

A CLOSEDNESS OF SET OF DIRICHLET SERIES SUMS

A.S. KRIVOSHEYEV, O.A. KRIVOSHEYEVA

Abstract. In the work we consider Dirichlet series. We study the problem of closedness for the set of the sums for such series in the space of functions holomorphic in a convex domain of a complex plane with a topology of uniform convergence on compact subsets. We obtain necessary and sufficient conditions under those each function in the closure of a linear span of exponents with positive indices is represented by a Dirichlet series. These conditions can be formulated only in terms of geometric characteristics of an index sequence and of the convex domain.

Keywords: exponential, convex domain, Dirichlet series, entire function, invariant subspace.

Mathematics Subject Classification: 41A05, 4130.

1. INTRODUCTION

Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be an unbounded strictly increasing sequence of positive numbers. In the work we consider the Dirichlet series

$$\sum_{k=1}^{\infty} d_k \exp(\lambda_k z). \quad (1)$$

It is known (see, for instance, [1, Ch. II, Sect. 1, Subsect. 4]) that under certain natural condition for the exponents λ_k , series (1) converges absolutely and uniformly on the compact sets in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < \gamma\}$ to an analytic function and diverges in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > \gamma\}$. The number γ called convergence abscissa is calculated by a formula being an analogue for Cauchy-Hadamard formula for power series (see, for instance, [1, Ch. II, Sect. 1, Subsect. 4, Th. 2.1.2]). We also note the expansion of a function into the Dirichlet series is always unique (see, for instance, [1, Ch. II, Sect. 1, Subsect. 3]).

Let D be a convex domain in \mathbb{C} and $H(D)$ stand for the space of analytic in D function with the topology of uniform convergence on compact subsets in D . The aim of this work is to find out the conditions under those the set of the sums of series (1), whose convergence half-planes contain domain D , is a closed subset in space $H(D)$.

This set contains a linear span of the system $\mathcal{E} = \{\exp(\lambda_k z)\}_{k=1}^{\infty}$ and it is a part of the subspace $W(\Lambda, D)$, which a closure of linear span \mathcal{E} in $H(D)$. Subspace $W(\Lambda, D)$ is closed and invariant w.r.t. the differentiation operator. System \mathcal{E} is the set of eigenfunctions for this operator in $W(\Lambda, D)$ and sequence Λ is its spectrum. The definition of space $W(\Lambda, D)$ implies immediately that it admits the spectral synthesis, i.e., each of its functions is a limit of linear combinations of the eigenfunctions. It is easy to see that the closedness of sums of series (1) in $H(D)$ is equivalent to the fact that each function in $W(\Lambda, D)$ is represented by series (1) which converges uniformly on compact subsets of domain D . If $W(\Lambda, D)$ is nontrivial (i.e., system \mathcal{E} is incomplete in $H(D)$) and the latter holds true, then one says that the fundamental principle holds true in subspace $W(\Lambda, D)$. The dual problem to that of

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fundamental principle in a nontrivial closed invariant subspace of $H(D)$ admitting the spectral synthesis is the interpolation problem in the space of entire functions of exponential kind whose adjoint diagrams lie in domain D . The studies of both the problems made first independently have a rich history. The survey of main results obtained within these studies was presented in works [2] and [3]. A criterion of the fundamental principle (and for the interpolation as well) in arbitrary nontrivial closed invariant subspaces admitting spectral synthesis was obtain in works [3] and [4]. However, this criterion is formulated in terms of existence of certain special family of entire function vanishing at the points $\lambda_k, k \geq 1$, and obeying appropriate lower estimates. In the general case, especially for unbounded domains D , there is still an open question on the conditions for sequence Λ and domain D guaranteeing the existence of such family.

In the present work we obtain the complete solution for the problem on closedness of sums of series (1) for an arbitrary convex domain D and, in particular, for the fundamental principle problems in the case of positive spectrum. In contrast to work [3], we employ simple geometric characteristics of sequence Λ .

In the second section we collect auxiliary results. In particular, we construct the aforementioned sequences of entire functions (Lemmata 7, 9) as well as an entire function in $W(\Lambda, \mathbb{C})$ which can not be represented by series (1) in any open subset (Lemma 3).

The main results of the work are given in the third section (Theorems 1-4). In particular, we prove that the set of the sums of the Dirichlet series converging in a given half-plane is closed if and only if $S_\Lambda > -\infty$. Quantity S_Λ was introduced in work [3] (its definition is given in the second section). It is similar to the classical Bernstein condensation index (see, for instance, [1, Ch. II, Sect. 6, Subsect. 2]) but in contrast to the latter, it is effective for each complex sequence.

2. PRELIMINARY RESULTS

We shall make use of some results from the theory of entire functions of exponential type, i.e., of the functions f satisfying the estimate: $\ln |f(z)| \leq A + B|z|, z \in \mathbb{C}$, where $A, B > 0$ depend on f . As the upper and lower indicators of f (subharmonic function $\ln |f|$), we call respectively the functions

$$h_f(\lambda) = \overline{\lim}_{t \rightarrow \infty} \frac{\ln |f(t\lambda)|}{t}, \quad \underline{h}_f(\lambda) = \lim_{\delta \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{\pi \delta^2} \int_{B(\lambda, \delta)} \frac{\ln |f(tz)|}{t} dx dy, \quad \lambda \in \mathbb{C},$$

where $z = x + iy$. By these definitions and Hartogs theorem on the upper limit of a family of subharmonic functions, it is easy to obtain the inequality $\underline{h}_f(\lambda) \leq h_f(\lambda), \lambda \in \mathbb{C}$. We say (cf. [5, Ch. III]) that f has (completely) regular growth if

$$h_f(\lambda) = \lim_{t \rightarrow \infty, t \notin E} \frac{\ln |f(t\lambda)|}{t}, \quad \lambda \in \mathbb{C},$$

where E is a set of zero relative measure on the ray $(0, +\infty)$, i.e., the Lebesgue measure of its intersection with the interval $(0, r)$ is infinitesimal w.r.t. r as $r \rightarrow +\infty$. There is a series of other equivalent definitions of regular growth. Let us provide one of them. A function f (cf. [6, Ch. 4, Def. 4.1]) is called function of regular growth if $\underline{h}_f(\lambda) = h_f(\lambda), \lambda \in \mathbb{C}$.

The upper indicator h_f is a convex function positive homogeneous of order one which coincides with the support function of a convex compact set K (more precisely, complex conjugate with K compact set) called adjoint diagram of f (see, for instance, [7, Ch. I, Sect. 5, Th. 5.4 (Pólya)]),

$$h_f(\lambda) = H_K(\lambda) = \sup_{z \in K} \operatorname{Re}(\lambda z), \quad \lambda \in \mathbb{C}.$$

Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be a sequence of complex numbers with the unique limiting point ∞ . By the symbol $n(\varphi_1, \varphi_2, r, \Lambda)$ we denote the number of points λ_k lying in the sector $\{\lambda = te^{i\varphi} : \varphi \in$

$(\varphi_1, \varphi_2), t \in (0, r)\}$. We say (see [5, Ch. II, Sect. 1]) that Λ has an angular density (as of order one) if for each φ_1, φ_2 except possible countable set there exists the limit

$$\tau(\varphi_1, \varphi_2, \Lambda) = \lim_{r \rightarrow \infty} \frac{n(\varphi_1, \varphi_2, r, \Lambda)}{r}.$$

Set Λ is called well-distributed if it has an angular density and there exists

$$\lim_{r \rightarrow \infty} \sum_{|\lambda_k| < r} \frac{1}{\lambda_k}.$$

According to Theorem 4 in Chapter III in book [5], a function f has a regular growth if and only if its zero set (counting multiplicities) is well-distributed. At that, if K is the adjoint diagram for f , then except possible a countable number of values φ_1, φ_2 , the identity

$$\tau(\varphi_1, \varphi_2, \Lambda) = \frac{1}{2\pi} s(\varphi_1, \varphi_2, K) \quad (2)$$

holds true [5, Ch. II, Sect. 1, Formula (2.07)], where $s(\varphi_1, \varphi_2, K)$ is the length of the arc on boundary ∂K between the support points $z(\varphi_1) \in \partial K$ and $z(\varphi_2) \in \partial K$ of respectively support lines $l(\varphi_1) = \{z : \operatorname{Re}(ze^{i\varphi_1}) = H_K(e^{i\varphi_1})\}$ and $l(\varphi_2) = \{z : \operatorname{Re}(ze^{i\varphi_2}) = H_K(e^{i\varphi_2})\}$. Except possible countable set of values φ (associated with straight parts of the boundary), support line $l(\varphi)$ has the only support point $z(\varphi)$. Among two arcs connecting points $z(\varphi_1)$ and $z(\varphi_2)$, we choose one so that each its point is the support one of some straight line $l(\varphi)$ (depending on it) with value of φ in the segment $[\varphi_1, \varphi_2]$. In the case K is the segment of length τ (and only in this case), the length of the arc $s(\varphi_1, \varphi_2, K)$, where φ_1 and φ_2 are not equal to two opposite numbers φ_0 and $-\varphi_0$, can take just one of three possible values: 0 if the interval (φ_1, φ_2) contains none of these numbers, τ if it contains just one of them, and 2τ if $-\varphi_0, \varphi_0 \in (\varphi_1, \varphi_2)$.

Let D be a convex domain in \mathbb{C} and $H^*(D)$ be the strongly adjoint space for $H(D)$. The Laplace transform $f(\lambda) = \nu(\exp(\lambda z))$ makes an isomorphism (see, for instance, [8, Ch. III, Sect. 12, Th. 12.3]) between $H^*(D)$ and space P_D consisting of entire functions of exponential type whose adjoint diagrams lie in domain D . By Hahn-Banach theorem, the incompleteness of the system $\mathcal{E} = \{\exp(\lambda_k z)\}_{k=1}^{\infty}$ in $H(D)$ (i.e., non-triviality of $W(\Lambda, D)$) is equivalent to the existence of a non-zero linear continuous functional $\nu \in H^*(D)$ vanishing at the elements of the system. Thus, the incompleteness of \mathcal{E} is equivalent to the existence of the function $f \in P_D$ vanishing at the points $\lambda_k, k = 1, 2, \dots$

Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be an unbounded strictly increasing sequence of positive numbers $n(r, \Lambda)$ and denotes the numbers of its terms lying in the semi-interval $(0, r]$. We sat that Λ has density $\tau(\Lambda)$ (is measurable) if there exists a limit

$$\tau(\Lambda) = \lim_{r \rightarrow \infty} \frac{n(r, \Lambda)}{r}.$$

As maximal density of sequence Λ , we call the quantity

$$\tau_0(\Lambda) = \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda) - n((1 - \delta)r, \Lambda)}{\delta r}.$$

We note that in accordance with Lemma in Section E3 of Chapter VI in book [9], the upper limit as $\delta \rightarrow 0$ in the definition of $\tau_0(\Lambda)$ can be replaced by the limit since it always exists. We let

$$\underline{\tau}(\Lambda) = \underline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda)}{r}, \quad \overline{\tau}(\Lambda) = \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda)}{r}.$$

Quantities $\underline{\tau}(\Lambda)$ and $\overline{\tau}(\Lambda)$ are called respectively lower and upper density for sequence Λ . The latter is measurable if and only if $\underline{\tau}(\Lambda) = \overline{\tau}(\Lambda)$. It is easy to see that inequalities

$$\underline{\tau}(\Lambda) \leq \overline{\tau}(\Lambda) \leq \tau_0(\Lambda) \quad (3)$$

hold true. The former follows immediately from the definitions. For the case $\bar{\tau}(\Lambda) < +\infty$, the latter is implied by the relations

$$\begin{aligned} \frac{\overline{\lim}_{r \rightarrow \infty} n(r, \Lambda) - n((1 - \delta)r, \Lambda)}{\delta r} &\geq \frac{\overline{\lim}_{r \rightarrow \infty} n(r, \Lambda)}{\delta r} - \frac{\overline{\lim}_{r \rightarrow \infty} n((1 - \delta)r, \Lambda)}{\delta r} = \\ &= \frac{\bar{\tau}(\Lambda)}{\delta} - (1 - \delta) \frac{\overline{\lim}_{r \rightarrow \infty} n((1 - \delta)r, \Lambda)}{(1 - \delta)\delta r} = \frac{\bar{\tau}(\Lambda)}{\delta} - (1 - \delta) \frac{\bar{\tau}(\Lambda)}{\delta} = \bar{\tau}(\Lambda), \delta > 0. \end{aligned}$$

If $\bar{\tau}(\Lambda) = +\infty$, then one can easily select a subsequence Λ' in sequence Λ with an arbitrarily large finite upper density. Then by the proven we obtain: $\bar{\tau}(\Lambda') \leq \tau_0(\Lambda') \leq \tau_0(\Lambda)$. Therefore, $\tau_0(\Lambda) = +\infty$, i.e., inequality (3) is valid in this case as well. Similar arguments show that in the case when the sequence has density $\tau(\Lambda)$, the identity $\tau_0(\Lambda) = \tau(\Lambda)$ holds true. In the general case, the second inequality in (3) can be strict. Indeed, consider the following example. Let $\Lambda = \cup_{m=1}^{\infty} \Lambda_m$, where $\Lambda_m = \{\lambda_k, k(m) \leq k < k(m+1)\}$, $\lambda_{k(m)+j} = 10^m + j$, $0 \leq j < k(m+1) - k(m)$, $m = 1, 2, \dots$, and $k(1) = 1$, $k(m+1) - k(m) = 10^{m-1}$ as $m > 1$. By direct calculations one can easily get the relations $\bar{\tau}(\Lambda) \leq 1/9$, $\tau_0(\Lambda) = 1$.

Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a complex sequence. Following work [3], we let

$$q_{\Lambda}^j(z, \delta) = \prod_{\lambda_k \in B(\lambda_j, \delta|\lambda_j|), k \neq j} \left(\frac{z - \lambda_k}{3\delta|\lambda_k|} \right).$$

Here $B(w, r)$ is an open circle centered at w with radius r . The absolute value of the function $q_{\Lambda}^j(z, \delta)$ can be interpreted as a measure of accumulation of the points $\lambda_k \in B(\lambda_j, \delta|\lambda_j|)$, $k \neq j$, at z . In the case of absence of such points, we assume that $q_{\Lambda}^j(z, \delta) \equiv 1$. We note that the absolute value of each of multipliers in the definition of q_{Λ}^j , in the circle $B(\lambda_j, \delta|\lambda_j|)$ is estimated from above by the quantity $2(3(1 - \delta))^{-1}$ (for $\delta \in (0, 1)$), i.e., as $\delta \in (0, 1/3)$, it does not exceeds one. Moreover, if $\delta_1 \leq \delta_2$, the number of multipliers in the definition of $q_{\Lambda}^j(z, \delta_1)$ does not exceeds the number of multipliers in the definition of $q_{\Lambda}^j(z, \delta_2)$ and each of the multipliers for $q_{\Lambda}^j(z, \delta_1)$ is not less in absolute value than the corresponding multiplier for $q_{\Lambda}^j(z, \delta_2)$. Thus, if $0 < \delta_1 \leq \delta_2 < 1/3$, then $|q_{\Lambda}^j(z, \delta_1)| \geq |q_{\Lambda}^j(z, \delta_2)|$, $z \in B(\lambda_j, \delta_2|\lambda_j|)$. We let $S_{\Lambda} = 0$ if Λ consists of a finite number of elements and

$$S_{\Lambda} = \lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\ln |q_{\Lambda}^k(\lambda_k, \delta)|}{|\lambda_k|}$$

otherwise. This definition is well-defined, since according to the latter inequality, the limit w.r.t. δ always exists. By the said above, $S_{\Lambda} \leq 0$. We note that the coefficient 3 in the definition of q_{Λ}^j is chosen just for convenience (cf. [3, Rem. 1 to Th. 5.1]). It ensures the non-positivity for quantity S_{Λ} . It is close by the meaning with the classical Bernstein condensation index, but it is effective for each complex sequence, not only for a measurable positive sequence and a complex sequence of zero density. Together with S_{Λ} , we introduce one more quantity

$$\tilde{S}_{\Lambda} = \lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\ln |q_{\Lambda}^k(\lambda_k, \delta)|}{\delta|\lambda_k|}.$$

As for S_{Λ} , the inequality $\tilde{S}_{\Lambda} \leq 0$ is valid. If \tilde{S}_{Λ} is finite, then, obviously, $S_{\Lambda} = 0$. As an example we consider the sequence of positive numbers $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ such that $\lambda_{k+1} - \lambda_k \geq h > 0$, $k = 1, 2, \dots$. Taking into consideration the inequality $n! \geq (n/3)^n$, we have

$$|q_{\Lambda}^k(\lambda_k, \delta)| \geq \prod_{\lambda_m \in B(\lambda_k, \delta\lambda_k), m \neq k} \left| \frac{\lambda_m - \lambda_k}{3\delta\lambda_m} \right| \geq \frac{(m(k, \delta)! h^{m(k, \delta)})^2}{(3\delta(1 + \delta)\lambda_k)^{2m(k, \delta)}} \geq \frac{(m(k, \delta)h)^{2m(k, \delta)}}{(9\delta(1 + \delta)\lambda_k)^{2m(k, \delta)}},$$

where $m(k, \delta)$ is the maximal integer satisfying inequality $m(k, \delta)h < \delta\lambda_k$. Therefore,

$$\tilde{S}_{\Lambda} \geq \lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \frac{2m(k, \delta) \ln(m(k, \delta)h/9\delta(1 + \delta)\lambda_k)}{\delta\lambda_k} \geq -\frac{2 \ln 9}{h}.$$

Lemma 1. *Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be an unbounded strictly increasing sequence of positive numbers. Suppose that $\tilde{S}_{\Lambda} > -\infty$. Then the maximal density $\tau_0(\Lambda)$ is finite.*

Proof. Let $\delta \in (0, 1)$. Employing the definition of q_{Λ}^j , we obtain

$$\begin{aligned} \ln |q_{\Lambda}^j(\lambda_j, \delta)| &= \ln \left| \prod_{\lambda_k \in B(\lambda_j, \delta\lambda_j), k \neq j} \left(\frac{\lambda_j - \lambda_k}{3\delta\lambda_k} \right) \right| \leq \ln \left| \prod_{\lambda_k \in B(\lambda_j, \delta\lambda_j), k \neq j} \left(\frac{\delta\lambda_j}{3\delta\lambda_k} \right) \right| \leq \\ &\leq \ln \left(\frac{\lambda_j}{3(1-\delta)\lambda_j} \right)^{(m(j, \delta) - 1)} = -(m(j, \delta) - 1) \ln(3(1-\delta)), \end{aligned}$$

where $m(j, \delta)$ is the number of points λ_k in the circle $B(\lambda_j, \delta\lambda_j)$, i.e., $m(j, \delta) = n((1+\delta)\lambda_j, \Lambda) - n((1-\delta)\lambda_j, \Lambda)$ if the point $(1+\delta)\lambda_j$ does not belong to Λ , and the number $m(j, \delta)$ is less by one otherwise. Therefore, we have

$$\begin{aligned} \tilde{S}_{\Lambda} &= \lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\ln |q_{\Lambda}^k(\lambda_k, \delta)|}{\delta\lambda_k} \leq \lim_{\delta \rightarrow 0} \lim_{j \rightarrow \infty} \frac{-\ln(3(1-\delta))m(j, \delta)}{\delta\lambda_j} = \\ &= -\overline{\lim}_{\delta \rightarrow 0} \ln(3(1-\delta)) \overline{\lim}_{j \rightarrow \infty} \frac{m(j, \delta)}{\delta\lambda_j} = -\ln 3 \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{j \rightarrow \infty} \frac{n((1+\delta)|\lambda_j|, \Lambda) - n((1-\delta)|\lambda_j|, \Lambda)}{\delta\lambda_j}. \end{aligned}$$

Let us show that the double upper limit in the latter identity is estimated from below by the quantity $\tau_0(\Lambda)$. If $\tau_0(\Lambda) = 0$, it is obvious. For each $\delta \in (0, 1)$, by $r_k(\delta)$, $k = 1, 2, \dots$ we denote the sequence implementing the upper limit as $r \rightarrow \infty$ in the definition of the maximal density. Let $\tau_0(\Lambda) > 0$. Then we can assume that each semi-interval $((1-\delta)r_k(\delta), r_k(\delta)]$ contains some nonzero number of sequence Λ . We choose arbitrarily one of them and by $j(k, \delta)$ we indicate its index. Since $\lambda_{j(k, \delta)} \leq r_k(\delta) \leq \lambda_{j(k, \delta)}/(1-\delta)$, it is easy to observe that the inclusion $((1-\delta)r_k(\delta), r_k(\delta)] \subset ((1-\tilde{\delta})\lambda_{j(k, \delta)}, (1+\tilde{\delta})\lambda_{j(k, \delta)})$ holds true, where $\tilde{\delta} = \delta/(1-\delta)$. Thus,

$$\begin{aligned} \tau_0(\Lambda) &= \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda) - n((1-\delta)r, \Lambda)}{\delta r} = \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{k \rightarrow \infty} \frac{n(r_k(\delta), \Lambda) - n((1-\delta)r_k(\delta), \Lambda)}{\delta r_k(\delta)} \leq \\ &\leq \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{j \rightarrow \infty} \frac{\tilde{\delta} n((1+\tilde{\delta})\lambda_j, \Lambda) - n((1-\tilde{\delta})\lambda_j, \Lambda)}{\tilde{\delta}\lambda_j} = \\ &= \overline{\lim}_{\tilde{\delta} \rightarrow 0} \overline{\lim}_{r \rightarrow \infty} \frac{n((1+\tilde{\delta})\lambda_j, \Lambda) - n((1-\tilde{\delta})\lambda_j, \Lambda)}{\tilde{\delta}\lambda_j}. \end{aligned}$$

Hence, in view of the above arguments, we obtain $\tilde{S}_{\Lambda} \leq -\ln 3\tau_0(\Lambda)$. It implies the required statement. The proof is complete. \square

Lemma 2. *Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be an unbounded strictly increasing sequence of positive numbers. If $S_{\Lambda} > -\infty$, then $\overline{\tau}(\Lambda) < +\infty$*

Proof. Suppose $\overline{\tau}(\Lambda) = +\infty$. Then for each $A > 0$ there exists a subsequence $\Lambda(A)$ of sequence Λ such that the upper density $\overline{\tau}(\Lambda(A))$ is finite and greater than A . We fix $\delta \in (0, 1)$ and $A > 0$. We have

$$\begin{aligned} \overline{\tau}(\Lambda(A)) &= \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda(A))}{r} \leq \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda(A)) - n((1-\delta)r, \Lambda(A))}{r} + \overline{\lim}_{r \rightarrow \infty} \frac{n((1-\delta)r, \Lambda(A))}{r} = \\ &= \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda(A)) - n((1-\delta)r, \Lambda(A))}{r} + (1-\delta)\overline{\tau}(\Lambda(A)). \end{aligned}$$

It follows that for each $A > 0$,

$$\delta A \leq \delta \overline{\tau}(\Lambda(A)) \leq \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda(A)) - n((1-\delta)r, \Lambda(A))}{r} \leq \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda) - n((1-\delta)r, \Lambda)}{r}.$$

Hence,

$$\lim_{r \rightarrow \infty} \frac{n(r, \Lambda) - n((1 - \delta)r, \Lambda)}{r} = +\infty, \quad \delta \in (0, 1).$$

Let $\delta \in (0, 1/2)$. We choose a sequence $r_j \rightarrow +\infty$ such that

$$\lim_{j \rightarrow \infty} \frac{n(r_j, \Lambda) - n((1 - \delta)r_j, \Lambda)}{r_j} = +\infty.$$

We can assume that for each $j \geq 1$, the semi-interval $((1 - \delta)r_j, r_j]$ contains some points of sequence Λ . Let λ_{k_j} be one of such points. Then the inclusion $((1 - \delta)r_j, r_j] \subset ((1 - 2\delta)\lambda_{k_j}, (1 + 2\delta)\lambda_{k_j})$ holds true. This is why

$$\lim_{j \rightarrow \infty} \frac{m(k_j, 2\delta)}{\lambda_{k_j}} = +\infty,$$

where $m(k_j, 2\delta)$ is defined in Lemma 1. As in this lemma, we obtain

$$\ln |q_\Lambda^{k_j}(\lambda_{k_j}, 2\delta)| \leq -(m(k_j, 2\delta) - 1) \ln(3(1 - 2\delta)).$$

Therefore, we have

$$\begin{aligned} S_\Lambda &= \lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\ln |q_\Lambda^k(\lambda_k, \delta)|}{\lambda_k} \leq \lim_{\delta \rightarrow 0} \lim_{j \rightarrow \infty} \frac{\ln |q_\Lambda^{k_j}(\lambda_{k_j}, 2\delta)|}{\lambda_{k_j}} \leq \\ &\leq \lim_{\delta \rightarrow 0} \ln(3(1 - 2\delta)) \lim_{j \rightarrow \infty} \frac{-(m(k_j, 2\delta) - 1)}{\lambda_{k_j}} = -\ln 3 \lim_{\delta \rightarrow 0} \lim_{j \rightarrow \infty} \frac{(m(k_j, 2\delta) - 1)}{\lambda_{k_j}} = -\infty. \end{aligned}$$

It contradicts the assumption. Thus, $\bar{\tau}(\Lambda) < +\infty$. The proof is complete. \square

By $S(z, r)$ we denote the circumference centered at z of radius $r > 0$. The proof of the next two statements is based on the ideas of the proof of Theorem 3.1 in work [10].

Lemma 3. *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers such that $S_\Lambda = -\infty$. Then there exists an entire function $g \in W(\Lambda, \mathbb{C})$ which does not expand into a Dirichlet series w.r.t. the system $\mathcal{E} = \{\exp(\lambda_k z)\}_{k=1}^\infty$ on any open subset in the plane.*

Proof. Suppose first that $\bar{\tau}(\Lambda) < +\infty$. By assumption, $S_\Lambda = -\infty$. Hence, there exist a sequence of positive numbers $\{\delta_p\}$ and a subsequence $\{\lambda_{k(p)}\}$ such that $\delta_p \rightarrow 0$ as $p \rightarrow \infty$ and

$$\lim_{p \rightarrow \infty} \frac{\ln |q_\Lambda^{k(p)}(\lambda_{k(p)}, \delta_p)|}{\lambda_{k(p)}} = -\infty. \quad (4)$$

We can assume that

$$\lambda_{k(p+1)} \geq 2\lambda_{k(p)}, \quad \delta_p < 1/4, \quad p \geq 1. \quad (5)$$

Consider the functions

$$g_p(z) = \frac{1}{2\pi i} \int_{S(\lambda_{k(p)}, 5\delta_p \lambda_{k(p)})} \frac{\exp(\lambda z) d\lambda}{(\lambda - \lambda_{k(p)}) q_\Lambda^{k(p)}(\lambda, \delta_p)}, \quad p = 1, 2, \dots$$

Let us find the upper estimate for $|g_p|$. We have

$$\begin{aligned} |q_\Lambda^{k(p)}(\lambda, \delta_p)| &= \left| \prod_{\lambda_k \in B(\lambda_{k(p)}, \delta_p \lambda_{k(p)}), k \neq k(p)} \frac{\lambda - \lambda_k}{3\delta_p \lambda_k} \right| \geq \\ &\geq \left(\frac{4\delta_p \lambda_{k(p)}}{3\delta_p (1 + \delta_p) \lambda_{k(p)}} \right)^{m(k(p), \delta_p)} \geq 1, \quad \lambda \in S(\lambda_{k(p)}, 5\delta_p \lambda_{k(p)}), \end{aligned}$$

where $m(k(p), \delta_p)$ is determined in Lemma 1. Thus, the inequalities

$$\begin{aligned} |g_p(z)| &= \left| \frac{1}{2\pi i} \int_{S(\lambda_{k(p)}, 5\delta_p \lambda_{k(p)})} \frac{\exp(\lambda z) d\lambda}{(\lambda - \lambda_{k(p)}) q_\Lambda^{k(p)}(\lambda, \delta_p)} \right| \leq \\ &\leq 5\lambda_{k(p)} \delta_p \sup_{\lambda \in S(\lambda_{k(p)}, 5\delta_p \lambda_{k(p)})} \left| \frac{\exp(\lambda z)}{(\lambda - \lambda_{k(p)})} \right| \leq \sup_{\lambda \in S(\lambda_{k(p)}, 5\delta_p \lambda_{k(p)})} |\exp(\lambda z)| \leq \\ &\leq \exp(\operatorname{Re}(\lambda_{k(p)} z) + 5\delta_p \lambda_{k(p)} |z|), \quad z \in \mathbb{C}, \end{aligned} \quad (6)$$

hold true.

Consider the function

$$g(z) = \sum_{p=1}^{\infty} c_p g_p(z), \quad (7)$$

where $c_p = \sqrt{|q_\Lambda^{k(p)}(\lambda_{k(p)}, \delta_p)|}$, $p \geq 1$. Let us show that this series converges uniformly on each compact subset of the plane. Let $R > 0$. By (4), there exists a number p_0 such that $|c_p| \leq \exp(-2R\lambda_{k(p)})$, $p \geq p_0$. Then in view of (6) we have

$$\sum_{p=1}^{\infty} |c_p| \max_{|z| \leq R} |g_p(z)| \leq A + \sum_{p=p_0}^{\infty} \exp(-2R\lambda_{k(p)} + R\lambda_{k(p)} + 5\delta_p R\lambda_{k(p)}) < +\infty.$$

This estimate holds true thanks to inequalities $\lambda_{k(p+1)} \geq 2\lambda_{k(p)}$, $p \geq 1$ and that $\delta_p \rightarrow 0$ as $p \rightarrow \infty$.

