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CONSTRUCTION OF GENERALIZED SOLUTION FOR FIRST ORDER DIVERGENCE TYPE EQUATION

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Abstract. We consider Cauchy problem for a first order divergence type equation with the right hand side independent of the unknown function and with a discontinuous initial condition. This equation was first mentioned by J.M. Burgers in 1940 and it is a model equation for the system of equations describing the non-stationary gas motion. Various properties of the solution to this problem were studied in works by O.A. Oleinik (1957), J. Whitham (1974), S.N. Kruzhkov (1970), E.Yu. Panov (2006). The original problem is reduced to Cauchy problem for Hamilton-Jacobi equation with a continuous initial condition. It is suggested to apply the method of singular characteristics to this problem, while this method was developed A.A. Melikyan for game problems. The effectiveness of technique is demonstrated by the example when in the original equation the derivative w.r.t. the spatial variable is applied to a cubic polynomial of the unknown function, and the boundary condition is specified as a “raising” step. The Hamiltonian in the modified problem is a third degree polynomial of a partial derivative for the unknown function, and the boundary condition is given by a piecewise linear convex function with a break in the origin. We single out the domains of the parameters for which the construction of a generalized solution is possible, and we describe the procedure of constructing the solution. It is shown that the solution involves nonsmooth singularities called the dispersal and equivocal surfaces according to the terminology of differential games. The constructing of the solution is illustrated by figures.

Keywords: Hamilton-Jacobi equation, generalized solution, method of characteristics.

Mathematics Subject Classification: 35R01, 49J20, 49N70.

1. Introduction

In many problems on waves propagation, one considers a continuous distribution of some matter or some media state. In the one-dimensional case, letting x to be time coordinate, and y to be the spatial coordinate, one can define the density $v(x, y)$ per unit of length and the expenditure $q(x, y)$ per unit of time. We define the flow rate $w(x, y)$ by the identity $w = q/v$. Supposing that the studied matter is conserved, we can assume that the rate of changing of its total amount in each interval $y_1 < y < y_2$ should be compensated by the total flow through the section y_1, y_2, i.e.,

$$\frac{d}{dx}\int\limits_{y_1}^{y_2} v(x,y)dy + q(x,y_2) - q(x,y_1) = 0.$$

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If $v(x,y)$ has continuous derivatives, we can pass to the limit as $y_1 \to y_2$ to obtain the conservation law

$$\frac{\partial v}{\partial x} + \frac{\partial q}{\partial y} = 0.$$

The simplest problem on waves propagation appears in the case when on the basis of theoretical or empirical arguments, one can postulate some functional relation between q and v as $q{=}\varphi(v)$. Then we obtain the conservation law in the form

$$\frac{\partial v}{\partial x} + \frac{\partial \varphi(v)}{\partial y} = 0. \tag{1}$$

In gas dynamics ([1], [2]), equation (1) is employed for approximate construction of discontinuous solution for flow of ideal gas without viscosity and heat conductivity.

Consider Cauchy problem for the first order equation

$$\begin{aligned} &\frac{\partial v}{\partial x} + \frac{\partial \varphi(v)}{\partial y} = f_1(x,y,v), \quad x \geq x_0, \\ &v(x_0,y) = \psi_1(y), \quad x,y \in \mathbb{R}^1, \quad f_1(x,y,v),\, \varphi(v) \in C^\infty. \end{aligned} \tag{2}$$

Here $\psi_1(y)$ is a bounded piecewise smooth function. If $f_1(x,y,v) \equiv 0$, in accordance with said above, this equation is the conservation law or the transport equation. If the free term $f_1(x,y,v)$ is independent of v, it can be regarded as an external source exciting waves ([2]).

Many physical problems leading one to problem (2) and its generalizations were considered in [2], [**?**]. First the equation in (2) was mentioned in the work by Burgers [4] and it is a model one for the system of equations describing non-stationary gas motion. In work by O.A. Oleinik [5] for the case $\varphi_{vv}(v) \neq 0$, the uniqueness of the generalized solution to problem (2) was proven. A further development for this approach was made in works by N.S. Kruzhkov [6] and E.Yu. Panov [7], where there were studied existence, uniqueness and stability of generalized solutions to equation (2). In physics, equation (2) is usually called quasilinear transport equation or non-homogeneous transport equation. Transport equation describes various processes related with particles propagation in a matter ([8]).

Definition 1.0. Suppose a function $v(x,y)$ defined on a domain Ω has several smoothness components Ω_1, Ω_2, $\dots\Omega_n$ and respectively, several curves of first kind discontinuity $\Gamma_1, \Gamma_2 \dots, \Gamma_k$, and

$$\Omega = \Big(\bigcup_{i=1}^{n} \Omega_i\Big) \bigcup \Big(\bigcup_{i=1}^{k} \Gamma_i\Big)$$

According to works [1], [9], [10], we call function $v(x,y)$ *a generalized solution* to equation (2) in domain Ω if the following conditions hold true,

1) in the smoothness domains Ω_i, $i = 1, \dots, n$, function $v(x,y)$ satisfies (2) in the classical sense;

2) on the discontinuity curves $y = y(x)$, Rankine-Hugoniot condition

$$y'(x){=}\frac{[\varphi(v)]}{[v]} \equiv \frac{\varphi(v_2(x)){-}\varphi(v_1(x))}{v_2(x) - v_1(x)}, \text{ where } v_1(x){=}v(x,y(x)-0),\ v_2(x){=}v(x,y(x)+0)$$

holds true except a finite number of the intersection points Γ_i, $i = 1, \dots, k$.

3) Discontinuity stability condition is valid: as $v_2 > v_1$ ($v_2 < v_1$), the graph of function $\varphi(v)$ lies not lower (respectively, not upper) than the chord connection the point of this graph with abscissae v_1, v_2. This condition can be written as the inequalities

$$\frac{\varphi(v_*) - \varphi(v_2)}{v_* - v_2} \leq y'(x) \leq \frac{\varphi(v_*) - \varphi(v_1)}{v_* - v_1}, \quad y = y(x),$$

which hold true for all v_* between the values v_1, v_2.

As $\varphi''_{vv}(v) \neq 0$, stability conditions for generalized solution are simplified and cast into the form
$$\varphi'_v(v_2) \leq y'(x) \leq \varphi'_v(v_1).$$
In work [10], for problem (2) with zero right hand side ($f_1(x,y,v) \equiv 0$), it was proven that the generalized condition in Definition 1.0 is entropic. In work [7], for problem (2) with zero right hand side under the assumptions that $y \in \mathbb{R}^n$, $\varphi \in C^1$, and the derivative $\varphi'(v)$ and the initial condition $\psi_1(y)$ belong to the class of locally-bounded functions $L^\infty_{loc}(\mathbb{R}^n)$ and satisfy the restrictions for the growth
$$\begin{aligned} &|\varphi'(v)| \leq C_0(1+|v|^{p-1}),\ p>1,\ C_0 = const, \\ &\alpha = (p-1)^{-1},\ |\psi_1(y)| \leq M(1+|y|^\alpha) \text{ a.e. on } \mathbb{R}^n, \end{aligned} \tag{3}$$
the existence and uniqueness of solution $v(x,y)$ were proven in the class of functions
$$\begin{aligned} &B_\alpha = \{v(x,y) \in L^\infty_{loc}(\Pi_T)|\ \exists M = M(x) \in L^\infty_{loc}([x_0, x_0+T)), \\ &|v(x,y)| \leq M(x)(1+y^\alpha) \text{ a.e. on } \Pi_T.\}, \end{aligned} \tag{4}$$
where $\Pi_T = (x_0, x_0+T) \times \mathbb{R}^n$, $x_0 < x_0 + T \leq +\infty$.

In work [7], for problem (2) with function $\varphi(v) = |v|^{p-1}v$, $p>1$, and $f_1(x,y,v) \equiv 0$, a family of non-zero generalized solutions not belonging to class B_α was constructed.

In the present work we assume that $f_1(x,y,v)$ is independent of v. Then we can consider Cauchy problem
$$\begin{cases} \dfrac{\partial u}{\partial x} + \varphi(\dfrac{\partial u}{\partial y}) = f(x,y), \quad u(x_0,y) = \psi(y), \quad x,y \in \mathbb{R}^1, \\ f(x,y) = \displaystyle\int_{y_0}^{y} f_1(x,y)dy, \quad \psi(y) = \int_{y_0}^{y} \psi_1(y)dy. \end{cases} \tag{5}$$
Here the value of y_0 is taken arbitrarily in the smoothness interval for $\psi_1(y)$, function $\psi(y)$ is continuous but not a smooth one. Differentiating its solution w.r.t. y, one can get a solution to problem (2).

