1.1 in monograph [15] that there exists a constant $A > 0$ such that $\left|c_n e^{\lambda_n z}\right| \leqslant A$ for each z, $|z| \leqslant B$, and each $n \in \mathbb{N}$.

It follows that $|c_n| \leqslant Ae^{-B|\lambda_n|}$. We have obtained that for each $x<0$

$$|f(x)| \leqslant Ae^{h(x)} \sum_{n=1}^{\infty} e^{(1-B)|\lambda_n|} = Ae^{h(x)} \sum_{n=1}^{\infty} e^{-(d+2\varepsilon)|\lambda_n|}.$$

Condition (8) implies that $(d+\varepsilon)\,|\lambda_n| \geqslant \ln n$, $n \geqslant n_0$, i.e.,

$$\sum_{n=1}^{\infty} e^{-(d+2\varepsilon)|\lambda_n|} < \infty.$$

The proven estimate shows that all the functions f in $\Sigma(\Lambda)$ have a regulated growth rate as $x \to -\infty$. But it means that the considered problem on simple interpolation by the functions in the kernel of this operator is unsolvable for the interpolation data having a greater rate than it is dictated by this estimate. We obtain the contradiction. The necessity of conditions in Assertion 1 is proven.

The necessity of conditions in Assertion 2 can be proven in the same way.

Let us prove the sufficiency of the conditions in Assertions 1 and 2.

Under the hypothesis of Assertion 1 it is sufficient to prove the representation

$$H(\mathbb{C}) = \Sigma(\Lambda) + \big(\psi_1\big),$$

where ψ_1 is an entire function having infinite zero set consisting of $\mu_{-k} \in \mathcal{M}^-$ of multiplicity m_{-k} and having at most finite set of zeroes which are all $\mu_k \in \mathcal{M}^+$ of multiplicity m_k.

Similarly, under the hypothesis of Assertion 2 we shall show that

$$H(\mathbb{C}) = \Sigma(\Lambda) + \big(\psi_2\big),$$

where ψ_2 is a some entire function having two infinite sets of zeroes $\mu_{-k} \in \mathcal{M}^-$ of multiplicity m_{-k} and $\mu_k \in \mathcal{M}^+$ of multiplicity m_k.

Here $\big(\psi_1\big)$, $\big(\psi_2\big)$ are the closed ideals in $H(\mathbb{C})$ defined in (9).

In a general situation, subspace $\Sigma(\Lambda)$ is not necessary closed in $H(\mathbb{C})$. We also observe the following. If the theorem holds for $\Lambda_0 \subset \Lambda$, it also holds for Λ. In what follows, we shall pass to a closed subspace in $\Sigma(\Lambda)$.

The results of monograph [10, p. 268] yield the following statement.

Each entire function in the closure of linear span of the system consisting of polynomial-exponential monomials with the exponents having a finite upper density counting multiplicities is represented by the series of exponents if and only if $\delta < \infty$, where δ is Bernstein-Leont'ev condensation index defined below.

In view of said above, in what follows, we pass to the subsequences of exponents in Λ.

Under the hypothesis of Assertion 1 we choose an infinite subsequence $\{t_\nu\} \in \Lambda \cap A_\alpha(\pi)$, $\nu \in \mathbb{N}$, such that the sequence $\{t_\nu\}_{\nu\in\mathbb{N}}$ satisfies the condition

$$|t_{\nu+1}| > 2|t_\nu|. \tag{10}$$

Under the hypothesis of Assertion 2, we choose two infinite subsequences, $\{t_{2n-1}\}$ in $\Lambda \cap A_\alpha(\pi)$ and $\{t_{2n}\}$ in $\Lambda \cap A_\alpha(0)$ so that sequence $\{t_\nu\}_{\nu\in\mathbb{N}}$ satisfies separation condition (10).

By G we denote the entire function with simple zeroes at t_ν,

$$G(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{t_\nu}\right),$$

where t_ν is any of two chosen subsequences.

Function G has the minimal type of growth for the order 1 and $\operatorname{Ker} M_G$ *consists of all entire functions $f(z)$ represented by the series of exponentials,*

$$f(z) = \sum_{\nu=1}^{\infty} c_\nu e^{t_\nu z},\ z \in \mathbb{C},$$

converging in the topology of space $H(\mathbb{C})$.

