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ON SOLVABILITY OF HOMOGENEOUS RIEMANN-HILBERT PROBLEM WITH COUNTABLE SET OF COEFFICIENT DISCONTINUITIES AND TWO-SIDE CURLING AT INFINITY OF ORDER LESS THAN 1/2

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Abstract. We consider the homogeneous Riemann-Hilbert problem in the complex upper half-plane with a countable set of coefficients' discontinuities and two-side curling at infinity. In the case the problem index has a power singularity of order less than 1/2, we obtain general solution and completely study the solvability of the problem in a special functional class.

Keywords: Riemann-Hilbert problem, curling at infinity, infinite index, entire functions.

Mathematics Subject Classification: 30E25, 30E20, 30D10.

1. Introduction

In the theory of analytic functions, the Hilbert boundary value problem for the half-plane is the problem on finding a function $F(z)$ analytic in the upper half-plane D and satisfying the prescribed boundary condition

$$a(t)\mathrm{Re}\,F(t) - b(t)\mathrm{Im}\,F(t) = c(t), \tag{1}$$

$a(t)$, $b(t)$, $c(t)$ are given functions of the variable t on the real axis L, $a^2(t) + b^2(t) \neq 0$. The complete description of solvability for problem (1) was obtained in the class of bounded up to the boundary functions provided the coefficients and the right hand side in the boundary conditions belong to $H_L(\mu)$ (see, for instance, [1, p. 150], [2, p. 280]), in the class of the functions with power singularities at the discontinuity points of the coefficients as $a(t)$, $b(t)$, $c(t)$ have a finite number of jumping discontinuities and are Hölder continuous in the intervals between the discontinuity points (see, for instance, [3]), in the class of bounded in D functions when the problem has continuous as $t \in (-\infty, +\infty)$ coefficients and a two-sided curling at infinite of power order $\rho < 1/2$ (see [4], [5]). The latter problem with a two-sided curling at infinity of power order less than 1 was formulated by I.E. Sandrygailo [6]. He obtained preliminary results on its solvability in the class of bounded functions and in the class of bounded functions decaying exponentially at infinity under certain restrictions for the properties of the singularity. However, the method by N.I. Muskhelishvili employed in this work together with the ideas and results of [7] did not allowed the author to choose the class of solutions in which it would be possible to study the solvability completely.

The Hilbert problem for the half-plane with a countable set of discontinuity points for the coefficients was studied first in [8], [9], where the formulae for the general solution in used classes

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were given and the methods for constructing the examples of entire functions with prescribed properties were developed. In the present paper we suggest the complete description for the solvability of the Hilbert problem with a countable set of discontinuity points for the coefficients and a two-sided curling at infinity of order $\rho < 1/2$.

2. Formulation and solving of homogeneous problem in class of functions with singularities at discontinuity points

Let L be the real axis in the complex plane of the variable $z = x + iy$, D be the half-plane $\operatorname{Im} z > 0$, t be a point on L. We should find a function $F(z)$ analytic in the domain D and satisfying prescribed boundary condition (1), where $a(t)$, $b(t)$, $c(t)$ are real functions of t defined on L, continuous everywhere except the points of jumping discontinuities t_j, $j = \pm1, \pm2, \cdots$, and $a^2(t) + b^2(t) \neq 0$ at all the continuity points of the coefficients, $0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, $\lim\limits_{k\to\infty} t_k = \infty$, $0 > t_{-1} > \cdots > t_{-k} > t_{-k-1} > \cdots$, $\lim\limits_{k\to\infty} t_{-k} = -\infty$. We shall assume that the boundary condition is satisfied everywhere in L except the points t_k, t_{-k}, $k = \overline{1,\infty}$. Later we shall specify certain aspects of the problem formulation.

If $c(t) \equiv 0$, the problem is called homogeneous, and if $c(t) \not\equiv 0$, it is inhomogeneous. In the present work we are interested in the homogeneous Hilbert problem.

We rewrite boundary condition (1) with $c(t) \equiv 0$ as

$$\operatorname{Re}[e^{-i\nu(t)} F(t)] = 0, \tag{2}$$

where the branch $\nu(t) = \arg[a(t) - ib(t)]$ is chosen on each interval of continuity of the coefficients in such a way that the number $\delta_j = \nu(t_j + 0) - \nu(t_j - 0)$ obeys the condition $0 \leqslant \delta_j < 2\pi$, $j = \pm1, \pm2, \ldots$

We introduce the function $\varphi(t) = \nu(t) - \beta(t)\pi$, where $\beta(t)$ is an integer-valued function taking the values β_k, β_{-k} in the intervals (t_k, t_{k+1}) and (t_{-k}, t_{-k-1}), $k = \overline{1,\infty}$, respectively. For the interval (t_{-1}, t_1), we choose the value $\beta_0 = 0$. We also choose the numbers β_k so that

$$0 \leqslant \varphi(t_k + 0) - \varphi(t_k - 0) < \pi,$$

and the numbers β_{-k} are chosen to satisfy

$$0 \leqslant \varphi(t_{-k} + 0) - \varphi(t_{-k} - 0) < \pi.$$

We denote

$$\kappa_j = \frac{\varphi(t_j + 0) - \varphi(t_j - 0)}{\pi}, \quad j = \pm1, \pm2, \cdots,$$

hence, we have $0 \leqslant \kappa_k < 1$, $0 \leqslant \kappa_{-k} < 1$, $k = \overline{1,\infty}$.

