ISSN 2304-0122 Ufa Mathematical Journal. Vol. 5. No 2 (2013). P. 3-11.

doi:10.13108/2013-5-2-3 UDC 517.968

APPROXIMATE SOLUTIONS OF NONLINEAR CONVOLUTION TYPE EQUATIONS ON SEGMENT

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Abstract. For various classes of integral convolution type equations with a monotone nonlinearity, we prove global solvability and uniqueness theorems as well as theorems on the ways for finding the solutions in real Lebesgue spaces. It is shown that the solutions can be found in space $L_2(0,1)$ by a Picard's type successive approximations method and we prove the estimates for the rate of convergence. The obtained results cover, in particular, linear integral convolution type equations. In the case of a power nonlinearity, it is shown that the solutions can be found by the gradient method in the space $L_p(0,1)$ and weighted spaces $L_p(\varrho)$.

Keywords: nonlinear integral equations, convolution type operator, potential operator, monotone operator.

Mathematics Subject Classification: 45G10, 47H05.

In work [1] for nonlinear integral convolution type equations

$$\lambda \cdot F(x, u(x)) + \int_0^1 \varphi(|x-t|)\, u(t)\, dt = f(x)\,, \tag{1}$$

$$u(x) + \lambda \int_0^1 \varphi(|x-t|)\, F[t, u(t)]\, dt = f(x)\,, \tag{2}$$

$$u(x) + \lambda \cdot F\left[x, \int_0^1 \varphi(|x-t|)\, u(t)\, dt\right] = f(x)\,, \tag{3}$$

the theorems on existence, uniqueness, and estimates for solutions in the real spaces $L_p(0,1)$, $1 < p < \infty$, were proven without any restriction for the absolute value of the parameter λ.

In the present work it is proven that in the case of the space $L_2(0,1)$, these solutions can be found by the method of Picard type successive approximations and at that, it is not needed the absolute value of the parameter λ to be "small". In contrast to [2], where similar equations with the kernel of potential type on the real axis were considered, here, by employing the method of *potential* monotone operators, we construct new successive approximations and improve substantially the estimates for the rate of convergence. Moreover, by the gradient method (method of steepest descent) we succeeded to solve equations with power nonlinearities not covered by the results in [2] both in $L_p(0,1)$ and the weighted spaces $L_p(\varrho)$.

S.N. Askhabov, A.L. Dzhabrailov, Approximate solutions of nonlinear convolution type equations on segment.

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The work is financially supported by RFBR, grant no. 13-01-00422-a.

Submitted May 10, 2012.

To simplify writing, we introduce following notations:

$$L_p(0,1) = L_p, \quad \|\cdot\|_{L_p(0,1)} = \|\cdot\|_p, \quad p' = \frac{p}{p-1},$$

$$\langle u, v\rangle = \int\limits_0^1 u(x)\, v(x)\, dx, \qquad (P_{01}^{\varphi} u)\,(x) = \int\limits_0^1 \varphi(|x-t|)\, u(t)\, dt\ .$$

Definition 1. *We shall say that a function $\varphi \in \Omega(0,1]$ if it is continuous, decreases, convex down in the segment $(0,1]$, and $\int\limits_0^1 \varphi(x)\, dx \geq 0$*

In what follows we shall make use of the following lemma playing an essential role in studying equations (1)–(3) and equations with power nonlinearities.

Lemma 1. *Suppose $1 < p \leqslant 2$ and $\varphi \in L_{p'/2} \bigcap \Omega(0,1]$. Then the convolution operator P_{01}^{φ} is a continuous one from L_p into $L_{p'}$, is potential and positive and $\forall u(x) \in L_p$ inequalities*

$$\|P_{01}^{\varphi} u\|_p \leqslant 2^{2/p'} \|\varphi\|_{p'/2} \|u\|_p\,, \tag{4}$$

$$\langle P_{01}^{\varphi} u, u\rangle = \int\limits_0^1 \left(\int\limits_0^1 \varphi(|x-t|)\, u(t)\, dt \right) u(x)\, dx \geq 0 \tag{5}$$

hold true.

