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ON A SPECTRAL PROPERTY OF IRREGULAR PENCILS

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Abstract. The present paper introduces the notion of a quasi-regular eigenvalue and a quasi-regular pencil spectrum of finite dimensional operator pencils. It is demonstrated that quasi-regular eigenvalues of irregular pencils are continuous with respect to perturbations of the pencil. Properties of quasi-regular eigenvalues are studied and formulae for calculating a quasi-regular spectrum are obtained.

Keywords: spectral theory of linear operators, irregular pencils, inverse spectral problems, regular spectrum of the operator pencil.

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1. Introduction, problem formulation

In an n-dimension Euklidean space $\mathbb{E}^n$ we consider the pencil of operators:

$$L(\mu,\varepsilon) = (A_0 + \varepsilon A_1) - \mu(B_0 + \varepsilon B_1) : \mathbb{E}^n \to \mathbb{E}^n, \tag{1}$$

where $\mu \in \overline{\mathbb{C}}$ is a spectral parameter, $\varepsilon \in \mathbb{C}$ is a parameter of perturbations.

Everywhere in what follows we assume that the pencil of operators $L(\mu,\varepsilon)$ is regular in some neighbourhood of the point $\varepsilon = 0$, and it is irregular in the point $\varepsilon = 0$ itself. This assumption holds for the following conditions:

$$\max_{\mu,\varepsilon\in\overline{\mathbb{C}}}\{rank L(\mu,\varepsilon)\} = n, \tag{2}$$

$$\max_{\mu\in\overline{\mathbb{C}}}\{rank L(\mu,0)\} = m < n. \tag{3}$$

According to the condition (2) the pencil $L(\mu,\varepsilon)$ has (due to order) n eigenvalues:

$$\mu_1(\varepsilon), \mu_2(\varepsilon), \ldots, \mu_n(\varepsilon) \in \overline{\mathbb{C}}, \tag{4}$$

which are zeroes of the characteristic polynomial

$$\det L(\mu,\varepsilon) = \sum_{k=0}^{n} l_k(\varepsilon)\mu^k. \tag{5}$$

Remark. *Let us consider that $\mu = \infty$ is an eigenvalue of some pencil $A - \mu B : E^n \to E^n$ if $det(B) = 0$.*

Every eigenvalue $\mu_k(\varepsilon)$ is an algebraic function from $\varepsilon \in \mathbb{C}$, meanwhile these values can be considered as functionals from the matrices A_0, B_0, A_1, B_1, i.e.

$$\mu_k(\varepsilon) = \mu_k(A_0, B_0, A_1, B_1, \varepsilon).$$

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It results from the general theory of finite dimensional linear operators (see, e.g., [2]) that the limiting values $\mu_k(\varepsilon)$ do not depend on A_1, B_1 and are functionals of the matrices A_0, B_0:

$$\lim_{\varepsilon\to 0}\mu_k(\varepsilon) = \mu_k(A_0, B_0)$$

under the condition

$$\max_{\mu\in\overline{\mathbb{C}}}\{rank L(\mu, 0)\} = n$$

In other words, the eigenvalues $\mu_k(\varepsilon) = \mu_k(A_0, B_0, A_1, B_1, \varepsilon)$ of a regular pencil are continuous functionals of the matrices A_0, B_0.

Let now the limiting pencil $L_0(\mu) = L(\mu, 0)$ be irregular, i.e. $\det L(\mu, 0) \equiv 0$. In this case the situation varies, in particular, we cannot already state that the limits of the eigenvalues $\mu_k(\varepsilon) = \mu_k(A_0, B_0, A_1, B_1, \varepsilon)$ depend only on A_0 and B_0 when $\varepsilon \to 0$. Namely, the limiting values $\mu_k(\varepsilon) = \mu_k(A_0, B_0, A_1, B_1, \varepsilon)$ can also depend on the direction of the pencil $L(\mu, \varepsilon)$ by which it approaches to the limiting pencil $L_0(\mu)$, i.e. from the pair of matrices (A_1, B_1).