Thus, function $g(z)$ is entire and belongs to space $W(\Lambda, \mathbb{C})$. Suppose it is represented by series (1) on some open subset $U \subset \mathbb{C}$ containing a point z_0 . Then by Abel theorem for Dirichlet series (see, for instance [1, Ch. II, Sect. 1, Subsect. 2], series (1) converges uniformly on compact subsets of the half-plane $\Pi = \{z \in \mathbb{C} : \operatorname{Re} z < \operatorname{Re} z_0\}$. On open subset $\Pi \cap U$ its sum is equal to $g(z)$. This is why it converges to $g(z)$ in the whole half-plane Π . Since upper density $\bar{\tau}(\Lambda)$ is finite, there exists (see, for instance, [1, Ch. IV, Sect. 1, Subsect. 1]) the biorthogonal to \mathcal{E} sequence of functionals $\{\nu_k\} \subset H^*(\Pi) \subset H^*(\mathbb{C})$, i.e., $\nu_k(\exp(\lambda_k z)) = 1$, $k \geq 1$, and $\nu_k(\exp(\lambda_j z)) = 0$, if $k \neq j$. Since series (1) in the topology of space $H(\Pi)$, the identities

$$\nu_k(g) = d_k, \quad k \geq 1 \quad (8)$$

are valid.

Employing the residua and the definition of function g_p , we obtain

$$g_p(z) = b_{k(p)} \exp(\lambda_{k(p)} z) + \sum_{\lambda_k \in ((1-\delta_p)\lambda_{k(p)}, (1+\delta_p)\lambda_{k(p)}), k \neq k(p)} b_k \exp(\lambda_k z),$$

where $b_{k(p)} = (q_\Lambda^{k(p)}(\lambda_{k(p)}, \delta_p))^{-1}$, $p \geq 1$. By (5), the intervals $((1-\delta_p)\lambda_{k(p)}, (1+\delta_p)\lambda_{k(p)})$ are mutually disjoint. Then, taking into consideration the convergence of series (7) in the topology of space $H(\mathbb{C})$ and identities (8), we obtain

$$|d_{k(p)}| = |\nu_{k(p)}(g)| = |c_p b_{k(p)}| = \left(\sqrt{|q_\Lambda^{k(p)}(\lambda_{k(p)}, \delta_p)|} \right)^{-1}, \quad p \geq 1.$$

It follows from (4) that for each $z \in \mathbb{C}$

$$|d_{k(p)}| \exp(\operatorname{Re}(\lambda_{k(p)} z)) \rightarrow +\infty, \quad p \rightarrow \infty.$$

It contradicts to the convergence of series (1) in half-plane Π . Thus, function $g \in W(\Lambda, \mathbb{C})$ does not expand into the series w.r.t. system \mathcal{E} on any open subset of the plane.

It remains to consider the situation when $\bar{\tau}(\Lambda) = +\infty$. In this case there exists no (see, for instance, [1, Ch. I, Sect. 1, Th. 1.1.2]) entire function of exponential type $f \in P_{\mathbb{C}}$ vanishing at

all points of λ_k . Therefore, system \mathcal{E} is complete in $H(\mathbb{C})$, i.e., $W(\Lambda, \mathbb{C}) = H(\mathbb{C})$. Let $\lambda_0 > 0$ differ from the points λ_k , $k \geq 1$. Suppose the function $\exp(\lambda_0 z) \in W(\Lambda, \mathbb{C})$ is represented by series (1) on some open subset of the complex plane. As above, this representation can be extended on some half-plane. Then in this half-plane, the identity

$$0 = \sum_{k=0}^{\infty} d_k \exp(\lambda_k z)$$

holds true, where $d_0 = -1$. In the beginning of the work we mentioned that the representation by the Dirichlet series is always unique. Thus, the relations $d_k = 0$, $k = 0, 1, \dots$ hold true. We obtain the contradiction. The proof is complete. \square

Corollary. *Let D be a convex domain in \mathbb{C} , $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be an unbounded strictly increasing sequence of positive numbers. Suppose the set of the sums of series (1) converging in domain D is closed in space $H(D)$. Then $S_{\Lambda} > -\infty$ and $\bar{\tau}(\Lambda) < +\infty$.*

Proof. If $S_{\Lambda} = -\infty$, by Lemma 3, there exists a function $g \in W(\Lambda, \mathbb{C}) \subset W(\Lambda, D)$ which does not expand into series (1) in domain D . It contradicts the assumption. Thus, $S_{\Lambda} > -\infty$. Then by Lemma 2, $\bar{\tau}(\Lambda) < +\infty$. The proof is complete. \square

Lemma 4. *Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be an unbounded strictly increasing sequence of positive numbers. Suppose $\tilde{S}_{\Lambda} = -\infty$. Then for each $\tau > 0$, there exists $\delta > 0$ and function $g \in W(\Lambda, G)$, where $G = (\{z : \operatorname{Re} z < \tau\delta\} \cap B(0, \tau)) \cup \{z : \operatorname{Re} z < 0\}$ is represented by series (1) with the convergence abscissa $\gamma = 0$.*

Proof. Let

$$g(z) = \sum_{p=1}^{\infty} c_p g_p(z), \tag{9}$$

where $c_p = \exp(-6\tau\delta\mu_{m(p)})$,

$$g_p(z) = \frac{1}{2\pi i} \int_{S(\mu_{m(p)}, 5\delta\mu_{m(p)})} \frac{\exp(\lambda z) d\lambda}{a_p(\lambda - \mu_{m(p)}) q_{\tilde{\Lambda}}^{m(p)}(\lambda, \delta)}, \quad p = 1, 2, \dots,$$

and the sequence $\tilde{\Lambda} = \{\mu_m\}_{m=1}^{\infty}$ is a part of Λ . We shall choose numbers $a_p \geq 1$ later. Now we determine number δ , construct $\tilde{\Lambda}$, and select $m(p)$. In order to do it, we first of all observe that according to the assumption, $\tilde{S}_{\Lambda} = -\infty$. This is why the exist $\delta \in (0, 1/4)$ and a subsequence $\{\lambda_{k(p)}\}_{p=1}^{\infty}$ of sequence Λ obeying condition

$$\ln |q_{\tilde{\Lambda}}^{k(p)}(\lambda_{k(p)}, \delta)| \leq -6\tau\delta\lambda_{k(p)}, \quad p = 1, 2, \dots \tag{10}$$

At that, we can assume that

$$\lambda_{k(p+1)} \geq 2\lambda_{k(p)}, \quad p = 1, 2, \dots \tag{11}$$

We shall seek sequence $\tilde{\Lambda} \subset \Lambda$ as the union $\cup_{p=1}^{\infty} \Lambda_p$. Fix $p = 1, 2, \dots$. If $n((1 + \delta)\lambda_{k(p)}, \Lambda) - n((1 - \delta)\lambda_{k(p)}, \Lambda) - 1 < 12\tau\delta\lambda_{k(p)} + 1$, as Λ_p , we take the set of all points in sequence Λ lying in the circle $B(\lambda_{k(p)}, \delta\lambda_{k(p)})$. Otherwise we put the point $\lambda_{k(p)}$ into Λ_p and as many points in Λ lying in the circle $B(\lambda_{k(p)}, \delta\lambda_{k(p)})$ as we need the number of points $l(p)$ of set Λ_p to satisfy the estimates

$$12\tau\delta\lambda_{k(p)} \leq l(p) - 1 < 12\tau\delta\lambda_{k(p)} + 1. \tag{12}$$

We note that by (11) and the choice of number δ , the circles $B(\lambda_{k(p)}, \delta\lambda_{k(p)})$, $p = 1, 2, \dots$, are mutually disjoint. This is why the sets Λ_p , $p = 1, 2, \dots$, are mutually disjoint, too. We shall assume that the elements μ_m of sequence $\tilde{\Lambda}$ are indexed in the ascending order. By $m(p)$, $p = 1, 2, \dots$, we denote the number for which $\mu_{m(p)} = \lambda_{k(p)}$. The inequalities

$$\ln |q_{\tilde{\Lambda}}^{m(p)}(\mu_{m(p)}, \delta)| \leq -6\tau\delta\mu_{m(p)}, \quad p = 1, 2, \dots \tag{13}$$

hold true. Indeed, the function $q_{\tilde{\Lambda}}^{m(p)}$ comprises the multipliers constructed by the points of set Λ_p . If it coincides with the set of points in Λ lying in circle $B(\lambda_{k(p)}, \delta\lambda_{k(p)})$, then $q_{\tilde{\Lambda}}^{m(p)}(\mu_{m(p)}, \delta) = q_{\Lambda}^{k(p)}(\lambda_{k(p)}, \delta)$ and (13) are implied by (11). Otherwise, taking into consideration (12) and inequality $\delta < 1/4$, as in Lemma 1, we get

$$\ln |q_{\tilde{\Lambda}}^{m(p)}(\mu_{m(p)}, \delta)| \leq -(l(p) - 1) \ln(3(1 - \delta)) \leq -(l(p) - 1)/2 \leq -6\tau\delta\mu_{m(p)}.$$

Let us show now that the upper density of sequence $\tilde{\Lambda}$ is finite. According to (12), the inequalities

$$\frac{n((1 + \delta)\mu_{m(p)}, \tilde{\Lambda}) - n((1 - \delta)\mu_{m(p)}, \tilde{\Lambda})}{\mu_{m(p)}} \leq 12\tau\delta + \frac{2}{\mu_{m(p)}}, \quad p = 1, 2, \dots \quad (14)$$

hold true. Let $r > 0$ and $p(r)$ be the maximal among the numbers $p = 1, 2, \dots$, for which the intervals $(0, r)$ and $((1 - \delta)\mu_{m(p)}, (1 + \delta)\mu_{m(p)})$ intersect. Then the inequality $r > (1 - \delta)\mu_{m(p(r))}$ is valid. Since all the points μ_m lie in the union $\cup_{p=1}^{\infty} ((1 - \delta)\mu_{m(p)}, (1 + \delta)\mu_{m(p)})$ and $\mu_{m(p)} = \lambda_{k(p)}$, $p = 1, 2, \dots$, due to (14) and (11) we get

$$\begin{aligned} \frac{n(r, \tilde{\Lambda})}{r} &\leq \sum_{p=1}^{p(r)} \frac{n((1 + \delta)\mu_{m(p)}, \tilde{\Lambda}) - n((1 - \delta)\mu_{m(p)}, \tilde{\Lambda})}{r} \leq \sum_{p=1}^{p(r)} \frac{12\tau\delta\mu_{m(p)} + 2}{r} \leq \\ &\leq \sum_{p=1}^{p(r)} \frac{12\tau\delta\mu_{m(p)} + 2}{(1 - \delta)\mu_{m(p(r))}} \leq \frac{p(r)}{(1 - \delta)2^{p(r)-2}\mu_{m(1)}} + \sum_{p=1}^{p(r)} \frac{12\tau\delta}{(1 - \delta)2^{p(r)-p}}. \end{aligned}$$

It follows that quantity $\bar{\tau}(\tilde{\Lambda})$ is finite.

Let us find the upper estimates for the absolute values of functions g_p . Since $a_p \geq 1$, we have

$$\begin{aligned} |a_p q_{\tilde{\Lambda}}^{m(p)}(\lambda, \delta)| &\geq |q_{\tilde{\Lambda}}^{m(p)}(\lambda, \delta)| = \left| \prod_{\mu_m \in B(\mu_{m(p)}, \delta\mu_{m(p)}), m \neq m(p)} \frac{(\lambda - \mu_m)}{3\delta\mu_m} \right| \geq \\ &\geq \prod_{\mu_m \in B(\mu_{m(p)}, \delta\mu_{m(p)}), m \neq m(p)} \frac{4\delta\mu_{m(p)}}{3\delta\mu_m} \geq 1, \quad \lambda \in S(\mu_{m(p)}, 5\delta\mu_{m(p)}). \end{aligned}$$

Therefore, the inequalities

$$\begin{aligned} |g_p(z)| &= \left| \frac{1}{2\pi i} \int_{\lambda \in S(\mu_{m(p)}, 5\delta\mu_{m(p)})} \frac{\exp(\lambda z) d\lambda}{a_p (\lambda - \mu_{m(p)}) q_{\tilde{\Lambda}}^{m(p)}(\mu_{m(p)}, \delta)} \right| \leq \\ &\leq \sup_{\lambda \in S(\mu_{m(p)}, 5\delta\mu_{m(p)})} |\exp(\lambda z)| \leq \exp(\operatorname{Re}(\mu_{m(p)} z) + 5\delta\mu_{m(p)} |z|), \quad z \in \mathbb{C}. \end{aligned}$$

hold true.

Let us show that series (9) converge uniformly on compact subsets of the domain $D = \{z : \operatorname{Re} z < \tau\delta\} \cap B(0, \tau)$. Let $z \in D$ and $|z| \leq \tau - \beta$. The latter estimates, the definition of coefficients c_p and (11) yield

$$|g(z)| \leq \sum_{p=1}^{\infty} \exp(\operatorname{Re}(\mu_{m(p)} z) + 5\delta\mu_{m(p)} |z| - 6\tau\delta\mu_{m(p)}) \leq \sum_{p=1}^{\infty} \exp(-5\beta\delta\mu_{m(p)}) < \infty.$$

The terms of series (9) are entire functions. Therefore, $g(z)$ is function analytic in domain D . We note that it is true for each choice of numbers $a_p \geq 1$, $p = 1, 2, \dots$. Let us show now that under a suitable choice of these numbers, function $g(z)$ can be expanded in the Dirichlet series

whose line of convergence coincides with the imaginary axis. Employing the theory of residues, for each $p = 1, 2, \dots$ we obtain

$$g_p(z) = a_p^{-1} \left(b_{m(p)} \exp(\mu_{m(p)} z) + \sum_{\mu_m \in \Lambda_p, m \neq m(p)} b_m \exp(\mu_m z) \right),$$

where $b_{m(p)} = (q_{\tilde{\Lambda}}^{m(p)}(\mu_{m(p)}, \delta))^{-1}$. Let $p = 1, 2, \dots$. For each number m such that $\mu_m \in \Lambda_p$, we let $d_m = c_p b_m a_p^{-1}$. By (13) and the definition of c_p , the inequality

$$\max_{m: \mu_m \in \Lambda_p} \frac{\ln |c_p b_m|}{\mu_{m(p)}} \geq \frac{-6\tau\delta\mu_{m(p)} - \ln |q_{\tilde{\Lambda}}^{m(p)}(\mu_{m(p)}, \delta)|}{\mu_{m(p)}} \geq 0$$

holds true. This is why there exists a number $a_p \geq 1$ so that

$$\max_{m: \mu_m \in \Lambda_p} \frac{\ln |d_m|}{\mu_{m(p)}} = \max_{m: \mu_m \in \Lambda_p} \frac{\ln |c_p b_m| - \ln a_p}{\mu_{m(p)}} = 0. \tag{15}$$

Consider the Dirichlet series

$$\sum_{m=1}^{\infty} d_m \exp(\mu_m z). \tag{16}$$

Since the upper density $\bar{\tau}(\tilde{\Lambda})$ is finite, it is obvious that the quantity $\lim_{m \rightarrow \infty} \ln m / \mu_m$ vanishes. This is why (see, for instance, [1, Ch. II, Sect. 1, Th. 2.1.2], [11]) Cauchy-Hadamard formula

$$\gamma = \overline{\lim}_{m \rightarrow \infty} \frac{\ln |d_m|}{\mu_m}$$

holds true for series (16), where γ is the convergence abscissa for this series. At that, in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < \gamma\}$, series (16) converges absolutely (see, for instance, [1, Ch. II, Sect. 1, Cor. of Th. 2.1.1], [11]). By (15),

$$\gamma = \overline{\lim}_{m \rightarrow \infty} \frac{\ln |d_m|}{\mu_m} = \overline{\lim}_{p \rightarrow \infty} \max_{m: \mu_m \in \Lambda_p} \frac{\ln |d_m|}{\mu_m} = 0.$$

Thus, the imaginary axis is the convergence line for series (16). Since it converges absolutely in the left half-plane, the sums of series (9) and (16) coincide in the intersection $\{z \in \mathbb{C} : \operatorname{Re} z < 0\} \cap D$. It means that function $g \in W(\Lambda, G)$ can be represented series (1) with convergence abscissa $\gamma = 0$. The proof is complete. \square

Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$, $\tilde{\Lambda} = \{\mu_n\}_{n=1}^{\infty}$. We say that Λ is a part of $\tilde{\Lambda}$ ($\Lambda \subset \tilde{\Lambda}$) or $\tilde{\Lambda}$ is the completion of Λ if there exists a subsequence $\{\mu_{n(k)}\}$ coinciding with $\{\lambda_k\}$. The well-known Pólya theorem (see, for instance, [9, Ch. VI, Sect. E3]) states that each sequence with a finite maximal density is a part of some measurable sequence with the same density. Since this result is important for further studies and understanding the whole picture, we provide its proof. The method employed for constructing the completion, in our opinion, is simpler than that in book [9].

