ISSN 2304-0122 Ufa Mathematical Journal. Vol. 5. No 1 (2013). P. 3-10.

UDC 517.518

COMPACTNESS CRITERION FOR FRACTIONAL INTEGRATION OPERATOR OF INFINITESIMAL ORDER

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Abstract. We obtain necessary and sufficient conditions of compactness for the operator

$$Kf(x) = \int_0^x \ln \frac{x}{x-s} \frac{f(s)}{s} ds$$

from $L_{p,v}$ in $L_{q,u}$ as $1 < p \leqslant q < \infty$ and $v(x) = x^{-\gamma}$, $\gamma > 0$, where $L_{q,u}$ is the set of all measurable on $(0,\infty)$ functions f with finite norm $\|uf\|_q$.

Keywords: compactness, fractional integration operator, Riemann-Liouville operator, singular operator, adjoint operator, Holder inequality, weighted inequalities.

1. Introduction

Let $1 < p \leqslant q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $R_+ = (0,\infty)$, $u, v : R_+ \to R$ be weight functions, i.e., nonnegative and measurable on R_+.

Starting from 70s of the last century, in the world mathematical literature a weighted estimate

$$\|uKf\|_q \leqslant C\|vf\|_p \tag{1}$$

is intensively studied for various classes of operators K, where $\|\cdot\|_p$ is the usual norm of the space $L_p \equiv L_p(R)$. In what follows by $L_{p,v}$ we indicate the set of the functions $f : R_+ \to R$ with finite norm $\|f\|_{p,v} = \|vf\|_p$. The survey of the studies of estimate (1) between 1970 and 1982 was presented in [1]. Some directions of studies of estimates (1) performed for integral operators before 2003 were provided in [2]. In paper [3] a sequence of the classes of nonnegative functions $K(\cdot,\cdot)$ was given as well as the complete description of the weights u and v for which the integral operator

$$Kf(x) = \int_0^x K(x,s)f(s)ds \tag{2}$$

obeys (1) if its kernel belongs to these classes. However, these results do not include operator (2) if its kernel $K(\cdot,\cdot)$ has a singularity, for instance, Riemann-Liouville operator

$$R_\alpha f(x) = \int_0^x \frac{f(s)ds}{(x-s)^{1-\alpha}} \tag{3}$$

as $0 < \alpha < 1$. Estimate (1) for operator (3) in the general case is still an open question. Nevertheless, the following cases were studied, $u \equiv v$ in [4], $v \equiv 1$ in [5, 6], and the case of non-increasing of one of weight functions u, v in [7].

A.M. Abylayeva, A.O. Baiarystanov, Compactness criterion for fractional integration operator of infinitesimal order.

Submitted December 23, 2011.

The operator of the form

$$Kf(x) = \int_0^x \ln \frac{x}{x-s} \frac{f(s)}{s} ds \tag{4}$$

is called a fractional integration operator of infinitesimal order (see [8], p. 34).

In [9] estimate (1) for operator (4) was studied in the case $v(x) = x^{-\gamma}$, $\gamma > 0$. It is equivalent to the estimate

$$\|uT_\gamma f\|_q \leqslant C\|f\|_p \tag{5}$$

for the operator

$$T_\gamma f(x) = \int_0^x s^{\gamma-1} \ln \frac{x}{x-s} f(s) ds.$$

Since

$$\ln \frac{x}{x-s} = \int_0^s \frac{dt}{x-t} \quad \text{for } x > s \geq 0,$$

the inequality

$$\frac{s}{x-s} > \ln \frac{x}{x-s} > \frac{s}{x}, \quad x > s > 0 \tag{6}$$

holds true. The function $\ln \frac{x}{x-s}$ decays w.r.t. x and increases w.r.t. s as $x > s \geq 0$, and the functions $x \ln \frac{x}{x-s}$, $\frac{1}{s} \ln \frac{x}{x-s}$ decay w.r.t. x and decrease w.r.t. s as $x > s > 0$. Indeed,