Since space $\operatorname{Ker} M_G$ admits the spectral synthesis, this statement is implied by Theorem 4.2.4 of monograph [10]. Let us show that as a result of choice of t_ν, we have $\delta = 0$, where

$$\delta = \limsup_{\nu\to\infty} \frac{1}{|t_\nu|} \ln \frac{1}{\big|G'(t_\nu)\big|}.$$

In both the cases, the set of zeroes $\{t_\nu\}$ of function G satisfies condition (10) and hence, function G has the minimal type for order 1. For such function we always have $\delta \geqslant 0$, since the derivative G' has also the zero type for order 1, i.e.,

$$\frac{1}{|G'(t_\nu)|} \geqslant e^{-\varepsilon|t_\nu|},\ \varepsilon > 0,\ \nu \geqslant \nu_0.$$

The estimates

$$|G'(t_\nu)| \geqslant \frac{1}{|t_\nu|} \prod_{j\neq\nu} \left(\left|1 - \frac{|t_\nu|}{|t_j|}\right| \right).$$

$$\ln|G'(t_\nu)| \geqslant \ln\frac{1}{|t_\nu|} + \sum_{j<\nu} \ln\Big(\frac{|t_\nu|}{|t_j|} - 1\Big) + \sum_{j>\nu} \ln\Big(1 - \frac{|t_\nu|}{|t_j|}\Big)$$

are valid. The second term is positive and then, in view of (10), we obtain

$$\frac{1}{|G'(t_\nu)|} \leqslant |t_\nu| e^{-A},\ A = \sum_{k=1}^{\infty} \ln\Big(1 - \Big(\frac{1}{2}\Big)^k\Big).$$

We see that $\delta \leqslant 0$. Hence, $\delta = 0$.

We denote $\Lambda^- = \Lambda \cap A_\alpha(\pi)$, $\Lambda^+ = \Lambda \cap A_\alpha(0)$. The sets can be finite, and one of them can be empty. In what follows, without loss of generality, we shall assume that under the hypothesis of Assertion 1 $\Lambda = \Lambda^-$, while under the hypothesis of Assertion 2 $\Lambda = \Lambda^- \cup \Lambda^+$, and moreover, the elements of sequence Λ satisfies separation condition (10) in the aforementioned sense. We denote by G_1, G_2 entire functions with zero sets Λ^-, Λ, respectively.

In view of these notations, the fact proved above implies that $\operatorname{Ker} M_{G_1} = \Sigma(\Lambda^-)$ and $\operatorname{Ker} M_{G_2} = \Sigma(\Lambda)$. Thus, to prove the sufficiency of the conditions in both the assertions of Theorem 1, it is sufficient to show that under each of their hypotheses, space $H(\mathbb{C})$ admits the corresponding Fischer representation,

$$H(\mathbb{C}) = \operatorname{Ker} M_{G_1} + \big(\psi_1\big).$$

$$H(\mathbb{C}) = \operatorname{Ker} M_{G_2} + \big(\psi_2\big).$$

Entire functions ψ_1, ψ_2 are determined in the beginning of the proof. It will be shown that for $k = 1$ and $k = 2$, the following two statements hold true.

(I) The subspace $\operatorname{Ker} M_{G_k} + \big(\psi_k\big)$ *is dense in space* $H(\mathbb{C})$*;*

(II) The subspace $\operatorname{Ker} M_{G_k} + \big(\psi_k\big)$ *is closed in space* $H(\mathbb{C})$.

In what follows we use the scheme of the proof given in work [13] which is based on the duality with employing Laplace transform $\mathcal{L}$ for the functionals in the strong dual space $H^*(\mathbb{C})$.

We define separately a continuous bilinear form $[\cdot,\cdot] : H(\mathbb{C}) \times P_{\mathbb{C}} \longmapsto \mathbb{C}$ by the formula $[\psi, \varphi] = \langle \mathcal{L}^{-1}\varphi, \psi\rangle$, $\psi \in H(\mathbb{C})$, $\varphi \in P_{\mathbb{C}}$. By means of the mapping $\varphi \longmapsto [\cdot, \phi] = \langle \mathcal{L}^{-1}\varphi, \cdot\rangle$, where $\mathcal{L}^{-1}\varphi \in H^*(\mathbb{C})$, we define an isomorphism between $P_{\mathbb{C}}$ and strong dual space $H^*(\mathbb{C})$. According to the introduced duality, each function in space $P_{\mathbb{C}}$ is in one-to-one correspondence with some linear continuous functional in $H^*(\mathbb{C})$.

Each function $G \in P_{\mathbb{C}}$, $G \not\equiv 0$, generates the convolution operator $M_G : P_{\mathbb{C}} \longmapsto P_{\mathbb{C}}$,

$$M_G[\psi](z) = \big[S_z\big(\psi(\lambda)\big), G_\lambda\big] = \big\langle (\mathcal{L}^{-1}G)_\lambda, \psi(z+\lambda)\big\rangle\,,$$

in the space of entire functions $H(\mathbb{C})$, where S_z is the shift operator $S_z\big(\psi(\lambda)\big) = \psi(\lambda + z)$.

It is known that M_G is linear continuous surjective operator. Its adjoint operator is the operator A_G of multiplication by the characteristic function G and it acts on the functions $\omega \in P_{\mathbb{C}}$ as $\omega \longmapsto G \cdot \omega$.