Lemma 5 (Pólya theorem). *Suppose the maximal density of a sequence $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ is finite. Then there exists its measurable completion $\tilde{\Lambda}$ such that $\tau(\tilde{\Lambda}) = \tau_0(\Lambda)$.*

Proof. We shall seek sequence $\tilde{\Lambda}$ as the union $\cup_{m=1}^{\infty} \Lambda_m$, where $\Lambda_m = \{\mu_n, n(m) \leq n < n(m+1)\}$, $m = 1, 2, \dots$ and $n(1) = 1$. We construct sets Λ_m by induction. Let $\alpha = 1/\tau_0(\Lambda)$ and $m = 1$. If the semi-interval $(0, \alpha]$ contains the points in Λ , we let $\mu_n = \lambda_n$, $1 \leq n < n(2)$, where $n(2)$ is the minimal of numbers k satisfying the inequality $\lambda_k > \alpha$. Otherwise we let $\mu_1 = \alpha$ and $n(2) = 2$. Suppose we have constructed sets Λ_m for $m < p$. Let us define Λ_p . By s_p we indicate the number of points in sequence Λ lying in the semi-interval $(\alpha(p-1), \alpha p]$ (it can happen that $s_p = 0$). If the total amount of points in the sets $\Lambda_1, \dots, \Lambda_{p-1}$ does not exceed $p-1$ and $s_p = 0$, we let $n(p+1) = n(p) + 1$ and $\Lambda_p = \{\mu_{n(p)}\}$, where $\mu_{n(p)} = \alpha p$. Otherwise, as Λ_p , we take the

set consisting of all points Λ lying in $(\alpha(p - 1), \alpha p]$ (if $s_p = 0$, set Λ_p is empty). At that, we let $n(p + 1) = n(p) + s_p$.

Let us show that $\tilde{\Lambda}$ is the desired sequence. Let λ_k be an arbitrary point in Λ and a number m is so that the semi-interval $(\alpha(m - 1), \alpha m]$ contains λ_k . Then $s_m \neq 0$ and by construction, set Λ_m together with sequence $\tilde{\Lambda}$ comprise λ_k . Therefore, $\Lambda \subset \tilde{\Lambda}$. Employing induction, let us prove the inequalities

$$n(\alpha m, \tilde{\Lambda}) \geq m, \quad m = 1, 2, \dots \tag{17}$$

By construction, each such semi-interval $(\alpha(m - 1), \alpha m]$ intersects with $\tilde{\Lambda}$ by the set Λ_m . This is why $n(\alpha m, \tilde{\Lambda})$ coincides with the total number of points in the sets $\Lambda_1, \dots, \Lambda_m$. Let $m = 1$. Then $n(\alpha, \tilde{\Lambda}) = 1$ if $s_1 = 0$, and $n(\alpha, \tilde{\Lambda}) = s_1 \geq 1$ otherwise. Suppose (17) is proven for all $m < p$. If $n(\alpha(p - 1), \tilde{\Lambda}) > p - 1$, then $n(\alpha p, \tilde{\Lambda}) \geq n(\alpha(p - 1), \tilde{\Lambda}) \geq p$. Let $n(\alpha(p - 1), \tilde{\Lambda}) \leq p - 1$. Then due to (17) we obtain $n(\alpha(p - 1), \tilde{\Lambda}) = p - 1$. In this case, by construction, $n(\alpha p, \tilde{\Lambda}) = n(\alpha(p - 1), \tilde{\Lambda}) + 1 = p$, if $s_p = 0$, and $n(\alpha p, \tilde{\Lambda}) = n(\alpha(p - 1), \tilde{\Lambda}) + s_p \geq p$ otherwise. Hence, inequality (17) is valid for all m . Taking it into account, we obtain

$$\underline{\tau}(\tilde{\Lambda}) = \lim_{r \rightarrow \infty} \frac{n(r, \tilde{\Lambda})}{r} \geq \lim_{p \rightarrow \infty} \frac{n(\alpha p(r), \tilde{\Lambda})}{\alpha p(r) + \beta(r)} = \lim_{p \rightarrow \infty} \frac{n(\alpha p, \tilde{\Lambda})}{\alpha p} \geq \frac{1}{\alpha} = \tau_0(\Lambda), \tag{18}$$

where $\beta(r) \in (0, \alpha]$ as $r \geq 1$. It remains to prove inequality $\bar{\tau}(\tilde{\Lambda}) \leq \tau_0(\Lambda)$. Suppose $\bar{\tau}(\tilde{\Lambda}) \geq \tau_0(\Lambda) + 3\varepsilon$ for some $\varepsilon > 0$. Then due to (2), $\tau_0(\tilde{\Lambda}) \geq \tau_0(\Lambda) + 3\varepsilon$. It was observed above that in the definition of the maximal density one can take the limit instead of the upper limit w.r.t. δ . This is why there exists $\delta_0 > 0$ such that

$$\lim_{r \rightarrow \infty} \frac{n(r, \tilde{\Lambda}) - n((1 - \delta)r, \tilde{\Lambda})}{\delta r} \geq \tau_0(\Lambda) + 2\varepsilon, \quad \delta \in (0, \delta_0). \tag{19}$$

Lessening $\delta_0 > 0$ if necessary, we can assume that the inequality

$$\lim_{r \rightarrow \infty} \frac{n(r, \Lambda) - n((1 - \delta)r, \Lambda)}{\delta r} \leq \tau_0(\Lambda) + \varepsilon, \quad \delta \in (0, \delta_0) \tag{20}$$

holds true as well. We fix $\delta \in (0, \delta_0)$. If for some $r > 0$, the semi-interval $((1 - \delta)r, r]$ contains no points $\tilde{\Lambda}$ different to Λ , then $n(r, \tilde{\Lambda}) - n((1 - \delta)r, \tilde{\Lambda}) = n(r, \Lambda) - n((1 - \delta)r, \Lambda)$. We note that by construction, all the points $\tilde{\Lambda}$ not belonging Λ read as $\mu_{n(p)} = \alpha p$, where p ranges in a some subsequence P of natural numbers. Thus, in view of (19) and (20), for each sufficiently large $r > 0$, there exists a maximal number $p(r) \in P$ such that $\alpha p(r) \in ((1 - \delta)r, r]$. Due to the maximal property $p(r)$, each non-empty semi-interval $(\alpha p(r), r]$ contains no points $\tilde{\Lambda}$ different to points Λ . This is why

$$n(r, \tilde{\Lambda}) - n(\alpha p(r), \tilde{\Lambda}) = n(r, \Lambda) - n(\alpha p(r), \Lambda).$$

While proving inequality (18), we have shown that $n(\alpha p(r), \tilde{\Lambda}) = p(r)$. Therefore,

$$\lim_{r \rightarrow \infty} \frac{n(r, \tilde{\Lambda}) - n((1 - \delta)r, \tilde{\Lambda})}{\delta r} \leq \lim_{j \rightarrow \infty} \frac{n(r_j, \Lambda) - n(\alpha p(r_j), \Lambda)}{\delta r_j} + \lim_{r \rightarrow \infty} \frac{p(r_j) - n((1 - \delta)r_j, \tilde{\Lambda})}{\delta r_j},$$

where $r_j \rightarrow \infty$ implements the left upper limit in this inequality. Since $\alpha p(r) \in ((1 - \delta)r, r]$, passing to a subsequence, we can assume that $\alpha p(r_j)/r_j$ converges to some $\alpha\gamma \in [1 - \delta, 1]$. Then in view of (20) we obtain

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{n(r_j, \Lambda) - n(\alpha p(r_j), \Lambda)}{\delta r_j} &\leq \lim_{j \rightarrow \infty} \frac{n(r_j, \Lambda) - n((\alpha\gamma - \tilde{\delta})r_j, \Lambda)}{\delta r_j} = \\ &= (1 - \alpha\gamma + \tilde{\delta}) \lim_{j \rightarrow \infty} \frac{n(r_j, \Lambda) - n((\alpha\gamma - \tilde{\delta})r_j, \Lambda)}{(1 - \alpha\gamma + \tilde{\delta})\delta r_j} \leq \frac{(1 - \alpha\gamma + \tilde{\delta})}{\delta} (\tau_0(\Lambda) + \varepsilon), \end{aligned}$$

where $\tilde{\delta} > 0$ is such that $\delta + \tilde{\delta} \in (0, \delta_0)$. Let m_j be a maximal natural number such that $\alpha m_j \leq (1 - \delta)r_j$. By (17) we have

$$\overline{\lim}_{j \rightarrow \infty} \frac{p(r_j) - n((1 - \delta)r_j, \tilde{\Lambda})}{\delta r_j} \leq \overline{\lim}_{j \rightarrow \infty} \frac{p(r_j) - n(\alpha m_j, \tilde{\Lambda})}{\delta r_j} = \frac{\gamma}{\delta} - \frac{(1 - \delta)}{\delta \alpha}.$$

Thus, we obtain

$$\begin{aligned} \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \tilde{\Lambda}) - n((1 - \delta)r, \tilde{\Lambda})}{\delta r} &\leq \frac{(1 - \alpha\gamma + \tilde{\delta})}{\delta} (\tau_0(\Lambda) + \varepsilon) + \frac{\gamma}{\delta} - \frac{(1 - \delta)}{\delta \alpha} = \\ &= \frac{\tilde{\delta}(\tau_0(\Lambda) + \varepsilon) + (1 - \alpha\gamma)\varepsilon}{\delta} + \tau_0(\Lambda) \leq \frac{\tilde{\delta}(\tau_0(\Lambda) + \varepsilon)}{\delta} + \varepsilon + \tau_0(\Lambda). \end{aligned}$$

Since $\tilde{\delta}$ can be regarded as infinitesimal, the latter inequality contradicts (19). Thus, our assumption is wrong, i.e., $\overline{\tau}(\tilde{\Lambda}) \leq \tau_0(\Lambda)$. Together with (18) it completes the proof. \square

Remark. The proof of the lemma remains true if instead of strictly increasing sequence Λ , one takes a non-decaying multiple sequence.

Lemma 6. *Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be an unbounded strictly increasing sequence of positive numbers with a finite maximal density $\tau_0(\Lambda)$. Then there exists an entire function f of exponential type and regular growth vanishing at the points λ_k , $k \geq 1$, whose adjoint diagram is the segment of imaginary axis $[-i\pi\tau_0(\Lambda), i\pi\tau_0(\Lambda)]$. At that, for all λ not lying on the real axis, the identity*

$$\lim_{t \rightarrow +\infty} \frac{\ln |f(t\lambda)|}{t} = \pi\tau_0(\Lambda) |Im\lambda|$$

holds true and the convergence is uniform the angle $\alpha < \varphi < \pi - \alpha$, $\alpha \in (0, \pi)$ for all $\lambda = \exp(i\varphi)$.

Proof. By Lemma 5, there exists the completion $\tilde{\Lambda} = \{\mu_n\}_{n=1}^{\infty}$ of sequence Λ such that $\tau(\tilde{\Lambda}) = \tau_0(\Lambda)$. Consider the function

$$f(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\mu_n^2}\right).$$

It vanishes at the points λ_k , $k \geq 1$. Since $\tilde{\Lambda}$ is measurable, it is easy to check that the zero set $f(\lambda)$ is well-distributed. This is why f has a regular growth. By Theorem 5.9 in book [7, Ch. I, Sect. 5], the required identity holds true that implies immediately that the segment $[-i\pi\tau_0(\Lambda), i\pi\tau_0(\Lambda)]$ is the adjoint diagram of function f . The proof is complete. \square

Let $R \subset \mathbb{C}$ and $\delta > 0$. By R^δ we denote the union of the circles $B(z, \delta|z|)$, where z ranges in set R .

Lemma 7. *Let D be a convex domain in \mathbb{C} , $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be an unbounded strictly increasing sequence of positive numbers with a finite maximal density $\tau_0(\Lambda)$. Suppose space $W(\Lambda, D)$ is nontrivial, $H_D(1) < +\infty$, and the intersection of the support line $\{z \in \mathbb{C} : Re z = H_D(1)\}$ with the boundary of domain D containing the segment of length $2\pi\tau_0(\Lambda)$. Then for each compact set $L \subset D$ and each $\delta > 0$, there exists function $f \in P_D$ vanishing at the points λ_k , $k \geq 1$, and such that for some $T(\delta) > 0$, the ray $(T(\delta), +\infty)$ lies on the set R^δ , where $R = \{z : \ln |f(z)| \geq H_L(z)\}$.*

Proof. By assumption, subspace $W(\Lambda, D)$ is nontrivial. This is why, as it was mentioned above, there exists an entire function $f_1 \in P_D$ vanishing at the points λ_k , $k \geq 1$. Let K_1 be the adjoint diagram of function f_1 . Since K_1 is a compact set in domain D , the inequality

$$H_{K_1}(z) < H_D(z), \quad z \neq 0 \tag{21}$$

holds true.