$$\frac{\partial}{\partial x}\left(x \ln \frac{x}{x-s}\right) = \ln \frac{x}{x-s} - \frac{s}{x-s} < 0,$$

$$\frac{\partial}{\partial s}\left(\frac{1}{s} \ln \frac{x}{x-s}\right) = \frac{1}{s^2}\left(\frac{s}{x-s} - \ln \frac{x}{x-s}\right) > 0$$

as $x > s > 0$. We observe that for a differentiable function f estimate (1) for operator (4) is equivalent to the inequality

$$\left(\int_0^\infty \left| u(x) \int_0^x \frac{f(x)-f(s)}{x-s} ds \right|^q dx\right)^{\frac{1}{q}} \leqslant C \left(\int_0^\infty |f'(x) x^{1-\gamma}|^p dx\right)^{\frac{1}{p}}. \tag{7}$$

In the paper we assume the following. The indeterminate forms $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ are assumed to be zero. The inequality of the form $A \leqslant \beta B$, where a positive constant β can depend on parameters p, q, and γ, will be written as $A \ll B$, while the relation $A \approx B$ will be indicated as $A \ll B \ll A$.

Let $\chi_{(a,b)}(\cdot)$ be the characteristic function of the interval (a,b), Z is the set of integers.

In work [9] the authors obtained the criteria of boundedness for the operator T_γ and its adjoint

$$T_\gamma^* g(s) = s^{\gamma-1} \int_s^\infty g(x) \ln \frac{x}{x-s} dx \tag{8}$$

acting from L_p into $L_{q,u}$.

In particular, the following theorems were proven.

Theorem A. *Let $1 < p \leqslant q < \infty$, $\gamma > \frac{1}{p}$. The operator T_γ is bounded as that from L_p in $L_{q,u}$ if and only if*

$$D_\gamma = \sup_{x>0} D_\gamma(x) < \infty, \quad where \;\; D_\gamma(x) = x^{\gamma+\frac{1}{p'}} \left(\int_x^\infty t^{-q} u(t) dt \right)^{\frac{1}{q}}.$$

At that, $\|T_\gamma\| \approx D_\gamma$.

Theorem B. *Let $1 < p \leqslant q < \infty$, $\gamma > 1 - \frac{1}{q}$. Then the operator T^*_γ is bounded as that from $L_{p,v}$ into L_q if and only if*

$$D^*_\gamma = \sup_{x>0} D^*_\gamma(x) \equiv \sup_{x>0} x^{\gamma+\frac{1}{q}} \left(\int_x^\infty t^{-p'} v^{1-p'}(t) dt \right)^{\frac{1}{p'}} < \infty.$$

At that, $\|T^*_\gamma\| \approx D^*_\gamma$.

In the present work we study the compactness of the operator T_γ acting from L_p into $L_{q,u}$.

2. Main result

Theorem 1. *Let $1 < p \leqslant q < \infty$, $\gamma > \frac{1}{p}$. The operator T_γ is compact from L_p in $L_{q,u}$ if and only if $D_\gamma < \infty$ and*

$$\lim_{x\to 0} D_\gamma(x) = \lim_{x\to\infty} D_\gamma(x) = 0. \tag{9}$$

Proof. Necessity. Let T_γ be a compact operator from L_p into $L_{q,u}$. By Theorem A $D_\gamma < \infty$.