As it is known, (M^*)-space $H(\mathbb{C})$ is reflexive, that is, its second strong dual space $H^{**}(\mathbb{C})$ is canonically isomorphic to space $H(\mathbb{C})$. This is why, in view of this canonical isomorphism, the mapping $\psi \longmapsto [\psi,\cdot]$ defines an isomorphism between (M^*)-space $H(\mathbb{C})$ and strongly adjoint one $P^*_{\mathbb{C}}$, and each function in $H(\mathbb{C})$ is in one-to-one correspondence with some linear continuous functional in strong dual space $P^*_{\mathbb{C}}$.

More precisely, this isomorphism is understood as follows: the canonical isomorphism between $H(\mathbb{C})$ and $H^{**}(\mathbb{C})$ reads as $\psi \longmapsto \Theta\psi = F_\psi$, $F_\psi \in P^*_{\mathbb{C}}$, $\langle F_\psi, \varphi\rangle = [\psi,\varphi] = \langle \mathcal{L}^{-1}\varphi,\, \psi\rangle$. Here $\psi \in H(\mathbb{C})$, $\varphi \in P_{\mathbb{C}}$.

Each function $\psi \in H(\mathbb{C})$, $\psi \not\equiv 0$ generates a convolution operator $\widetilde{M}_\psi : P_{\mathbb{C}} \longmapsto P_{\mathbb{C}}$, $\widetilde{M}_\psi[\varphi](z) = \big[(\Theta\psi)_\lambda, S_z\big(\varphi(\lambda)\big)\big]$, in space $P_{\mathbb{C}}$ of entire functions of exponential type. Here S_z is the shift operator, $S_z\big(\varphi(\lambda)\big) = \varphi(\lambda+z)$, $\lambda \in \mathbb{C}$.

Then we obtain

$$\widetilde{M}_\psi[\varphi](z) = \big\langle(\mathcal{L}^{-1}S_z\varphi)_\lambda,\, \psi(\lambda)\big\rangle = \big\langle e^{z\lambda}(\mathcal{L}^{-1}\varphi)_\lambda,\, \psi(\lambda)\big\rangle = \big\langle(\mathcal{L}^{-1}\varphi)_\lambda,\, e^{z\lambda}\psi(\lambda)\big\rangle,\ \varphi \in P_{\mathbb{C}}.$$

By employing the well-known formula for Borel inverse transformation ([10]), we then get that convolution operator $\widetilde{M}_\psi$ reads as

$$\widetilde{M}_\psi[\varphi](z) = \frac{1}{2\pi i}\int\limits_C \psi(\lambda)e^{z\lambda}\gamma_\varphi(\lambda)\,d\lambda,\ \varphi \in P_{\mathbb{C}},$$

where γ_φ is the function associated with function φ in the sense of Borel, and C is a rectifiable closed contour enveloping all singularity points of function γ_φ. An entire function ψ is called characteristic function of convolution operator $\widetilde{M}_\psi$.

It is known that $\widetilde{M}_\psi$ is a linear continuous surjective operator. Denote $\operatorname{Ker}\widetilde{M}_\psi = \{f \in P_{\mathbb{C}} : \widetilde{M}_\psi[f] = 0\}$.

Operator $\widetilde{M}_\psi$ is adjoint to the operator $\widetilde{A}_\psi$ of multiplication by an entire function ψ in space $H(\mathbb{C})$, which acts on functions $g \in H(\mathbb{C})$ as follows: $g \longmapsto \psi\cdot g$. Operator $\widetilde{A}_\psi$ is linear and continuous and its image coincides with the closed ideal (ψ).

If X_1 is a subspace in a topological vector space X, by X_1^0 we denote its polar (or annihilator), that is, the set of the functionals in X^* vanishing on X_1.

In view of the introduced duality, the polar of set $\big(\operatorname{Ker} M_G\big)^0$ coincides with the ideal defined as

$$\big(G\big)_{P_{\mathbb{C}}} = \{h \in P_{\mathbb{C}} : h = G\cdot r;\ r \in P_{\mathbb{C}}\},$$

and $\big(G\big)_{P_{\mathbb{C}}} = \big(G\big) \cap P_{\mathbb{C}}$, and the ideal $\big(G\big)_{P_{\mathbb{C}}}$ is a closed subspace in $P_{\mathbb{C}}$. The proof of these facts will be adduced later. In view of the introduced duality, the polar of set $\big((\psi)\big)^0$ coincides with $\operatorname{Ker}\widetilde{M}_\psi$.

Since $\big(\operatorname{Ker} M_G + (\psi)\big)^0 = \big(\operatorname{Ker} M_G\big)^0 \cap \big((\psi)\big)^0$, in view of the duality, we have proven that $\big(\operatorname{Ker} M_G + (\psi)\big)^0 = \big(G\big)_{P_{\mathbb{C}}} \cap \operatorname{Ker}\widetilde{M}_\psi$.