According to Lemma 6, there exists a function \tilde{f}_2 of exponential type vanishing at the points λ_k , $k \geq 1$ whose adjoint diagram is $K = [-\pi\tau_0(\Lambda), \pi\tau_0(\Lambda)]$. Moreover, for all z not lying on the real axis, we have

$$\lim_{t \rightarrow +\infty} \frac{\ln |\tilde{f}_2(tz)|}{t} = \pi\tau_0(\Lambda)|\operatorname{Im}z|, \quad (22)$$

and the convergence is uniform in the angle $\alpha < \varphi < \pi - \alpha$, $\alpha \in (0, \pi)$ for all $z = \exp(i\varphi)$. The hypothesis implies that for some $w_0 \in \mathbb{C}$, the segment $K_2 = K + w_0$ lies on the boundary of domain D . Therefore, the inequality

$$H_{K_2}(z) \leq H_D(z), \quad z \in \mathbb{C}, \quad (23)$$

holds true. Compact set K_2 is the adjoint diagram of $f_2(z) = \tilde{f}_2(z) \exp(w_0z)$, which by Lemma 6 is an entire function of exponential type and regular growth. Moreover, it vanishes at the points λ_k , $k \geq 1$. Consider the function

$$f_\Lambda(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\lambda_k}\right) \exp\left(\frac{z}{\lambda_k}\right).$$

It is entire, has the first order of growth and possibly infinite type (see, for instance, [1, Ch. I, Sect. 1, Thms. 1.1.3 and 1.1.5]). Since f_1 and f_2 are divisible by f_Λ , the functions $\ln |f_1| - \ln |f_\Lambda|$ and $\ln |f_2| - \ln |f_\Lambda|$ are subharmonic in the plane and has the first order of growth (see [5, Ch. I, Sect. 9, Cor. of Thm. 12]). Then by Theorem 5 in work [12], for each $\tau \in (0, 1)$, there exist an entire function $\varphi_\tau(z)$, constant $C > 0$, and an exceptional set $E \subset \mathbb{C}$ such that

$$|\ln |\varphi_\tau(z)| - \psi_\tau(z)| \leq C \ln |z|, \quad z \in \mathbb{C} \setminus E, \quad (24)$$

where $\psi_\tau(z) = \tau(\ln |f_1(z)| - \ln |f_\Lambda(z)|) + (1 - \tau)(\ln |f_2(z)| - \ln |f_\Lambda(z)|)$. At that, E can be covered by the circles $B_i = B(z_i, \gamma_i)$, $i \geq 1$, such that $\sum \gamma_i < \infty$. Let $f_\tau(z) = \varphi_\tau(z)f_\Lambda(z)$, $\tau \in (0, 1)$. Then f_τ is an entire function.

We fix a compact set $L \subset D$ and a number $\delta > 0$ and let us show that as the required function f we can take f_τ for some $\tau \in (0, 1)$. First of all, we observe that f_τ vanish at points λ_k , $k \geq 1$. Then, since K_1 and K_2 are adjoint diagrams of functions f_1 and f_2 , respectively, then in view of the aforementioned Pólya theorem for the indicators in (21), (23), (24) and ‘‘smallness’’ of exceptional set E , it is easy to obtain the estimate (see, for instance, [3, Th. 4.3])

$$h_{f_\tau}(z) < H_D(z), \quad z \neq 0, \quad \tau \in (0, 1).$$

It means that the adjoint diagram of f_τ lies in domain D , i.e., $f_\tau \in P_D$ for each $\tau \in (0, 1)$. It remains to choose $\tau \in (0, 1)$ so that for some $T(\delta) > 0$, the ray $(T(\delta), +\infty)$ lies in set R^δ , if in the definition of R one takes f_τ as f .

By the theorem on the lower estimate for an entire function of finite order and type on the circles (see, for instance, [1, Ch. I, Sect. 1, Thm. 1.1.9]), there exists a number $a > 0$ and an unbounded increasing sequence of positive numbers $\{r_p\}_{p=1}^{\infty}$ such that

$$r_{p+1} \leq (1 + \delta/2)r_p, \quad \ln |f_1(z)| \geq -a|z|, \quad |z| = r_p, \quad p \geq 1. \quad (25)$$

Since the segment $K_2 = K + w_0$ lies on the boundary of domain D , and segment K contains the origin, point w_0 lies in the intersection of the boundary of domain D and the support line $\{z \in \mathbb{C} : \operatorname{Re} z = H_D(1)\}$. Therefore, due to the fact that L is a compact set in D , there exist $\varepsilon > 0$ and $\tilde{\delta} \in (0, \delta/2)$ so that the inequality

$$\operatorname{Re}(w_0z) > H_L(z) + 4\varepsilon|z|, \quad z \in B(1, \tilde{\delta}), \quad (26)$$

holds true. Finally, we choose $\tau \in (0, 1)$ so that

$$-\tau \operatorname{Re}(w_0z) > -\varepsilon|z|, \quad -\tau a \geq -\varepsilon. \quad (27)$$

Since the sum of radii of exceptional circles B_i , $i \geq 1$ is finite, there exists an index p_0 such that for all $p \geq p_0$, there exists a point $z_p \notin E$ in the arc of the circumference $|z| = r_p$ lying

in the upper half plane and the ring $B(r_p, \tilde{\delta}r_p) \setminus B(r_p, \tilde{\delta}r_p/2)$. At that, one can assume that the inequalities $C \ln |z_p| < \varepsilon |z_p|$, $p \geq p_0$ hold true, and by (22), the inequality $\ln |\tilde{f}_2(z_p)| > \pi \tau_0(\Lambda) |Im z_p| - \varepsilon |z_p|$, $p \geq p_0$ hold true as well. Then due to (24)-(27), we obtain

$$\begin{aligned} \ln |f_\tau(z_p)| &\geq \tau \ln |f_1(z_p)| + (1 - \tau) \ln |f_2(z_p)| - \varepsilon |z_p| \geq -\tau a |z_p| + \\ &+(1 - \tau)(\ln |\tilde{f}_2(z_p)| + \operatorname{Re}(w_0 z_p)) - \varepsilon |z_p| \geq -3\varepsilon |z_p| + (1 - \tau) \ln |\tilde{f}_2(z_p)| + \operatorname{Re}(w_0 z_p) \geq \\ &\geq H_L(z_p) + 4\varepsilon |z_p| - 3\varepsilon |z_p| - (1 - \tau)\varepsilon |z_p| > H_L(z_p), \quad p \geq p_0. \end{aligned} \quad (28)$$

We let $T(\delta) = r_{p_0}$ and $z \in (T(\delta), +\infty)$. We choose an index $p \geq p_0$ such that $r_p \leq z < r_{p+1}$. Then by (25) and the choice of $\tilde{\delta}$, we have

$$|z_p - z| \leq |z_p - r_p| + |r_p - z| < \tilde{\delta}r_p + r_{p+1} - r_p < \delta r_p = \delta |z_p|.$$

Thus, in accordance with (28), we get $z \in B(z_p, \delta |z_p|) \subset R^\delta$. The proof is complete. \square

We let

$$q_\Lambda(z, \lambda_k, \delta) = q_\Lambda^k(z, \delta) \frac{z - \lambda_k}{3\delta \lambda_k}, \quad k \geq 1.$$

As for functions $q_\Lambda^j(z, \delta)$, we have the inequality

$$|q_\Lambda(z, \lambda_k, \delta_1)| \geq |q_\Lambda(z, \lambda_j, \delta_2)|, \quad z \in B(\lambda_j, \delta_2 \lambda_j), \quad (29)$$

if $0 < \delta_1 \leq \delta_2 < 1/3$ and $B(\lambda_k, \delta_1 \lambda_k) \subset B(\lambda_j, \delta_2 \lambda_j)$.

Lemma 8. *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers such that $S_\Lambda = 0$. Then for $\varepsilon > 0$ there exists $\delta \in (0, 1/3)$ such that for each $\gamma, \theta \in (0, 1]$ and some index $k_0 = k_0(\varepsilon, \delta, \gamma, \theta)$, the inequalities*

$$\ln |q_\Lambda(z, \lambda_k, \delta)| \geq -\varepsilon |z|, \quad z \in B(\lambda_k, \delta |\lambda_k|) \setminus \bigcup_{\lambda_j \in B(\lambda_k, \delta \lambda_k)} B(\lambda_j, \theta \rho_j(\gamma)), \quad k \geq k_0, \quad (30)$$

hold true, where $\rho_j(\gamma) = \min\{\gamma/2, (\lambda_j - \lambda_{j-1})/2, (\lambda_{j+1} - \lambda_j)/2\}$, $j \geq 1$ and $\lambda_0 = 0$.

Proof. Fix $\varepsilon > 0$. By the hypothesis, $S_\Lambda = 0$. Thus, by the definition of S_Λ , there exist $\delta \in (0, 1/9)$ and an index k_1 such that

$$\ln |q_\Lambda^j(\lambda_j, 3\delta)| / \lambda_j \geq -\frac{\varepsilon}{6}, \quad j \geq k_1. \quad (31)$$

As in Lemma 1, we obtain

$$\ln |q_\Lambda^j(\lambda_j, 3\delta)| \leq -(m(j, 3\delta) - 1) \ln(3(1 - 3\delta)),$$

where $m(j, 3\delta)$ is the number of points λ_k lying in the circle $B(\lambda_j, 3\delta \lambda_j)$. Thus, increasing index k_1 , if necessary, due to (31) and the choice of number δ we have

$$m(j, 3\delta) \leq \frac{\varepsilon \lambda_j}{6 \ln 2} + 1 \leq \frac{\varepsilon \lambda_j}{3}, \quad j \geq k_1. \quad (32)$$

Let $\gamma, \theta \in (0, 1]$ and $z \in S(\lambda_j, \theta \rho_j(\gamma))$. Then by the definition of numbers $\rho_j(\gamma)$, for each $l \neq j$, the inequality $|z - \lambda_l| \geq |\lambda_j - \lambda_l|/2$ holds true. Together with (31), (32) it implies

$$\ln |q_\Lambda^j(z, 3\delta)| \geq \ln |q_\Lambda^j(\lambda_j, 3\delta)| - m(j, 3\delta) \ln 2 \geq -\varepsilon \lambda_j / 2, \quad z \in S(\lambda_j, \theta \rho_j(\gamma)), \quad j \geq k_1. \quad (33)$$

If $\rho_j(\gamma) = \gamma/2$, there exists an index $k_2 \geq k_1$ such that

$$\frac{1}{\lambda_j} \ln \left| \frac{z - \lambda_j}{9\delta \lambda_j} \right| \geq \frac{1}{\lambda_j} \ln \frac{\gamma \theta}{12\delta \lambda_j} \geq -\varepsilon / 4, \quad z \in S(\lambda_j, \theta \rho_j(\gamma)), \quad j \geq k_2.$$

Consider now the case $\rho_j(\gamma) = (\lambda_{j+1} - \lambda_j)/2$. Since $\rho_j(\gamma) \leq 1/2$, for some $k_3 \geq k_2$ and all the indices $j \geq k_3$ corresponding to this case, the circle $B(\lambda_{j+1}, 3\delta \lambda_{j+1})$ contains point λ_j . This is why one of the terms in the sum determining quantity $\ln |q_\Lambda^{j+1}(z, 3\delta)|$ is $\ln |(z - \lambda_j) / 9\delta \lambda_j|$. As it

was mentioned above, all these terms are non-positive in the circle $B(\lambda_{j+1}, 3\delta\lambda_{j+1})$. Therefore, inequality

$$\ln \left| \frac{z - \lambda_j}{9\delta\lambda_j} \right| \geq \ln |q_\Lambda^{j+1}(z, 3\delta)|, \quad z \in B(\lambda_{j+1}, 3\delta\lambda_{j+1}), \quad j \geq k_3,$$

holds true. Hence, by (31),

$$\ln \frac{\lambda_{j+1} - \lambda_j}{9\delta\lambda_j} \geq -\frac{\varepsilon\lambda_{j+1}}{6}, \quad j \geq k_3.$$

It follows that for some $k_4 \geq k_3$

$$\begin{aligned} \ln \left| \frac{z - \lambda_j}{9\delta\lambda_j} \right| &= \ln \frac{\theta\rho_j(\gamma)}{9\delta\lambda_j} = \ln \frac{\theta}{2} + \ln \frac{\lambda_{j+1} - \lambda_j}{9\delta\lambda_j} \geq \ln \frac{\theta}{2} - \frac{\varepsilon\lambda_{j+1}}{6} = \\ &= \ln \frac{\theta}{2} - \frac{\varepsilon(\lambda_j + 2\rho_j(\gamma))}{6} \geq \ln \frac{\theta}{2} - \frac{\varepsilon\lambda_j}{6} - \frac{1}{6} \geq -\frac{\varepsilon\lambda_j}{4}, \quad z \in S(\lambda_j, \theta\rho_j(\gamma)), \quad j \geq k_4. \end{aligned}$$

The case $\rho_j(\gamma) = (\lambda_j - \lambda_{j-1})/2$ can be considered in the same way. Thus, we can suppose that in all cases

$$\ln \left| \frac{z - \lambda_j}{9\delta\lambda_j} \right| \geq -\frac{\varepsilon\lambda_j}{4}, \quad z \in S(\lambda_j, \theta\rho_j(\gamma)), \quad j \geq k_4.$$

Together with (33) and the definition of function $q_\Lambda(z, \lambda_j, 3\delta)$ it yields

$$\ln |q_\Lambda(z, \lambda_j, 3\delta)| \geq -3\varepsilon\lambda_j/4, \quad z \in S(\lambda_j, \theta\rho_j(\gamma)), \quad j \geq k_4. \quad (34)$$

Since $\delta < 1/3$, for each point $\lambda_j \in B(\lambda_k, \delta|\lambda_k|)$, the circle $B(\lambda_j, 3\delta|\lambda_j|)$ contains the circle $B(\lambda_k, \delta|\lambda_k|)$. We choose an index $k_0 \geq k_4$ such that $j \geq k_4$, if $\lambda_j \in B(\lambda_k, \delta|\lambda_k|)$ and $k \geq k_0$. We can suppose that $S(\lambda_j, \theta\rho_j(\gamma)) \subset B(\lambda_j, 3\delta|\lambda_j|)$ as $j \geq k_4$. Then in accordance with (29) and (34) we have

$$\ln |q_\Lambda(z, \lambda_k, \delta)| \geq -3\varepsilon\lambda_j/4, \quad z \in S(\lambda_j, \theta\rho_j(\gamma)), \quad \lambda_j \in B(\lambda_k, \delta|\lambda_k|), \quad k \geq k_0.$$

Therefore, for each $k \geq k_0$ and j such that $\lambda_j \in B(\lambda_k, \delta|\lambda_k|)$ the inequality

$$\ln |q_\Lambda(z, \lambda_k, \delta)| \geq -3\varepsilon(1 + \delta)\lambda_k/4 \geq -5\varepsilon\lambda_k/6, \quad z \in S(\lambda_j, \theta\rho_j(\gamma))$$

is valid. Moreover, as $\delta < 1/3$, the inequality

$$\ln |q_\Lambda(z, \lambda_k, \delta)| \geq 0, \quad z \in S(\lambda_k, 5\delta|\lambda_k|), \quad k \geq 1,$$

is valid as well. Then by the maximum principle in the domain $B(\lambda_k, 5\delta|\lambda_k|) \setminus \cup_{\lambda_j \in B(\lambda_k, \delta|\lambda_k|)} B(\lambda_j, \theta\rho_j(\gamma))$, the harmonic function $\ln |q_\Lambda(z, \lambda_k, \delta)|$ satisfies the estimate

$$\ln |q_\Lambda(z, \lambda_k, \delta)| \geq -5\varepsilon\lambda_k/6, \quad k \geq k_0.$$

Therefore,

$$\ln |q_\Lambda(z, \lambda_k, \delta)| \geq -\frac{5\varepsilon|z|}{6(1 - \delta)} \geq -\varepsilon|z|, \quad z \in B(\lambda_k, \delta|\lambda_k|) \setminus \cup_{\lambda_j \in B(\lambda_k, \delta|\lambda_k|)} B(\lambda_j, \theta\rho_j(\gamma)), \quad k \geq k_0.$$

The proof is complete. \square

Lemma 9. *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers such that $S_\Lambda = 0$. Then for each $\tilde{\varepsilon} > 0$, there exists its completion $\Lambda_{\tilde{\varepsilon}} = \{\mu_n\}_{n=1}^\infty \subset (0, +\infty)$ ($\Lambda_{\tilde{\varepsilon}}$ is strictly increasing) and a number $\gamma_0 \in (0, 1)$ such that the inequalities*

$$\left| \ln |f(\lambda)| - \frac{\pi |\operatorname{Im} \lambda|}{\gamma_0} \right| \leq \tilde{\varepsilon} |\lambda|, \quad \lambda \in \mathbb{C} \setminus (E_1 \cup E_2 \cup B(0, T)) \quad (35)$$

hold true, where f is an entire function of exponential type,

$$f(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\mu_n^2} \right), \quad E_1 = \bigcup_{k=1}^{\infty} B(\pm\lambda_k, \theta\rho_k(\gamma_0)), \quad E_2 = \bigcup_{n=1}^{\infty} B(\pm\tilde{\mu}_n(\gamma_0), \theta\gamma_0/2),$$

$\rho_k(\gamma_0)$, $k \geq 1$, are defined in Lemma 8, $\tilde{\mu}_n(\gamma_0) = \gamma_0 n - \gamma_0/2$, $n \geq 1$, $\theta \in (0, 1)$ and $T > 0$ depend on $\theta, \tilde{\varepsilon}$.