In connection with the above we accept the following definition.

Definition 1. *Let $\mu_k(A_0, B_0, A_1, B_1, \varepsilon)$ be an eigenvalue of the pencil*

$$L(\mu, \varepsilon) = (A_0 + \varepsilon A_1) - \mu(B_0 + \varepsilon B_1) : \mathbb{E}^n \to \mathbb{E}^n,$$

satisfying (2)-(3).

If the limiting value of the functional

$$\mu^* := \lim_{\varepsilon\to 0}\mu_k(A_0, B_0, A_1, B_1, \varepsilon) \in \overline{\mathbb{C}}$$

in this case does not depend on the operators A_1 and B_1, then we call it a ***quasi-regular*** *eigenvalue of the pencil $L_0(\mu) = A_0 - \mu B_0$. The set of all the quasi-regular eigenvalues μ_k^* is called a quasi-spectre of the pencil.*

Let us note that if μ^* is a quasi-regular eigenvalue of the pencil $L_0(\mu)$, then as opposed to a regular eigenvalue it may appear that $rank L_0(\mu^*) = \max_{\mu\in C} rank L_0(\mu)$.

The basic objective of the present paper is to investigate properties of a quasi-spectre of the singular pencil $L_0(\mu)$. Introducing the notion of a quasi-spectre of a pencil was naturally motivated in the theory of multi-parameter reverse spectral problems (MPRSP) by the problem of solving (MPRSP), which are perturbations-resistant. Let us note that similar problems, in particular, the notion of a regular spectre of pencils of operators were considered in ([4]–[7]).

Due to the objectives of the research (without loss of generality) we make the following two suppositions.

Firstly, we consider that

$$\mathrm{rank} B_0 = \max_{\mu\in\overline{\mathbb{C}}}\mathrm{rank}(A_0 - \mu B_0) = m. \tag{6}$$

If it turns out that the $\mathrm{rank} B_0 < \max_{\mu\in\overline{\mathbb{C}}}\mathrm{rank}(A_0 - \mu B_0)$, then we can move over the pencil satisfying the condition (6)with the help of a fractional-linear transformation of the spectral parameter μ.

Indeed, it results from (3) that there exists $\mu^* \in \mathbb{C}$ such that $\mathrm{rank}(A_0 + \mu^* B_0) = m$. Then

$$(A_0 + \varepsilon A_1) - \mu(B_0 + \varepsilon B_1) = \frac{\mu^* - \mu}{\mu^*}\left[A_0 + \varepsilon A_1 - \frac{\mu}{\mu - \mu^*}\left(A_0 + \varepsilon A_1 - \mu^*(B_0 + \varepsilon B_1)\right)\right].$$

Now, assuming that

$$s = \frac{\mu}{\mu - \mu^*}, \quad \hat{B}(\varepsilon) = A_0 + \varepsilon A_1 - \mu^*(B_0 + \varepsilon B_1), \quad \hat{A}(\varepsilon) = A_0 + \varepsilon A_1,$$

we proceed to research of the equivalent pencil $\hat{L}(s,\varepsilon)=\hat{A}(\varepsilon)-s\hat{B}(\varepsilon)$, where rank$\hat{B}(0)=m$. The quasi-regular spectres of pencils $\hat{L}(s,\varepsilon)$ and $L(\mu,\varepsilon)$ are connected by the fractional-linear transformation $s=\frac{\mu}{\mu-\mu^*}$ in this transformation.

Secondly, we consider that $B=B^*\geq 0$. We can achieve this condition, if we proceed to the strictly equivalent pencil $\tilde{L}(\mu,\varepsilon)=UL(\mu,\varepsilon)V$, where $U,V:E^n\to E^n$ are unitary matrices, included into the singular expansion of the matrix B, i.e. $\tilde{B}=BUV$.