Problem (5) is a boundary value problem for Hamilton-Jacobi equation appearing in control theory, mechanics, physics. In control theory, the equation in (5) is the basis for dynamic programming and is called basic equation or Bellman-Isaacs equation. For a wide class of the problems [11], [12], it was proven that the viscous solution of Cauchy problem for Hamilton-Jacobi equation is identical to optimal result function of the problem (Bellman-Isaacs function, game price). This is why the method of singular characteristic ([13]) can be applied for solving problem (5).

2. Definition of continuous generalized solution

Consideration of non-autonomous control problems and differential games leads one to the boundary value problem
$$\begin{cases} H(x, S(x), p) = 0,\ p = \partial S/\partial x = S_x,\ x \in \Omega \subset \mathbb{R}^n, \\ S(x) = w(x),\ x \in M \subset \partial\Omega,\ x, p \in \mathbb{R}^n, \end{cases} \tag{6}$$
where function H and set M read as
$$\begin{cases} H = p_1 + H^*(x_1, \dots, x_n,\ S,\ p_2, \dots, p_n), \\ M = \{x \in \mathbb{R}^n :\ x_1 = c_1 = const\}. \end{cases} \tag{7}$$
Set Ω is the half-space (or a layer) on the right or on the left of set M. Functions H^*, w are continuous w.r.t. their variables on the sets $\Omega \times \mathbb{R}^n$ and M, respectively. Equation $H = 0$ in

(6) with function H like in (7) is usually referred as Hamilton-Jacobi equation. Problem (6) can have no classical solution $S(x) \in C^1(\Omega)$ even functions H w are smooth.

Problem (6), (7) is called initial (terminal), if

$$\Omega = \{x \in \mathbb{R}^n : x_1 > c_1\} \ (\{x \in \mathbb{R}^n : x_1 < c_1\}). \tag{8}$$

Let us provide the definition of generalized solution by M.G. Crandall and P.L. Lions [14] for boundary value problem (6), (7).

Definition 1.1. A continuous function $S : \Omega \to \mathbb{R}^1$ is called *generalized (viscous)* solution to initial problem (6)–(8) if for each test function $\varphi(x) \in C^1(\Omega)$ the inequality

$$H(x_0, S(x_0), \varphi_x(x_0)) \geq 0 \ \ (H(x_0, S(x_0), \varphi_x(x_0)) \leq 0) \tag{9}$$

holds true at the points of local minimum (maximum) of the difference $S(x) - \varphi(x)$. In the case of the terminal problem, inequalities (9) are opposite.

In work [11], the definition of minimax solution was given and the equivalency of minimax and generalized (viscous) solution was proven. Under the condition that function $H(x, s, p)$ for problem (6)-(8) is non-increasing w.r.t. s and is Lipshitz w.r.t. p, the existence and uniqueness of minimax solution was proven.

It was shown in work [15], that a generalized solution to Cauchy problem for quasilinear equation (2) is the superdifferential discriminator of minimax solution to Cauchy problem for Hamilton-Jacobi equation (5). In particular, given a smooth minimax solution to problem (5) with a Hamiltonian independent on the unknown function, its derivative w.r.t. the phase variable is a generalized solution to problem (2) in the sense of Definition 1.0. The questions on relations between two different definitions of generalized solutions are treated also in works [1], [16].

In work [6], for problem (2) in the strip $x \in [0, T], y \in \mathbb{R}^1$, the unique solvability in the class of bounded measurable functions was proven. Problem (13), (14) considered below obey the restrictions formulated in that work.

According to said above, in order to construct a generalize solution to problem (2) where the right hand side is independent of the unknown functions, by integrating equation and initial conditions w.r.t. y one can reduce problem (2) to problem (5) which is a particular case of problem (6)-(8) with a Hamiltonian independent of the unknown function. In the obtained problem, one can employ the method of singular characteristics for constructing generalized solution in the sense of Definition 1.1.

3. Characteristics methods. Singular manifolds

For local constructing of a classical solution to problem (6) by characteristics method, the existence of second derivatives for functions $S(x)$, $H(x, S, p)$ ([18]) is sufficient. Then construction of classical solution to problem (6) is reduced to integrating the system of regular characteristics

$$\dot{x} = H_p, \quad \dot{S} = \langle p, H_p \rangle, \quad \dot{p} = -H_x - pH_S, \tag{10}$$

with initial conditions

$$x_1 = c_1, \ \ x_i = z_{i-1}, \ \ p_i(c_1, z) = \frac{\partial w(c_1, z)}{\partial z_i}, \ \ i = 2, \ldots, n, \ \ z = (z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1},$$

$$p_1 = -H^*(c_1, z_1 \ldots, z_{n-1}, w(c_1, y), p_2, \ldots, p_n), \ \ S(c_1, z) = w(c_1, z).$$

As the differentiation parameter in equations (10), one can regard coordinate x_1. In the vicinity of the points where functions S, H do not have the required smoothness, the mentioned procedure of constructing the solution, generally speaking, does not work.

Definition 2.1. By regular point of a generalized solution to equation (6) we call any internal point x^0 of domain Ω for solution $S(x)$ such that in its neighborhood D function $S(x)$ is twice differentiable and satisfies main equation $H(x, S(x), p) = 0$ in (6, where function $H(x, S, p)$ is twice differentiable in the vicinity of the point $(x^0, S(x^0), p^0) \in \mathbb{R}^{2n+1}$, $p^0 = S_x(x^0)$. All points not being regular are called singular. Singular set consists of singular points ([13, p. 57]).

In the case singular sets are surfaces, they can be classified by the behavior of regular characteristics and the smoothness of functions $S(x)$, $H(x, S, p)$ in its neighborhood. Let us provide briefly its classification. Hereinafter, while describing various behaviors, we mean the behavior of its phase components.

Dispersal surface. Regular characteristics approach on both sides, $S(x) \notin C^1$.

Equivocal surface. Regular characteristics approach on one side and leave on the other, $S(x) \notin C^1$. For $H \in C^1$, the characteristics leave with tangency.

Switching surface. It is similar with the equivocal one, but $S(x) \in C^1$, $H \notin C^1$.

Universal surface. Regular characteristics leave on both sides, $S(x) \in C^1$ and $H(x, S, p) \notin C^1$.

Focal surface. It is similar with the universal one, $S(x) \notin C^1$. As $H \in C^1$, the characteristics leave with tangency. If a focal surface is degenerate to a point, we obtain the vertex of an integral funnel.

At the points of singular surface, the following lemma holds true ([13, p. 60]).

Lemma 1. *Let $S(x)$ be a generalized solution to problem (6), (7), which can be represented by the identity*

$$S(x) = min\left[S^+(x),\ S^-(x)\right] \quad S^+(x), S^-(x) \in C^1(D) \tag{11}$$

in a neighborhood D of a singular surface. Then on surface Γ, testing function $h(\tau)$ satisfies the condition

$$\begin{aligned} &h(\tau) = H(x, S(x), p^+(1+\tau)/2 + p^-(1-\tau)/2) \le 0,\ |\tau| \le 1,\ x \in \Gamma, \\ &p^s = \partial S^s/\partial x,\ s = +, -, \quad h(-1) = h(1) = 0. \end{aligned} \tag{12}$$

If problem (6),(7) is terminal or generalized solution $S(x)$ can be represented as $S(x) = max\,[S^+(x),\ S^-(x)]$, inequality (12) changes the sign.

To prove Lemma 1, it is sufficient to take the test function $\varphi(x) = S^+(1+\tau)/2 + S^-(1-\tau)/2$ in (9).

4. Formulation of problem. General form of function $h(\tau)$

Consider Cauchy problem

$$v_x + \varphi_y(v) = f, \quad x \ge 0, \quad \varphi(v) = av^3 + bv^2 + cv + d; \quad x, y \in \mathbb{R}^1, \tag{13}$$

$$v(0, y) = \psi_1(y) = \begin{cases} \rho_1, & y > 0 \\ \rho_2 & y < 0. \end{cases} \tag{14}$$

for various parameters a, b, c, d, e, f. If we let $\rho_1 = g+1$, $\rho_2 = g-1$, following the procedure described in Introduction, we obtain Hamilton-Jacobi initial problem

$$H = p + \varphi(q) - fy = 0,\ \varphi(q) = aq^3 + bq^2 + cq + d; \tag{15}$$

$$S(0, y) = |y| + gy, p = \partial S/\partial x,\ q = \partial S/\partial y, \quad x > 0. \tag{16}$$

Solution to problem (13), (14) as $a = 0$, $f = 0$ has two kinds of singularities, see, for instance, [2].

As $\rho_1 > \rho_2$, $b > 0$, there is a space between two characteristics with different boundary conditions, and this space is covered by a fan of characteristics. This is the first kind of singularities.

For values $\rho_2 > \rho_1$, $b > 0$ there appears the second kind of singularities which is a shock wave, the wave breaks and the characteristics intersect.