Proof. Since $S_\Lambda = 0$, by Lemma 2, upper density $\bar{\tau}(\Lambda)$ is finite. Let $\gamma > 0$. First we construct the completion $\Lambda(\gamma) = \{\mu_n(\gamma)\}_{n=1}^\infty$ of sequence Λ . We shall seek it as the union $\cup_{m=1}^\infty \Lambda_m(\gamma)$, where $\Lambda_m(\gamma) = \{\mu_n(\gamma), n(m) \leq n < n(m+1)\}$, $m = 1, 2, \dots$, and $n(1) = 1$. We construct sets $\Lambda_m(\gamma)$ by induction. Let $m = 1$. If the semi-interval $(0, \gamma]$ contains points in Λ , we let $\mu_n(\gamma) = \lambda_n$, $1 \leq n < n(2)$, where $n(2)$ is the minimal of indices k satisfying the inequality $\lambda_k > \gamma$. Otherwise we let $\mu_1(\gamma) = \gamma/2$ and $n(2) = 2$. Suppose we have already constructed sets $\Lambda_m(\gamma)$ for all $m < p$. Let us define $\Lambda_p(\gamma)$. We indicate s_p the number of the points of sequence Λ lying in the semi-interval $(\gamma(p-1), \gamma p]$ (it can happen that $s_p = 0$). If the total number of points in the sets $\Lambda_1(\gamma), \dots, \Lambda_{p-1}(\gamma)$ does not exceed $p-1$ and $s_p = 0$, we let $n(p+1) = n(p) + 1$ and $\Lambda_p(\gamma) = \{\mu_{n(p)}(\gamma)\}$, where $\mu_{n(p)}(\gamma) = \gamma p - \gamma/2$. Otherwise as $\Lambda_p(\gamma)$, we take the set consisting of all points Λ lying in $(\gamma(p-1), \gamma p]$ (if $s_p = 0$, then $\Lambda_p(\gamma)$ is empty). Here we let $n(p+1) = n(p) + s_p$.

Let λ_k be an arbitrary point in Λ , and the index m is so that the semi-interval $(\gamma(m-1), \gamma m]$ contains λ_k . Then $s_m \neq 0$ and by construction, set $\Lambda_m(\gamma)$ together with sequence $\Lambda(\gamma)$ contain λ_k . Therefore, $\Lambda \subset \Lambda(\gamma)$. Employing the induction, let prove inequalities

$$n(\gamma m, \Lambda(\gamma)) \geq m, \quad m = 1, 2, \dots \tag{36}$$

By construction, each semi-interval $(\gamma(m-1), \gamma m]$ intersects $\Lambda(\gamma)$ by set $\Lambda_m(\gamma)$. This is why $n(\gamma m, \Lambda(\gamma))$ coincides with the total amount of the points in sets $\Lambda_1(\gamma), \dots, \Lambda_m(\gamma)$. Let $m = 1$. Then $n(\gamma, \Lambda(\gamma)) = 1$, if $s_1 = 0$, and $n(\gamma, \Lambda(\gamma)) = s_1 \geq 1$ otherwise. Suppose (36) is proven for all $m < p$. If $n(\gamma(p-1), \Lambda(\gamma)) > p-1$, then $n(\gamma p, \Lambda(\gamma)) \geq n(\gamma(p-1), \Lambda(\gamma)) \geq p$. Let $n(\gamma(p-1), \Lambda(\gamma)) \leq p-1$. Then by induction assumption we get $n(\gamma(p-1), \Lambda(\gamma)) = p-1$. In this case by construction, $n(\gamma p, \Lambda(\gamma)) = n(\gamma(p-1), \Lambda(\gamma)) + 1 = p$, if $s_p = 0$, and $n(\gamma p, \Lambda(\gamma)) = n(\gamma(p-1), \Lambda(\gamma)) + s_p \geq p$ otherwise. Thus, inequality (36) is valid for all m .

By construction, each of groups $\Lambda_m(\gamma)$ either comprises only some of points λ_k , or is empty, or consists just of one point $\mu_{n(m)}(\gamma)$ not involved in sequence Λ . Let us show that as $\gamma < 1/\bar{\tau}(\Lambda)$, there exists infinitely many of the groups of latter kind. Suppose the opposite. Then for some $r_0 > 0$, an infinite interval $(r_0, +\infty)$ contains no points of sequence $\Lambda(\gamma)$ different to λ_k . This is why

$$\overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda(\gamma))}{r} = \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda) - n(r_0, \Lambda) + n(r_0, \Lambda(\gamma))}{r} = \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda)}{r} = \bar{\tau}(\Lambda).$$

On the other hand, by (36) we have

$$\overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda(\gamma))}{r} \geq \overline{\lim}_{m \rightarrow \infty} \frac{n(\gamma m, \Lambda(\gamma))}{\gamma m} \geq \frac{1}{\gamma} > \bar{\tau}(\Lambda).$$

We obtain the contradiction. Therefore, there exist infinitely many groups of mentioned kind. Let $I(\gamma)$ be the union of all semi-intervals $(\gamma m, \gamma(m+1)]$ associated with such groups. Then $J(\gamma) = (0, +\infty) \setminus I(\gamma)$ is the union of all bounded semi-intervals $(\gamma m_j(\gamma), \gamma p_j(\gamma)]$, $j \geq 1$.

Let $j \geq 1$. According to the definition of $I(\gamma)$, group $\Lambda_{m_j(\gamma)}(\gamma)$ consists of the only point $\mu_{n(m_j(\gamma))}(\gamma)$ not involved in sequence Λ . Then by construction, we have $n(\gamma(m_j(\gamma)-1), \Lambda(\gamma)) \leq m_j(\gamma)-1$ and $s_{m_j(\gamma)} = 0$. Therefore, $n(\gamma m_j(\gamma), \Lambda(\gamma)) \leq m_j(\gamma)$. Together with (36) it implies the identity

$$n(\gamma m_j(\gamma), \Lambda(\gamma)) = m_j(\gamma), \quad j \geq 1. \tag{37}$$

Group $\Lambda_{p_j(\gamma)+1}(\gamma)$ is analogous to group $\Lambda_{m_j(\gamma)}(\gamma)$. Then by construction we have also the identity $n(\gamma p_j(\gamma), \Lambda(\gamma)) \leq p_j(\gamma)$. By (36) it yields

$$n(\gamma p_j(\gamma), \Lambda(\gamma)) = p_j(\gamma), \quad j \geq 1. \tag{38}$$

Let $\delta' > 0$. According to the definition of $J(\gamma)$, all the points λ_k lie in set $J(\gamma)$ and none of the points of sequence $\Lambda(\gamma)$ different to λ_k , $k \geq 1$, belongs to $J(\gamma)$. Therefore, by (37) and (38), for all $j \geq 1$ we have

$$p_j(\gamma) - m_j(\gamma) = n(\gamma p_j(\gamma), \Lambda(\gamma)) - n(\gamma m_j(\gamma), \Lambda(\gamma)) = n(\gamma p_j(\gamma), \Lambda) - n(\gamma m_j(\gamma), \Lambda). \quad (39)$$

Since Λ has a finite upper density, for some $C > 0$ we have the inequality $n(r, \Lambda) \leq Cr$. This is why it follows from (39) that

$$p_j(\gamma) - m_j(\gamma) \leq n(\gamma p_j(\gamma), \Lambda) \leq C\gamma p_j(\gamma) \leq \delta' p_j(\gamma), \quad j \geq 1,$$

for all $\gamma < \min\{1/\bar{\tau}(\Lambda), \delta'/C\} = \gamma(\delta')$. It implies easily the estimate

$$p_j(\gamma) - m_j(\gamma) \leq \delta'' m_j(\gamma), \quad j \geq 1, \quad \gamma < \gamma(\delta'), \quad (40)$$

where $\delta'' = \delta'/(1 - \delta')$.

We fix $\tilde{\varepsilon} > 0$ and $\varepsilon \in (0, \tilde{\varepsilon}/5)$. Let us show that for some $\gamma > 0$, as the required sequence Λ_ε we can take $\Lambda(\gamma)$. First of all, according to Lemma 8, given $\varepsilon > 0$, we find $\delta \in (0, 1/3)$ such that (30) holds true.

We introduce a new sequence $\tilde{\Lambda}(\gamma) = \{\tilde{\lambda}_k(\gamma)\}_{k=1}^\infty$ of positive numbers taken in the ascending order. It is the union of the groups $\tilde{\Lambda}_j(\gamma)$, $j \geq 1$, where $\tilde{\Lambda}_j(\gamma) = \{\gamma p_j(\gamma) + \gamma/2, \gamma(p_j(\gamma) + 1) + \gamma/2, \dots, \gamma m_j(\gamma) - \gamma/2\}$. Consider the functions

$$L(\lambda) = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_k^2}\right), \quad L(\lambda, \gamma) = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda^2}{(\tilde{\lambda}_k(\gamma))^2}\right).$$

Sequence Λ has a finite upper density. By construction, $\bar{\tau}(\tilde{\Lambda}(\gamma)) \leq 1/\gamma$. This is why (see, for instance, [1, Ch. I, Sect. 1, Thm. 1.1.5]) $L(\lambda)$ and $L(\lambda, \gamma)$ are entire functions of exponential type. Let us compare their behavior. In order to do it, we employ the result from work [13]. By (39) and (40), sequence $\tilde{\Lambda}(\gamma)$ is δ'' -close to Λ , i.e.,

$$|\lambda_k - \tilde{\lambda}_k(\gamma)| \leq \delta'' |\lambda_k|, \quad k \geq 1, \quad (41)$$

and thus, it is asymptotically δ'' -close to sequence Λ (in the terminology of work [13]). Since in our case the sums of inverses to zeroes of functions $L(\lambda)$ and $L(\lambda, \gamma)$ lying in the circle $B(0, r)$ equal to zero for each $r > 0$, as $\delta'' \in (0, 1/2)$, by Theorem in work [13] (where we let $\alpha = 1/2, \beta = 1$), the inequality

$$|\ln |L(\lambda)| - \ln |L(\lambda, \gamma)|| \leq A\sqrt{\delta''}|\lambda|, \quad \lambda \in \mathbb{C} \setminus E(\delta'') \quad (42)$$

holds true, where $A > 0$ is independent of δ'' . Set $E(\delta'')$ is the union of the circles $B_i = B(\xi_i, \sigma_i)$, $i \geq 1$ and it has the linear density not exceeding $\sqrt[4]{\delta''}$,

$$\overline{\lim}_{r \rightarrow \infty} \frac{1}{r} \sum_{|\xi_i| \leq r} \sigma_i \leq \sqrt[4]{\delta''}, \quad (43)$$

and it is centered with the union of zero sets for functions $L(\lambda)$ and $L(\lambda, \gamma)$, i.e., each point of these sets lies in at least of circles B_i , and each of circles B_i contains at least one point of this union. We can obviously assume that $E(\delta'')$ is symmetric w.r.t. the origin.

We choose $\delta' > 0$ such that for $\delta'' = \delta'/(1 - \delta')$, the inequality

$$A\sqrt{\delta''} < \varepsilon, \quad \sqrt[4]{\delta''} < \delta/12 \quad (44)$$

are valid. We fix $\gamma_0 \in (0, \gamma(\delta'))$, $\gamma_0 < 1$, and let us show that sequence $\Lambda_\varepsilon = \Lambda(\gamma_0)$ is the required one. Let $\theta \in (0, 1)$ and k_0 be the index for each (30) is satisfied with $\gamma = \gamma_0$. According to (43) and (44), we find $r_0 > 0$ such that

$$\frac{1}{r} \sum_{|\xi_i| \leq r} \sigma_i \leq \delta/11, \quad r \geq r_0. \quad (45)$$

We choose a number $k_1 \geq k_0$ to have $\lambda_k \geq r_0$ and $\tilde{\lambda}_k(\gamma_0) \geq r_0$ for each $k \geq k_1$, and let B_i be an arbitrary circle in set $E(\delta'')$ containing either some point λ_k or $\tilde{\lambda}_k(\gamma_0)$ with an index $k \geq k_1$. If $\lambda_k \in B_i$, according to (45), in the case $|\xi_i| \leq \lambda_k$ the inequality $\sigma_i \leq \delta\lambda_k/11$ holds true, while in the case $|\xi_i| \geq \lambda_k$ the inequality $\sigma_i \leq \delta|\xi_i|/11$ is valid, and then in view of the inclusions $\lambda_k \in B_i$, $\delta \in (0, 1/3)$ we obtain $\lambda_k \geq |\xi_i|(1-\delta/11)$ and $\sigma_i \leq \delta|\xi_i|/11 \leq \delta\lambda_k/11(1-\delta/11) \leq 3\delta\lambda_k/32$. The situation $\tilde{\lambda}_k(\gamma_0) \in B_i$ can be treated in the same way. Thus, by (41) and (44), we have

$$\sigma_i \leq \max\{3\delta\lambda_k/32, 3\delta\tilde{\lambda}_k(\gamma_0)/32\} \leq 3\delta(1+\delta'')\lambda_k/32 \leq 13\delta\lambda_k/128 \leq \delta\lambda_k/9.$$

It follows that B_i lies either in the circle $B(\lambda_k, 2\delta\lambda_k/9)$ or in the circle $B(\tilde{\lambda}_k(\gamma_0), 2\delta\lambda_k/9)$. Therefore, in view of (41) and (44), the inclusion $B_i \subset B(\lambda_k, \delta\lambda_k/3)$ holds true.