Let us prove the validity of the conditions (9). For $0 < s < \infty$ we consider a family of the functions

$$f_s(x) = \chi_{(0,s)}(x) s^{-\frac{1}{p}}, \quad x > 0, \tag{10}$$

with norm

$$\|f\|_{L_p} = \left(\int_0^\infty |f_s(x)|^p dx \right)^{\frac{1}{p}} = \left(\int_0^s s^{-1} dx \right)^{\frac{1}{p}} = s^{-\frac{1}{p}} \left(\int_0^s dx \right)^{\frac{1}{p}} = 1. \tag{11}$$

Let us show that family of the functions (10) converges weakly to zero in L_p. By Theorem [10] on the general form of linear continuous functionals in a Lebesgue space, the linear continuous functional in L_p reads as

$$\int_0^\infty f(x) g(x) dx, \quad \text{where} \;\; g \in L_{p'}.$$

Employing Hölder inequality, we deduce

$$\int_0^\infty f_s(x) g(x) dx = \int_0^s s^{-\frac{1}{p}} g(x) dx \leqslant$$

$$\leqslant s^{-\frac{1}{p}} \left(\int_0^s dx \right)^{\frac{1}{p}} \left(\int_0^s |g(x)|^{p'} dx \right)^{\frac{1}{p'}} = \left(\int_0^s |g(x)|^{p'} dx \right)^{\frac{1}{p'}}. \tag{12}$$

For each $g \in L_{p'}$ the latter integral in (12) tends to zero as $s \to 0$ that implies the weak convergence $f_s \to 0$ in L_p as $s \to 0$. Then by the properties of compact operators in a Banach space

$$\lim_{s\to 0} \|T_\gamma f_s\|_{q,u} = 0. \tag{13}$$

Since $\ln \frac{x}{x-t} \geq \frac{t}{x}$ as $0 < t < x$, we have

$$\|T_\gamma f_s\|_{q,u} = \left(\int_0^\infty u(x) \left|\int_0^x t^{\gamma-1} \ln \frac{x}{x-t} f_s(t)dt\right|^q dx\right)^{\frac{1}{q}} \geq$$

$$\geq \left(\int_s^\infty u(x) \left|\int_0^s t^{\gamma-1} s^{-\frac{1}{p}} \ln \frac{x}{x-t} dt\right|^q dx\right)^{\frac{1}{q}} \geq$$

$$\geq s^{-\frac{1}{p}} \left(\int_0^s t^\gamma dt\right) \left(\int_s^\infty x^{-q} u(x)\right)^{\frac{1}{q}} = \frac{1}{\gamma+1} D_\gamma(s). \tag{14}$$

It follows from (13) and (14) that $\lim\limits_{s\to 0} D_\gamma(s) = 0$, i.e., the first relation in (9). Let us show that the second relation in (9) holds true. The compactness of the operator T_γ yields the same for adjoint operator T_γ^* (8) acting from $L_{q',u^{1-q'}}$ into $L_{p'}$. For $0 < s < \infty$ we introduce the family of the functions

$$g_s(x) = \chi_{(s,\infty)}(x) \left(\int_s^\infty t^{-q} u(t)dt\right)^{-\frac{1}{q'}} u(x) x^{1-q}. \tag{15}$$

The conditions $D_\gamma < \infty$ implies that the integral in (15) is convergent. Let us show that $g_s \in L_{q',u^{1-q'}}$ for each $s > 0$.

Indeed,

$$\|g_s\|_{q',u^{1-q'}} = \left(\int_0^\infty |g_s x|^{q'} u^{1-q'}(x)dx\right)^{\frac{1}{q'}} =$$

$$= \left(\int_s^\infty t^{-q} u(t)dt\right)^{-\frac{1}{q'}} \left(\int_s^\infty \left(u(x)x^{1-q}\right)^{q'} u^{1-q'}(x)dx\right)^{\frac{1}{q'}} = 1. \tag{16}$$

By (16) for all $f \in L_{q,u}$

$$\int_0^\infty g_s(x)f(x)dx = \int_s^\infty g_s(x)f(x)dx \leqslant \left(\int_s^\infty |g_s(x)|^{q'} u^{1-q'}(x)dx\right)^{\frac{1}{q'}} \times$$

$$\times \left(\int_s^\infty |f(x)|^q u(x)dx\right)^{\frac{1}{q}} = \left(\int_s^\infty |f(x)|^q u(x)dx\right)^{\frac{1}{q}}.$$