Proof. Inequalities (4) and (5) were proven in [1]. Thus, the operator P_{01}^{φ} is a continuous one from L_p into $L_{p'}$ and is positive. Since $\varphi(|x-t|) = \varphi(|t-x|)$, the operator P_{01}^{φ} is symmetric. Therefore, (see, for instance, [3] or [4], Example 1.2), the operator P_{01}^{φ} is potential, and its potential is calculated by the formula $p(u) = \frac{1}{2}\langle P_{01}^{\varphi} u, u\rangle$. □

It should be noted that as $p = 2$, under additional restrictions (differentiability and non-negativity) for functions $\varphi(x)$, the positivity of the operator P_{01}^{φ} was proven earlier by A.M. Nakhushev [5].

We proceed to the study of nonlinear equations (1)–(3) involving convolution type operator P_{01}^{φ}. We denote by $\mathbf{N}$ the set of all natural numbers. Hereafter we assume that the function $F(x,t)$ generating Nemytski operator $Fu = F[x, u(x)]$ is defined for $x \in [0,1]$, $t \in (-\infty, \infty)$ and satisfies Caratheodory conditions: it is measurable w.r.t. x for each fixed t and continuous w.r.t. t for almost each x.

In what follows we shall make use of the following theorem (see [4], p. 16, where its proof was given) being a corollary of more general results of [6].

Theorem 1 ([6]). *Suppose H is a real Hilbert space with a scalar product $(\cdot,\cdot)$ and a norm $\|\cdot\|_H$, an operator A acts in H and is potential. If there exist constants $m > 0$ and $M > 0$ $(M > m)$ such that for each $u, v \in H$ the inequalities*

$$\|Au - Av\|_H \leqslant M \cdot \|u - v\|_H\,, \qquad (Au - Av, u - v) \geq m \cdot \|u - v\|_H^2$$

hold true, the equation $Au = f$ has the unique solution $u^* \in H$ for each $f \in H$. This solution can be found by the method of successive approximations by the formula ($n \in \mathbf{N}$)

$$u_n = u_{n-1} - \frac{2}{M+m}(Au_{n-1} - f), \tag{6}$$

with the estimate for the error

$$\|u_n - u^*\|_H \leqslant \frac{2}{M+m} \cdot \frac{\alpha^n}{1-\alpha} \|Au_0 - f\|_H, \tag{7}$$

where $\alpha = (M-m)/(M+m)$, $u_0 \in H$ is an initial approximation.

We note that estimate (7) ensures a higher rate of convergence of successive approximations in comparison with estimate (16) in [2] obtained without the assumption of potentiality of operator A.

Theorem 2. *Suppose $\varphi \in \Omega(0,1]$ and for almost each fixed $x \in [0,1]$ and each $t_1, t_2 \in (-\infty, \infty)$ the nonlinearity $F(x,t)$ satisfy the assumptions*

1). $|F(x,t_1) - F(x,t_2)| \leqslant M \cdot |t_1 - t_2|$, *where* $M > 0$;

2). $\Big(F(x,t_1) - F(x,t_2)\Big) \cdot (t_1 - t_2) \geq m \cdot |t_1 - t_2|^2$, *where* $m > 0$.

Then for each $\lambda > 0$ and $f(x) \in L_2$, equation (1) has the unique solution $u^*(x) \in L_2$. This solution can be found by the iteration method via the scheme

$$u_n = u_{n-1} - \mu_1 \cdot (\lambda \cdot F u_{n-1} + P_{01}^{\varphi} u_{n-1} - f), \tag{8}$$

with the estimate for the error

$$\|u_n - u^*\|_2 \leqslant \mu_1 \cdot \frac{\alpha_1^n}{1-\alpha_1} \cdot \|\lambda \cdot F u_0 + P_{01}^{\varphi} u_0 - f\|_2, \tag{9}$$

where $\mu_1 = 2/(M + m + 2\,\|\varphi\|_1)$, $\alpha_1 = (M - m + 2\,\|\varphi\|_1)/(M + m + 2\,\|\varphi\|_1)$, $u_0(x) \in L_2$ is an initial approximation.