We suppose that the discontinuity points satisfy the conditions

$$\sum_{k=1}^{\infty} \frac{1}{t_k} < \infty, \quad \sum_{k=1}^{\infty} \frac{1}{-t_{-k}} < \infty. \tag{3}$$

We consider the infinite products

$$P_+(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{t_k}\right)^{\kappa_k}, \quad P_-(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{t_{-k}}\right)^{\kappa_{-k}}, \tag{4}$$

where by $\arg(1 - z/t_j)$ we mean the single-valued branch vanishing as $z = 0$ and continuous in the z-plane cut along a part of the real axis connecting the points $t = t_j$, $t = +\infty$ as $j > 0$ and the points $t=-\infty$, $t=t_j$ as $j<0$. Thus, for the upper sides of the cuts we have

$$\begin{aligned} \arg P_+(t) &= -\sum_{j=1}^{k}\kappa_j\pi, \quad t_k<t<t_{k+1}, \\ \arg P_-(t) &= \sum_{j=1}^{k}\kappa_{-j}\pi, \quad t_{-k-1}<t<t_{-k}, \end{aligned} \qquad k=\overline{1,\infty}, \tag{5}$$

$$\arg P_+(t)=0, \quad t<t_1, \qquad \arg P_-(t)=0, \quad t>t_{-1}. \tag{6}$$

We write boundary condition (2) as

$$\operatorname{Re}\,[e^{-i\varphi_1(t)}F(t)P_+(t)P_-(t)]=0, \tag{7}$$

where

$$\varphi_1(t)=\varphi(t)+\arg P_+(t)+\arg P_-(t), \tag{8}$$

the functions $\arg P_+(t)$, $\arg P_-(t)$ are determined by the formulae (5), (6).

Introducing a new unknown function $F_1(z)=F(z)P_+(z)P_-(z)$, we write condition (7) as

$$\operatorname{Re}\,[e^{-i\varphi_1(t)}F_1(t)]=0,$$

and due to formulae (8), (5), the function $\varphi_1(t)$ is continuous at all finite points of L. Thus, we have arrived to the Hilbert boundary value problem for the function $F_1(z)$. We shall assume that the function $\varphi_1(t)$ satisfies the condition

$$\varphi_1(t)=\begin{cases} \nu^- t^\rho+\tilde{\nu}(t), & t>0, \\ \nu^+|t|^\rho+\tilde{\nu}(t), & t<0, \end{cases} \tag{9}$$

$(\nu^-)^2+(\nu^+)^2\neq 0$, where ν^-, ν^+, ρ are constants, $0<\rho<1/2$, $\tilde{\nu}(t)$ is a function continuous on L including the point at infinity and satisfying the Hölder condition with the exponent μ, $0<\mu\leqslant 1$ on L including the point at infinity ([1, pp. 18, 67]) that is the condition $H_L(\mu)$ (see [7, p. 113]).

We shall seek a solution $F(z)$ of the homogeneous problem in the class of functions such that the product

$$F(z)P_+(z)P_-(z)=F_1(z)$$

is a bounded in D function. Hence, the problem is reduced to one considered in [5]. Once the function $F_1(z)$ is found, by formula

$$F(z)=F_1(z)/P_+(z)P_-(z)$$

we determine the solution to the homogeneous boundary value problem associated with problem (2). At a point t_j, this solution, generally speaking, becomes the infinity of order κ_j.

Under additional restrictions, we shall study the behavior of this solution to the homogeneous boundary value problem on the ray $z=re^{i\theta}$, $r>0$, $\theta=\text{const}$, $0<\theta<\pi$, $r\to\infty$. As in [9], (see also [8, p. 112]), we introduce the functions

$$n_-^*(x)=\begin{cases} 0, & 0\leqslant x<-t_{-1}, \\ \sum\limits_{j=1}^{k-1}\kappa_{-j}, & -t_{-k+1}\leqslant x<-t_{-k}, \end{cases} \qquad n_+^*(x)=\begin{cases} 0, & 0<x<t_1, \\ \sum\limits_{j=1}^{k-1}\kappa_j, & t_{k-1}\leqslant x<t_k, \end{cases}$$

and assume the conditions

$$\lim_{x\to+\infty}\frac{n_+^*(x)}{x^{\kappa_+}}=\Delta_+, \qquad \lim_{x\to+\infty}\frac{n_-^*(x)}{x^{\kappa_-}}=\Delta_-, \tag{10}$$

with positive constants Δ_+, Δ_- and $0 < \kappa_- < 1/2$, $0 < \kappa_+ < 1/2$. In [9], [8, p. 112], the structure formulae

$$\ln P_+(z) = \frac{\pi\Delta_+ e^{-i\kappa_+\pi}}{\sin\pi\kappa_+} z^{\kappa_+} + I_+(z),\ I_+(z) = -z\int\limits_0^{+\infty} \frac{n_+^*(\tau) - \Delta_+\tau^{\kappa_+}}{\tau(\tau - z)} d\tau \tag{11}$$

were obtained for $0 < \arg z < 2\pi$ and

$$\ln P_-(z) = \frac{\pi\Delta_-}{\sin\pi\kappa_-} z^{\kappa_-} + I_-(z), \quad I_-(z) = z\int\limits_0^{+\infty} \frac{n_-^*(\tau) - \Delta_-\tau^{\kappa_-}}{\tau(\tau + z)} d\tau \tag{12}$$

if $-\pi < \arg z < \pi$. Expressing the limiting values of Cauchy type integrals by the Sokhotski formulae and extracting the real parts from (11), (12), we obtain the identities

$$\begin{gathered} \ln|P_+(t)| = \frac{\pi\Delta_+\cos(\kappa_+\pi)}{\sin(\kappa_+\pi)} t^{\kappa^+} + I_+(t),\ t > 0, \\ I_+(t) = -\int\limits_0^{+\infty} \frac{n_+^*(\tau) - \Delta_+\tau^{\kappa_+}}{\tau(\tau - t)} d\tau,\ t > 0,\ t \neq t_k,\ k = \overline{1,\infty}, \end{gathered} \tag{13}$$

$$\begin{gathered} \ln|P_-(-t)| = \frac{\pi\Delta_-\cos(\kappa_-\pi)}{\sin(\kappa_-\pi)} t^{\kappa^-} + I_-(-t), \\ I_-(-t) = -\int\limits_0^{+\infty} \frac{n_-^*(\tau) - \Delta_-\tau^{\kappa_-}}{\tau(\tau - t)} d\tau,\ t > 0,\ t \neq -t_{-k},\ k = \overline{1,\infty}. \end{gathered} \tag{14}$$

It follows from formulae (4), (13), (14) that the function $|\exp\{I_+(z)\}|$ ($|\exp\{I_-(z)\}|$) has a zero of order κ_k (of order κ_{-k}) at the point t_k (t_{-k}), $k = \overline{1,\infty}$.