In view of the duality, Lemma 2 in paper [16] implies that the space $\operatorname{Ker} M_G + \big(\psi\big)$ is closed in $H(\mathbb{C})$ if and only if the space $\big(\operatorname{Ker} M_G\big)^0 + \big((\psi)\big)^0 = \big(G\big)_{P_{\mathbb{C}}} + \operatorname{Ker}\widetilde{M}_\psi$ is closed in $P_{\mathbb{C}}$.

Hence, we have obtained that for $k = 1$ and $k = 2$, Statements (I) and (II) are equivalent to the following two dual statements in (LN^*)-space $P_{\mathbb{C}}$.

(I^*) The identity $\big(G_k\big)_{P_{\mathbb{C}}} \cap \operatorname{Ker}\widetilde{M}_{\psi_k} = \{0\}$ holds true.

(II^*) The space $\big(G_k\big)_{P_{\mathbb{C}}} + \operatorname{Ker}\widetilde{M}_{\psi_k}$ is closed in space $P_{\mathbb{C}}$.

An important aspect in the proof of dual statements (I^*) and (II^*) is the following well-known fact (see, for instance, [17]).

Let f be an arbitrary function with the zero set $\{t_k\}$ and the multiplicities of t_k are p_k, $k \in \mathbb{N}$. The closed subspace $\operatorname{Ker}\widetilde{M}_f$ in space $P_{\mathbb{C}}$ is the linear span of all monomials $\{z^\nu e^{t_k z}\}$,

$\nu = 0, 1, \cdots, p_k - 1$, $k \in \mathbb{N}$, *i.e., it consists only of polynomials of exponentials,*

$$\operatorname{Ker} \widetilde{M}_f = \{p \in P_{\mathbb{C}} : p(z) = \sum_{k=1}^{u_p} a_k(z) e^{t_k z}\}.$$

Here, for each $k \in \mathbb{N}$, functions a_k are arbitrary polynomials of degree at most $p_k - 1$, respectively.

This is the fundamental principle $\operatorname{Ker} \widetilde{M}_\psi$ in space $P_{\mathbb{C}}$ and it be proved rather easily.

Let us prove Statement (I^*) under the hypotheses of Assertions 1 and 2 of Theorem 1.

Under the hypotheses of both of these items, suppose $p \in \operatorname{Ker} \widetilde{M}_{\psi_k}$, $p \not\equiv 0$, then function p is a polynomial of exponentials

$$p(z) = \sum_{\mathrm{Fin}_{\mathcal{M}^-}} a_{-j}(z) e^{\mu_{-j} z} + \sum_{\mathrm{Fin}_{\mathcal{M}^+}} a_k(z) e^{\mu_k z}$$

of the form (1). In the right hand side, we have finite sums over finite subsets $\mathrm{Fin}_{\mathcal{M}^-} \subset \mathcal{M}^-$, $\mathrm{Fin}_{\mathcal{M}^+} \subset \mathcal{M}^+$. In this representation we take into account the possibility of the case $\mathrm{Fin}_{\mathcal{M}^-} = \varnothing$ or $\mathrm{Fin}_{\mathcal{M}^+} = \varnothing$ and this is why we adopt the convention that for an arbitrary sequence $\{b_k\}$ we have $\sum_{\varnothing} b_k = 0$.

Let us show that under the hypotheses of Assertions 1 and 2, the polynomial of exponentials $p \not\equiv 0$ can not belong to $(G_1)_{P_{\mathbb{C}}}$. This fact follows from Lemma 3. Indeed, according to the hypothesis of Theorem 1, $\Lambda^- \subset A_\alpha(\pi)$ and it implies the sequence $v_{-k} = \lambda_{-k}$ satisfies condition (6) of Lemma 3.

Moreover, we need to show that $(G_1)_{P_{\mathbb{C}}} = I_{\mathcal{V}^-}$. By definition, $(G_1)_{P_{\mathbb{C}}} \subset I_{\mathcal{V}^-}$. Then, by Lindelöf theorem for the functions in space $P_{\mathbb{C}}$, we have $\big(G\big)_{P_{\mathbb{C}}} = \big(G\big) \cap P_{\mathbb{C}}$, i.e., the inverse inclusion holds true as well.

Ideal $I_{\mathcal{V}^-}$ in Lemma 3 is closed since the topology in $P_{\mathbb{C}}$ is stronger than that of uniform convergence on compact sets. It is proven that ideal $(G_1)_{P_{\mathbb{C}}}$ is closed in $P_{\mathbb{C}}$.

