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ON SOLUTIONS OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS WITH TWO INDEPENDENT VARIABLES

S. BAIZAEV, D.A. VOSITOVA

Abstract. In the paper we consider first order linear elliptic and hyperbolic systems with constant coefficients and two independent variables. For such systems we study the variety of all the solutions and that of the solutions growing at infinity at most as a power function.

Keywords: elliptic and hyperbolic system, moderately growing solutions, solutions of power growth, dimension of space of solutions.

Mathematics Subject Classification: 35C11.

1. Consider a system of linear partial differential equations

$$A_1U_x + A_2U_y + A_3U = F(x, y), \quad (1)$$

where $U = (u_1, u_2, \dots, u_n)^T$ is the unknown vector-function, A_1, A_2, A_3 are constant real matrices of order n, $F = (f_1, f_2, \dots, f_n)^T$ is a given vector-function.

As it is known, (see, for instance, [1]), system (1) is called *elliptic* if its principal symbol $P_0(\xi, \eta) = i\xi A_1 + i\eta A_2$ is non-degenerate as $(\xi, \eta) \neq (0, 0)$, i.e.,

$$Q(\xi, \eta) \equiv det(\xi A_1 + \eta A_2) \neq 0 \ \forall\, (\xi, \eta) \neq \ (0, 0). \quad (2)$$

System (1) is called *hyperbolic* if for each $\eta \in R$ all the solutions of the equation $Q(\xi, \eta) = 0$ w.r.t. ξ are real.

In the paper we study the variety of all the solutions and that of the solutions with at most power growth at infinity for systems (1) being elliptic or hyperbolic. Solutions of elliptic and hyperbolic equations and systems defined on whole plane were studied in the works of V.S. Vinogradov, E. Mukhamadiev, V.P. Palamodov, N.E. Tovmasyan and others (see, for instance, [2-4]). In work [5], for elliptic systems (1) as $n = 2$ moderately growing solutions and solutions of power growth were studied.

We shall assume that system (1) is either elliptic or hyperbolic. In this case it is easy to prove that $detA_1 \neq 0$ and in the case of ellipticity $detA_2 \neq 0$. This is why system (1) can be rewritten as

$$U_x + A_1^{-1}A_2U_y + A_1^{-1}A_3U = A_1^{-1}F(x, y).$$

Because of this in what follows we write system (1) as

$$U_x + AU_y + BU = f(x, y), \quad (3)$$

where $A = A_1^{-1}A_2$, $B = A_1^{-1}A_3$, $f = A_1^{-1}F$. Then ellipticity condition (2) becomes the inequality

$$det(\xi E + \eta A) \neq 0 \ \forall(\xi, \eta) \neq (0, 0),$$

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which is equivalent to the absence of real eigenvalues for the matrix A. The hyperbolicity condition of system (3) is equivalent to the presence of only real eigenvalues for the matrix A.

Suppose that matrices A and B commute. In system (3) we make a change of the unknown function $U = e^{-Bx}CV$, where C is a non-degenerate matrix. Then we have

$$-e^{-Bx}BCV + e^{-Bx}CV_x + Ae^{-Bx}CV_y + Be^{-Bx}CV = f(x, y)$$

or

$$V_x + C^{-1}e^{Bx}Ae^{-Bx}CV_y = C^{-1}e^{Bx}f(x, y). \tag{4}$$

Due to commuting of the matrices A and B, the identity $e^{Bx}Ae^{-Bx} = A$ holds true. Hence, system (4) casts into the form

$$V_x + C^{-1}ACV_y = g(x, y), \tag{5}$$

where $g(x, y) = C^{-1}e^{Bx}f(x, y)$.