Lemma 1. *Suppose (10) and let δ be a given small positive number and $z = re^{i\theta}$. Then the asymptotic estimates*

$$I_+(re^{i\theta}) = \mathrm{o}(\mathrm{r}^{\kappa_+}),\ \mathrm{r} \to +\infty,\ \delta < \theta < 2\pi - \delta,$$
$$I_-(re^{i\theta}) = \mathrm{o}(\mathrm{r}^{\kappa_-}),\ \mathrm{r} \to +\infty,\ -\pi + \delta < \theta < \pi + \delta$$

hold true.

Proof. Let δ be a sufficiently small positive number, then for each θ, $\delta < \theta < 2\pi - \delta$ the inequality $|\tau - z| \geqslant (\tau + r)\sin(\delta/2)$ is obvious. Given an arbitrary small positive number ε and $\tau > r_1(\varepsilon)$, by (10) we have $n_+^*(\tau) - \Delta_+\tau^{\kappa_+} < \varepsilon\tau^{\kappa_+}$ and thus

$$|I_+(re^{i\theta})| < \frac{r^{\kappa_+}}{\sin(\delta/2)} \left(r^{1-\kappa_+} \int\limits_{t_1}^{r_1(\varepsilon)} \frac{n_+^*(\tau)}{\tau(\tau + r)} d\tau + \right.$$
$$\left. + \Delta_+ r^{1-\kappa_+} \int\limits_0^{r_1(\varepsilon)} \frac{d\tau}{\tau^{1-\kappa_+}(\tau + r)} + \varepsilon r^{1-\kappa_+} \int\limits_{r_1(\varepsilon)}^{\infty} \frac{d\tau}{\tau^{1-\kappa_+}(\tau + r)} \right),$$

for $z = re^{i\theta}$, $\delta < \theta < 2\pi - \delta$. For a given ε, the first two integrals are of order $\mathrm{O}(\frac{1}{r})$, while for the latter we have

$$r^{1-\kappa_+} \int\limits_{r_1(\varepsilon)}^{\infty} \frac{1}{\tau^{1-\kappa_+}(\tau + r)} d\tau < r^{1-\kappa_+} \int\limits_0^{\infty} \frac{1}{\tau^{1-\kappa_+}(\tau + r)} d\tau = \frac{\pi}{\sin(\pi\kappa_+)}.$$

Hence (cf. [10, p. 178], [8, p. 115]),

$$I_+(re^{i\theta}) = \mathrm{o}(\mathrm{r}^{\kappa_+}), \quad \mathrm{r} \to +\infty,\ \delta < \theta < 2\pi - \delta.$$

The estimate for $I_-(re^{i\theta})$ is not more complicated. □

It follows from Lemma 1 and the restrictions $\kappa_+ < 1/2$, $\kappa_- < 1/2$ that as $\delta < \theta < \pi - \delta$, the solution $F(re^{i\theta})$ to homogeneous problem (7) satisfies the condition $F(re^{i\theta}) \to 0$, $r \to \infty$.

As in [5], we introduce the function

$$P(z) + iQ(z) = le^{i\alpha} r^\rho e^{i\theta\rho}, \tag{15}$$

where l, α are real constants, $r = |z|$, $\theta = \arg z$ is a single-valued continuous in D branch of $\arg z$ satisfying the relation $0 \leqslant \theta \leqslant \pi$. This function is analytic in the domain D, and on the boundary L, it reads as

$$P(t) + iQ(t) = l|t|^\rho[\cos(\alpha + \theta\rho) + i\sin(\alpha + \theta\rho)],$$

where $\theta = 0$ as $t > 0$ and $\theta = \pi$ as $t < 0$.

We choose the numbers l, $l > 0$, α, $0 \leqslant \alpha < 2\pi$ so that

$$l\cos\alpha = \nu^-, \quad l\cos(\alpha + \pi\rho) = \nu^+,$$

i.e., in order to get

$$l\cos\alpha = \nu^-, \quad l\sin\alpha = \frac{\nu^-\cos(\pi\rho) - \nu^+}{\sin(\pi\rho)}.$$

We observe that at the same time

$$l\sin(\alpha + \pi\rho) = \frac{\nu^- - \nu^+\cos(\pi\rho)}{\sin(\pi\rho)}, \quad l = \frac{[(\nu^-)^2 + (\nu^+)^2 - 2\nu^-\nu^+\cos(\pi\rho)]^{1/2}}{\sin(\pi\rho)},$$

$$P(t) = \begin{cases} \nu^- t^\rho, & t > 0, \\ \nu^+|t|^\rho, & t < 0, \end{cases}$$

$$Q(t) = \begin{cases} (\nu^-\cos(\pi\rho) - \nu^+)t^\rho/\sin(\pi\rho), & t > 0, \\ (\nu^- - \nu^+\cos(\pi\rho))|t|^\rho/\sin(\pi\rho), & t < 0, \end{cases} \tag{16}$$

and

$$Q(re^{i\theta}) = r^\rho \frac{\nu^-\cos(\rho(\pi - \theta)) - \nu^+\cos(\rho\theta)}{\sin(\pi\rho)}.$$

Then by the formula

$$\Gamma(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \tilde{\nu}(t)\frac{dt}{t - z}$$

we define an analytic bounded in D function such that on the boundary its imaginary part satisfies the identity $\varphi_1(t) - P(t) = \tilde{\nu}(t)$.

On the contour L this function takes the values $\Gamma^+(t) = \Gamma(t) + i\tilde{\nu}(t)$, where

$$\Gamma(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \tilde{\nu}(t_1)\frac{dt_1}{t_1 - t}.$$

We write boundary condition (7) as

$$\operatorname{Im}\left\{ie^{-\Gamma^+(t)}e^{-iP(t)+Q(t)}F(t)P_+(t)P_-(t)\right\} = 0, \tag{17}$$

transforming, under additional restrictions (3), (9), (10), the boundary conditions of the homogeneous problem to the form of problem (17).