Statement (I^*) under the hypothesis of Assertion 1 is proven. To prove it under the hypothesis of Assertion 2, it remains to observe that ideal $(G_2)_{P_{\mathbb{C}}}$ is contained in ideal $(G_1)_{P_{\mathbb{C}}}$.

We have obtained, under the hypotheses of both of Assertions 1 and 2, that we have algebraic direct sums $(G_k)_{P_{\mathbb{C}}} \oplus \operatorname{Ker} \widetilde{M}_{\psi_k}$, $k = 1, 2$. Under the hypothesis of each of Assertions 1 and 2, let us prove the closedness of these subspaces in $P_{\mathbb{C}}$ (i.e., we shall prove Statement (II^*)). As it is known ([19]), the closedness of each subspace X in (LN^*)-space $P_{\mathbb{C}}$ is equivalent to its sequential closedness.

Under the hypothesis of Assertion 1, let us consider an arbitrary sequence $\{g_l\}$, $l \in \mathbb{N}$, of functions in the algebraic direct sum $(G_1)_{P_{\mathbb{C}}} \oplus \operatorname{Ker} \widetilde{M}_{\psi_1}$ and suppose that it converges to a function $g \in P_{\mathbb{C}}$ in space $P_{\mathbb{C}}$. Let us show that limiting function g belongs to $(G_1)_{P_{\mathbb{C}}} \oplus \operatorname{Ker} \widetilde{M}_{\psi_1}$.

The convergence of $\{g_l\}$ in (LN^*)-topology of space $P_{\mathbb{C}}$ means the following:

1. $\{g_l\}$ converges to g in the topology of space $H(\mathbb{C})$;
2. There exist $A > 0$, $B > 0$ such that for each $l \in \mathbb{N}$, the estimate

$$|g_l(z)| \leqslant A e^{B|z|},\ z \in \mathbb{C}, \tag{11}$$

holds true.

Sequence $\{g_l\}$ consists of the functions $g_l = p_l + R_l$, where $R_l \in (G_1)_{P_{\mathbb{C}}}$, i.e., $R_l|\Lambda^- = 0$, and $p_l \in \operatorname{Ker} \widetilde{M}_{\psi_1}$.

If sequence $\{g_l\}$ contains infinitely many terms with $R_l \equiv 0$, then $g \in \operatorname{Ker} \widetilde{M}_{\psi_1}$. If sequence $\{g_l\}$ contains infinitely many terms with $p_l \equiv 0$, then $g \in (G_1)_{P_{\mathbb{C}}}$. For such sequences $\{g_l\}$ we have $g \in (G_1)_{P_{\mathbb{C}}} \oplus \operatorname{Ker} \widetilde{M}_{\psi_1}$.

Thus, in what follows we can assume that sequence $\{g_l\}$ is so that $R_l \not\equiv 0$, $p_l \not\equiv 0$ for each l.

By the fundamental principle for the kernel $\operatorname{Ker}\widetilde{M}_{\psi_1}$ in space $P_{\mathbb{C}}$,

$$p_l(z) = \sum_{\mathrm{Fin}^{(l)}_{\mathcal{M}^-}} a^{(l)}_{-j}(z)e^{\mu_{-j}z} + \sum_{\mathrm{Fin}^{(l)}_{\mathcal{M}^+}} a^{(l)}_{k}(z)e^{\mu_{k}z}. \tag{12}$$

If in this representation we have $\mathrm{Fin}^{(l)}_{\mathcal{M}^-} \neq \varnothing$ for some fixed l, i.e., there exists at least one $a^{(l)}_{-j} \not\equiv 0$ associated with the exponent μ_{-j}, by q_l we denote the index of the minimal among all such μ_{-j}. If in this representation $\mathrm{Fin}^{(l)}_{\mathcal{M}^+} \neq \varnothing$ for some fixed l, i.e., there exists at least one $a^{(l)}_{k} \not\equiv 0$ associated with the exponent μ_k, we denote by u_l the index of the maximal among all such μ_k.

Let sequence $\{g_l\}$ be so that the set $\{q_l\}$ is infinite. Let us show that it is bounded.

By the hypothesis of Assertion 1, for an arbitrary sequence $\{g_l\}$, the set $\{u_l\}$ is either empty or bounded. Then without loss of generality we can assume in the representations by polynomials of exponentials $\mathrm{Fin}^{(l)}_{\mathcal{M}^-} \neq \varnothing$ for each l. Indeed, it is sufficient to consider $\widetilde{g_l} = g_l \cdot e^{-az}$ for some $a \in \mathbb{R}^+$. Then $a^{(l)}_{-q_l} \not\equiv 0$ for each l.