As C we take the matrix reducing the matrix A to Jordan form. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$, $(m \leqslant n)$ be the eigenvalues of the matrix A. Then system (5) splits into the systems of smaller dimension

$$\frac{\partial V_k}{\partial x} + \Lambda_k \frac{\partial V_k}{\partial y} = g_k(x, y), \;\; k = 1, \ldots, m, \tag{6}$$

where

$$\Lambda_k = \begin{pmatrix} \lambda_k & 1 & 0 & \ldots & 0 \\ 0 & \lambda_k & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \lambda_k & 1 \\ 0 & \ldots & \ldots & 0 & \lambda_k \end{pmatrix}$$

is the Jordan block of order s_k, $s_1+s_2+\ldots+s_m = n$, $[V_1, V_2, \ldots, V_m]^T = V$, $[g_1, g_2, \ldots, g_m]^T = g$.

For the sake of convenience, we denote the coordinates of the vector V_k by $w_1, w_2, \ldots, w_\nu$, $(\nu = s_k)$, and the coordinates of the vector g_k are $h_1, h_2, \ldots, h_\nu$. Then the system obtained from (6) by fixing k can be written as

$$\frac{\partial w_1}{\partial x} + \lambda_k \frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial y} = h_1(x, y), \tag{7}$$

$$\frac{\partial w_2}{\partial x} + \lambda_k \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial y} = h_2(x, y), \tag{8}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial w_{\nu-1}}{\partial x} + \lambda_k \frac{\partial w_{\nu-1}}{\partial y} + \frac{\partial w_\nu}{\partial y} = h_{\nu-1}(x, y), \tag{9}$$

$$\frac{\partial w_\nu}{\partial x} + \lambda_k \frac{\partial w_\nu}{\partial y} = h_\nu(x, y). \tag{10}$$

Consider the equation

$$u_x + \lambda u_y = h(x, y), \tag{11}$$

where $\lambda \in C$ and conditions for function $h(x, y)$ will be given later (see Lemma 2).

Lemma 1. *The general solution to the homogeneous equation*

$$u_x + \lambda u_y = 0 \tag{12}$$

read as

$$u(x, y) = \varphi(\lambda x - y), \tag{13}$$

where $\varphi(z)$ is an arbitrary function in $C^1(R)$, if λ is a real number, and is the derivative of an analytic function of complex variable z, if λ is a complex number.

Proof. The statement of the lemma is obvious for real λ. Let $\lambda = \alpha + i\beta$ be complex and $\varphi(z)$ be an arbitrary analytic in z function. Then

$$u_x = \varphi_z(\lambda x - y) \cdot \frac{\partial}{\partial x}(\lambda x - y) = \lambda \varphi_z(\lambda x - y),$$

$$u_y = \varphi_z(\lambda x - y) \cdot \frac{\partial}{\partial y}(\lambda x - y) = -\varphi_z(\lambda x - y).$$

It follows that the function $u(x, y) = \varphi(\lambda x - y)$ solves equation (12).

Let us show now that each solution $u(x, y)$ of equation (12) can be represented as (13) with an analytic function $\varphi(z)$. Let $\zeta = \lambda x - y$. Then

$$x = \frac{1}{2i\beta}(\zeta - \overline{\zeta}),\ y = \frac{1}{2i\beta}(\overline{\lambda}\zeta - \lambda\overline{\zeta})$$

and substituting these expressions into function $u(x, y)$, we obtain a function of the variable $\zeta : u(\zeta) = u[x(\zeta),\ y(\zeta)]$. Let us calculate the derivative $u_{\overline{\zeta}}$:

$$u_{\overline{\zeta}} = u_x \cdot x_{\overline{\zeta}} + u_y \cdot y_{\overline{\zeta}} = u_x(\frac{i}{2\beta}) + u_y(\frac{i\lambda}{2\beta}) = \frac{i\lambda}{2\beta}(u_x + \lambda u_y) = 0,$$

since $u(x, y)$ is a solution to equation (12). It yields that function $u(\zeta)$ is analytic in ζ. Hence, there exists an analytic function $\varphi(z)$ such that $u = \varphi(\zeta) \equiv \varphi(\lambda x - y)$. The proof is complete. □

Lemma 2. *Let λ be real and function $h(x, y)$ is continuous in x and has a continuous derivative w.r.t. y. Then the function*

$$w(x, y) = \int_{\lambda x - y}^{x} h[t,\ \lambda(t - x) + y]dt \tag{14}$$

is a particular solution to non-homogenous equation (11).