In the brackets in formula (17), there is a boundary value of the analytic in D function

$$\Phi(z) = ie^{-\Gamma^+(z)}e^{-iP(z)+Q(z)}F(z)P_+(z)P_-(z), \tag{18}$$

which in view of (12) can be written as follows,

$$\begin{gathered}\Phi(z) = ie^{-\Gamma^+(z)}\exp\{Q(z)-iP(z)\}F(z)e^{I_+(z)}e^{+I_-(z)}\times \\ \times\exp\left\{\frac{\pi\Delta_+e^{-i\kappa_+\pi}}{\sin\pi\kappa_+}z^{\kappa_+}\right\}\exp\left\{\frac{\pi\Delta_-}{\sin\pi\kappa_-}z^{\kappa_-}\right\}.\end{gathered} \tag{19}$$

According to (17), for the boundary values of the function $\Phi(z)$ we have

$$\operatorname{Im}\Phi_+(t) = 0 \tag{20}$$

everywhere on L. It means that the function $\Phi(z)$ can be analytically continued in the half-plane $\operatorname{Im} z < 0$ and for each point z in this half-plane

$$\Phi(z) = \overline{\Phi(\overline{z})}, \quad \operatorname{Im} z < 0. \tag{21}$$

Let $\tilde{B}$ be the class of solutions $F(z)$ to homogeneous problem (17), for which the product $|F(z)||z-t_j|^{\kappa_j}$ is bounded in a vicinity of the point t_j for each $j = \pm1, \pm2, \ldots$

For such solutions, we have to regard the condition $\operatorname{Im}\Phi_+(t) = 0$ as satisfied also at the points t_k, t_{-k}, $k = \overline{1,+\infty}$. Indeed, continuing the function $\Phi(z)$ into the half-plane $\operatorname{Im} z < 0$ through the segments $t_{k-1}t_k$ and t_kt_{k+1} as it was described above, we obtain the same function due to (21). For the class $\tilde{B}$ of solutions $\Phi(z)$ to problem (17), the quantity $|\Phi(z)|$ is bounded in a vicinity of t_k and this is why the point t_k is a removable singularity of the function $\Phi(z)$ obtained by the analytic continuation. Therefore, we can assume $\operatorname{Im}\Phi^+(t_k) = 0$. Thus, as a result of the aforementioned analytic continuation, we obtain the entire function $\Phi(z)$.

The above arguments lead us to Theorem 1.

Theorem 1. *Homogeneous boundary value problem* (17) *has a solution $F(z)$ in the class $\tilde{B}$ if and only if this function satisfies formula* (18)*, where $\Phi(z)$ is an arbitrary entire function with real values on L.*

In what follows we shall seek a solution to homogeneous problem (17) in the class B_* of the functions $F(z)$ for which the product $|F(z)|e^{\operatorname{Re} I_+(z)}e^{\operatorname{Re} I_-(z)}$ is a bounded analytic in D function. It is clear that $B_* \subset \tilde{B}$.

In view of the symmetry of formula (18) for the aforementioned entire function $\Phi(z)$ (as (20), (21) hold true),

$$M(r) := \max_{0\leqslant\theta\leqslant2\pi}|\Phi(re^{i\theta})| = \max_{0\leqslant\theta\leqslant\pi}|\Phi(re^{i\theta})|. \tag{22}$$

According to (19),

$$\begin{gathered}|\Phi(re^{i\theta})| = \exp\{Q(re^{i\theta}) - \operatorname{Re}\Gamma(re^{i\theta}) + \operatorname{Re} I_+(re^{i\theta}) + \operatorname{Re} I_-(re^{i\theta})\}|F(re^{i\theta})|\times \\ \times\exp\left\{\frac{\pi\Delta_+r^{\kappa_+}}{\sin(\pi\kappa_+)}\cos(\kappa_+(\theta-\pi)) + \frac{\pi\Delta_-r^{\kappa_-}}{\sin(\pi\kappa_-)}\cos(\kappa_-\theta)\right\}, \quad 0\leqslant\theta\leqslant\pi.\end{gathered} \tag{23}$$

Since $\operatorname{Re}\Gamma(z)$, $|F(z)|e^{\operatorname{Re}(I_+(z)+I_-(z))}$ are bounded in D functions, we have

$$\max_{0\leqslant\theta\leqslant\pi}\left\{\exp\{-\operatorname{Re}\Gamma(re^{i\theta})\}|F(re^{i\theta})|\exp\{\operatorname{Re}(I_+(re^{i\theta}) + I_-(re^{i\theta}))\}\right\} \leqslant C, \tag{24}$$

$C = \text{const} > 0$, and moreover, in accordance with (15) $Q(re^{i\theta}) \leqslant lr^\rho$. Hence, by (22), (23) we get

$$M(r) \leqslant C\exp\left\{lr^\rho + \frac{\pi\Delta_+}{\sin(\pi\kappa_+)}r^{\kappa_+} + \frac{\pi\Delta_-}{\sin(\pi\kappa_-)}r^{\kappa_-}\right\},$$

$$\ln M(r) \leqslant \ln C + lr^\rho + \frac{\pi\Delta_+}{\sin(\pi\kappa_+)}r^{\kappa_+} + \frac{\pi\Delta_-}{\sin(\pi\kappa_-)}r^{\kappa_-}. \tag{25}$$

We shall distinguish the cases $\rho > \max\{\kappa_+, \kappa_-\}$, $\rho = \max\{\kappa_+, \kappa_-\}$, and $\rho < \max\{\kappa_+, \kappa_-\}$.