According to Section 3, the vertex of an integral fun and a dispersal surface correspond to to, see [17, p. 1664-1673]. In the same work [17, p. 1664-1673], problem (13), (14) was considered for the case $\rho_2 > \rho_1$ and it was shown that apart a dispersal surface, a generalized solution to the appropriate Hamilton-Jacobi initial problem has also an equivocal surface. Let us show that in the case $\rho_1 > \rho_2$ the solution has similar features.

Since the function $S(0,y)$ in (16) is represented as

$$S(0,y){=}max\left[y(-1{+}g),y(1{+}g)\right],$$

one should expect that in the vicinity of singular surfaces

$$S(x,y){=}max\left[S^{+}(x,y),\ S^{-}(x,y)\right].$$

The further construction of the solution justified this assumption.

We note that for problem (15),(16), function $h(\tau)$ in (12) reads as

$$h\left(\tau\right)\ =\ \frac{1}{4}\left(\tau^{2}-1\right)\left(\alpha\tau+\beta\right)(q^{+}-q^{-})^{2},\quad \tau\in[-1;1], \tag{17}$$

$$\alpha=a(q^{+}-q^{-})/2,\quad \beta=3a\left(q^{+}+q^{-}\right)/2+b. \tag{18}$$

The sigh change for function $h(\tau)$ happens only due to the linear factor w.r.t. τ. This is why the condition $h(\tau)\geq 0$ is equivalent to inequality

$$\mu(\tau)=\alpha\tau+\beta\leq 0,\quad \tau\in[-1;1]. \tag{19}$$

Below we shall find out for which relations between the parameters the existence of various singular surfaces is possible and we shall provide the procedure of constructing the solution.

5. Primary solution. Dispersal surface

Equations for regular characteristic (10) of problem (15), (16) read as

$$\begin{gathered}\dot{x}=H_p=1,\quad \dot{y}=H_q=\varphi_q(q)=3aq^2+2bq+c,\\ \dot{p}=-H_x=0,\quad \dot{q}=-H_y=f,\quad \dot{S}=pH_p+qH_q.\end{gathered} \tag{20}$$

Here as the differentiation parameter one can regard coordinate x. Employing identities (15), (16) and differentiating function $S(0,y)$, we obtain initial conditions for system (20) at an arbitrary point $(0,y_0)$ of axis y,

$$x=0,\quad y=y_0,\quad p=-\varphi(q_0)+fy_0,\quad q=sgn\,y_0+g,\quad S=|y_0|+gy_0. \tag{21}$$

It follows that all the regular characteristics of problem (20) are cubic parabolas on the plane x,y

$$y_q(x,x_0,y_0){=}a\,(x{-}x_0)^3f^2{+}(3\,aq{+}b)(x{-}x_0)^2f{+}(3\,aq^2{+}2\,bq{+}c)(x{-}x_0)+y_0. \tag{22}$$

According to (20)–(22), in the vicinity of the boundary, which axis y, for boundary values

$$x_0=0,\quad q_0=\begin{cases} q_{01}=g+1, & y_0>0\\ q_{02}=g-1, & y_0<0\end{cases} \tag{23}$$

we obtain two families of curves, the upper and lower one, respectively,

$$\left.\begin{aligned} y_{xak1}(x,0,y_0)&=y_0+ax^3f^2+\left(3\,a\,(g{+}1)+b\right)x^2f+\\ &+\left(3\,a\,(g{+}1)^2+2\,b\,(g{+}1)+c\right)x,\\ y_{xak2}(x,0,y_0)&=y_0+ax^3f^2+\left(3\,a\,(g{-}1)+b\right)x^2f+\\ &+\left(3\,a\,(g{-}1)^2+2\,b\,(g{-}1)+c\right)x.\end{aligned}\right\} \tag{24}$$

We observe that as $a=0$, $f\neq 0$, regular characteristics in (22), (24) are quadratic in x functions, but the solution to the problem can be constructed by analogy with solution studied by Witham for the case $a=0$, $f=0$ with linear regular characteristic. As $a=0$, $f\neq 0$, there appear no new singularities and the behavior of solution is preserved. This is why in what follows we assume that $a \neq 0,\ f \neq 0$. We provide the case $a = 0,\ f \neq 0$ as a simple situation while constructing the integral funnel.

Consider the difference of regular characteristics (24) leaving from the origin,

$$y_{xak1}(x,0,0) - y_{xak2}(x,0,0) = 6\,fax^2 + (12\,ag + 4\,b)\,x. \tag{25}$$

It follows from (25) that if the parameters of the problem satisfy the conditions

$$\left[\begin{array}{l} 3ag + b < 0, \\ 3ag + b = 0, \quad fa < 0, \end{array}\right. \tag{26}$$

then two families of regular trajectories with values $q_0 = \pm 1 + g$ intersect at origin and there is no domain not covered by regular characteristics. In this and the subsequent section conditions (26) are supposed to be satisfied.

Integrating expression $\dot{S} = pH_p + qH_q$ along the trajectories of system (20) and substituting then values $x_0 = 0$, q_0 from (23) and y_0 from (22), letting $y = y_q(x, x_0, y_0)$, we obtain function $S(x,y)$ called primary solution to problem (15), (16),

$$\left.\begin{array}{l} S(x,y) = max\left[S_1(x,y),\ S_2(x,y)\right], \\ S_i(x,y) = q_{0i}y - \dfrac{a}{4}f^3x^4 - \left(aq_{0i} + \dfrac{b}{3}\right)f^2x^3 - \\ -\left(3\,aq_{0i}{}^2 + 2bq_{0i} + c\right)\dfrac{fx^2}{2} - \left(aq_{0i}{}^3 + bq_{0i}{}^2 + cq_{0i} + d\right)x + fxy, \\ q_{0i} = g - (-1)^i,\ i = 1,2. \end{array}\right\} \tag{27}$$

In what follows we shall provide the examples of domains for which the primary solution is a *generalized* one to problem (15),(16). Identity $S = S_1$, ($S = S_2$) holds true above (below) the cubic parabola determined by the continuity condition $S_1 = S_2$,

$$y_{disp}(x) = ax^3f^2 + (3\,ag + b)\,x^2f + \left(3\,ag^2 + 2\,bg + a + c\right)x. \tag{28}$$

For dispersal surface (28) the relations

$$\left.\begin{array}{ll} q^+ = q_1(x) = g + 1 + fx, & q^- = q_2(x) = g - 1 + fx, \\ \alpha = a(q^+ - q^-)/2 = a, & \beta = 3\,a(q^+ + q^-)/2 + b = 3a(g + fx) + b \end{array}\right\} \tag{29}$$

are valid.

By (29) and Lemma 1 we obtain the necessary condition for existing surface (curve) (28)

$$\max_{\tau \in [-1;1]} \mu(\tau) = |a| + 3a(g + fx) + b \le 0 \tag{30}$$

Introducing function $disp(x)$

$$disp(x) = |a| + 3a(g + fx) + b \tag{31}$$

and observing that $disp(0) = |a| + 3ag + b$, we obtain that dispersal curve (28) exists in the vicinity of the origin if and only if one of the conditions

$$|a| + 3ag + b < 0 \quad \text{or} \quad \left[\begin{array}{l} |a| + 3ag + b = 0, \\ fa \le 0. \end{array}\right. \tag{32}$$

holds true.

Conditions (32) are stronger than condition (26). If condition (26) is satisfied and conditions (32) are not, an equivocal surface leave from the origin. The method of constructing this surface is discussed in the next section.