Let $B_j \cap B(\lambda_k, \delta\lambda_k) \neq \emptyset$. Then by (45), $|\xi_i| \leq (1+\delta)\lambda_k/(1-\delta/11) \leq 11\delta\lambda_k/8$. This is why, employing (45) once again, we find that the total sum of all the radii of circles B_j intersecting $B(\lambda_k, \delta\lambda_k)$ does not exceed $\delta\lambda_k/4$. It means that for $t \in (1/2, 1)$, the circle $S(\lambda_k, t\delta\lambda_k)$ does not intersect set $E(\delta'')$. Then due to (42) and (44) we have the estimate

$$\ln |L(\lambda)| \geq \ln |L(\lambda, \gamma_0)| - \varepsilon|\lambda|, \quad \lambda \in S(\lambda_k, t\delta\lambda_k), \quad k \geq k_1.$$

We recall that the absolute value of the function $q_\Lambda(\lambda, \lambda_k, \delta)$ does not exceeding one in the circle $B(\lambda_k, \delta|\lambda_k|)$. This is why for each $\lambda \in S(\lambda_k, t\delta\lambda_k)$ the inequality

$$\ln |L(\lambda, \gamma_0)| \leq \ln |L(\lambda)| - \ln |q_\Lambda(\lambda, \lambda_k, \delta)| + \varepsilon(1+t\delta)\lambda_k \leq \ln |L(\lambda)/q_\Lambda(\lambda, \lambda_k, \delta)| + 2\varepsilon\lambda_k$$

holds true, and it can be extended to the circle $B(\lambda_k, t\delta\lambda_k)$, since $\ln |L(\lambda)/q_\Lambda(\lambda, \lambda_k, \delta)|$ is a harmonic in $B(\lambda_k, \delta|\lambda_k|)$ function and $\ln |L(\lambda, \gamma_0)|$ is subharmonic in the same domain. Then by (30) we get

$$\ln |L(\lambda, \gamma_0)| \leq \ln |L(\lambda)| + 2\varepsilon\lambda_k + \varepsilon|\lambda| \leq \ln |L(\lambda)| + 4\varepsilon|\lambda|,$$

$$\lambda \in B(\lambda_k, t\delta\lambda_k) \setminus \bigcup_{\lambda_j \in B(\lambda_k, \delta\lambda_k)} B(\lambda_j, \theta\rho_j(\gamma_0)), \quad k \geq k_1, \quad (46)$$

At that, the union of the circles $B(\pm\lambda_k, t\delta\lambda_k)$, $k \geq k_1$ cover all the circles B_i in set $E(\delta'')$ each of those consists some of the points $\pm\lambda_k$ or $\pm\tilde{\lambda}_k(\gamma_0)$ with an index $k \geq k_1$. It is easy to see that the remaining part of set $E(\delta'')$ (not covered by the mentioned union) lies in the circle $B(0, T_1)$ for some $T_1 > 0$. Thus, it follows from (46) that

$$\ln |L(\lambda)| \geq \ln |L(\lambda, \gamma_0)| - 4\varepsilon|\lambda|, \quad \lambda \in \mathbb{C} \setminus \bigcup_{k \geq 1} B(\pm\lambda_k, \theta\rho_k(\gamma_0)) \cup B(0, T_1). \quad (47)$$

We also note that the zeroes of the function $L(\lambda, \gamma_0)$ are separated by the distance at least γ_0 . This is why, according to the example discussed before Lemma 1, the quantity $\tilde{S}_{\tilde{\Lambda}(\gamma_0)}$ is finite and thus $S_{\tilde{\Lambda}(\gamma_0)} = 0$. Therefore, as to sequence Λ , we can apply Lemma 8 to sequence $\tilde{\Lambda}(\gamma_0)$ and arguing as above, we can obtain the estimate

$$\ln |L(\lambda, \gamma_0)| \geq \ln |L(\lambda)| - 4\varepsilon|\lambda|, \quad \lambda \in \mathbb{C} \setminus \bigcup_{k \geq 1} B(\pm\tilde{\lambda}_k(\gamma_0), \theta\gamma_0/2) \cup B(0, T_2). \quad (48)$$

for some $T_2 > 0$.

Consider the functions

$$f(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{(\mu_n(\gamma_0))^2}\right), \quad \tilde{L}(\lambda, \gamma_0) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{(\tilde{\mu}_n(\gamma_0))^2}\right),$$

where $\tilde{\mu}_n(\gamma_0) = \gamma_0 n - \gamma_0/2$, $n \geq 1$. It is easy to show that $\Lambda(\gamma_0)$ has a finite upper density not exceeding $\bar{\tau}(\Lambda) + 1/\gamma_0$. This is why (see, for instance., [1, Ch. I, Sect. 1, Thm. 1.1.5]) $f(\lambda)$ is an entire function of exponential type. Sequence $\tilde{\Lambda}(\gamma_0)$ has the density $1/\gamma_0$ and is a regular set (see [5, Ch. II, Sect. 1]). Therefore, $\tilde{L}(\lambda, \gamma_0)$ has a regular growth and its adjoint

diagram coincides with the segment of imaginary axis $[-i\pi/\gamma_0, i\pi/\gamma_0]$ (see [1, Ch. I, Sect. 2, Thm. 1.2.9]). Since its zero set $\tilde{L}(\lambda, \gamma_0)$ is regular, the inequality

$$\ln |\tilde{L}(\lambda, \gamma_0)| = \frac{\pi |Im \lambda|}{\gamma_0} + o(\lambda), \quad \lambda \in \mathbb{C} \setminus \bigcup_{n \geq 1} B(\pm \tilde{\mu}_n(\gamma_0), \theta \gamma_0/2), \tag{49}$$

holds true (see [5, Ch. II, Sect. 1, Thm. 5]), $o(\lambda)$ depends on $\theta \in (0, 1]$ and $o(\lambda)/|\lambda| \rightarrow 0$ as $|\lambda| \rightarrow \infty$.

By construction,

$$\ln |f(\lambda)| - \ln |\tilde{L}(\lambda, \gamma_0)| = \ln |L(\lambda)| - \ln |L(\lambda, \gamma_0)|, \quad \lambda \in \mathbb{C}.$$

Moreover, each point $\tilde{\lambda}_k(\gamma_0)$ coincides with one of the points $\tilde{\mu}_n(\gamma_0)$. This is why in view of the choice of number $\varepsilon > 0$ and by (47)–(49) we get (35). The proof is complete. \square

Lemma 10. *Let D be a convex domain in \mathbb{C} , $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers. Suppose the set of sums of series (1) converging in domain D is closed in space $H(D)$. Then the system $\mathcal{E} = \{\exp(\lambda_k z)\}_{k=1}^\infty$ is incomplete in $H(D)$.*

Proof. If the hypothesis of the lemma holds true, by the corollary of Lemma 3 the inequality $\bar{\tau}(\Lambda) < \infty$ holds true. Then, as above, there exists an entire function f of exponential type vanishing at the points $\lambda_k, k \geq 1$. Suppose some shift $z_0 + K$ of the adjoint diagram K for function f lies in domain D . Then the function $\exp(z_0 \lambda) f(\lambda)$ belongs to space P_D and vanishes at the points $\lambda_k, k \geq 1$. It means that system \mathcal{E} is incomplete in $H(D)$. It happens, for instance, when D is the plane of a half-plane.

Suppose domain D is neither the plane nor a half-plane and assume system \mathcal{E} is complete in space $H(D)$. Then by the hypothesis, each function in $H(D)$ is represented by series (1) in domain D . By Abel theorem for Dirichlet series (see, for instance, [1, Ch. II, Sect. 1, Subsect. 4]), each of such series converges uniformly on compact sets in the half-plane $\{z : \text{Re } z < H_D(1)\}$ (which can coincide with the plane once $H_D(1) = +\infty$). Therefore, each function in $H(D)$ can be analytically continued into this half-plane that is impossible. Thus, \mathcal{E} is incomplete in $H(D)$. The proof is complete. \square

3. MAIN RESULTS

Let D be an unbounded convex domain and denote

$$J(D) = \{\lambda \in \mathbb{C} : H_D(\lambda) = +\infty\}.$$

If $D = \mathbb{C}$, then $J(D) = \mathbb{C} \setminus \{0\}$. In the case D is a half-plane $\{z \in \mathbb{C} : \text{Re}(ze^{i\varphi}) < a\}$, set $J(D)$ is the plane with the cut along the ray $\{\lambda = te^{i\varphi} : t \geq 0\}$. If D is the strip $\{z \in \mathbb{C} : \text{Re}(ze^{i\varphi}) < a, \text{Re}(ze^{i(\varphi+\pi)}) < b\}$, then $J(D) = \mathbb{C} \setminus \{\lambda = te^{i\varphi} : t \in \mathbb{R}\}$. In all other cases, domain D contains no straight lines. However, D always contains some ray $\{z = z_0 + te^{i\varphi}, t \geq 0\}$. At, set $J(D)$ is an angle strictly less than 2π and it involves an open angle of size π which the half-plane $\{\lambda = te^{i\psi} : -\varphi - \frac{\pi}{2} < \psi < -\varphi + \frac{\pi}{2}, t > 0\}$. We shall say that D is narrow, if D is a strip or $J(D)$ coincides with an open half-plane. Otherwise we shall say D is wide.

For a narrow domain D , there exists the unique $\psi \in [0, \pi)$, such that $H_D(e^{i\psi}) < +\infty$ and $H_D(e^{i(\psi+\pi)}) < +\infty$. It corresponds to the boundary rays $\{\lambda = te^{i\psi} : t > 0\}$ and $\{\lambda = te^{i(\psi+\pi)} : t > 0\}$ of set $J(D)$. By means of a shift, a convex compact set can be put inside narrow domain K if and only if the inequality

$$H_D(e^{i\psi}) + H_D(e^{i(\psi+\pi)}) > H_K(e^{i\psi}) + H_K(e^{i(\psi+\pi)})$$

holds true.

If D is a wide domain, then for each convex compact set K there exists a shift which puts this set inside domain D . Thus, if the sequence $\Lambda = \{\lambda_k\}_{k=1}^\infty$ is so that $\bar{\tau}(\Lambda) < \infty$, then, as in Lemma 10, system $\mathcal{E} = \{\exp(\lambda_k z)\}_{k=1}^\infty$ is incomplete in $H(D)$.

Theorem 1. *Suppose $\Lambda = \{\lambda_k\}_{k=1}^\infty$ is an unbounded strictly increasing sequence of positive numbers and D is an unbounded convex domain in \mathbb{C} such that the positive semi-axis lies on the boundary of set $J(D)$ but does not belong to it. Then the following statements are equivalent*

1. *Each function in $W(\Lambda, D)$ can be represented by series (1) in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < H_D(1)\}$ (i.e., it is analytically continued into the half-plane and is represented there by the series);*
2. $S_\Lambda = 0$.

Proof. Suppose $z_0 \in D$. By the hypothesis, the positive semi-axis lies on the boundary of set $J(D)$. Therefore, one of the rays $l_1 = \{z = z_0 + te^{i\pi/2}, t \geq 0\}$, $l_2 = \{z = z_0 + te^{-i\pi/2}, t \geq 0\}$ (possibly, both) lies in domain D . For clarity's sake, let it be ray l_1 . We choose $\varepsilon_0 > 0$ such that the circle $B(z_0, 2\varepsilon_0)$ lies inside D . Then domain $l_1 + B(z_0, 2\varepsilon_0)$ is contained in D as well. By Lemma 9, there exists an entire function f of exponential type vanishing at the points λ_k , $k \geq 1$, and a number $\gamma_0 > 0$ satisfying inequality (35) (as $\tilde{\varepsilon} = \varepsilon_0$). Let K be the adjoint diagram for function f . Then according to (35), for all points λ not lying on the real axis we have

$$H_K(\lambda) = h_f(\lambda) \leq \pi |\operatorname{Im} \lambda| / \gamma_0 + \varepsilon_0 |\lambda|.$$

Since the support function is continuous, this inequality can be continued on the whole plane. Let $t_0 > \pi/\gamma_0$. Consider the function $f_0(\lambda) = f(\lambda) \exp(\lambda \tilde{z}_0)$, where $\tilde{z}_0 = z_0 + t_0 e^{i\pi/2}$. Its adjoint diagram is the compact set $K + \tilde{z}_0$ which, as it is easy to see, lies in domain $l_1 + B(z_0, 2\varepsilon_0)$ and thus in D . Hence, f_0 belongs to space P_D and vanishes at the points λ_k , $k \geq 1$. It means that system $\mathcal{E} = \{\exp(\lambda_k z)\}_{k=1}^\infty$ is incomplete in $H(D)$.

Now we can apply Theorem 5.1 in work [3]. If Item 1 holds true, by this theorem we have the identity $S_\Lambda = 0$ (under the hypothesis of the theorem the quantity $S_\Lambda(D)$ involved in the cited theorem coincides with S_Λ).

Let us prove the opposite. Suppose Item 2 holds true and let us show that in this case Item 5 in Theorem 5.1 of work [3] is valid. Let L be an arbitrary convex set in domain D . Then $H_D(\lambda) > H_L(\lambda)$, $\lambda \neq 0$. In accordance with the definition of the support function and the continuity of the support function of a compact set, we find a point $z_1 \in D$ and numbers $\tilde{\varepsilon}, \tilde{\delta} > 0$ satisfying inequality

$$\operatorname{Re} z_1 - \tilde{\varepsilon} > H_L(\lambda), \quad \lambda \in B(1, \tilde{\delta}). \tag{50}$$

Lessening $\tilde{\varepsilon} > 0$, if necessary, we can assume that the circle $B(z_1, 2\tilde{\varepsilon})$ lies in domain D . Then D contains also the domain $l_1 + B(z_1, 2\tilde{\varepsilon})$. Let the a number $\gamma_0 \in (0, 1)$ and a function $f(\lambda)$ satisfy inequality (35). We indicate $f_1(\lambda) = f(\lambda) \exp(\lambda \tilde{z}_1)$, where $\tilde{z}_1 = z_1 + t_1 e^{i\pi/2}$ and $t_1 > \pi/\gamma_0$. Function f_1 vanishes at the points λ_k , $k \geq 1$. Moreover, its adjoint diagram lies in domain $l_1 + B(z_1, 2\tilde{\varepsilon})$, and this, in D , i.e., $f_1 \in P_D$.

Suppose $\delta > 0$. Let us show that for some $T(\delta) > 0$ each point λ_k obeying $|\lambda_k| > T(\delta)$ belongs to the set R^δ , where $R = \{z : \ln |f_1(\lambda)| \geq H_L(\lambda)\}$.

We denote $T(\delta) = \max\{T, \gamma_0/\tilde{\delta}, \gamma_0/\delta\}$, where $T > 0$ is the same as in (35). Let $|\lambda_k| > T(\delta)$. Then $|\lambda_k - i\gamma_0 - \lambda_k| \leq \delta \lambda_k < \delta |\lambda_k - i\gamma_0|$ and by (35) we have

$$\ln |f(\lambda_k - i\gamma_0)| \geq \frac{|\operatorname{Im}(\lambda_k - i\gamma_0)|}{\gamma_0} - \tilde{\varepsilon} |\lambda_k - i\gamma_0| = \pi - \tilde{\varepsilon} |\lambda_k - i\gamma_0| \geq -\tilde{\varepsilon} \lambda_k.$$

By (50) and the homogeneity of the support function we obtain

$$\begin{aligned} \ln |f_1(\lambda_k - i\gamma_0)| &= \ln |f(\lambda_k - i\gamma_0)| + \operatorname{Re}(\tilde{z}_1(\lambda_k - i\gamma_0)) \geq -\tilde{\varepsilon} \lambda_k + \lambda_k \operatorname{Re} \tilde{z}_1 + \gamma_0 \operatorname{Im} \tilde{z}_1 = \\ &= -\tilde{\varepsilon} \lambda_k + \lambda_k \operatorname{Re} z_1 + \gamma_0 \operatorname{Im} \tilde{z}_1 \geq H_L(\lambda_k - i\gamma_0). \end{aligned}$$

Here we suppose that $\operatorname{Im} \tilde{z}_1 \geq 0$. If $\operatorname{Im} \tilde{z}_1 < 0$, then instead of $\lambda_k - i\gamma_0$, one should take $\lambda_k + i\gamma_0$.