Let $\rho \geqslant \max\{\kappa_+, \kappa_-\}$. Then by formula (25) we obtain

$$\ln\ln M(r) \leqslant \rho \ln r + \ln\left[l + \frac{\ln C}{r^\rho} + \frac{\pi\Delta_+}{r^{\rho-\kappa_+}\sin(\pi\kappa_+)} + \frac{\pi\Delta_-}{r^{\rho-\kappa_-}\sin(\pi\kappa_-)}\right],$$

$$\overline{\lim_{r\to\infty}} \frac{\ln\ln M(r)}{\ln r} \leqslant$$

$$\overline{\lim_{r\to\infty}} \left\{\rho + \frac{1}{\ln r}\ln\left[l + \frac{\ln C}{r^\rho} + \frac{\pi\Delta_+}{r^{\rho-\kappa_+}\sin(\pi\kappa_+)} + \frac{\pi\Delta_-}{r^{\rho-\kappa_-}\sin(\pi\kappa_-)}\right]\right\} = \rho.$$

Therefore, the order $\rho_\Phi = \overline{\lim\limits_{r\to\infty}}[\ln\ln M(r)/\ln r]$ of the entire function $\Phi(z)$ determined by formulae (18), (20), (21) does not exceed ρ.

Theorem 2. *Boundary value problem* (17) *has a solution $F(z)$ in the class B_* as $\rho \geqslant \max\{\kappa_+, \kappa_-\}$ if and only if the function $F(z)$ satisfies formula* (18), *where $\Phi(z)$ is an arbitrary entire function of order $\rho_\Phi \leqslant \rho$ satisfying condition* (20) *and, on the contour L for all sufficiently large t, the inequalities*

$$|\Phi(t)| \leqslant C\exp\left\{\frac{\nu^-\cos(\pi\rho) - \nu^+}{\sin(\pi\rho)}t^\rho + \frac{\pi\Delta_+\cos(\kappa_+\pi)}{\sin(\pi\kappa_+)}t^{\kappa_+} + \frac{\pi\Delta_-}{\sin(\pi\kappa_-)}t^{\kappa_-}\right\} \tag{26}$$

if $t > 0$ and

$$|\Phi(t)| \leqslant C\exp\left\{\frac{\nu^- - \nu^+\cos(\pi\rho)}{\sin(\pi\rho)}|t|^\rho + \frac{\pi\Delta_+|t|^{\kappa_+}}{\sin(\pi\kappa_+)} + \frac{\pi\Delta_-\cos(\kappa_-\pi)}{\sin(\pi\kappa_-)}|t|^{\kappa_-}\right\} \tag{27}$$

if $t < 0$. Here $C = \mathrm{const} > 0$.

Proof. The necessity. Suppose $F(z)$ is a solution to boundary value problem (17) in the class B_*. As it was shown above, then relations (18), (20) hold true, where $\Phi(z)$ with condition (21) is an entire function of order $\rho_\Phi \leqslant \rho$. Letting $\theta = 0$ in (23) and then $\theta = \pi$, in view of (24) we obtain respectively inequalities (26), (27).

The sufficiency. Suppose a function $F(z)$ satisfies formula (18) and $\Phi(z)$ is an entire function of order $\rho_\Phi \leqslant \rho$ obeying condition (20) and inequalities (26), (27); then the function $F(z)$ determined by formula (18) solves problem (17). In view of the latter inequalities and formulae (23), (16) for the analytic in D function $F(z)e^{I_+(z)+I_-(z)}$ as t is large enough, and in view of the fact that $\mathrm{Re}\,\Gamma(z)$ is a bounded in D function, i.e.,

$$|\mathrm{Re}\,\Gamma(re^{i\theta})| \leqslant q, \quad q = \mathrm{const}, \quad r \geqslant 0, \quad 0 \leqslant \theta \leqslant \pi, \tag{28}$$

we get

$$|F(t)e^{I_+(t)+I_-(t)}| \leqslant Ce^q, \quad \text{as} \quad t > 0 \quad \text{and} \quad t < 0.$$

Therefore, the inequality

$$|F(t)e^{I_+(t)+I_-(t)}| \leqslant \tilde{C}, \quad \tilde{C} = \mathrm{const} > 0$$

holds true everywhere in L.

Due to (23) and (15), we have

$$|F(re^{i\theta})\exp\{I_+(re^{i\theta}) + I_-(re^{i\theta})\}| = |\Phi(re^{i\theta})|\exp\left\{-lr^\rho\sin(\alpha+\rho\theta) + \mathrm{Re}\,\Gamma(re^{i\theta})\right\} \times$$

$$\times\exp\left\{-\frac{\pi\Delta_+r^{\kappa_+}}{\sin(\pi\kappa_+)}\cos(\kappa_+(\theta-\pi)) - \frac{\pi\Delta_-r^{\kappa_-}}{\sin(\pi\kappa_-)}\cos(\kappa_-\theta)\right\}, \quad 0 \leqslant \theta \leqslant \pi.$$

By (22) and (28) it implies

$$\max_{0\leqslant\theta\leqslant\pi}|F(re^{i\theta})\exp\{I_+(re^{i\theta}) + I_-(re^{i\theta})\}| \leqslant M(r)e^{lr^\rho+q}\exp\left\{\frac{\pi\Delta_+r^{\kappa_+}}{\sin(\pi\kappa_+)} + \frac{\pi\Delta_-r^{\kappa_-}}{\sin(\pi\kappa_-)}\right\}.$$

Since for each $\varepsilon > 0$ the inequality

$$M(r) < \exp\{r^{\rho_\Phi+\varepsilon}\}$$

holds true for all $r > r_\varepsilon$, choosing the numbers ε, ρ_1 so that $\rho < \rho_1 < 1$, $\rho_\Phi + \varepsilon < \rho_1$, by the former relation we get

$$\max_{0\leqslant\theta\leqslant\pi} |F(re^{i\theta}) \exp\{I_+(re^{i\theta}) + I_-(re^{i\theta})\}| < \exp\{r^{\rho_1}\}$$

for all sufficiently large r obeying

$$r > r_\varepsilon, \quad \frac{r^{\rho_\Phi+\varepsilon}}{r^{\rho_1}} + \frac{lr^\rho}{r^{\rho_1}} + \frac{q}{r^{\rho_1}} + \frac{\pi\Delta_+ r^{\kappa_+}}{r^{\rho_1}\sin(\pi\kappa_+)} + \frac{\pi\Delta_- r^{\kappa_-}}{r^{\rho_1}\sin(\pi\kappa_-)} < 1.$$