Suppose that the set of numbers $\{q_l\}$ is unbounded. All p_l are of the form (1). Since $a^{(l)}_{-q_l} \not\equiv 0$, employing estimate (4) in Lemma 2 and estimate (11), we obtain the following estimate for the function $R_l = g_l - p_l$, $R_l \not\equiv 0$,

$$|R_l(z)| \geqslant |p_l(z)| - |g_l(z)| \geqslant c_3 e^{(|\mu_{-q_l}|\cos\alpha)|z|} - Ae^{B|z|}$$

for each z in domain $A_\alpha(\pi)$, $|z| > r$. Here $r = r(l)$. By assumption, there exists $\mu_{-q_{l_0}}$ such that $|\mu_{-q_{l_0}}| > \dfrac{B}{\cos\alpha}$.

We see that $|R_{l_0}(z)| > 0$ for each z in domain $A_\alpha(\pi)$, $|z| > r_1(l_0)$. We obtain the contradiction since by the hypothesis of Assertion 1, domain $A_\alpha(\pi)$, $|z| > r_1(l_0)$, contains an infinite discrete sequence from Λ^- and we know that $R_{l_0}|_{\Lambda^-} = 0$.

We have proven the following: if sequence of functions $g_l = p_l + R_l$ in $(G_1)_{P_{\mathbb{C}}} \oplus \operatorname{Ker}\widetilde{M}_{\psi_1}$ converges in $P_{\mathbb{C}}$, then $|\mu_{-q_l}| \leqslant \dfrac{B}{\cos\alpha}$ for each l and the set of numbers $\{q_l\}$ in the representations of the polynomials of exponentials is bounded. The set of numbers $\{u_l\}$ is finite or empty for each sequence $\{g_l\}$ by the hypothesis of Assertion 1.

Therefore, sequence $\{p_l\}$ of polynomials of exponentials belongs to some finite-dimensional subspace $X \subset \operatorname{Ker}\widetilde{M}_{\psi_1}$. Statement (I^*) means that all the terms of the converging sequence $g_l = p_l + R_l$ lie in the algebraically direct sum $X \oplus (G_1)_{P_{\mathbb{C}}}$.

In each topological vector space, an algebraically direct sum of a finite-dimensional subspace and a closed subspace is a closed subspace [20, p. 41] Hence, the limiting function g of sequence $g_l = p_l + R_l$ belongs to $\operatorname{Ker}\widetilde{M}_{\psi_1} \oplus (G_1)_{P_{\mathbb{C}}}$. Statement (II^*) is proven.

Statements (I^*) and (II^*) imply Assertion 1 of Theorem 1.

Under the hypothesis of Assertion 2, let us prove that the algebraically direct sum $(G_2)_{P_{\mathbb{C}}} \oplus \operatorname{Ker}\widetilde{M}_{\psi_2}$ is a closed subspace in $P_{\mathbb{C}}$. By the hypothesis of Assertion 2, there exist two infinite sets of interpolation nodes μ_k of multiplicity m_k and of interpolations nodes μ_{-k} with multiplicity m_{-k}. Moreover, the angles $A_\alpha(\pi)$ and $A_\alpha(0)$ comprise two infinite sequences of points in Λ^- and Λ^+, respectively.

Consider an arbitrary converging sequence of functions $\{g_l\} \subset \operatorname{Ker} M_{G_2} \oplus \big(\psi_2\big)$ of the form $g_l = R_l + p_l$, $l \in \mathbb{N}$, where $p_l \in \operatorname{Ker}\widetilde{M}_{\psi_1}$, $R_l \in (G_2)_{P_{\mathbb{C}}}$. As in the above proof of closedness for $(G_1)_{P_{\mathbb{C}}} \oplus \operatorname{Ker}\widetilde{M}_{\psi_1}$, we can assume that $p_l \not\equiv 0$, $R_l \not\equiv 0$ for each l. The polynomials of exponentials $p_l \not\equiv 0$ satisfy representation (12). If sequence $\{g_l\}$ is so that the set of numbers $\{u_l\}$ is finite or empty, it has been proven above that the set of numbers $\{q_l\}$ is bounded.

Suppose that sequence $\{g_l\}$ is so that set $\{u_l\}$ is infinite.

If sequence $\{p_l\}$ is such that it comprises an infinite set of terms with negative exponents in their representations, we can assume that $a^{(l)}_{-q_l} \not\equiv 0$ for all terms of such sequence. This can be achieved by passing to a subsequence. Then, as in the proof of Assertion 1, we show that the set of numbers $\{q_l\}$ associated with all such terms is bounded. Moreover, for the further purposes, we mention that for each sequence of such type without loss of generality we can assume that $a^{(l)}_{u_l} \not\equiv 0$ for each l.

If sequence $\{p_l\}$ contains at most finitely many of such terms, we can assume that $a^{(l)}_{u_l} \not\equiv 0$ for each l considering $\widetilde{g}_l = g_l \cdot e^{az}$ for some $a \in \mathbb{R}^+$.