Passing in the latter inequality to the limit as $s \to \infty$, we see that $g_s \to 0$ weakly in $L_{q',u^{1-q'}}$ as $s \to \infty$. Hence, $T_\gamma^* g_s$ (by compactness of T_γ^*) converges to zero as $s \to \infty$ in the sense of $L_{p'}$-norm, i.e.,

$$\lim_{s\to\infty} \|T_\gamma^* g_s\|_{p'} = 0. \tag{17}$$

We have

$$\|T^*_\gamma g_s\|_{p'} = \left(\int\limits_0^\infty \left| t^{\gamma-1}\int\limits_t^\infty g_s(x)\ln\frac{x}{x-t}dx\right|^{p'} dt\right)^{\frac{1}{p'}} \geq$$

$$\geq \left(\int\limits_0^s t^{p'(\gamma-1)}\left(\int\limits_s^\infty \frac{u(x)}{x^{q-1}}\ln\frac{x}{x-t}dx\right)^{p'} dt\right)^{\frac{1}{p'}}\left(\int\limits_s^\infty x^{-q}u(x)dx\right)^{-\frac{1}{q'}} \geq$$

(we again employ inequality $\ln\frac{x}{x-t} \geq \frac{t}{x}$)

$$\geq \left(\int\limits_s^\infty x^{-q}u(x)dx\right)^{\frac{1}{q}}\left(\int\limits_0^s t^{p'(\gamma-1)}t^{p'}dt\right)^{\frac{1}{p'}} = \left(\frac{1}{p'\gamma+1}\right)^{\frac{1}{p'}} D_\gamma(s).$$

Together with (17) it implies the second relation in (9). The proof of the necessity is completed.

Sufficiency. Let $0 < a < b < \infty$ and

$$P_a f = \chi_{(0,a)}f,\ \ P_{ab}f = \chi_{[a,b)}f,\ \ Q_b f = \chi_{[b,\infty)}f.$$

Then for the operator T_γ

$$T_\gamma f = P_{ab}T_\gamma P_{ab} + P_a T_\gamma P_a f + P_{ab}T_\gamma P_a f + Q_b T_\gamma f. \tag{18}$$

Let us show that the operator $P_{ab}T_\gamma P_{ab}$ is compact from L_p into $L_{q,u}$. Since $P_{ab}T_\gamma P_{ab}f(x) = P_{ab}T_\gamma \chi_{[ab)}(x)f(x) = 0$ for $x \notin [a,b)$, it is sufficient to show that the operator $P_{ab}T_\gamma P_{ab}$ is compact from $L_p(a,b)$ into $L_{q,u}(a,b)$, and, in its turn, it is equivalent to the compactness of the operator $Tf(x) = \int\limits_a^b K(x,s)f(s)ds$ with kernel $K(x,s) = u^{\frac{1}{q}}(x)\chi_{(a,b)}(x-s)s^{\gamma-1}\ln\frac{x}{x-s}$ from $L_p(a,b)$ into $L_q(a,b)$ that by the local integrability of the function u satisfies the condition

$$\int\limits_a^b\left(\int\limits_a^b |K(x,s)|^{p'}ds\right)^{\frac{q}{p'}}dx = \int\limits_a^b u(x)\left(\int\limits_a^x\left(s^{\gamma-1}\ln\frac{x}{x-s}\right)^{p'}ds\right)^{\frac{q}{p'}} \leqslant$$

(we employ the inequality $\frac{s}{x-s} \geq \ln\frac{x}{x-s}$ as $x > s > 0$)

$$\leqslant \int\limits_a^b u(x)\left(\int\limits_a^x s^{p'\gamma}\left(\frac{1}{x-s}\right)^{p'}ds\right)^{\frac{q}{p'}} \leqslant \left(\int\limits_a^b s^{p'\gamma}ds\right)^{\frac{q}{p'}}\int\limits_a^b u(x)x^{-q}dx < \infty.$$