Proof. Assumption 1) implies that the Nemytski operator F is an operator in L_2 and satisfies Lipshitz condition

$$\|Fu - Fv\|_2 \leqslant M \cdot \|u - v\|_2, \qquad \forall u, v \in L_2, \tag{10}$$

and Assumption 2) yields that it is strongly monotone,

$$(Fu - Fv, u - v) \geq m \cdot \|u - v\|_2^2, \qquad \forall u, v \in L_2\,. \tag{11}$$

Moreover, under Assumption 1), the Nemytski operator F is potential and its potential g is calculated by the formula (see [3]),

$$g(u) = g_0 + \int_0^1 \left[\int_0^{u(x)} F(x,t)\, dt \right] dx,$$

where $g_0 = \text{const}$.

Let $u, v \in L_2$ be arbitrary functions. We write equation (1) in the operator form, $Au = f$, where $A = \lambda \cdot F + P_{01}^{\varphi}$. We note that by inequalities (4) and (10), the operator A is a continuous one in L_2 and is potential (as a sum of two potential operators $\lambda \cdot F$ and P_{01}^{φ}). Employing then Minkowski inequality and by inequalities (4) and (10), on one hand, we have $\|Au - Av\|_2 \leqslant (\lambda \cdot M + 2\,\|\varphi\|_1) \cdot \|u - v\|_2$, and on the other, using inequalities (5) and (11), we get $(Au - Av, u - v) \geq \lambda \cdot m \cdot \|u - v\|_2^2$. Therefore, by Theorem 1, the equation $Au = f$ has the unique solution $u^* \in L_2$ and this solution can be found by the scheme (8) implied by formula (6) with estimate (9) for the error following from inequality (7). □

It is more complicated to study nonlinear equations (2) and (3) by the method of potential monotone operators. Here we succeed to construct the successive approximations only in terms of the inverse operator F^{-1}.

Theorem 3. *Suppose $\varphi \in \Omega(0,1]$ and the nonlinearity $F(x,t)$ satisfies Assumptions 1) and 2) of Theorem 2. Then for each $\lambda > 0$ and $f(x) \in L_2$, nonlinear equation (2) has the unique solution $u^* \in L_2$. This solution can be found by the iteration method via the scheme*

$$u_n = F^{-1} v_n, \quad v_n = v_{n-1} - \mu_2 \cdot (F^{-1} v_{n-1} + \lambda \cdot P_{01}^{\varphi} v_{n-1} - f), \tag{12}$$

with the estimate for the error

$$\|u_n - u^*\|_2 \leqslant \frac{\mu_2}{m} \cdot \frac{\alpha_2^n}{1-\alpha_2} \cdot \|u_0 + \lambda \cdot P_{01}^{\varphi} F u_0 - f\|_2, \tag{13}$$

where $n \in \mathbf{N}$, $\mu_2 = 2/(m^{-1} + m\, M^{-2} + 2\,\lambda \cdot \|\varphi\|_1)$,

$$\alpha_2 = (m^{-1} - m\, M^{-2} + 2\,\lambda \cdot \|\varphi\|_1)/(m^{-1} + m\, M^{-2} + 2\,\lambda \cdot \|\varphi\|_1),$$

F^{-1} *is the inverse operator for* F, $v_0 = Fu_0$, $u_0 \in L_2$ *is an initial approximation.*

Proof. Since the operator F satisfies inequalities (10) and (11), by Theorem 1.3 in [4], there exists the inverse operator F^{-1} such that

$$\|F^{-1}u - F^{-1}v\|_2 \leqslant \frac{1}{m}\|u - v\|_2, \qquad \forall u, v \in L_2\,, \tag{14}$$

$$(F^{-1}u - F^{-1}v, u - v) \geq \frac{m}{M^2}\|u - v\|_2^2, \qquad \forall u, v \in L_2\,. \tag{15}$$

We note ([6], p. 137) that the operator F^{-1} is potential as the inverse to the monotone potential operator F. We write equation (2) in the operator form,

$$u + \lambda \cdot P_{01}^{\varphi} F u = f. \tag{16}$$

It is straightforward to check that if v^* solves the equation

$$Bv \equiv F^{-1}v + \lambda \cdot P_{01}^{\varphi} v = f, \tag{17}$$

$u^* = F^{-1}v^*$ solves equation (16).