By assumption, for sequence $\{p_l\}$ of each of the latter two types of sets, the set $\{u_l\}$ is unbounded. Sequence $\{g_l\}$ satisfies estimate (11). Since $a^{(l)}_{u_l} \not\equiv 0$ for each l, employing estimate (5) in Lemma 2 and estimate (11), we obtain that for each z in domain $A_\alpha(0)$, $|z| > r$, $r = r(l)$,

$$|R_l(z)| \geqslant c_3 e^{(\mu_{u_l} \cos\alpha)|z|} - Ae^B|z|.$$

Since set Λ^+ is infinite, as in the proof of Assertion 1, it is easy to get a contradiction and we conclude that the set of numbers $\{u_l\}$ associated to an arbitrary converging sequence $\{u_l\}$ is bounded. The above arguments yield both the sets $\{q_l\}$ and $\{u_l\}$ involved in the representation of an arbitrary converging sequence $\{g_l\}$ of polynomials of exponentials are bounded. We complete the proof by the same arguments as in the proof of Assertion 1.

Assertion 2 is proven, and thus the same is for Theorem 1. □

4. Discussion of hypotheses and examples

Let us show that there can not be the uniqueness of the interpolation under the assumptions of the problem.

Suppose that set Λ satisfies condition (8). Moreover, let set Λ and set of nodes $\mathcal{M} \subset \mathbb{R}$ satisfy the hypothesis of Theorem 1, then the problem on interpolation with multiplicity by series of exponentials f in $\Sigma(\Lambda)$ with nodes $\mathcal{M}$ is solvable. Let $\psi = \psi_{\mathcal{M}}$ be an entire function with the zero set $Z_\psi = \mathcal{M}$ taken counting the multiplicities.

Proposition 1. *Subspace $\Sigma(\Lambda) \cap \big(\psi\big)$ is nonzero and infinite-dimensional.*

Proof. By Theorem 1, the representation $\operatorname{Ker} M_\varphi + \big(\psi\big) = H(\mathbb{C})$ holds true. Let us show that $\Sigma(\Lambda) \cap \big(\psi\big) \neq \{0\}$.

There exist $g \in \big(\psi\big)$, $g \not\equiv 0$,$g \notin \Sigma(\Lambda)$ and $x_0 \in \mathbb{R}$ such that $g(x_0) \neq 0$, $\psi(x_0) \neq 0$. We indicate by ψ_1 an entire function with zero set $Z_{\psi_1} = \mathcal{M} \cup \{x_0\}$, then ideal $\big(\psi_1\big)$ is a subspace of $\big(\psi\big)$.

By Theorem 1, $H(\mathbb{C}) = \Sigma(\Lambda) + \big(\psi_1\big)$, in particular, $g = f + \psi_1 \cdot r$, where $f \in \Sigma(\Lambda)$, and $f \not\equiv 0$ since $g \notin \big(\psi_1\big)$. Function r belongs to $H(\mathbb{C})$ and $r \not\equiv 0$ since $g \notin \Sigma(\Lambda)$.

By the relations $r \not\equiv 0$ and $f = g - \psi_1 \cdot r$ we see that $f \in \Sigma(\Lambda) \cap \big(\psi\big)$ and $f \not\equiv 0$. The first statement is proven.

We note that $f \notin \Sigma(\Lambda) \cap \big(\psi_1\big)$, since $f(x_0) = g(x_0) \neq 0$. It has been proven that the strict inclusion $\Sigma(\Lambda) \cap \big(\psi_1\big) \subset \Sigma(\Lambda) \cap \big(\psi\big)$ holds true.

Suppose that the subspace $\Sigma(\Lambda) \cap \big(\psi\big)$ is finite-dimensional. Continuing the above procedure, we obtain the sequence of strict inclusions $\Sigma(\Lambda) \cap \big(\psi_{k+1}\big) \subset \Sigma(\Lambda) \cap \big(\psi_k\big)$. In finitely many steps we obtain $\Sigma(\Lambda) \cap \big(\psi_{l_0}\big) = \{0\}$. It contradicts to what has been above. The proof is complete. □

In conclusion we address a more general situation described before Lemma 3. Consider an infinite discrete sequence of complex numbers $\mathcal{V} = \mathcal{V}^- \cup \mathcal{V}^+$, where $\mathcal{V}^+ = \{v_j\}$, $\mathcal{V}^- = \{v_{-j}\}$ are two infinite discrete sequence and $\operatorname{Re} v_j > 0$, $\operatorname{Re} v_{-j} < 0$. Suppose that conditions (6) and (7) in Lemma 3 hold and assume that sets $\mathcal{V}^-$, $\mathcal{V}^+$ are the sets of zeroes for entire functions Φ_1, Φ_2 of exponential type, respectively.

For an arbitrary discrete set $\Omega = \{\omega_k\}$, $\omega_k \in \mathbb{R}$, with multiplicity $\{n_k\}$, by ψ_Ω we denote an entire function having simple zeroes at the points ω_k with multiplicities n_k and no zero at any other point.