Therefore, the order of the function $F(z)e^{I_+(z)+I_-(z)}$ inside the angle $0 \leqslant \theta \leqslant \pi$ does not exceed ρ_1 (see, for instance, [11, p. 69]). Thus, according to the Phragmén-Lindelöf principle, we have $|F(z)e^{I_+(z)+I_-(z)}| < \tilde{C}$ everywhere in D, i.e., $F(z)$ belongs to the class B_*. □

Theorem 3. *The general solution in the class B_* to boundary value problem* (17) *as $\rho \geqslant \max\{\kappa_+, \kappa_-\}$ is given by the formula*

$$F(z) = -ie^{\Gamma(z)}e^{i[P(z)+iQ(z)]}\Phi(z)[P_+(z)P_-(z)]^{-1} \tag{29}$$

or

$$F(z) = -ie^{-I_+(z)-I_-(z)}e^{\Gamma(z)}e^{i[P(z)+iQ(z)]}\Phi(z)\exp\left\{-\frac{\pi\Delta_+e^{-i\pi\kappa_+}z^{\kappa_+}}{\sin(\pi\kappa_+)} + \frac{\pi\Delta_- z^{\kappa_-}}{\sin(\pi\kappa_-)}\right\},$$

where $\Phi(z)$ is an arbitrary entire function of order $\rho_\Phi \leqslant \rho$ satisfying condition (20) *and inequalities* (26), (27) *for sufficiently large $|t|$.*

Proof. Indeed, the function $F(z)$ determined by formula (29) satisfies relation (18) that implies the statement of the theorem. □

3. Solvability of homogeneous Hilbert problem

In this section we give the complete description for the solvability of the homogeneous Hilbert problem in the class B_*.

Theorem 4. *Let $\rho > \max\{\kappa_+, \kappa_-\}$, $\rho < 1/2$.*

a) If

$$\nu^-\cos(\pi\rho) - \nu^+ < 0 \quad or \quad \nu^- - \nu^+\cos(\pi\rho) < 0,$$

homogeneous boundary value problem (17) *has no nontrivial solutions in the class B_*.*

b) If

$$\left\{\begin{array}{l} \nu^-\cos(\pi\rho) - \nu^+ = 0, \\ \nu^- - \nu^+\cos(\pi\rho) > 0 \end{array}\right. \quad or \quad \left\{\begin{array}{l} \nu^-\cos(\pi\rho) - \nu^+ > 0, \\ \nu^- - \nu^+\cos(\pi\rho) = 0, \end{array}\right. \tag{30}$$

homogeneous boundary value problem (17) *has the solution in the class B_* determined by formula* (29) *where $\Phi(z)$ is arbitrary function of order ρ_Φ, $\rho_\Phi \leqslant \rho$ satisfying condition* (20) *and the inequality*

$$|\Phi(t)| \leqslant \left\{\begin{array}{ll} C\exp\left\{\frac{\pi\Delta_+\cos(\kappa_+\pi)}{\sin(\pi\kappa_+)}t^{\kappa_+} + \frac{\pi\Delta_-}{\sin(\pi\kappa_-)}t^{\kappa_-}\right\}, & t > 0, \\ C\exp\left\{\frac{\nu^- - \nu^+\cos(\pi\rho)}{\sin(\pi\rho)}|t|^\rho + \frac{\pi\Delta_+|t|^{\kappa_+}}{\sin(\pi\kappa_+)} + \frac{\pi\Delta_-\cos(\kappa_-\pi)}{\sin(\pi\kappa_-)}|t|^{\kappa_-}\right\}, & t < 0, \end{array}\right. \tag{31}$$

or, respectively,

$$|\Phi(t)| \leqslant \begin{cases} C\exp\left\{\dfrac{\nu^-\cos(\pi\rho)-\nu^+}{\sin(\pi\rho)}t^\rho + \dfrac{\pi\Delta_+\cos(\kappa_+\pi)}{\sin(\pi\kappa_+)}t^{\kappa_+} + \dfrac{\pi\Delta_- t^{\kappa_-}}{\sin(\pi\kappa_-)}\right\}, t>0, \\ C\exp\left\{\dfrac{\pi\Delta_+}{\sin(\pi\kappa_+)}|t|^{\kappa_+} + \dfrac{\pi\Delta_-\cos(\kappa_-\pi)}{\sin(\pi\kappa_-)}|t|^{\kappa_-}\right\}, \quad t<0, \end{cases}$$

for sufficiently large $|t|$.

c) If

$$\begin{cases} \nu^-\cos(\pi\rho)-\nu^+>0, \\ \nu^- - \nu^+\cos(\pi\rho)>0, \end{cases}$$

homogeneous boundary value problem (17) *has the solution in the class B_* determined by formula* (29) *where $\Phi(z)$ is arbitrary function of order ρ_Φ, $\rho_\Phi \leqslant \rho$ satisfying condition* (20) *and also inequalities* (26), (27) *as $\rho_\Phi = \rho$.*

Proof. a) Let $\rho < 1/2$ and the inequality

$$\nu^-\cos(\pi\rho)-\nu^+<0$$

holds true. Then by (26), $\lim_{t\to+\infty}|\Phi(t)| = 0$, and due to the Phragmén-Lindelöf principle applied for the plane cut along the positive semi-axis we obtain $\Phi(z)\equiv 0$. It is clear that we get the same in the case $\rho<1/2$ and $\nu^- - \nu^+\cos(\pi\rho) < 0$.

b) Suppose the first condition in (30). For the definiteness we assume $\kappa_+ \geqslant \kappa_-$. According to Theorem 3 and by (20), the entire function $\Phi(z)$ involved in formula (29) for the general solution should be real on the real axis, should satisfy conditions (26), (27) which by the identity $\nu^-\cos(\pi\rho)-\nu^+=0$ become inequality (31), and should have the order $\rho_\Phi\leqslant\rho<1/2$. The existence of such entire functions follows from the constructions made in all the details in [5], see also [8, p. 100]. Namely, we take the entire function

$$\Phi_0(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{r_k e^{i\theta_0}}\right)\left(1-\frac{z}{r_k e^{-i\theta_0}}\right), \tag{32}$$

where $0\leqslant\theta_0\leqslant\pi$, r_k is a increasing sequence of numbers that we define later. Here $\arg(1-z/r_k e^{i\theta_0})$ indicates the branch being continuous and single-valued in the plane cut along the ray $z=re^{i\theta_0}$, $r>r_k$ and vanishing as $z=0$; $\arg(1-z/r_k e^{-i\theta_0})$ indicates the branch being continuous and single-valued in the plane cut along the ray $z=re^{i\theta_0}$, $r>r_k$ and vanishing as $z=0$.