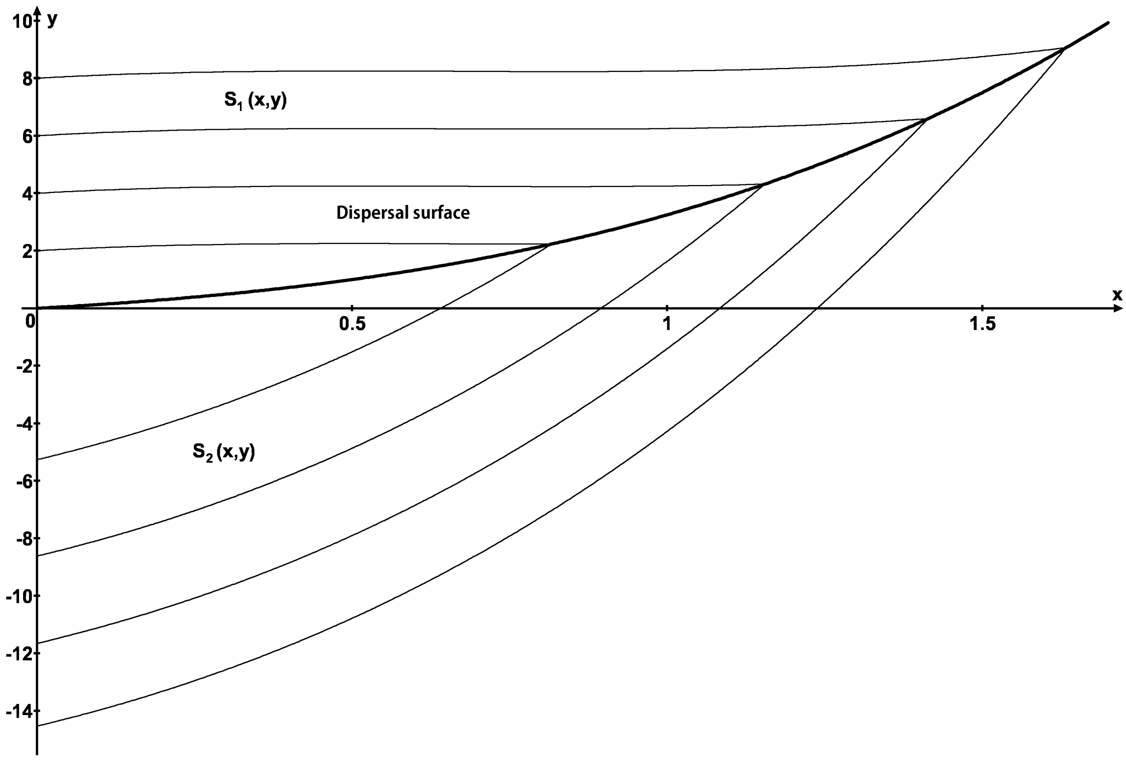

FIGURE 1. Dispersal surface

For values

$$\left[\begin{array}{l} |a|+3ag+b\le 0, \\ fa\le 0, \end{array}\right. \tag{33}$$

$disp(x){\le}0$ as $x{\in}[0,+\infty)$ and the dispersal surface (curve) goes from the origin to infinity. The procedure of constructing the dispersal curve and generalized solution in case (33) is clarified by Figure 1. We perform the construction for the values

$$a{=}1,\ \ b{=}{-}a{-}3\,a\,g{=}1/2,\ \ c{=}0,\ d{=}0,\ \ g{=}{-}1/2,\ \ f{=}{-}1.$$

The bold curve is the dispersal curve separating the half-plane $(x,y), x>0$ into two domains. In each of these domain the solution to problem (15), (16) is given by functions $S_1(x,y)$, $S_2(x,y)$ as it is indicated on the figure. The solution to problem (13), (14) in these domain is determined by the relations

$$v_1(x,y)=g+1+f\,x,\quad v_2(x,y)=g-1+f\,x \tag{34}$$

The thin lines show regular characteristics. We obtain

Lemma 2. *For values of the parameters (33), the solution to problem (15), (16) is given by the relations (27), and the solution to problem (13), (14) as* $\rho_1=g+\ 1$, $\rho_2=g-1$ *is given by the relations*

$$v(x,y)=\left\{\begin{array}{ll} g+1+f\,x, & y>y_{disp}(x) \\ g-1+f\,x, & y<y_{disp}(x). \end{array}\right. \tag{35}$$

For the values

$$\left[\begin{array}{l} |a|+3ag+b<0, \\ fa>0, \end{array}\right. \tag{36}$$

$disp(x)$ vanishes at zero 0 as

$$x^*=-\frac{|a|+3ag+b}{3af}.$$

The value x^* as $a>0$ is associated with the point (x_1^*, y_1^*), while as $a<0$, it is associated with the point (x_2^*, y_2^*) on the dispersal curve (28)

$$
\begin{aligned}
&x_i^* = \frac{a(-1)^i - 3\,ag - b}{3af}, \quad i=1,2,\\
&y_i^* {=} {-}\frac{((-1)^{i+1}a{+}b{+}3\,ag)\,(9\,a^2g^2{+}6\,agb{+}(-1)^i 3\,a^2g{-}2\,b^2{+}(-1)^i ba{+}10\,a^2{+}9\,ac)}{27\,a^2 f}.
\end{aligned}
\tag{37}
$$

Exactly at these points parabola (28) touches one of the characteristics of primary solution (24). As $a>0$, one of the characteristics (critical) of the upper family of parabolas (24) touches parabola (28) at the point (x_1^*, y_1^*). As $a<0$, one of the characteristics (critical) of the lower family of parabolas (24) touches parabola (28) at the point (x_2^*, y_2^*). The bifurcation of the dispersal curve into another kind of singularity can happen exactly at these points.

Since $x_2^* - x_1^* = 2/(3f)$ and $x_1^* \neq x_2^*$ for each $a \neq 0$ and $f \neq 0$, the dispersal surface can not bifurcate into the focal one or the integral funnel and vise versa.

It follows from said above that the condition of one-side touching at the points (x_1^*, y_1^*) as $a>0$ and $f>0$, (x_2^*, y_2^*) as $a<0$ and $f<0$ select equivocal surface among aforementioned singular surfaces.

6. Construction of equivocal surface

In this section, condition (36) is assumed to hold true. In the general case, on an equivocal surface we have three necessary conditions which are identity (6), touching condition, and continuity condition

$$
H(x, S(x), p) = 0, \quad \big\langle H_p, p - \partial S^+/\partial x \big\rangle = 0, \quad F_1(x,S) \equiv S - S^+(x) = 0. \tag{38}
$$

Here $S^+(x)$ is a smooth function coinciding with the solution on the side of the surface where the regular characteristics do not touch the surface. It was shown in work [19] that in the general case (6), the equivocal surface (curve) for $H \in C^1$ is constructed by integrating the system,

$$
\begin{aligned}
&\dot{x} = H_p, \quad \dot{S} = \langle p, H_p\rangle, \quad \dot{p} = -H_x - pH_S - \frac{\{\{H, F_1\}, H\}}{\{\{F_1, H\}, F_1\}}\left(p - \frac{\partial S^+}{\partial x}\right),\\
&F_1(x,S) \equiv S - S^+(x), \quad \{F, H\} = \langle F_x + pF_s, H_p\rangle - \langle H_x + pH_s, F_p\rangle\,.
\end{aligned}
\tag{39}
$$

The left hand sides of identities (38) are the first integrals for system (39). In notations of problem (15), (16), $S^+(x_1, x_2) = S^+(x,y)$. Employing (15), (27), (38), (39), we obtain differential equation with initial conditions for the equivocal curve,

$$
\begin{aligned}
&\frac{dy}{dx} {=} H_q, \ \frac{dq}{dx} {=} {-}\frac{S^+_{xx} {+} f\varphi_q(q)}{(S^+_y {-} q)\varphi_{qq}(q)},\\
&x{=}x_1^*,\ y(x_1^*){=}y_1^*,\ S^+(x,y){=}S_2(x,y),\ q{=}g{+}1{+}f\,x_1^* \quad \text{as} \quad a{>}0, f{>}0,\\
&x{=}x_2^*,\ y(x_2^*){=}y_2^*,\ S^+(x,y){=}S_1(x,y),\ q{=}g{-}1{+}f\,x_2^* \quad \text{as} \quad a{<}0, f{<}0.
\end{aligned}
\tag{40}
$$

Equation $\dot{x}{=}H_p{=}1$ and equation for p here are omitted, since p is not involved in (40). After integration of equations (40), the values of p for both equivocal curves are determined by the identity $H=0$, and quantity S can be found by the integration after determination of other variables.

Case a¿0, f¿0. By help of (40) we can obtain that the equivocal curve leaving from the point (x_1^*, y_1^*) as $a>0$ and $f>0$ is determined by the solution to Cauchy problem

$$
\begin{aligned}
&\frac{dy}{dx} = \varphi_q, \quad \frac{dq}{dx} = \frac{(3\,afx - 3\,a + 2\,b + 3\,ag + 3\,aq)\,f}{2(3\,aq + b)},\\
&x = x_1^*, \qquad y = y_1^*, \qquad q = q_1^* = q(x_1^*) = g + 1 + fx_1^* = \frac{2a-b}{3a}.
\end{aligned}
\tag{41}
$$