Let us prove that equation (17) has the unique solution $v^* \in L_2$. Employing inequalities (4), (5), (14), and (15), we get

$$\|Bu - Bv\|_2 \leqslant (m^{-1} + 2\,\lambda \cdot \|\varphi\|_1)\,\|u - v\|_2, \quad (Bu - Bv, u - v) \geq \frac{m}{M^2}\|u - v\|_2^2\,.$$

Moreover, the operator B is potential as a sum of two potential operators F^{-1} and $\lambda \cdot P_{01}^{\varphi}$. Hence, by Theorem 1, equation $Bv = f$ has the unique solution $v^* \in L_2$ that can be found by the scheme

$$v_n = v_{n-1} - \mu_2 \cdot (Bv_{n-1} - f), \tag{18}$$

with the estimate for error

$$\|v_n - v^*\|_2 \leqslant \mu_2 \cdot \frac{\alpha_2^n}{1 - \alpha_2}\|Bv_0 - f\|_2, \tag{19}$$

where μ_2 and α_2 are determined above in the formulation of Theorem 3. But then equation (16) has the unique solution $u^* = F^{-1}v^* \in L_2$ that can be found by scheme (12) yielded by (18) with estimate (13) for the error implied by (19) taking into consideration identity $Bv = F^{-1}v + \lambda \cdot P_{01}^{\varphi} v$ and estimate $\|u_n - u^*\|_2 = \|F^{-1}v_n - F^{-1}v^*\|_2 \leqslant \frac{1}{m}\|v_n - v^*\|_2$. □

Theorem 4. *Suppose* $\varphi(x) \in \Omega(0, 1]$ *and the nonlinearity* $F(x, t)$ *satisfies Assumptions 1) and 2) of Theorem 2. Then for each* $\lambda > 0$ *and* $f(x) \in L_2$ *nonlinear equation (3) has the unique solution* $u^*(x) \in L_2$*. This solution can be found by iteration methods via the scheme*

$$u_n = u_{n-1} + \lambda \cdot \mu_2 \cdot \left(F^{-1}\left(\lambda^{-1}(f - u_{n-1})\right) - P_{01}^{\varphi} u_{n-1}\right), \tag{20}$$

with the estimate for the error

$$\|u_n - u^*\|_2 \leqslant \lambda \cdot \mu_2 \cdot \frac{\alpha_2^n}{1 - \alpha_2} \cdot \|F^{-1}\left(\lambda^{-1}(f - u_0)\right) - P_{01}^{\varphi} u_0\|_2, \tag{21}$$

where $n \in \mathbf{N}$, μ_2, *and* α_2 *are defined in the formulation of Theorem 3,* F^{-1} *is the inverse operator for* F, $u_0 \in L_2$ *is an initial approximation.*

Proof. We write equation (3) in the operator form

$$u + \lambda \cdot F P_{01}^{\varphi} u = f \ . \tag{22}$$

We let $f - u = \lambda \cdot v$. Then equation (22) casts into the form $F P_{01}^{\varphi}(f - \lambda \cdot v) = v$. Applying the operator F^{-1} (its existence was proven in Theorem 3) to both sides of the latter equation, we arrive at the equation

$$Bv \equiv F^{-1} v + \lambda \cdot P_{01}^{\varphi} v = P_{01}^{\varphi} f \ . \tag{23}$$

It is straightforward to check that v^* is a solution to equation (23), then $u^* = f - \lambda \cdot v^*$ solves equation (22).

Since (23) is of the same form as equation (17), repeating arguments in Theorem 3, we make sure that equation (23) has the unique solution $v^* \in L_2$ and it can be found by a scheme like (18),

$$v_n = v_{n-1} - \mu_2 (B v_{n-1} - P_{01}^{\varphi} f), \tag{24}$$

with the estimate for the error like (19),

$$\|v_n - v^*\| \leqslant \mu_2 \cdot \frac{\alpha_2^n}{1 - \alpha_2} \|B v_0 - P_{01}^{\varphi} f\|_2 \ . \tag{25}$$

Bearing in mind that $v = \lambda^{-1}(f - u)$, from (24) and (25) we obtain immediately iteration scheme (20) and estimate (21) for the error. $\square$

Theorems 2-4 cover, in particular, equations with kernel like potential $|x - t|^{\alpha - 1}$, $0 < \alpha < 1$, and logarithmic potential $-\ln |x - t|$, as well as corresponding linear equations and some equations with monotone nonlinearities (for instance, $(u(x) + 2\, u^3(x))/(1 + u^2(x)))$. However, these theorems do not cover power nonlinearities leading out the space L_2.