Now, in addition to the conditions (2), (3), we consider that

$$\operatorname{rank}(A_0-\mu B_0)=\operatorname{rank}(B_0)=m,\quad B_0^*=B_0>0. \tag{7}$$

2. An unperturbed pencil

In this section we consider an irregular unperturbed pencil

$$L_0(\mu)=A-\mu B:\mathbb{E}^n\to\mathbb{E}^n, \tag{8}$$

satisfying the conditions

$$\max_{\mu\in\overline{\mathbb{C}}}\operatorname{rank}(A-\mu B)=\operatorname{rank}(B)=m<n,\quad B=B^*\geq 0. \tag{9}$$

Let P be a selfconjugated projection on the subspace $V_2=KerB$, $V_1=V_2^{\perp}$. Then the pencil $L_0(\mu)$ in a suitable basis can be presented in the form

$$L_0(\mu)=\begin{pmatrix} A_{11}-\mu B & A_{12}\\ A_{21} & A_{22}\end{pmatrix} \tag{10}$$

where $A_{11}=(I-P)A(I-P):V_1\to V_1$, $A_{12}=(I-P)AP:V_2\to V_1$,
$A_{21}=PA(I-P):V_1\to V_2$ and $A_{22}=PAP:V_2\to V_2$.

Note that the pencil $L_{11}(\mu)=A_{11}-\mu B:V_1\to V_1$ is regular.

Theorem 1. *Let the conditions (9) hold, then*

$$A_{22}-A_{12}(A_{11}-\mu B)^{-1}A_{21}\equiv 0 \tag{11}$$

in the representation (10) of the pencil $L_0(\mu)$ for any $\mu\in\mathbb{C}$.

Proof. The matrix

$$F(\mu)=\begin{pmatrix} I & -(A_{11}-\mu B)^{-1}A_{12}\\ 0 & I\end{pmatrix}, \tag{12}$$

by virtue of regularity of the pencil $A_{11}-\mu B:V_1\to V_1$, exists, is bounded and degenerated with all $\mu\in\mathbb{C}$ except for zeroes of the characteristic polynomial $\det(A_{11}-\mu B)$. Consequently, rank$L_0(\mu)=\operatorname{rank}(L_0(\mu)F(\mu))$. Hence

$$L_0(\mu)F(\mu)=\begin{pmatrix} A_{11}-\mu B & 0\\ A_{21} & A_{22}-A_{21}(A_{11}-\mu B)^{-1}A_{12}\end{pmatrix}.$$

Now, using a block-diagonal matrix $L_0(\mu)F(\mu)$ we obtain

$$\operatorname{rank}(L_0(\mu)F(\mu))=\operatorname{rank}(A_{11}-\mu B)+\operatorname{rank}[A_{22}-A_{21}(A_{11}-\mu B)^{-1}A_{12}].$$

It results from (9) and (10) that $\operatorname{rank}(A_{11}-\mu B)=m$, therefore $\operatorname{rank}(A_{22}-A_{21}(A_{11}-\mu B)^{-1}A_{12})\equiv 0$ with all $\mu\in\mathbb{C}$ with the exception of zeros of the characteristic polynomial $\det(A_{11}-\mu B)$. The latest is possible only in the case when $A_{22}-A_{21}(A_{11}-\mu B)^{-1}A_{12})\equiv 0$, hence we prove the Theorem.

The proved Theorem denotes that further we can consider pencils of the form

$$L_0(\mu)=\begin{pmatrix} A_{11}-\mu B & A_{12}\\ A_{21} & 0\end{pmatrix}, \tag{13}$$

which block matrices satisfy the identity $A_{22}-A_{12}(A_{11}-\mu B)^{-1}A_{21}\equiv 0$.

The following property of the considered pencil results from this identity.

Theorem 2. *Let $\mu^* \in \mathbb{C}$ be an arbitrary simple eigenvalue of the pencil $L_{11}(\mu) = A_{11} - \mu B : V_1 \to V_1$ and*

$$A_{11}\vec{x}^* = \mu^* B\vec{x}^*, \quad A_{11}^*\vec{y}^* = \mu^* B\vec{y}^*. \tag{14}$$

Then either $A_{21}(\vec{x})^* = 0$ or $A_{12}^*(\vec{y})^* = 0$.