We have shown that $\lambda_k \in R^\delta$, if $|\lambda_k| > T(\delta)$. Thus, the hypothesis of Theorem 5.1 in work [3] is satisfied and as well as Item 5 of this Theorem. This is why, Item 2 of this theorem holds true as well, and in accordance with it, each function in $W(\Lambda, D)$ is represented by series

(1) in domain D . Then by Abel theorem for Dirichlet series (see, for instance, [1, Ch. II, Sect. 1, Subsect. 2,4]), each function in $W(\Lambda, D)$ is represented by series (1) in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < H_D(1)\}$. The proof is complete. \square

We note that each half-plane $D = \{z : \operatorname{Re} z < a\}$ satisfies the hypothesis of the theorem. This is why the following statement holds true.

Corollary. *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers and $D = \{z : \operatorname{Re} z < a\}$. Each function in $W(\Lambda, D)$ is represented by series (1) in half-plane D if and only if $S_\Lambda = 0$.*

Theorem 2. *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers and D be an unbounded convex domain in \mathbb{C} such that the positive semi-axis belongs to set $J(D)$. Then the following statements are equivalent*

1. *Each function in $W(\Lambda, D)$ is represented by series (1) in the whole plane;*
2. $S_\Lambda > -\infty$.

Proof. As we have mentioned above, under the hypothesis of the theorem, system \mathcal{E} is incomplete in $H(D)$. Then by Theorem 5.1 in work [3], Item 1 of the present theorem implies Item 2.

Let us prove the opposite and check Item 5 of Theorem 5.1 in work [3]. Inequality $S_\Lambda > -\infty$ holds due to Item 2. In our case, other statements of Item 5 of this theorem hold true trivially, since all the points λ_k belong to the real semi-axis lying in $J(D)$. Thus, by Theorem 5.1 in work [3], each function in $W(\Lambda, D)$ is represented by series (1) in domain D . Since the support function of D is unbounded on the real semi-axis, by Abel theorem for Dirichlet series, each of such series converges in the whole plane. The proof is complete. \square

In the particular case $D = \mathbb{C}$ we obtain

Corollary. *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers. There following statements are equivalent*

1. *Each function in $W(\Lambda, \mathbb{C})$ is represented by series (1) in the whole plane;*
2. $S_\Lambda > -\infty$;
3. *Each function in $W(\Lambda, \mathbb{C})$ is represented by series (1) on some open subset of the plane.*

Proof. Items 1 and 2 are equivalent by Theorem 2. The implication Item 1 \Rightarrow Item 3 is trivial. The implication Item 3 \Rightarrow Item 2 follows from Lemma 3. The proof is complete. \square

Theorem 3. *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers, D is an unbounded convex domain in \mathbb{C} such that the real semi-axis belongs to set $J(D)$ and $\psi \in [0, \pi)$ is so that $H_D(e^{i\psi}) < +\infty$ and $H_D(e^{i\psi+\pi}) < +\infty$. Then the following statements are equivalent*

1. *Each function in $W(\Lambda, D)$ is represented by series (1) in the whole plane;*
2. $S_\Lambda > -\infty$ and

$$|\sin \psi| \lim_{a \rightarrow +\infty} \frac{1}{\ln a} \overline{\lim}_{r \rightarrow +\infty} \sum_{r \leq \lambda_k < ar} \operatorname{Re} \frac{1}{\lambda_k} < \frac{1}{2\pi} (H_D(e^{i\psi}) + H_D(e^{i\psi+\pi})). \tag{51}$$

Proof. If Item 1 holds, then by Lemma 10, system \mathcal{E} is incomplete in $H(D)$. Then, as one easily see, Theorem 2 in work [14] yields (51). Applying then Theorem 5.1 in work [3], we obtain Item 2.

And vice versa, if (51) holds, by Theorem 2 in work [14], system \mathcal{E} is incomplete in $H(D)$. Then by Theorem 5.1 in work [3], the implication Item 2 \Rightarrow Item 1 holds true. The proof is complete. \square

Theorem 4. *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers, D be a convex domain in \mathbb{C} such that function $H_D(\lambda)$ is bounded in the vicinity of $\lambda = 1$. Then the following statements are equivalent*

1. Each function in $W(\Lambda, D)$ can be represented by series (1) in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < H_D(1)\}$;
2. System \mathcal{E} is incomplete in $H(D)$, $S_\Lambda = 0$, $\tau_0(\Lambda) < +\infty$, and the intersection of the support line $\{z \in \mathbb{C} : \operatorname{Re} z = H_D(1)\}$ with the boundary of the domain D contains the segment of length $2\pi\tau_0(\Lambda)$.

Proof. Suppose Item 2 holds true. Let us show that the hypothesis of Theorem 3.6 in work [3] holds true. Conditions 1 and 2 are equivalent, i.e., $S_\Lambda = 0$. Conditions 3 and 4 are trivially satisfied since all the points λ_k belong to the positive real semi-axis, in the vicinity of which the support function of domain D is bounded. It remains to check Condition 5. Such check has already been made in Lemma 7. In accordance with this lemma, condition 5 of Theorem 3.6 in work [3] is also satisfied. Then by this theorem and Proposition 2.10 of work [3], each function $W(\Lambda, D)$ is represented by series (1) in domain D . As above, by Abel theorem for Dirichlet series, we obtain Item 1 of the present theorem.

Suppose Item 1 holds true and let $\tilde{S}_\Lambda = -\infty$. Then by Lemma 4, for each $\tau > 0$, there exist $\delta > 0$ and function $g_\tau \in W(\Lambda, G)$, where $G = (\{z : \operatorname{Re} z < \tau\delta\} \cap B(0, \tau)) \cup \{z : \operatorname{Re} z < 0\}$ represented by series (1) with the convergence abscissa $\gamma = 0$. We let $X = \partial D \cap \{z : \operatorname{Re} z = H_D(1)\}$. By assumption, $H_D(\lambda)$ is bounded in the vicinity of $\lambda = 1$. This is why X is either a point or a segment. If X consists of the only point, we denote it by z_0 otherwise this symbols stands for the center of segment X . Let $\tau > 0$ be strictly greater than the length of X (possibly being zero). Then for some $\tilde{\delta} \in (0, \delta)$, the domain $\tilde{G} = G + z_0 - \tilde{\delta}$ contains D . We let $g(z) = g_\tau(z - z_0 + \tilde{\delta})$. Function g belongs to space the $W(\Lambda, \tilde{G}) \subset W(\Lambda, D)$ and is represented by series (1) with convergence abscissa $\gamma = \operatorname{Re} z_0 - \tilde{\delta} = H_D(1) - \tilde{\delta}$. Since the expansion into the Dirichlet series is unique, it contradicts Item 1.

Thus, $\tilde{S}_\Lambda > -\infty$. As it has already been mentioned, it yields $S_\Lambda = 0$. Moreover, by Lemma 1, the maximal density $\tau_0(\Lambda)$ is finite and by Lemma 10-, system \mathcal{E} is incomplete in $H(D)$. It remains to show that the length of set X is at least $2\pi\tau_0(\Lambda)$.

Suppose the length X equals $2\pi\tau' < 2\pi\tau_0(\Lambda)$. We choose $\alpha > 0$ so that $\tau' + \alpha < \tau_0(\Lambda)$. Consider the domains

$$D'' = \{z : \operatorname{Re} z < H_D(1)\} \cap \{z : \operatorname{Re}(ze^{-i\beta}) < \operatorname{Re}((z_0 + i(\tau' + \alpha))e^{-i\beta})\} \cap \\ \cap \{z : \operatorname{Re}(ze^{i\beta}) < \operatorname{Re}((z_0 - i(\tau' + \alpha))e^{i\beta})\}, \quad D' = D'' \cap \{z : \operatorname{Re} z > b\}.$$

It is easy to see that domain D'' is unbounded and convex and it contains D for some $\beta > 0$, and domain $D' \subset D''$ is an isosceles trapezium (as $b < H_D(1)$) whose bases are parallel to the imaginary axis. One of them lies on the support line $\{z : \operatorname{Re} z = H_D(1)\}$ of domain D , contains X , and has the length $2\pi(\tau' + \alpha)$. The length of the other is strictly greater $2\pi(\tau' + \alpha)$, and it increases as b decreases. We choose $b \in \mathbb{R}$ such that this length becomes strictly greater than $2\pi\tau_0(\Lambda)$. Then domain D' contains some shift of the segment $[-i\pi\tau_0(\Lambda), i\pi\tau_0(\Lambda)]$. Let f be a function whose existence is stated in Lemma 6. Since f vanishes at points λ_k , $k \geq 1$, system \mathcal{E} is incomplete in $H(D')$. This is why space $W(\Lambda, D')$ is non-trivial.

Suppose each function in $W(\Lambda, D')$ is represented by series (1) in domain D' . Since it is bounded, by Theorem 5.2 in work [3], there exists an entire function φ of exponential type vanishing at points λ_k , $k \geq 1$, having regular growth everywhere in the plane, and its adjoint diagram coincides with the closure of domain D' . Then by (2), the identity

$$\tau(-\varphi, \varphi, \tilde{\Lambda}) = \frac{1}{2\pi}s(-\varphi, \varphi, D') \tag{52}$$

holds true except at most a countable set of values of φ . Here $\tilde{\Lambda}$ is the zero set of f , $s(-\varphi, \varphi, D')$ is the length of the arc $\gamma(\varphi)$ on boundary $\partial D'$ connecting the support points $z(\varphi), z(-\varphi) \in \partial D'$ of the support lines $l(\pm\varphi) = \{z : \operatorname{Re}(ze^{\pm\varphi i}) = H'_D(e^{\pm\varphi i})\}$, respectively. Since Λ is a part of a

measurable set $\tilde{\Lambda}$, we have

$$\begin{aligned} \tau_0(\Lambda) &= \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda) - n((1 - \delta)r, \Lambda)}{\delta r} \leq \overline{\lim}_{\delta \rightarrow 0} \overline{\lim}_{r \rightarrow \infty} \frac{n(-\varphi, \varphi, r, \Lambda) - n(-\varphi, \varphi, (1 - \delta)r, \Lambda)}{\delta r} = \\ &= \overline{\lim}_{\delta \rightarrow 0} \left(\overline{\lim}_{r \rightarrow \infty} \frac{n(-\varphi, \varphi, r, \Lambda)}{\delta r} - \overline{\lim}_{r \rightarrow \infty} \frac{n(-\varphi, \varphi, (1 - \delta)r, \Lambda)}{\delta r} \right) = \\ &= \overline{\lim}_{\delta \rightarrow 0} \left(\frac{\tau(-\varphi, \varphi, \tilde{\Lambda})}{\delta} - \frac{(1 - \delta)\tau(-\varphi, \varphi, \tilde{\Lambda})}{\delta} \right) = \tau(-\varphi, \varphi, \Lambda) \end{aligned}$$

except at most a countable set of values of φ . As φ is small enough, the arc $\gamma(\varphi)$ coincides with the basis of trapezium D' of length $2\pi(\tau' + \alpha)$. This is why by (52), inequality $\tau_0(\Lambda) \leq \tau' + \alpha$ should hold that contradicts the choice of number α .

Hence, there exists a function $g' \in W(\Lambda, D')$ which can not be represented by (1) in domain D' . Now consider the domain $\tilde{D}' = D' \cap \{z : \operatorname{Re} z < a\}$, where $b < a < H_D$. It lies in D' and is an isosceles trapezium whose bases are parallel to the imaginary axis. One of them coincides with that of D' having the length strictly greater than $2\pi\tau_0(\Lambda)$. We choose a number a such that the other basis of \tilde{D}' has a length strictly greater than $2\pi\tau_0(\Lambda)$. System \mathcal{E} is incomplete in $H(\tilde{D}')$ due to the same reasons as in $H(D')$. Then by the proven implication Item 2 \Rightarrow Item 1, function $g' \in W(\Lambda, \tilde{D}')$ is represented by series (1) in the half-plane $\{z : \operatorname{Re} z < a\}$. The union of the latter and domain D' contains D'' . This is why $g' \in W(\Lambda, \tilde{D}') \cap H(D'')$. Since domain D'' is unbounded, by Theorem 8.1 in work [15], the inclusion $g' \in W(\Lambda, D'')$ holds true as well. Domain D lies in D'' . Therefore, $g' \in W(\Lambda, D)$. Due to Item 1, function g' is represented by series (1) in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < H_D(1)\}$ containing D' . It contradicts to the choice of g' .

Thus, the length of segment X is at least $2\pi\tau_0(\Lambda)$. The proof is complete. □

Remark. If sequence Λ has the density $\tau(\Lambda)$ (and then $\tau(\Lambda) = \tau_0(\Lambda)$), system \mathcal{E} is incomplete in $H(D)$ if and only domain D contains some vertical segment of length $2\pi\tau_0(\Lambda)$ (see [16, Ch. III, Sect. 1, Thm. 3.1.6]). If domain D is unbounded, then, as above, the condition of incompleteness for system \mathcal{E} is incomplete in $H(D)$ can be either excluded or replaced by (51).

Corollary 1. *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers, D be an unbounded wide convex domain in \mathbb{C} such that function $H_D(\lambda)$ is bounded in the vicinity of $\lambda = 1$. Then the following statements are equivalent*

1. *Each function in $W(\Lambda, D)$ can be represented by series (1) in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < H_D(1)\}$;*
2. *$S_\Lambda = 0$, $\tau_0(\Lambda) < +\infty$ and the intersection of the support line $\{z \in \mathbb{C} : \operatorname{Re} z = H_D(1)\}$ with the boundary of domain D contains the segment of length $2\pi\tau_0(\Lambda)$.*

Corollary 2. *Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ be an unbounded strictly increasing sequence of positive numbers, D be an unbounded convex domain in \mathbb{C} such that function $H_D(\lambda)$ is bounded in the vicinity of $\lambda = 1$ and $\psi \in [0, \pi)$ is so that $H_D(e^{i\psi}) < +\infty$ and $H_D(e^{i\psi+\pi}) < +\infty$. Then the following statements are equivalent*

1. *Each function in $W(\Lambda, D)$ can be represented by series (1) in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < H_D(1)\}$;*
2. *There holds (51), $S_\Lambda = 0$, $\tau_0(\Lambda) < +\infty$, and the intersection of the support line $\{z \in \mathbb{C} : \operatorname{Re} z = H_D(1)\}$ and of the boundary of domain D contains a segment of length $2\pi\tau_0(\Lambda)$.*

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