Proof. By formula (14) and the rule for differentiating of integrals with variable limits we get

$$w_x = h(x, y) - \lambda h[\lambda x - y,\ \lambda(\lambda x - y - x) + y] - \lambda \int_{\lambda x - y}^{x} h_y[\xi,\ \lambda(\xi - x) + y]d\xi,$$

$$w_y = h[\lambda x - y,\ \lambda(\lambda x - y - x) + y] + \int_{\lambda x - y}^{x} h_y[\xi,\ \lambda(\xi - x) + y]d\xi.$$

It implies

$$w_x + \lambda w_y = h(x, y),$$

i.e., the function $w(x, y)$ is a particular solution to equation (11). □

In the case of real λ, the formula

$$u(x, y) = \varphi(\lambda x - y) + \int_{\lambda x - y}^{x} h[\xi,\ \lambda(\xi - x) + y]d\xi$$

gives the general solution of non-homogeneous equation (11), where φ is an arbitrary function in C^1.

2. Suppose system (3) is hyperbolic. We shall solve system (7)–(10) upwards. By Lemma 2, equation (10) implies

$$w_\nu(x, y) = \varphi_\nu(\lambda x - y) + Lh_\nu(x, y),$$

where $Lh_\nu(x,y) = \int\limits_{\lambda x-y}^{x} h_\nu[\xi, \lambda(\xi - x) + y]d\xi$, φ_ν is an arbitrary function in C^1. Substituting w_ν into equation (9), we find $w_{\nu-1}$,

$$w_{\nu-1}(x,y) = \varphi_{\nu-1}(\lambda x - y) + L[\varphi'_\nu(\lambda x - y) - \frac{\partial}{\partial y}(Lh_\nu)] + Lh_{\nu-1},$$

where $\varphi_{\nu-1}$ is an arbitrary function in C^1.

Next, we have

$$L\varphi'_\nu(\lambda x - y) = \int\limits_{\lambda x-y}^{x} \varphi'_\nu[\lambda\xi - \lambda(\xi - x) - y]d\xi = \varphi'_\nu(\lambda x - y)(x - \lambda x + y),$$

$$\frac{\partial}{\partial y}(Lh_\nu) = h_\nu[\lambda x - y,\ \lambda(\lambda x - y - x) + y] + \int\limits_{\lambda x-y}^{x} \frac{\partial}{\partial y} h_\nu[\xi,\ \lambda(\xi - x) + y]d\xi.$$

Hence,

$$w_{\nu-1}(x,y) = \varphi_{\nu-1}(\lambda x - y) + \varphi'_\nu(\lambda x - y)(x - \lambda x + y) + h_\nu[\lambda x - y,\ (\lambda - 1)(\lambda x - y)] +$$

$$+ \int\limits_{\lambda x-y}^{x} \frac{\partial}{\partial y} h_\nu[\xi,\ \lambda(\xi - x) + y]d\xi + Lh_{\nu-1}.$$

Repeating this procedure, we find $w_{\nu-2}, \ldots, w_1$. It should be noted that the arbitrary functions φ_j appearing while integrating equations (7)–(10) should have an appropriate smoothness, namely, the function φ_j as $1 \leqslant j \leqslant \nu$ should belong to C^j.