Remark 1. Lemma 3 implies the following two statements.

Subspaces $\operatorname{Ker} M_{\Phi_1} + \big(\psi_\Omega\big)$ *and* $\operatorname{Ker} M_{\Phi_2} + \big(\psi_\Omega\big)$ *are dense in space* $H(\mathbb{C})$.

Indeed, $I_{\mathcal{V}^-} = (\Phi_1)_{P_\mathbb{C}}$. This has been shown in the proof of Theorem 1. Therefore, by Lemma 3, $\big(\Phi_1\big)_{P_\mathbb{C}} \cap \operatorname{Ker} \widetilde{M}_{\psi_\Omega} = \{0\}$, since each function in $\operatorname{Ker} \widetilde{M}_{\psi_\Omega}$ satisfies (1) due to the fundamental principle for $\operatorname{Ker} \widetilde{M}_{\psi_\Omega}$ in space $P_\mathbb{C}$. This statement by duality is equivalent to the first one. The second statement is obtained by the transformation $z \to -z$ of plane $\mathbb{C}$.

Let us discuss two examples related to the essentiality of the conditions imposed for the set of exponents for exponentials Λ and interpolation nodes $\mathcal{M}$.

Example 1 adduced below shows that conditions (6) and (7) in the statements in Remark 1 are close to the necessary ones. In order to simplify this example, we note that after the transformation $z \to -z$ of plane $\mathbb{C}$, Assertion 1 of Theorem 1 becomes

$1'$. *Suppose that the set of nodes $\mathcal{M}^-$ is finite or empty. The problem on interpolation with multiplicity by the series of exponentials in $\Sigma(\Lambda)$ with the set of nodes $\mathcal{M}$ is solvable in space $H(\mathbb{C})$ if and only if the set $\Lambda \cap A_\alpha(0)$ is infinite for some $\alpha \in (0, \frac{\pi}{2})$.*

Example 1. *Let $\varphi(z) = z - e^z$ and the set of interpolation nodes $\mathcal{M}$ contains some set of nodes $\mu_k \geqslant 1$, $k \in \mathbb{N}$. The problem on simple interpolation by the functions in* $\operatorname{Ker} M_\varphi$ *with the set of nodes $\mathcal{M}$ is unsolvable, and generally speaking, the subspace $M_\varphi + \big(\psi_\mathcal{M}\big)$ is not dense in space $H(\mathbb{C})$.*

It is easy to show that the function $\varphi(z) = z - e^z$ has infinitely many zeroes $\Lambda = \{\lambda_n,\ \lambda_n = x_n + iy_n\}$. Since $\lambda_n = e^{\lambda_n}$, we obtain, in particular, that $x_n - \ln|\lambda_n| = 0$. It follows that $\operatorname{Re}\lambda_n \to +\infty$ and λ_n satisfy neither the hypothesis of Theorem 1 nor condition (7). Moreover, $\varphi'(\lambda_n) = 1 - e^{\lambda_n} = 1 - \lambda_n$. Hence, all the zeroes of function φ are simple and the condensation index δ is zero for function φ. Therefore, $\operatorname{Ker} M_\varphi = \Sigma(\Lambda)$.

In particular, if the set of nodes $\mathcal{M}$ contains $\mu_0 = 0$ of multiplicity $m_0 = 2$ and $\mu_1 = 1$ of multiplicity $m_1 = 1$, then the polynomial of exponents $p = \varphi$ belongs to $\operatorname{Ker} \widetilde{M}_{\psi_\mathcal{M}} \cap \big(\varphi\big)_{P_\mathbb{C}}$, i.e., by the equivalent dual statement (I^*), the subspace $M_\varphi + \big(\psi_\mathcal{M}\big)$ is not dense in space $H(\mathbb{C})$.

Our second example shows that there exist convolution operators in the considered class such that the problem on interpolation by the functions in the kernel of the convolution operator with the interpolation nodes μ_k is in general unsolvable.

Example 2. *Suppose that the set of interpolation nodes comprises points $\mu_1 \in \mathbb{R}$, $\mu_2 = \mu_1 + i \in \mathbb{C}$, and $\varphi(z) = 1 - e^{iz}$. Then the problem on simple interpolation by the entire functions in* $\operatorname{Ker} M_\varphi = \{f \in H(\mathbb{C}) : f(z) = f(z+i)\}$ *is unsolvable.*

In this example $\lambda_n = 2\pi n \in \mathbb{R}$. All the functions in kernel M_φ are periodic with the period i, and thus there is no possibility to impose arbitrary interpolation data in the nodes $\mu_1 \in \mathbb{R}$, $\mu_2 = \mu_1 + i \in \mathbb{C}$.

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