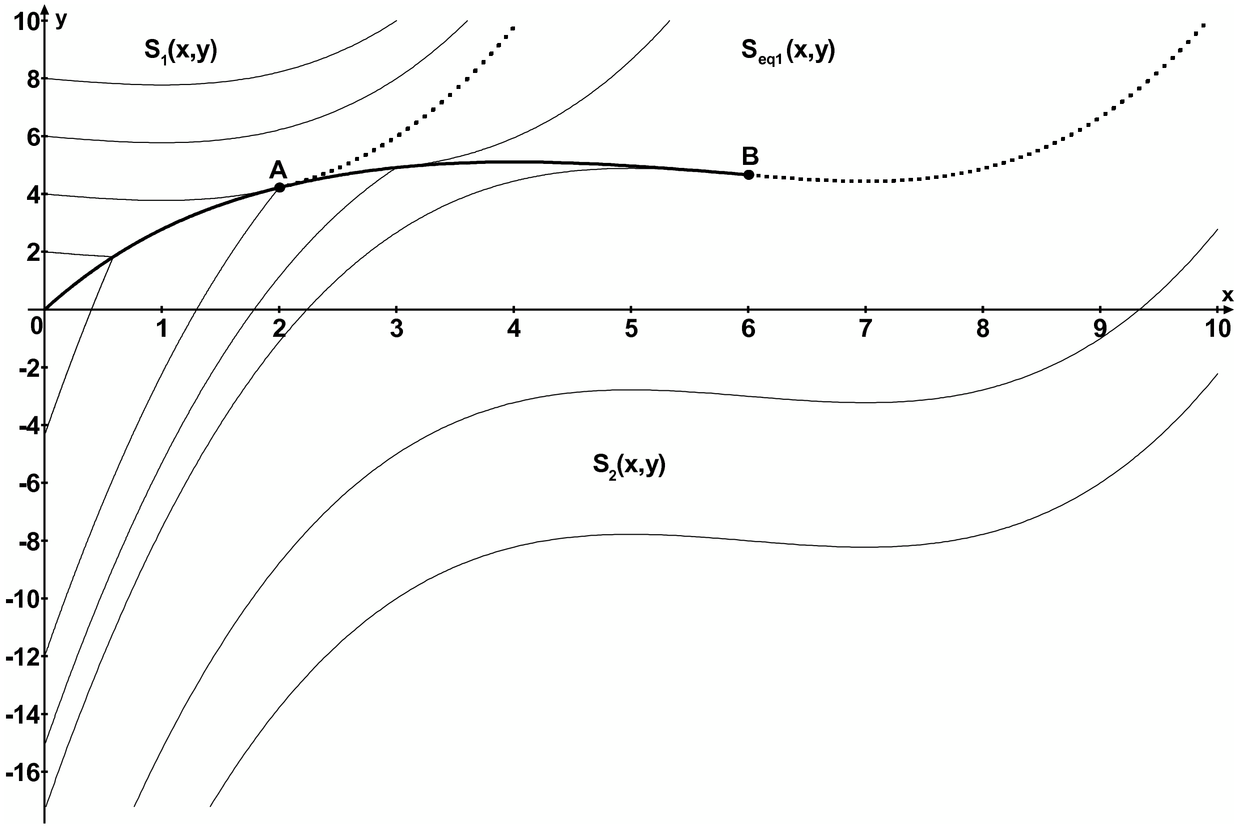

FIGURE 2. $0A$ is the dispersal surface, AB is the equivocal surface

System (41) has an analytic solution satisfying Lemma 1 on the segment $[x_1^*, x_{m1}]$,

$$\begin{aligned} &y_{eq1}(x) = y_1^* - \frac{2(\varphi(q) - \varphi(q_1^*))}{f}, \quad q_{eq1}(x) = q_1^* - \frac{f}{2}(x - x_1^*) = \frac{1 - g - fx}{2} - \frac{b}{2a}, \\ &x_{m1} = -\frac{3\,ag - 3\,a + b}{3fa}, \quad y_{m1} = y_{eq1}(x_{m1}), \quad x \in [x_1^*, x_{m1}]. \end{aligned} \tag{42}$$

Indeed, the testing function $\mu(\tau)$ for equivocal surface (42) reads as

$$\mu(\tau) = \mu_1(\tau) = \frac{(1-\tau)\,(3\,xfa - 3\,a + 3\,ag + b)}{4} \le 0, \quad \tau \in [-1; 1]. \tag{43}$$

Identity $\mu_1(\tau) = 0$ as $\tau \neq 1$ is attained for $x = x_{m1}$, where x_{m1} is the abscissa of the point (x_{m1}, y_{m1}) in (42) where it ends since as $x > x_{m1}$, the inequality $\mu_1(\tau) \le 0$ is false for $\tau \neq 1$. At this point, the numerator and denominator of the right hand side of the second equation in (41) vanish and exactly at this point we have the coincidence of the values of $q_{eq1}(x)$, $q_2(x)$, as well as the values of the derivatives for functions $y_{xak2}(x, x_{m1}, y_{m1})$ and $y_{eq1}(x)$ in (42), that is, a characteristics from the lower family of parabolas touches the equivocal surface. Then the equivocal curve becomes a regular characteristics.

The construction of the dispersal and equivocal surfaces for the case when $a > 0$ and $f > 0$ is demonstrated by Figure 2 for the values of the parameters $a = 1$, $b = -1$, $c = d = 0$, $g = -2/3$, $f = 1/3$.

The bold solid line and bold dashed line in Figure 2 separate the half-plane $x > 0$ into three domains, in each of those the generalized solution is determined by an appropriate formula. In the domains adjacent to axis y, the generalized solution to problem (15), (16) is determined by the formulae $S_1(x, y)$ and $S_2(x, y)$ for the domains lying respectively above and below the dispersal curve $0A$.

Let us describe the procedure of constructing the generalized solution $S(x,y) = S_{eq1}(x,y)$ in the domain above equivocal curve AB. We shall denote the coordinate of a point on equivocal curve AB by ξ, η, i.e., $\xi = x$, $\eta = y$. The values of the function $S_{eq1}(\xi,\eta)$ on the equivocal curve (42) are determined by the relations

$$S_{eq1}(\xi,\eta) = S_2(x_1^*, y_1^*) + \int_{x_1^*}^{\xi} (p + q\, H_q)\, dx =$$
$$= S_2(x_1^*, y_1^*) + \int_{x_1^*}^{\xi} (-\varphi(q_{eq1}(x)) + f\eta(x) + q_{eq1}(x)\, \varphi_q(q_{eq1}(x))\, dx, \tag{44}$$
$$\eta = y_1^* + \int_{x_1^*}^{\xi} H_q\, dx = y_1^* + \int_{x_1^*}^{\xi} \varphi_q(q_{eq1}(x))\, dx, \qquad \xi \in [x_1^*, x_{m1}].$$

From the domain adjacent from below to equivocal curve AB, the regular characteristics come on it. The generalized solution $S(x,y)$ coincides with function $S_2(x,y)$ in this domain. In the domain adjacent from above to equivocal curve AB, we construct the family of characteristics $y_{eq1x}(x,\xi)$, according to equation (20) this family is tangential to equivocal surface AB,

$$y_{eq1x}(x,\xi) = a\,(x-\xi)^3 f^2 + (3\,aq_{eq1}(\xi)+b)(x-\xi)^2 f +$$
$$+(3\,aq_{eq1}^2(\xi)+2\,bq_{eq1}(\xi)+c)(x-\xi) + \eta(\xi), \quad \xi \in [x_1^*, x_{m1}], \quad \xi \le x. \tag{45}$$

We denote the solution in this domain by $S_{eq1}(x,y)$. The family of characteristics $y_{eq1x}(x,\xi)$ for Figure 2 is defined by the formulae

$$y_{eq1x}(x,\xi) = \frac{x^3}{9} - \frac{x^2\,\xi}{2} + \frac{3\,x\,\xi^2}{4} - \frac{\xi^3}{3} + x^2 - 3\,x\,\xi + \frac{3\,\xi^2}{2} + \frac{8\,x}{3} + \frac{2}{3}.$$

The construction of function $s_{eq1}(x,y)$ for problem (15), (16) and of function $v_{eq1}(x,y)$ for appropriate problem (13), (14) is made by integration along the family of characteristics $y_{eq1x}(x,\xi)$

$$S_{eq1}(x,y) = S_{eq1}(\xi,\eta(\xi)) + \int_{\xi}^{x} (p + q\, H_q)\, dx' = S_{eq1}(\xi,\eta(\xi)) +$$
$$+ \int_{x_1^*}^{\xi} (f y_{eq1x}(x',\xi) - \varphi(q_{eq1x}(x',\xi)) + q_{eq1x}(x',\xi)\, \varphi_q(q_{eq1x}(x',\xi))\, dx', \tag{46}$$
$$v_{eq1}(x,y) = q_{eq1x}(x,\xi), \;\; q_{eq1x}(x,\xi) = q_{eq1}(\xi) + f\,(x-\xi) = \frac{1-g-3\,f\,\xi}{2} - \frac{b}{2a} + f\,x,$$
$$x \in [\xi, x_{m1}], \;\; y = y_{eq1x}(x,\xi), \xi \in [x_1^*, x_{m1}].$$

Case a¡0, f¡0. The equivocal curve leaving from the point (x_2^*, y_2^*) as $a < 0$ and $f < 0$ is determined by the solution to Cauchy problem

$$\frac{dy}{dx} = \varphi_q, \quad \frac{dq}{dx} = \frac{(3\,afx + 3\,a + 2\,b + 3\,ag + 3\,aq)\,f}{2(3\,aq + b)},$$
$$x = x_2^*, \qquad y = y_2^*, \qquad q = q_2^* = q(x_2^*) = g - 1 + fx_2^* = -\frac{2a+b}{3a}. \tag{47}$$