Consider now the elliptic case. Let us find the general solution to the homogeneous system associated with (7)–(10). By equation (10) and Lemma 1 we obtain

$$w_\nu(x,y) = \varphi_\nu(\lambda_k x - y),$$

where $\varphi_\nu(z)$ is an analytic in z function. Then equation (9) becomes

$$\frac{\partial w_{\nu-1}}{\partial x} + \lambda_k \frac{\partial w_{\nu-1}}{\partial y} = \varphi'_\nu(\lambda_k x - y). \tag{15}$$

The function $w = x\varphi'_\nu(\lambda_k x - y)$ is a particular solution of equation (15). Thus, the general solution of this equation reads as follows,

$$w_{\nu-1} = x\varphi'_\nu(\lambda_k x - y) + \varphi_{\nu-1}(\lambda_k x - y),$$

where $\varphi_{\nu-1}(z)$ is an analytic in z function. In the same way we find

$$w_{\nu-2} = x^2\varphi''_\nu(\lambda_k x - y) + x\varphi'_{\nu-1}(\lambda_k x - y) + \varphi_{\nu-2}(\lambda_k x - y),$$

where $\varphi_{\nu-2}(z)$ is an analytic in z function. Repeating this procedure, we find $w_{\nu-3}, \ldots, w_1$.

Suppose matrix A has n different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $z_1, z_2, \ldots, z_n$ are the associated eigenvectors. If the matrices A and B commute, as it is well-known (see, for instance, [6]), $z_1, z_2, \ldots, z_n$ are the eigenvectors of the matrix B as well. By $\mu_1, \mu_2, \ldots, \mu_n$ we denote the eigenvalues of the matrix B associated with the eigenvectors $z_1, z_2, \ldots, z_n$.

The following theorem holds true.

Theorem 1. *Suppose a matrix A has n different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $z_1, z_2, \ldots, z_n$ are the associated eigenvectors. Suppose matrices A and B commute. Then the general solution to the homogeneous system*

$$U_x + AU_y + BU = 0 \tag{16}$$

reads as follows,

$$U(x,y) = C(e^{-\mu_1 x}\varphi_1(\lambda_1 x - y), \dots, e^{-\mu_n x}\varphi_n(\lambda_n x - y))^T, \tag{17}$$

where C is the matrix whose columns are the eigenvectors z_1, z_2, ..., z_n of matrix A, μ_j are the eigenvalues of matrix B introduced above, $\varphi_j(z)$ are functions in C^1 in the case of hyperbolic system (16) and analytic in z in the case of elliptic system (16).

Proof. Since $C = [z_1, z_2, \dots, z_n]$, the change $U = e^{-Bx}CV$ reduces system (16) to

$$V_x + \Lambda V_y = 0,$$

where $\Lambda = \operatorname{diag}[\lambda_1, \dots, \lambda_n]$. By Lemma 1, the general solution of the latter system is

$$V(x,y) = [\varphi_1(\lambda_1 x - y), \dots, \varphi_n(\lambda_n x - y)]^T,$$

where $\varphi_1(z), \dots, \varphi_n(z)$ are functions in C^1 in the case of hyperbolic system (16) and analytic in z in the case of elliptic system (16). Hence,

$$U(x,y) = e^{-Bx}C[\varphi_1(\lambda_1 x - y), \dots, \varphi_n(\lambda_n x - y)]^T. \tag{18}$$

Since $z_1, \dots, z_n$ are the eigenvectors of the matrix B associated with the eigenvalues $\mu_1, \dots, \mu_n$, matrix C reduces the matrix B to the diagonal form $M = \operatorname{diag}[\mu_1, \dots, \mu_n]$, i.e., $C^{-1}BC = M$. Therefore, by the property of the exponential of a matrix the identity

$$e^{-Bx}C = e^{-CMC^{-1}x}C = Ce^{-Mx} = C\operatorname{diag}[e^{-\mu_1 x}, \dots, e^{-\mu_n x}]$$

holds true. Together with (18) it implies the formula

$$U = C\operatorname{diag}[e^{-\mu_1 x}, \dots, e^{-\mu_n x}][\varphi_1(\lambda_1 x - y), \dots, \varphi_n(\lambda_n x - y)]^T,$$

that yields (17). The proof is complete. □

In the case of elliptic system (16) general solution (17) can be also represented as

$$U(x,y) = C(e^{-iRe(\lambda_1\overline{\mu_1}x - \mu_1 y)/\operatorname{Im}\lambda_1}\psi_1(\lambda_1 x - y), \dots, e^{-iRe(\lambda_n\overline{\mu_n}x - \mu_n y)/\operatorname{Im}\lambda_n}\psi_n(\lambda_n x - y))^T. \tag{19}$$

In order to get it, in formula (17) one should let

$$\varphi_j(z) = e^{i\gamma_j z}\psi_j(z),$$

where $\gamma_j = -\operatorname{Re}\mu_j/\operatorname{Im}\lambda_j$, $\psi_j(z)$ are analytic in z functions.