Proof. Let us consider two alternative cases:

- $\|(A_{11} - \mu B)^{-1}A_{12}\|$ is bounded when $\mu \to \mu^*$;
- $\|(A_{11} - \mu B)^{-1}A_{12}\|$ is not bounded when $\mu \to \mu^*$.

In the first case all the elements of the matrix $(A_{11} - \mu B)^{-1}A_{12}$ are rational functions and are bounded in the neighbourhood of the point $\mu = \mu^*$ and elements of the matrix $(A_{11} - \mu B)^{-1}A_{12}$ do not have specifications in the point $\mu = \mu^*$. Consequently, elements of the matrix $A_{12}^*(A_{11}^* - sB)^{-1}$ are also analytical functions in the neighbourhood of the point $s = \bar{\mu}^*$.

On the other hand, there is a vector $\vec{e}$ such that the eigenvector $\vec{y}^*$ of the pencil $A_{11}^* - \mu B$ can be presented in the form

$$\vec{y}^* = \frac{1}{2\pi i} \oint\limits_{|S-\bar{\mu}^*|=\delta} (A_{11}^* - sB)^{-1}\vec{e}ds.$$

But then

$$A_{12}^*\vec{y}^* = \frac{1}{2\pi i} \oint\limits_{|S-\bar{\mu}^*|=\delta} A_{12}^*(A_{11}^* - sB)^{-1}\vec{e}ds = 0,$$

by virtue of the fact that $A_{12}^*(A_{11}^* - sB)^{-1}$ is an analytical function in the point $S = \bar{\mu}^*$.

In the second case it results from the unboundedness of the operator $(A_{11} - sB)^{-1}A_{12}$ that any nontrivial vector

$$\vec{x}_* = \frac{1}{2\pi i} \oint\limits_{|s-\mu^*|=\delta} (A_{11} - sB)^{-1}A_{12}\vec{\varphi}ds$$

is an eigenvector of the pencil $A_{11} - sB$. Further, taking into account the Theorem 1, we obtain

$$A_{21}\vec{x}_* = \frac{1}{2\pi i} \oint\limits_{|S-\mu^*|=\delta} A_{21}(A_{11} - sB)^{-1}A_{12}\vec{\varphi}ds = 0$$

for any $\vec{\varphi} \in \mathbb{E}^n$. The proof of the Theorem results from the considered cases.

We formulate the general statement of the present section for the pencils of the form

$$C_0(\mu) = \begin{pmatrix} C_{11} - \mu I_1 & C_{12} \\ C_{21} & 0 \end{pmatrix} : \mathbb{E}^n \to \mathbb{E}^n, \tag{15}$$

where $\text{rank}C_0(\mu) = \text{rank}(C_{11} - \mu I_1) = m < n$, $V_2 = KerI_1$, $V_1 = V_2^{\perp}$.

The pencils $L_0(\mu)$ and $C_0(\mu)$ are equivalent due to

$$C_0(\mu) = \begin{pmatrix} B^{-1} & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} A_{11} - \mu B & A_{12} \\ A_{21} & 0 \end{pmatrix}, \tag{16}$$

$C_{11} = B^{-1}A_{11}$, $C_{12} = B^{-1}A_{12}$, $C_{21} = A_{21}$, and I_1 and I_2 are unique operators in the subspaces V_1 and V_2, respectively.

Theorem 3. *Let the range of the pencil $C_0(\mu)$ of the form (15) be equal to $m < n$, and all the eigenvalues of the matrix C_{11} be simple. Then*

$$q(\mu) = det(C_{11} - \mu I) \tag{17}$$

is a general divisor of all the minors of the m-th order of the pencil $(C_0(\mu))^2$.