System (47) has analytic solution satisfying Lemma 1 on the segment $[x_2^*, x_{m2}]$,

$$
\begin{aligned}
&y_{eq2}(x)=y_2^* - \frac{2(\varphi(q)-\varphi(q_2^*))}{f}, \quad q_{eq2}(x)=q_2^*-\frac{f}{2}(x-x_2^*)=\frac{-1-g-fx}{2}-\frac{b}{2a},\\
&x_{m2} = -\frac{3\,ag+3\,a+b}{3fa}, \quad y_{m2}=y_{eq2}(x_{m2}).
\end{aligned} \tag{48}
$$

Indeed, testing function $\mu(\tau)$ for equivocal surface (48) reads as

$$
\mu(\tau)=\mu_2(\tau)=\frac{(1+\tau)\,(3\,xfa+3\,a+3\,ag+b)}{4}\le 0, \quad \tau\in[-1;1]. \tag{49}
$$

Identity $\mu_2(\tau)=0$ as $\tau\ne -1$ is attained for $x=x_{m2}$, where x_{m2} is the abscissa of the point (x_{m2}, y_{m2}), where it ends since as $x>x_{m2}$, the inequality $\mu_2(\tau)\le 0$ is false for $\tau\ne -1$. At this point, the numerator and the denominator of the right hand side in the second equation in (47) vanish and exactly at this point the values of $q_1(x)$, $q_{eq2}(x)$ coincide and the same happen for the derivatives of function $y_{xak1}(x)$ $y_{eq2}(x)$ in (48), i.e., a characteristics in the upper family of parabolas touches the equivocal curve. Then the equivocal curve becomes a regular characteristics. Construction of functions $S_{eq2}(\xi,\eta)$, $S_{eq2}(x,y)$, $v_{eq2}(x,y)$ for problems (15), (16), and (13), (14) is done by analogy with the case $a>0$, $f>0$,

$$
\begin{aligned}
&S_{eq2}(\,\xi,\,\eta)=S_2(x_2^*,y_2^*)+\int\limits_{x_2^*}^{\xi}(-\varphi(q_{eq2}(x))+f\eta(x)+q_{eq2}(x)\,\varphi_q(q_{eq2}(x))\,dx,\\
&\eta=y_2^*+\int\limits_{x_2^*}^{\xi}\varphi_q(q_{eq2}(x))\,dx, \quad S_{eq2}(x,y)=S_{eq2}(\xi,\eta(\xi))+\\
&+\int\limits_{x_2^*}^{\xi}(fy_{eq2x}(x',\xi)-\varphi(q_{eq2x}(x',\xi))+q_{eq2x}(x',\xi)\,\varphi_q(q_{eq2x}(x',\xi))\,dx',\\
&v_{eq2}(x,y)=q_{eq2x}(x,\xi), \;\; q_{eq2x}(x,\xi)=q_{eq2}(\xi)+f\,(x-\xi), \;\; x\in[\xi,x_{m2}],\\
&y=y_{eq2x}(x,\xi), \xi\in[x_2^*,x_{m2}].
\end{aligned} \tag{50}
$$

We summarize the above facts as

Lemma 3. *For the values of parameters (36), the half-plane $x\ge 0$ is separated into three domains by the curves whose parts are the ordinate axis, dispersal curve (28)*

$$
y=y_{disp}(x), \;\; x\in[0,x^*], \quad x^*=\left\{\begin{array}{l} x_1^*, \;\; a>0, \;\; f>0\\ x_2^*, \;\; a<0, \;\; f<0\end{array}\right., \;\; y^*=y_{disp}(x^*),
$$

the equivocal curve in (42), (48)

$$
y=y_{eq}(x), \;\; x\in[x^*,x_m], \quad y_{eq}(x)=\left\{\begin{array}{l} y_{eq1}(x), \;\; a>0, \;\; f>0\\ y_{eq2}(x), \;\; a<0, \;\; f<0\end{array}\right.,
$$

$$
x_m=\left\{\begin{array}{l} x_{m1}, \;\; a>0, \;\; f>0\\ x_{m2}, \;\; a<0, \;\; f<0\end{array}\right.,
$$

and two regular characteristics $y_1(x)$, $x\in[x^*,+\infty)$, $y_2(x)$, $x\in[x_m,+\infty)$, $y_m=y_{eq}(x_m)$,

$$
y_1(x)=\left\{\begin{array}{l} y_{xak1}(x,x^*,y^*), \;\; a>0, \;\; f>0\\ y_{xak2}(x,x^*,y^*), \;\; a<0, \;\; f<0\end{array}\right.,
$$

$$
y_2(x)=\left\{\begin{array}{l} y_{xak2}(x,x_m,y_m), \;\; a>0, \;\; f>0\\ y_{xak1}(x,x_m,y_m), \;\; a<0, \;\; f<0\end{array}\right..
$$

In two domains adjacent to axis y, respectively for the upper and the lower domain, the generalized solution to problem (15), (16) is determined by the formulae $S_1(x, y)$ and $S_2(x, y)$ in (27), and the solution to problem (13), (14) as $\rho_1{=}g{+}1$, $\rho_2{=}g{-}1$ is given by relations (34). In third domain adjacent to the equivocal curve, the generalized solution to problem (15), (16) and (13), (14) is determined by the formulas in (46) where $a > 0$, $f > 0$ and (49) for $a < 0$, $f < 0$.

7. Absence of focal surface

Let us show that in problems described by equation (15) the focal surface is absent no matter what the boundary condition are. On a focal surface, necessary conditions should be satisfied. These are equation (6), the continuity condition $w(x, y){=}S^{(1)}(x, y){-}S^{(2)}(x, y){=}0$, the condition of touching the surface $w(x, y){=}0$ by the regular characteristics, i.e., there should be touching condition for both sides. Writing down these conditions, we arrive at the equations system for the vectors (p_1, q_1), (p_2, q_2), $(p_i = \partial S^{(i)}/\partial x,\ q_i = \partial S^{(i)}/\partial y,\ i = 1, 2)$,

$$\left.\begin{array}{c} p_1 + \varphi(q_1) - fy = 0; \quad p_2 + \varphi(q_2) - fy = 0, \\ p_1 - p_2 + (q_1 - q_2)\varphi_q(q_i) = 0, \\ \varphi_q(q) = 3aq^2 + 2bq + c, \quad i = 1, 2. \end{array}\right\} \tag{51}$$

A focal surface can be constructed by solving equations system ([13])

$$\begin{cases} \dot{y} = H_{q_1} = \bar{H}_{q_2}, \quad \dot{S} = q_1 H_{q_1} - H^* = q_2 \bar{H}_{q_2} - \bar{H}^*, \quad H^* = \varphi(q_1) - fy \\ \dot{q_1} = K(x, y, q_1, q_2), \quad \dot{q_2} = K(x, y, q_2, q_1), \\ K(x, y, q_1, q_2) = \dfrac{[H_x + H_y\, H_{q_1}]^*}{(q_1 - q_2) H_{q_1 q_1}} - \dfrac{H_{q_1 x} + H_{q_1 y}\, H_{q_1}}{H_{q_1 q_1}}. \end{cases} \tag{52}$$

The overline indicates the replacement of arguments p_1, q_1 by p_2, q_2, square brackets with asterisk denotes the jump of the function on the required focal surface Γ_f, i.e., $[f]^* = f(p_1, q_1) - f(p_2, q_2)$.

Considering system (51), we arrive at two equations for q_1, q_2,

$$(q_1 - q_2)^2\,(2\,aq_1 + b + aq_2) = 0$$

$$-\,(q_1 - q_2)^2\,(aq_1 + b + 2\,aq_2) = 0,$$

having just coinciding solutions $q_1 \equiv q_2$. This is why system (51) has the only solutions $p_1 \equiv p_2$, $q_1 \equiv q_2$, i.e., conditions (51) lead to the smoothness of required solution $S(x, y)$ and the focal surface is absent.

8. Integral funnel

Suppose now condition (26) is false, i.e.,

$$\left[\begin{array}{l} 3ag + b > 0, \\ 3ag + b = 0, \quad fa > 0. \end{array}\right. \tag{53}$$

As it was noticed in Section 5, among regular trajectories with values $q_0 = \pm 1 + g$ in the vicinity of the origin, there is a domain not filled by regular trajectories. We shall try to cover this domain by regular trajectories (22) not satisfying (23). In Figure 3 we show an example of such successful filling of this domain by the regular characteristics with values $q_0 \in [-1 + g, +1 + g]$ for the case $a = 0$. Under the figure we provide the corresponding values of the parameters. For values $a = 0$, it is always possible to cover whole the space by the regular characteristics leaving from ordinate axis.