Therefore, by Kantorovich test [10], the operator T_γ is compact from $L_p(a,b)$ into $L_q(a,b)$, that is equivalent to the compactness from L_p into $L_{q,u}$ of the operator $P_{ab}T_\gamma P_{ab}$. It follows from (18) that

$$\|T_\gamma - P_{ab}T_\gamma P_{ab}\| \leqslant \|P_a T_\gamma P_a\| + \|P_{ab}T_\gamma P_a\| + \|Q_b T_\gamma\|. \tag{19}$$

Let us show that the right hand side in (19) tends to zero as $a \to 0$ and $b \to \infty$, then the operator T_γ will be compact from L_p into $L_{q,u}$ as the uniform limit of compact operator ([11], VI.12).

Let $u_a = P_a u$, then by Theorem A we have

$$\|P_a T_\gamma P_a f\|_{q,u_a} \leqslant \|P_a T_\gamma f\|_{q,u_a} = \left(\int\limits_0^\infty u_a(x)\left|\int\limits_0^x s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}} \ll$$

$$\ll \sup_{z>0} z^{\gamma+\frac{1}{p'}} \left(\int_z^\infty u_a(x)x^{-q}dx\right)^{\frac{1}{q}} \|f\|_p.$$

Thus,

$$\|P_aT_\gamma P_a\| \ll \sup_{z>0} z^{\gamma+\frac{1}{p'}} \left(\int_z^\infty u_a(x)x^{-q}dx\right)^{\frac{1}{q}} = \sup_{0<z<a} z^{\gamma+\frac{1}{p'}} \left(\int_z^a u(x)x^{-q}dx\right)^{\frac{1}{q}} \leqslant$$

$$\leqslant \sup_{0<z<a} z^{\gamma+\frac{1}{p'}} \left(\int_z^\infty u(x)x^{-q}dx\right)^{\frac{1}{q}} = \sup_{0<z<a} D_\gamma(z).$$

It yields

$$\lim_{a\to 0} \|P_aT_\gamma P_a\| \ll \overline{\lim_{z\to 0}} D_\gamma(z) = \lim_{z\to 0} D_\gamma(z) = 0. \tag{20}$$

The estimate $\|P_{ab}T_\gamma P_a\|$ for is

$$\|P_{ab}T_\gamma P_a f\|_{q,u} = \left(\int_a^b u(x)\left|\int_0^x s^{\gamma-1}\ln\frac{x}{x-s}(P_af)(s)ds\right|^q dx\right)^{\frac{1}{q}} \leqslant$$

$$\leqslant \left(\int_a^\infty u(x)\left(\int_0^a s^{\gamma-1}\ln\frac{x}{x-s}|f(s)|ds\right)^q dx\right)^{\frac{1}{q}} \leqslant$$

(we employ Hölder inequality and the properties of the function $x\ln\frac{x}{x-s}$)

$$\leqslant \left(\int_a^\infty u(x)\left(\int_0^a \left|s^{\gamma-1}\ln\frac{x}{x-s}\right|^{p'} ds\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}} \left(\int_0^a |f(s)|^p ds\right)^{\frac{1}{p}} \leqslant$$

$$\leqslant \left(\int_a^\infty u(x)x^{-q}\left(\int_0^a \left|s^{\gamma-1}x\ln\frac{x}{x-s}\right|^{p'} ds\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}} \|f\|_p \leqslant$$

$$\leqslant \left(\int_a^\infty u(x)x^{-q}dx\right)^{\frac{1}{q}} \left(\int_0^a \left|s^{\gamma-1}a\ln\frac{a}{a-s}\right|^{p'} ds\right)^{\frac{1}{p'}} \|f\|_p \leqslant$$