3. For homogeneous system (16) let us consider the solutions defined on whole plane and satisfying the growth condition

$$\|U(x,y)\| \leqslant K(1 + |x|^N + |y|^N) \tag{20}$$

as $|x| + |y| \to \infty$, where $\|U\| = |u_1| + |u_2| + \ldots + |u_n|$, N is an non-negative integer, K is constant depending on U. The variety of such solutions is a real linear space which we denote by $\mathcal{P}_N$.

Suppose system (16) is hyperbolic. Then the number λ_j, μ_j, $j = 1, \dots, n$ are real. By (20), formula (17) implies the estimate

$$|e^{-\mu_j x}\varphi_j(\lambda_j x - y)| \leqslant K\|C^{-1}\|(1 + |x|^N + |y|^N) \tag{21}$$

as $|x| + |y| \to \infty$. As $x = 0$, $|y| \to \infty$ it follows

$$|\varphi_j(y)| \leqslant K\|C^{-1}\|(1 + |y|^N), \tag{22}$$

and as $y = 0$, $|x| \to \infty$ it yields

$$|e^{-\mu_j x}\varphi_j(\lambda_j x)| \leqslant K\|C^{-1}\|(1 + |x|^N). \tag{23}$$

Estimate (22) implies that function $\varphi_j(t)$ grows not faster than a power as $|t| \to \infty$. Then by estimate (23) we obtain that either $\mu_j = 0$ or $\varphi_j = 0$. Hence, in the hyperbolic case, if all the eigenvalues of the matrix B are non-zero, problem (16), (20) has only trivial solution. And if one of the eigenvalues $\mu_{j_0} = 0$, taking in (17) as the function $\varphi_{j_0}(t)$ a function in C^1 growing at most as a power as $|t| \to \infty$, and letting $\varphi_j = 0$ for $\mu_j \neq 0$, we obtain non-zero solutions to problem (16), (20). In this case the space $\mathcal{P}_{\mathcal{N}}$ is infinite-dimensional.

Suppose now system (16) is elliptic. By (20), formula (19) yields the estimate

$$|\psi_j(\lambda_j x - y)| \leqslant K\|C^{-1}\|(1 + |x|^N + |y|^N)$$

as $|x| + |y| \to \infty$. Since the function $\psi_j(z)$ is analytic in z, by Liouville theorem, it is a polynomial w.r.t. z of the degree at most N. Thus, the solutions to problem (16), (20) read as

$$U(x, y) = C(e^{-iRe(\lambda_1 \overline{\mu_1} x - \mu_1 y)/\operatorname{Im}\lambda_1} p_{1N}(\lambda_1 x - y), \ldots,$$
$$e^{-iRe(\lambda_n \overline{\mu_n} x - \mu_n y)/\operatorname{Im}\lambda_n} p_{nN}(\lambda_n x - y))^T,$$

where $p_{jN}(z)$ are polynomials w.r.t. z of degree at most N. Then the space $\mathcal{P}_{\mathcal{N}}$ is finite-dimensional and its dimension is $(N + 1)n$.

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Sattor Baizaev,
Sibai institute (branch) of Bashkir State University,
Belova str., 21,
453838, Sibai, Russia
E-mail: `baisat54@rambler.ru`

Dilorom Abdurasulovna Vositova,
Khudzhand State University,
Mavlonbekov str., 2,
735700, Khudzhand, Republic of Tadzhikistan
E-mail: `rasuli@mail.ru`