For approximate solving of equations with power nonlinearities in wider spaces we shall make use of the following well-known theorem. Before we formulate it, we give necessary notations and definitions.

Let X be a real Banach space and X^* is adjoint space. We denote by $\langle y, x \rangle$ the value of a linear continuous functional $y \in X^*$ on an element $x \in X$, and the symbols $\| \cdot \|$ and $\| \cdot \|_*$ denote the norms in X and X^*, respectively.

Definition 2. *Let $u, v \in X$ be arbitrary elements. Operator $A : X \to X^*$ (i.e., acting from X into X^*) is called*

uniformly monotone *if $\langle Au - Av, u - v \rangle \geq \beta(\|u - v\|)$, where β is an increasing on $[0, \infty)$ function such that $\beta(0) = 0$;*

boundedly Lipshitz-continuous *if $\|Au - Av\|_* \leqslant \mu(r) \cdot \|u - v\|$, where μ is increasing on $[0, \infty)$ function, and $r = \max(\|u\|, \|v\|)$.*

Theorem 5 ([6]). *Let X be a real Banach space and $A : X \to X^*$ be a hemicontinuous uniformly monotone coercive operator. Then the equation $Au = f$ has the unique solution $u^* \in X$ for each $f \in X^*$. Moreover, if X and X^* are strictly convex space and operator is potential and boundedly Lipshitz-continuous, the sequence $u_{n+1} = u_n - \delta_n \cdot J^*(Au_n - f)$, where $\delta_n = \min\{1, 2/[\varepsilon + \mu(\|u_n\| + \|Au_n - f\|_*)]\}$, $n = 0, 1, 2, 3, \dots$, $J^* : X^* \to X$ is the dualizing mapping for X^*, $\varepsilon > 0$ is an arbitrary number, converges to u^* in the norm of space X.*

The existence and uniqueness of solution u^* in Theorem 5 follow from Browder-Minty theorem (main theorem of theory of monotone operators [6]), and the strong convergence of the sequence $\{u_n\}$ to u^* by the mentioned scheme is followed by Theorem 4.2 and Remark 4.13 in [6], since each uniformly monotone operator is strictly monotone and possesses (S)-property [6]. The way of finding u^* given in Theorem 5 is known as the method of *steepest* descent (or *gradient* method) [6] .

Lemma 2. *Let $2 < p < \infty$, $\varphi \in \Omega(0,1]$ and $b(x) \in L_{2p/(p-2)}$. Then the operator*

$$(B_{01}^{\varphi}u)(x) = b(x) \int_0^1 b(t)\, \varphi(|x-t|)\, u(t)\, dt$$

is a continuous one from L_p into $L_{p'}$, is positive and potential, and $\forall u(x) \in L_p$ inequalities

$$\|B_{01}^{\varphi}u\|_{p'} \leqslant 2\, \|b\|_{2p/(p-2)}^2 \cdot \|\varphi\|_1 \cdot \|u\|_p\,, \quad \langle B_{01}^{\varphi}u, u\rangle \ge 0 \tag{26}$$

hold true.

Proof. Let $u(x) \in L_p$ be an arbitrary function. By Hölder inequality, $\|b \cdot u\|_2 \leqslant \|b\|_{2p/(p-2)} \|u\|_p$. Hence, employing estimate (4), we have $\|P_{01}^{\varphi}(b \cdot u)\|_2 \leqslant 2\, \|\varphi\|_1 \|b \cdot u\|_2 \leqslant 2\, \|b\|_{2p/(p-2)} \|\varphi\|_1 \|u\|_p$. Since $B_{01}^{\varphi}u = b \cdot P_{01}^{\varphi}(b \cdot u)$ and by Hölder inequality $\|B_{01}^{\varphi}u\|_{p'} \leqslant \|b\|_{2p/(p-2)} \|P_{01}^{\varphi}(b \cdot u)\|_2 \leqslant 2\, \|b\|_{2p/(p-2)}^2 \cdot \|\varphi\|_1 \cdot \|u\|_p$, the operator B_{01}^{φ} is a continuous one from L_p into $L_{p'}$ and is potential as a symmetric operator, at that, the first inequality in (26) holds true. Finally, employing inequality (5), we obtain $\langle B_{01}^{\varphi}u, u\rangle = \langle P_{01}^{\varphi}(b \cdot u), (b \cdot u)\rangle \ge 0$ which is equivalent to the second inequality in (26), i.e., the operator B_{01}^{φ} is positive. □