We suppose that the order of the function $\Phi_0(z)$ is equal to κ_0 (i.e., the exponent of the convergence for the sequence of its zeroes is equal to κ_0, see [12, p. 278]). We assume in addition that for the number of zeroes $n(r)$ of this entire function lying in a closed disk $|z|\leqslant r$, there exists the limit

$$\lim_{r\to\infty}\frac{n(r)}{r^{\kappa_0}} = \Delta_0, \quad 0<\Delta_0<\infty. \tag{33}$$

Under the above assumptions for the sequence of zeroes of entire function (32), in [5], the formula

$$\ln\Phi_0(z) = -ze^{-i\theta_0}\int_0^\infty \frac{n(\tau)-\tau^{\kappa_0}\Delta_0}{\tau(\tau-ze^{-i\theta_0})}d\tau - ze^{i\theta_0}\int_0^\infty\frac{n(\tau)-\tau^{\kappa_0}\Delta_0}{\tau(\tau-ze^{i\theta_0})}d\tau +$$

$$+\begin{cases} \Delta_0\pi(re^{i\theta})^{\kappa_0}2\cos((\theta_0-\pi)\kappa_0)/\sin(\pi\kappa_0), \quad 0\leqslant\theta<\theta_0, \\ \Delta_0\pi e^{-i\kappa_0\pi}(re^{i\theta})^{\kappa_0}2\cos(\theta_0\kappa_0)/\sin(\pi\kappa_0), \quad \theta_0\leqslant\theta<2\pi-\theta_0, \\ \Delta_0\pi e^{-2i\kappa_0\pi}(re^{i\theta})^{\kappa_0}2\cos((\theta_0-\pi)\kappa_0)/\sin(\pi\kappa_0), 2\pi-\theta_0\leqslant\theta\leqslant 2\pi. \end{cases} \tag{34}$$

is proven for the function $\ln\Phi_0(z)$, $z = re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$.

If we choose the sequence of the zeroes $z_k = r_k e^{i\theta_0}$ of the function $\Phi_0(z)$ so that

$$r_k = \left(\frac{2k-1}{2\Delta_0}\right)^{1/\kappa_0}, \quad n(x) = \begin{cases} k, & r_k \leqslant x < r_{(k+1)}, \\ 0, & 0 \leqslant x < r_0, \end{cases}$$

both integrals in the right hand side of formula (34) are of order $o(r^{\kappa_0})$ as $r \to +\infty$, $0 \leqslant \theta \leqslant \pi$ (see [5]). On the real axis, the function $\ln\Phi_0(z)$ takes real values

$$\ln\Phi_0(t) = 2I_0(t,\theta_0) + \begin{cases} \dfrac{\Delta_0 \pi 2\cos\Big((\theta_0-\pi)\kappa_0\Big)}{\sin(\pi\kappa_0)} t^{\kappa_0}, & t > 0, \\ \dfrac{\Delta_0\pi 2\cos(\theta_0\kappa_0)}{\sin(\pi\kappa_0)}|t|^{\kappa_0}, & t < 0, \end{cases}$$

where

$$I_0(t,\theta_0) = \int\limits_0^\infty (n(x) - x^{\kappa_0}\Delta_0)\frac{t^2 - tx\cos(\theta_0)}{x(t^2 - 2tx\cos(\theta_0) + x^2)}dx.$$

Thus, condition (31) for the entire function $\Phi_0(z)$ is satisfied for each Δ_0, θ_0 if we take $\kappa_0 < \kappa_+$, while in the case $\kappa_0 = \kappa_+$ the numbers $\Delta_0 > 0$, $\theta_0 \in (0,\pi)$ should be chosen so that the inequality

$$\Delta_0 2\cos\Big((\theta_0-\pi)\kappa_0\Big) \leqslant \Delta_+\cos(\pi\kappa_+)$$

holds true.

c) One can construct entire functions of order $\rho_\Phi \leqslant \rho$ satisfying condition (20) and inequalities (26), (27) in the same way as in Item b) resigning the condition $\kappa_0 \leqslant \kappa_+$ and choosing $\kappa_0 \leqslant \rho$. If we take $\kappa_0 < \rho$, then Δ_0, θ_0 in formulae (32), (33), (34) are arbitrary, and if $\kappa_0 = \rho$, then Δ_0, θ_0 should satisfy the system of inequalities

$$\begin{cases} \Delta_0 2\cos\Big((\theta_0-\pi)\kappa_0\Big) \leqslant \nu^-\cos(\pi\rho) - \nu^+, \\ \Delta_0 2\cos(\theta_0\kappa_0) \leqslant \nu^- - \nu^+\cos(\pi\rho). \end{cases}$$

Here the identities hold true if we take

$$\tan(\theta_0\kappa_0) = \frac{\nu^-\cos(\pi\rho)-\nu^+}{(\nu^- - \nu^+\cos(\pi\rho))\sin(\pi\kappa_0)} - \cot(\pi\kappa_0), \quad \Delta_0 = \frac{\nu^-\cos(\pi\rho)-\nu^+}{2\cos(\theta_0\kappa_0)}.$$