As $a \neq 0$, there can be situations, when between regular trajectories leaving from origin with values $q_0{=}{\pm}1{+}g$, there appear a space which can not be covered by the regular trajectories (22). An example of such situation is demonstrated in Figure 4. The parameters are chosen so

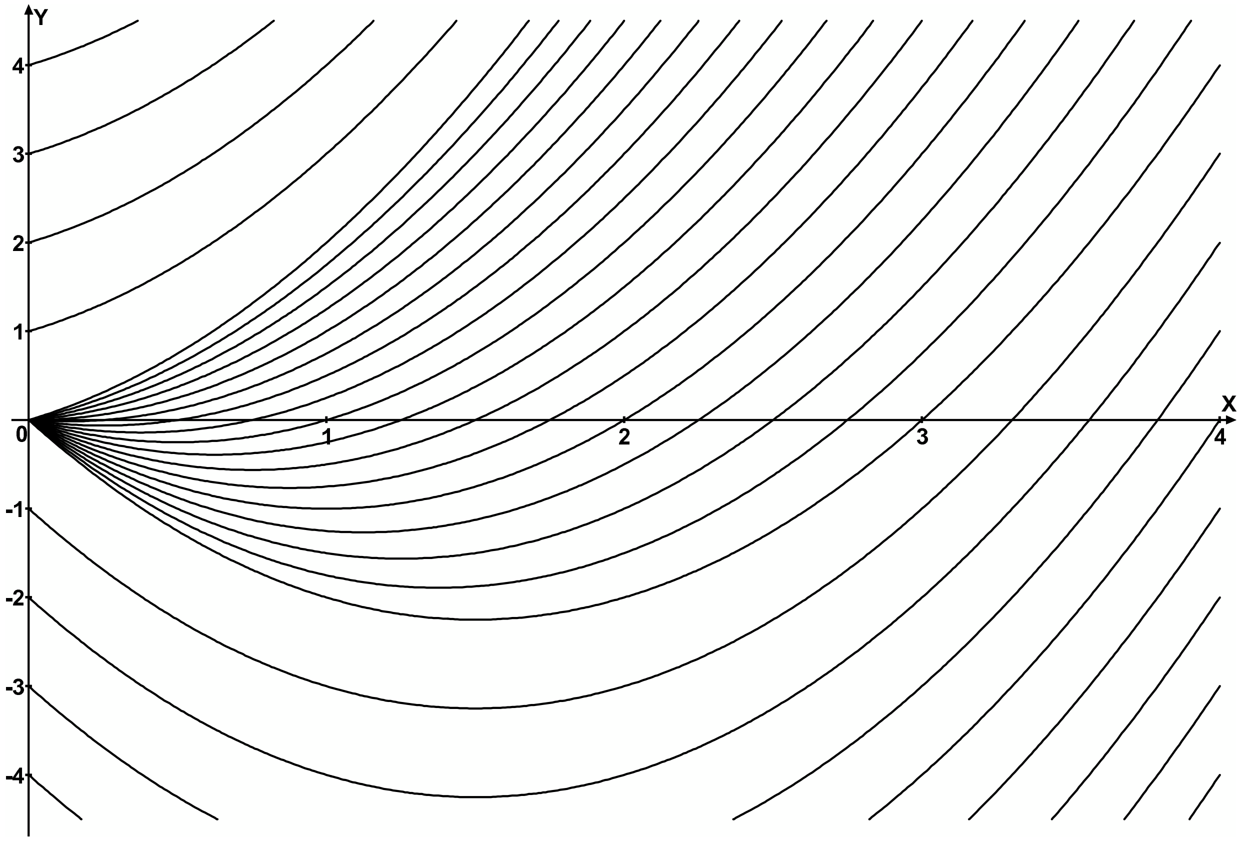

FIGURE 3. Integral funnel: $a = 0$, $b = 1$, $c = d = 0$, $f = 1$, $g = -0.5$

that the linear in x term in (25) vanishes, and the quadratic one is positive, i.e., $3ag + b = 0$, $fa > 0$. For these values of the parameters, the regular characteristics with values $q_0 = \pm 1 + g$ have a joint tangential at the origin and there are no other regular characteristics with the same tangential, since in (22), the linear in x term involves the qudratic in q_0 factor. It follows that in this case, by a choice of q_0, it is impossible to cover whole the domain $x > 0$ by regular characteristics leaving from origin. In Section 6, there was proven the absence of focal surface. This is why, for such values of the parameters, the approach employed in the present work does not give the solution.

It is obvious that if the coefficients at the linear and quadratic in x terms in (22) are increasing w.r.t. q_0 functions on the segment $[-1+g, 1+g]$, then the regular characteristics leaving from the origin with values $q_0 \in [-1+g, +1+g]$ cover the domain formed by the regular characteristics leaving from the origin with the values $q_0 = \pm 1 + g$. The next lemma describes the construction of the solution in this case.

Lemma 4. *Suppose the parameters of the problem satisfy the conditions*

$$\left[\begin{array}{lll} g > +1 - \dfrac{b}{3\,a} & \textit{as} & a > 0, \quad f > 0, \\[2ex] g < -1 - \dfrac{b}{3\,a} & \textit{as} & a < 0, \quad f < 0. \end{array}\right.$$

Then regular characteristics leaving from the ordinate axis cover uniquely whole the domain $x > 0$. The half-plane $x \geq 0$ is separated into three domains by the curves whose parts are the ordinate axis and two regular curves $y_{xak1}(x,0,0)$, $y_{xak2}(x,0,0)$. In two domains adjacent to the ordinate axis, respectively for in the upper and lower domains, the generalized solution to problem (15), (16) is determined by the formulae $S_1(x,y)$ and $S_2(x,y)$ in (27), and the solution to problem (13), (14) as $\rho_1 = g + 1$, $\rho_2 = g - 1$ given by the relations (34). In the

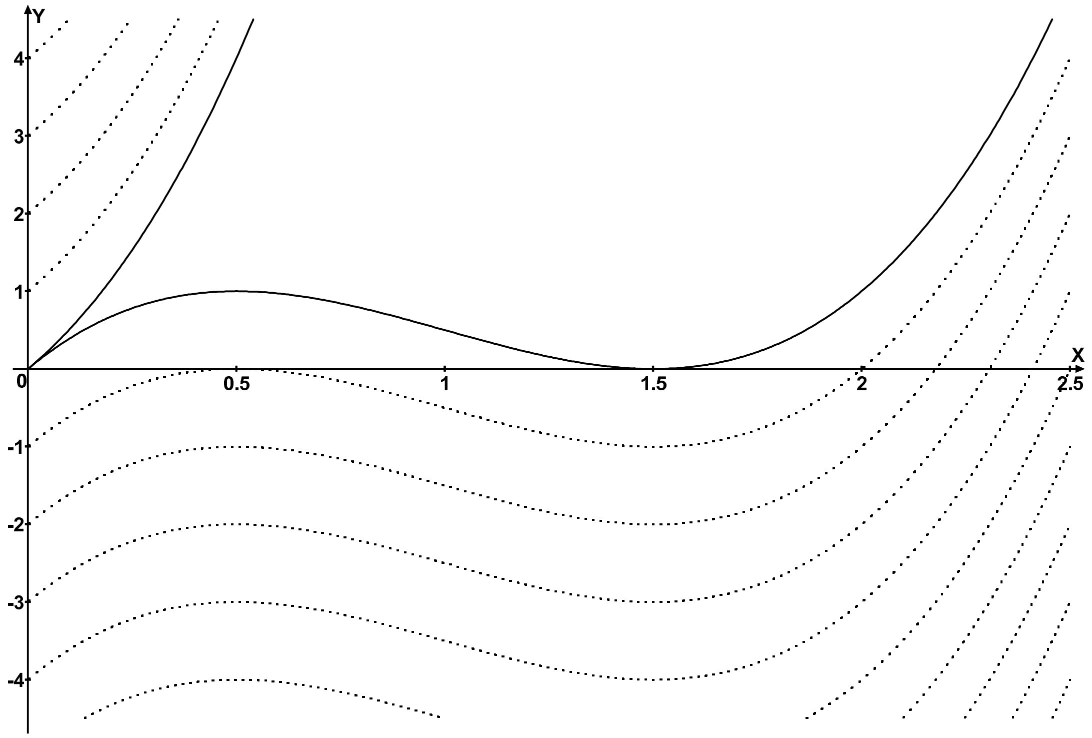

FIGURE 4. $a = 2, b = 3, c = 0, f = 1, g = -0.5$

third domain located between the curves $y_{xak1}(x, 0, 0)$, $y_{xak2}(x, 0, 0)$, *the generalized solutions to problems (15), (16) and (13), (14) are given by the formulae* $S_3(x, y)$, $v_3(x, y)$ *implicitly*

$$S_3(x,y)=\int\limits_0^x (p+q\,H_q)\,dx=\int\limits_0^x (f y_3(x,q_0)-\varphi(q_3(x,q_0))+q_3(x,q_0)\,\varphi_q(q_3(x,q_0)))\,dx,$$

$$v_3(x,y) = q_3(x,q_0),\quad q_3(x,q_0)=q_0+f\,x,$$

$$y=y_3(x,q_0)=a\,x^3 f^2+(3\,aq_0+b)x^2 f+(3\,aq_0{}^2+2\,bq_0+c)x,\quad q_0\in[g-1,g+1]$$

On Figure 5 we provide an example of constructing an integral funnel for the considered case. We note that in the case the hypothesis of Lemma 4 is false and conditions (36) hold true, behind the equivocal surface, the integral funnel (depression wave) follows the equivocal surface.