Theorem 6. *Suppose $p \ge 4$ is an even number, $\varphi \in \Omega(0,1]$, and $b(x) \in L_{2p/(p-2)}$. Then the equation*

$$u^{p-1}(x) + b(x) \int_0^1 b(t)\, \varphi(|x-t|)\, u(t)\, dt = f(x) \tag{27}$$

has the unique solution $u^* \in L_p$ for each $f \in L_{p'}$. This solution can be found by the successive approximations method as follows,

$$u_{n+1} = u_n - \delta_n \cdot \|Au_n - f\|_{p'}^{2-p'} \cdot |Au_n - f|^{p'-2} \cdot (Au_n - f)\,, \tag{28}$$

where $n = 0, 1, 2, 3, \dots$, $u_0(x) \in L_p$ is an initial approximation, $Au = u^{p-1} + B_{01}^{\varphi}u$,

$$\delta_n = \min\left(1, \frac{2}{\varepsilon + (p-1) \cdot \Big(\|u_n\|_p + \|Au_n - f\|_{p'}\Big)^p + 2\, \|b\|_{2p/(p-2)}^2 \|\varphi\|_1}\right),$$

$\varepsilon > 0$ is an arbitrary number.

Proof. We write equation (27) in the operator form $Au = f$, where $Au = u^{p-1} + B_{01}^{\varphi}u$. It is obvious that operator A is a continuous one from L_p into $L_{p'}$ and coercive since $\langle Au, u\rangle = \langle u^{p-1}, u\rangle + \langle B_{01}^{\varphi}u, u\rangle \ge \|u\|_p^p$ and $p \ge 4$.

Let us show that A is uniformly monotone operator. Employing Lemma 2 and inequality $(t^{p-1} - s^{p-1}) \cdot (t - s) \ge 2^{2-p}|t - s|^p$ being valid for each $t, s \in (-\infty, \infty)$, we have

$$\langle Au - Av, u - v\rangle \ge \int_0^1 [u^{p-1}a(x) - v^{p-1}(x)] \cdot [u(x) - v(x)]\, dx \ge$$

$$\ge 2^{2-p} \cdot \|u - v\|_p^p = \beta(\|u - v\|_p)\,, \quad \forall u, v \in L_p\,,$$

where $\beta(s) = 2^{2-p} \cdot s^p$ is a strictly increasing on $[0, \infty)$ function such that $\beta(0) = 0$, i.e., A is an uniformly monotone operator.

Thus, by Browder-Minty theorem, equation (27) has the unique solution $u^* \in L_p$. It remains to prove that sequence (28) converges to $u^*(x)$ in the norm of space L_p. Let us employ Theorem 5. It is known [3] that the spaces L_p, $1 < p < \infty$, are strictly convex and the dualizing mapping J^* for the space $L_{p'}$ reads as follows,

$$(J^*w)(x) = \|w\|_{p'}^{2-p'} \cdot |w(x)|^{p'-2} \cdot w(x)\,. \tag{29}$$

Let us show that the operator A is boundedly Lipshitz-continuous. For each $u, v \in L_p$ we have

$$\|Au - Av\|_{p'} \leqslant \|u^{p-1} - v^{p-1}\|_{p'} + \|B_{01}^{\varphi}(u-v)\|_{p'} = I_1 + I_2 \; .$$

Since $|t^{p-1} - s^{p-1}| \leqslant \frac{p-1}{2} \cdot |t-s| \cdot (t^{p-2} + s^{p-2}) \, , \quad \forall t, s \in (-\infty, \infty)$, we have

$$I_1 \leqslant \frac{p-1}{2} \left(\int_0^1 |u(x) - v(x)|^{p'} |u^{p-2}(x) + v^{p-2}(x)|^{p'} dx \right)^{1/p'} \leqslant$$

(we apply first Hölder inequality with exponents p/p' and $p/(p-p')$,
and then we apply Minkowski inequality to the second factor)

$$\leqslant \frac{p-1}{2} \|u-v\|_p \left(\|u\|_p^{p-2} + \|v\|_p^{p-2} \right) \leqslant (p-1) \cdot r^{p-2} \cdot \|u-v\|_p,$$

where $r = \max\left(\|u\|_p, \|v\|_p\right)$. Thus, estimating I_2 by means of the first inequality in (26), we have $\|Au - Av\|_{p'} \leqslant \mu(r) \cdot \|u-v\|_p$, where $\mu(r) = (p-1) \cdot r^{p-2} + 2 \, \|b\|^2_{2p/(p-2)} \|\varphi\|_1$ is an increasing on $[0, \infty)$ function. Hence, A is a boundedly Lipshitz-continuous operator.