Therefore, the system is compatible. □

Theorem 5. *Suppose $\kappa_+ = \kappa_- = \rho < 1/2$. Then homogeneous boundary value problem* (17)
a) has no nontrivial solutions in the class B_* if

$$(\nu^- + \pi\Delta_+)\cos(\pi\rho) - (\nu^+ - \pi\Delta_-) < 0, \quad \textit{or} \quad \nu^- + \pi\Delta_+ - (\nu^+ - \pi\Delta_-)\cos(\pi\rho) < 0;$$

b) has the solution $F(z) = Ae^{\Gamma(z)}e^{i[P(z)+iQ(z)]}/P_+(z)P_-(z)$ in the class B_*, where A is an arbitrary real constant, if

$$\begin{cases} (\nu^- + \pi\Delta_+)\cos(\pi\rho) = \nu^+ - \pi\Delta_-, \\ \nu^- + \pi\Delta_+ > (\nu^+ - \pi\Delta_-)\cos(\pi\rho), \end{cases} \quad \textit{or} \quad \begin{cases} (\nu^- + \pi\Delta_+)\cos(\pi\rho) > \nu^+ - \pi\Delta_-, \\ \nu^- + \pi\Delta_+ = (\nu^+ - \pi\Delta_-)\cos(\pi\rho); \end{cases}$$

c) has the solutions in the class B_* determined by the formula (29)*, where $\Phi(z)$ is arbitrary entire function of order $\rho_\Phi \leqslant \rho$ satisfying condition* (20) *and also inequalities* (26)*,* (27) *as*

$\rho_\Phi = \rho$ *for sufficiently large* $|t|$, *if*

$$\begin{cases} (\nu^- + \pi\Delta_+)\cos(\pi\rho) - (\nu^+ - \pi\Delta_-) > 0, \\ \nu^- + \pi\Delta_+ - (\nu^+ - \pi\Delta_-)\cos(\pi\rho) > 0. \end{cases}$$

Proof. Let us prove Case b). Suppose the conditions

$$\begin{cases} (\nu^- + \pi\Delta_+)\cos(\pi\rho) - (\nu^+ - \pi\Delta_-) = 0, \\ \nu^- + \pi\Delta_+ - (\nu^+ - \pi\Delta_-)\cos(\pi\rho) > 0 \end{cases}$$

hold true. According to Theorem 3, the general solution of problem (17) is determined by the formula (29) involving an arbitrary entire function $\Phi(z)$ of order $\rho_\Phi \leqslant \rho$ obeying condition (20) and inequalities (26), (27). Under the assumptions of Theorem 5, the latter casts into the form

$$|\Phi(t)| \leqslant C\exp\left\{\frac{(\nu^- + \pi\Delta_+)\cos(\pi\rho) + \pi\Delta_- - \nu^+}{\sin(\pi\rho)}t^\rho\right\}, \quad t > 0, \tag{35}$$

$$|\Phi(t)| \leqslant C\exp\left\{\frac{\nu^- + \pi\Delta_+ - (\nu^+ - \pi\Delta_-)\cos(\pi\rho)}{\sin(\pi\rho)}|t|^\rho\right\}, \quad t < 0,$$

and by the first assumption in b), inequality (35) becomes $|\Phi(t)| \leqslant C$ $t > 0$. The symmetry of the function $\Phi(z)$ implies that $\Phi(z)$ is bounded on the both sides of the cut made along the positive semi-axis, and since $\rho_\Phi < 1/2$, the function $\Phi(z)$ is bounded on the whole plane, i.e., it is constant.

Cases a) and c) can be proven in the same way as in Theorem 4. □

Suppose now $\rho < \max\{\kappa_+, \kappa_-\} < 1/2$. As $\kappa_+ \geqslant \kappa_-$, by (25) we have

$$\ln\ln M(r) \leqslant \kappa_+ \ln r + \ln\left[\frac{\pi\Delta_+}{\sin(\pi\kappa_+)} + \frac{\pi\Delta_-}{\sin(\pi\kappa_-)r^{\kappa_+-\kappa_-}} + \frac{\ln C}{r^{\kappa_+}} + \frac{l}{r^{\kappa_+-\rho}}\right],$$

hence,

$$\rho_\Phi = \overline{\lim_{r\to\infty}}\frac{\ln\ln M(r)}{\ln r} \leqslant \kappa_+.$$

Therefore, the order of the entire function $\Phi(z)$ determined by the formulae (18), (20), (21) does not exceed κ_+ as $\kappa_+ \geqslant \kappa_-$. In the same way we obtain the inequality $\rho_\Phi \leqslant \kappa_-$ in the case $\kappa_- > \kappa_+$.

Since the coefficients at t^{κ_+}, t^{κ_-} in the right hand side of inequalities (26), (27) are always strictly positive, as $\rho_\Phi < \max\{\kappa_+, \kappa_-\}$, these inequalities are satisfied immediately. This is why the next theorem holds true.

Theorem 6. *Homogeneous boundary value problem* (17) *has a solution* $F(z)$ *in the class* B_* *as* $\rho < \max\{\kappa_+, \kappa_-\}$ *if and only if formula* (18) *is satisfied, where* $\Phi(z)$ *is an arbitrary entire function of order* $\rho_\Phi \leqslant \max\{\kappa_+, \kappa_-\}$ *satisfying condition* (20) *and also conditions* (26), (27) *as* $\rho_\Phi = \max\{\kappa_+, \kappa_-\}$.

Theorem 7. *As* $\rho < \max\{\kappa_+, \kappa_-\}$, *the general solution to homogeneous boundary value problem* (17) *in the class* B_* *is determined by formula* (29), *where* $\Phi(z)$ *is an arbitrary entire function of order* $\rho_\Phi \leqslant \max\{\kappa_+, \kappa_-\}$ *taking real values on* L *and also satisfying* (26), (27) *for sufficiently large* $|t|$ *as* $\rho_\Phi = \max\{\kappa_+, \kappa_-\}$.

Theorem 8. *Suppose* $\rho < \max\{\kappa_+, \kappa_-\} < 1/2$. *Then homogeneous boundary value problem* (17) *has solutions in the class* B_* *determined by formula* (29), *where* $\Phi(z)$ *is an arbitrary entire function of order* $\rho_\Phi \leqslant \max\{\kappa_+, \kappa_-\}$ *taking real values on* L *and also satisfying inequalities* (26), (27) *for sufficiently large* $|t|$ *as* $\rho_\Phi = \max\{\kappa_+, \kappa_-\}$.

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