9. SUMMARY

The Cauchy problem for a known quasilinear first order equation with the right hand side independent of the unknown function and a discontinuous initial condition is reduced to Cauchy problem for Hamilton-Jacobi equation with a continuous initial condition. It is proposed to apply the method of singular characteristics to this problem; this method was developed by A.A. Melikyan for game and control problems. The efficiency of this approach is demonstrated for quasilinear problem (3.1), (3.2) in the case when the original function φ involved in the equation is a cubic polynomial w.r.t. the unknown function, while the initial condition is introduced as a “raising” step. We find the domains of the parameters, for which it is possible to construct the generalized solution to quasilinear problem (3.1), (3.2) and we write down a similar procedure for the constructing the solution. The appropriate formulae for constructing the solution to problem (3.1), (3.2) as well as for constructing the solution to the auxiliary initial Hamilton-Jacobi problem (3.3),(3.4) are provided in Lemmata 2-4.

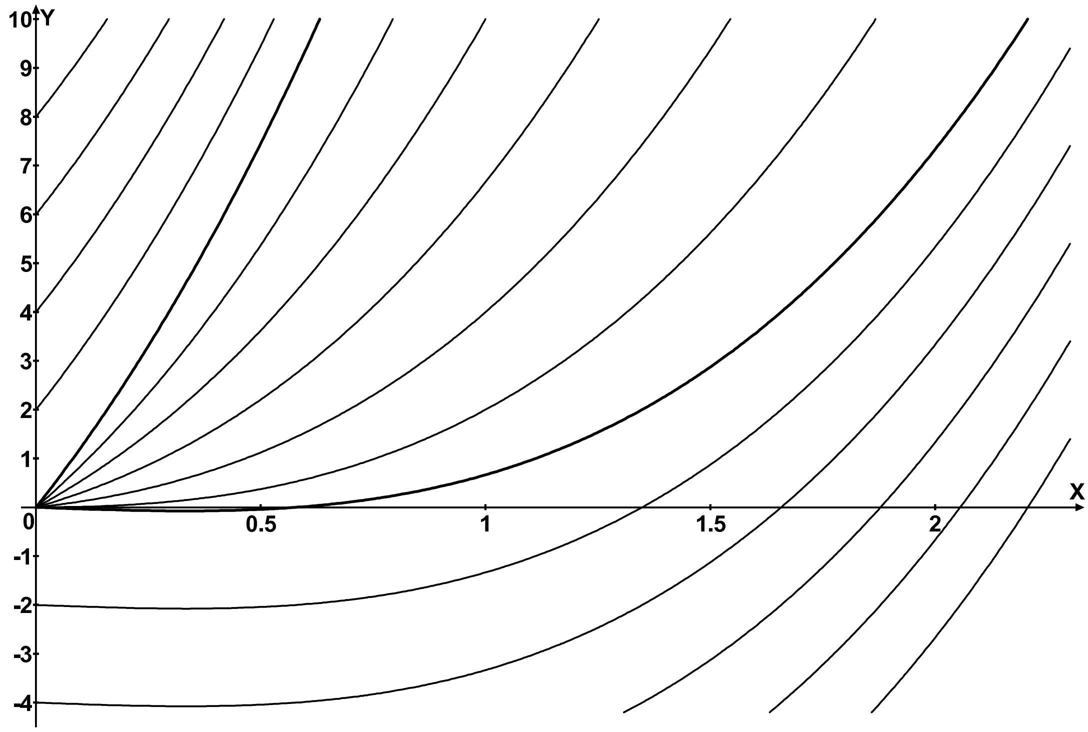

FIGURE 5. Integral funnel: $a = 1$, $b = 1$, $c = d = 0$, $f = 1$, $g = 2/3$

The described approach of constructing the generalized solution was employed in works [20], [21].

BIBLIOGRAPHY

1. B.L. Rozhdestvenskii, N.N. IAnenko. *Systems of quasilinear equations and their applications to gas dynamics.* Nauka, Moscow. 1968. [Amer. Math. Soc., Providence, R.I. 1983.]
2. G. Whitham. *Linear and nonlinear waves.* Wiley, New York. 1974.
3. .M. Il'in. *Matching of asymptotic expansions of solutions of boundary value problems.* Nauka, Moscow. 1989. [Amer. Math. Soc. Providence, R.I. 1992.]
4. J. Burgers *Application of a model system to illustrate some points of the statistical theory of free turbulence* // Nederl. Alcad. Wefensh. Proc. Acad. Sci.Amsterdam. 1940. V. 43. P. 3-12.
5. O.A. Oleinik. *Discontinuous solutions of non-linear differential equations* // Uspekhi matem. nauk. 1957. V. 12(75), No. 3. P. 3-73. (in Russian).
6. S.N. Kružkov. *First order quasilinear equations in several independent variables* // Matem. sbornik. 1970. V. 81, No. 2. P. 228-255. [Math. USSR Sb. 1970. V. 10, No. 2. P. 217-243.]
7. E.Yu. Panov. *On well-posedness classes of locally bounded generalized entropy solutions of the Cauchy problem for quasilinear first-order equations* // Fundamen. prikl. matem. 2006. V. 12, No. 5, P. 17-188. [J. Math. Sci. 2008. V. 150, No. 6. P. 2578-2587.]
8. N.A. Berkov, A.I. Martynenko, V.B. Minoscev, E.A. Pushkar', O.E. Shishanin, ed. V.B. Minoscev. *A course of higher mathematics. Textbook for technical colleges. Part 3.* Moscow State Industrial University, Moscow. 2007. (in Russian).
9. A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux *Handbook of First Order Partial Differential Equations.* Taylor & Francis, London. 2002.
10. A.Yu. Goristkii, S.N. Kružkov, G.A. Chechkin. *First order partial differential equations: tutorial.* Izd. Mosk. Univ., Moscow. 1999. (in Russian).
11. A.I. Subbotin. *Generalized solutions of first-order PDEs. The dynamical optimization perspective.* Moscow-Izhevsk: Institute for comptuter studies. 2003. [Systems & Control: Foundations & Appl. Birkhauser, Boston. 1995.]

12. P.L. Lions and P.E. Souganidis *Differential Games, Optimal Control and Directional Derivatives of Viscosity Solutions of Bellman's and Isaacs' Equations* // SIAM Journal of Control and Optimization. 1985. V. 23, No 4. P. 566-583.
13. A.A. Melikyan *Generalized characteristics of first order PDEs.* Applications in Optimal Control and Differential Games. Birkhauser, Boston. 1998.
14. M.G. Crandall, P.L. Lions. *Viscosity solutions of Hamilton-Jacobi equations* // Trans. Amer. Math. Soc. 1983. V. 277, No. 1. P. 1-42.
15. E.A. Kolpakova. *A generalized method of characteristics in the theory of Hamilton-Jacobi equations and conservation laws* // Tridy IMM UrO RAN. 2010. V. 16, No. 5. P. 95-102. (in Russian).
16. N.N. Kuznetsov, B.L. Rozhdestvenskii. *Construction of the generalized solution of the Cauchy problem for a quasi-linear equation* // Uspekhi matem. nauk. 1959. V. 14(86), No. 2. P. 211-215. (in Russian).
17. V.A. Korneev. *Construction of a generalized solution of an equation in divergence form by the methods of characteristics* // Differen. urav. 2007. V. 43. No. 12. P. 1664-1673. [Diff. Equats. V. 43, No. 12. P. 1705-1714.]
18. R. Courant. *Partial differential equations.* Moscow: Mir. 1964. [Interscience, New York. 1962.]
19. A.A. Melikyan. *Construction of weak discontinuities in optimal control problems and differential games* // Izv. AN SSSR. Tekhn. kibern. 1984. No. 1. P. 45-50. [Eng. Cybern. 1984. V. 22, No. 1. P. 129-133.]
20. V.A. Korneev, A.A. Melikyan. *Construction of the generalized solution to the two-dimensional Hamilton-Jacobi equation by the method of characteristics* // Izv. RAN. Teor. sistem uprav. 1995, No. 6. P. 168-177. (in Russian).
21. V.A. Korneev. *Numerical construction of a generalized solution to a two-dimensional Hamilton-Jacobi equation* // Izv. RAN. Teor. sistem uprav. 1998. No 1. P. 92-98. [J. Comp. Syst. Sci. Inter. 1998. V. 37, No. 1. P. 94-101.]

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