Next, since $Fu = u^{p-1}$ is a potential operator, taking into consideration Lemma 2, we obtain that the operator A is potential as well.

Hence, due to Theorem 5, sequence (28) converges to $u^*(x)$ in the norm of space L_p. □

We introduce weighted spaces $L_p(\varrho)$. Let $\varrho(x)$ be a non-negative Lebesgue measurable on $[0,1]$ function being almost everywhere finite and non-zero. We denote by $L_p(\varrho)$, $1 < p < \infty$, the set of all Lebesgue measurable on $[0,1]$ functions $u(x)$ with a finite norm

$$\|u\|_{p,1} = \left(\int_0^1 \varrho(x) \, |u(x)|^p dx \right)^{1/p} .$$

It is known [7] that $L_p(\varrho)$ is a reflexive Banach space and its adjoint space is $L_{p'}(\varrho^{1-p'})$ with the norm $\|\cdot\|_{p', 1-p'}$, $p' = p/(p-1)$. In the case $\varrho(x) = 1$ we shall write as usually L_p and $\|\cdot\|_p$.

In the weighted space $L_p(\varrho)$, consider the equation

$$\varrho(x) \cdot u^{p-1}(x) + \int_0^1 \varphi(|x-t|) \, u(t) \, dt = f(x) \, . \tag{30}$$

We impose the following restriction for the weight $\varrho(x)$,

$$c(\varrho) = \left(\int_0^1 [\rho(x)]^{2/(2-p)} dx \right)^{(p-2)/(2p)} < \infty \, . \tag{31}$$

Lemma 3. *Suppose $2 < p < \infty$, $\varphi \in \Omega(0,1]$, and condition (31) is satisfied. Then the convolution operator P_{01}^{φ} acts from $L_p(\rho)$ into $L_{p'}(\rho^{1-p'})$ and is a continuous potential positive operator,*

$$\|P_{01}^{\varphi} u\|_{p', \, 1-p'} \leqslant 2 \, c^2(\varrho) \cdot \|\varphi\|_1 \cdot \|u\|_{p, \, 1} \, , \qquad \forall u \in L_p(\varrho) \, . \tag{32}$$

Proof. Let $u(x) \in L_p(\varrho)$ be an arbitrary function. Since, by Hölder inequality,

$$\|u\|_2 = \left(\int_0^1 [\varrho(x)]^{-2/p} [\varrho(x)]^{2/p} |u(x)|^2 dx \right)^{1/2} \leqslant c(\varrho) \cdot \|u\|_{p, \, 1} \, , \tag{33}$$

the space $L_p(\varrho)$ is continuously embedded into L_2.

Similarly, for each $\psi(x) \in L_2$ we have

$$\|\psi\|_{p',\,1-p'} = \left(\int\limits_0^1 [\varrho(x)]^{1-p'} |\psi(x)|^{p'} dx \right)^{1/p'} \leqslant c(\varrho) \cdot \|\psi\|_2 \,. \tag{34}$$

Inequalities (33) (34) imply that the continuous embedding

$$L_p(\varrho) \subset L_2 \subset L_{p'}(\varrho^{1-p'}) \tag{35}$$

holds true. Since, by inequality (4), $\|P_{01}^{\varphi} u\|_2 \leqslant 2\,\|\varphi\|_1 \cdot \|u\|_2$, employing estimates (33) and (34), we get

$$\|P_{01}^{\varphi} u\|_{p',\,1-p'} \leqslant c(\varrho) \cdot \|P_{01}^{\varphi} u\|_2 \leqslant 2\,c(\varrho) \cdot \|\varphi\|_1 \cdot \|u\|_2 \leqslant 2\,c^2(\varrho) \cdot \|\varphi\|_1 \cdot \|u\|_{p,\,1} \,.$$