$$\leqslant (\beta_p)^{\frac{1}{p'}} a^{\gamma+\frac{1}{p'}} \left(\int_a^\infty u(x)x^{-q}dx\right)^{\frac{1}{q}} \|f\|_p \leqslant (\beta_p)^{\frac{1}{p'}} D_\gamma(a)\|f\|_p,$$

where $\beta_p = \int_0^1 \left|s^{\gamma-1}\ln\frac{1}{1-s}\right|^{p'} ds \leqslant \ln^{p'} 2\int_0^{\frac{1}{2}} s^{p'(\gamma-1)}ds + \max\left\{1, 2^{-p'(\gamma-1)}\right\}\int_{\ln 2}^\infty t^{p'}e^{-t}dt$. Hence, $\|P_{ab}T_\gamma P_a\| \ll D_\gamma(a)$. Therefore,

$$\lim_{b\to\infty}\lim_{a\to 0} \|P_{ab}T_\gamma P_a\| \ll \lim_{a\to 0} D_\gamma(a) = 0. \tag{21}$$

Let $u_b = Q_b u$, then due to Theorem A we get

$$\|Q_bT_\gamma f\|_{q,u} = \left(\int\limits_0^\infty u_b(x)\left|\int\limits_0^x s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}} \ll$$

$$\ll \sup_{z>0} z^{\gamma+\frac{1}{p'}}\left(\int\limits_z^\infty u_b(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p.$$

It follows

$$\|Q_bT_\gamma\| \ll \sup_{z>0} z^{\gamma+\frac{1}{p'}}\left(\int\limits_z^\infty u_b(x)x^{-q}dx\right)^{\frac{1}{q}} =$$

$$= \sup_{z\geq b} z^{\gamma+\frac{1}{p'}}\left(\int\limits_z^\infty u(x)x^{-q}dx\right)^{\frac{1}{q}} = \sup_{z\geq b} D_\gamma(z).$$

Therefore,

$$\lim_{b\to\infty}\|Q_bT_\gamma\| \ll \overline{\lim_{z\to\infty}} D_\gamma(z) = \lim_{z\to\infty} D_\gamma(z) = 0. \qquad (22)$$

It follows from (19), (20), (21), and (22) that the right hand side of (19) tends to zero as $a\to 0$ and $b\to\infty$. Theorem 1 is proven. □

Passing to the adjoint operator and applying Theorem 1, we obtain

Theorem 2. *Let $1 < p \leqslant q < \infty$, $\gamma > 1 - \frac{1}{q}$. Then operator (8) is compact from $L_{p,v}$ into L_q if and only if*

$$D^*_\gamma < \infty, \ \ \textit{and} \ \lim_{x\to 0} D^*_\gamma(x) = \lim_{x\to\infty} D^*_\gamma(x) = 0.$$

From Theorem 1 we immediately obtain

Theorem 3. *Let $1 < p \leqslant q < \infty$ and $v(x) = x^{-\gamma}$. Fractional integration operator of infinitesimal order (4) is compact from $L_{p,v}$ into $L_{q,u}$ if and only if $D_\gamma < \infty$ and (9) holds true.*

In the case $q < p$ we have

Theorem 4. *Let $1 < q < p < \infty$, $v(x) = x^{-\gamma}$, $\gamma > \frac{1}{p}$. Fractional integration operator of infinitesimal order (4) is compact from $L_{p,v}$ in $L_{q,u}$ if and only if*

$$E_\gamma = \left(\int\limits_0^\infty \left[\left(\int\limits_x^\infty \frac{u(t)}{t^q}dt\right)^{\frac{1}{q}} x^{\gamma+\frac{1}{p'}}\right]^{\frac{pq}{p-q}} dx\right)^{\frac{p-q}{pq}} < \infty.$$

Theorem 2 follows directly from Theorem 2 in work [9], since by Ando theorem ([12], § 5) for $1 < q < p < \infty$ each bounded integral operator from L_p in $L_{q,u}$ is compact.

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