Thus, the operator P_{01}^{φ} is a continuous one from $L_p(\rho)$ into $L_{p'}(\rho^{1-p'})$ and inequality (32) holds true. The potentiality and positivity of the operator P_{01}^{φ} follows from Lemma 1 since embeddings (35) hold true. □

Theorem 7. *Let $p \geq 4$ be an even number, $\varphi \in \Omega(0,1]$, and condition (31) be satisfied. Then equation (30) has the unique solution $u^*(x) \in L_p(\varrho)$ for each $f(x) \in L_{p'}(\varrho^{1-p'})$. This solution can be found by the successive approximations method as follows,*

$$u_{n+1} = u_n - \delta_n \cdot \|Bu_n - f\|_{p',\,1-p'}^{2-p'} \cdot \varrho^{1-p'} \cdot |Bu_n - f|^{p'-2} \cdot (Bu_n - f)\,, \tag{36}$$

where $u_0(x) \in L_p(\varrho)$ is an initial approximation, $Bu = \varrho \cdot u^{p-1} + P_{01}^{\varphi} u$,

$$\delta_n = \min\left(1, \frac{2}{\varepsilon + (p-1) \cdot \Big(\|u_n\|_{p,\,1} + \|Bu_n - f\|_{p',\,1-p'} \Big)^{p-2} + 2\,c^2(\varrho) \cdot \|\varphi\|_1} \right),$$

$\varepsilon > 0$ is an arbitrary number.

Proof. Since the proof follows the same lines as that of Theorem 6, we restrict ourselves just by main milestones. We write equation (30) in the operator form $Bu = f$, where $Bu = \rho \cdot u^{p-1} + P_{01}^{\varphi} u$. Since $\varrho \cdot u^{p-1} \in L_{p'}(\varrho^{1-p'})$, $\forall u \in L_p(\varrho)$, employing Lemma 3, we obtain that the operator B acts from $L_p(\varrho)$ into $L_{p'}(\rho^{1-p'})$. It is straightforward to check that the dualizing mapping J^* for the space $L_{p'}(\varrho^{1-p'})$ reads as

$$(J^* w)(x) = \|w\|_{p',1-p'}^{2-p'} \cdot \varrho^{1-p'}(x) \cdot |w(x)|^{p'-2} \cdot w(x)\,.$$

Next, $\forall u, v \in L_p(\varrho)$ we have

$$\|Bu - Bv\|_{p',\,1-p'} \leqslant \|\varrho \cdot (u^{p-1} - v^{p-1}\|_{p',\,1-p'} + \|P_{01}^{\varphi}(u - v)\|_{p',\,1-p'} = I_1 + I_2\,.$$

As in the proof of Theorem 6, we get

$$I_1 \leqslant \frac{p-1}{2} \left(\int\limits_0^1 \varrho^{p'-1}(x) \Big| u(x) - v(x) \Big|^{p'} \varrho^{2-p'}(x) \Big| u^{p-2}(x) + v^{p-2}(x) \Big|^{p'} dx \right)^{1/p'} \leqslant$$

$$\leqslant \frac{p-1}{2}\, \|u - v\|_{p,\,1} \left(\|u\|_{p,\,1}^{p-2} + \|v\|_{p,\,1}^{p-2} \right) \leqslant (p-1) \cdot r^{p-2} \cdot \|u - v\|_{p,\,1}\,,$$

where $r = \max\left(\|u\|_{p,\,1}, \|v\|_{p,\,1}\right)$. Thus, employing Lemma 3 to estimate I_2, we have $\|Bu - Bv\|_{p',\,1-p'} \leqslant \mu(r) \cdot \|u - v\|_{p,\,1}$, where $\mu(r) = (p-1) \cdot r^{p-2} + 2\,c^2(\varrho)\,\|\varphi\|_1$ is an increasing on $[0, \infty)$ function. Therefore, B is a boundedly Lipshitz-continuous operator. Finally, exactly as in the proof of Theorem 6, one can prove that B is an uniformly monotone (with $\beta(s) = 2^{2-p} \cdot s^p$) potential operator. □

In conclusion we note that similar results can be obtained for nonlinear singular integral equations and nonlinear Wiener-Hopf equations with special kernels considered in [4], [8], [9].

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