Proof. Let μ^* be an arbitrary simple eigenvalue of the matrix C_{11} and

$$C_{11}\vec{x}^* = \mu^*\vec{x}^*, \quad C_{11}^*\vec{y}^* = \bar{\mu}^*\vec{y}^* \tag{18}$$

According to the statement of the Theorem 2 we have $\|C_{11}\vec{x}^*\| \cdot \|C_{11}^*\vec{y}^*\| = 0$. Let us consider all the possible cases for the eigenvectors $\vec{x}^*$ and $\vec{y}^*$ resulting in the specified equality.

Let us assume that $C_{21}\vec{x}^* = 0$ and $C_{12}^*\vec{y}^* = 0$. Since μ^* is a simple eigenvalue of the matrix C_{11}, then for any $\vec{\varphi} \in \mathbb{E}^m$

$$\vec{x}^* = \frac{1}{2\pi i} \oint\limits_{|\mu-\mu^*|=\delta} (C_{11} - \mu I)^{-1}\vec{\varphi}d\mu$$

is an eigenvector of the matrix C_{11}. Therefore, due to the supposition $C_{21}\vec{x}^* = 0$ we have

$$C_{21}\vec{x}^* = \frac{1}{2\pi i} \oint\limits_{|\mu-\mu^*|=\delta} C_{21}(C_{11} - \mu I)^{-1}\vec{\varphi}d\mu = 0$$

for any $\vec{\varphi} \in \mathbb{E}^m$. And it denotes that $C_{21}(C_{11} - \mu I)^{-1}$ is bounded in the neighbourhood $\mu = \mu^*$ and the point $\mu = \mu^*$ is a singular point of an eliminable type. Hence, $(C_{11}^* - \mu I)^{-1}C_{21}^*$ is also analytical in the neighbourhood of the point $\mu = \bar{\mu}^*$. Let us now show that $Ker(C_0(\mu^*))^*$ contains not less than $n - m + 1$ linear-independent vectors.

Let us consider the vectors

$$\vec{\varphi}_0 = \begin{pmatrix} \vec{y}^* \\ 0 \end{pmatrix}, \quad \vec{\varphi}_k = \begin{pmatrix} -(C_{11}^* - \mu I)^{-1}C_{21}^*\vec{e}_k \\ \vec{e}_k \end{pmatrix}, \quad k = \overline{1, n-m},$$

where $\vec{e}_k$ make a unique basis of the space V_2, and $\vec{y}^*$, by the condition, is an eigenvector of the matrix C_{11}^*, corresponding to the simple eigenvalue $\bar{\mu}^*$.

Taking into account that $C_{11}\vec{y}^* = \bar{\mu}^*\vec{y}^*$, $C_{12}^*\vec{y}^* = 0$, $C_{12}^*(C_{11}^* - \mu I)^{-1}C_{21}^* \equiv 0$, it is easy to show that $(C_0(\mu_*))^*\vec{\varphi}_k = 0$ for all $k = \overline{0, n-m}$.

Since $dimKer(C_0^*(\mu_*)) \geq n - m + 1$, then $rankC_0(\mu_*) \leqslant m - 1$.

Therefore, $rankC_0(\mu) = m$ when $\mu \neq \mu_*$ and $rankC_0(\mu_*) = m$ is possible only under the condition that $\mu - \mu_*$ is a divisor of all the minors of m-th matrix $C_0(\mu)$.

Assume now that $C_{21}\vec{x}_* \neq 0$ and $C_{12}^*\vec{y}_* = 0$. Let us first show that in this case $(C_{11} - \mu I)^{-1}C_{12}$ is bounded when $\mu \to \mu_*$.

Indeed, on the contrary, there is a vector $\vec{\varphi}$ such that

$$\frac{1}{2\pi i} \oint\limits_{|\mu-\mu_*|=\delta} (C_{11} - \mu I)^{-1}C_{12}\vec{\varphi}d\mu = \vec{\psi} \neq 0.$$

Since $\vec{\psi}$ is an eigenvector corresponding to the eigenvalue μ_*, then we can consider that $\vec{x}^* = \frac{1}{2\pi i} \oint\limits_{|\mu-\mu_*|=\delta} (C_{11} - \mu I)^{-1}C_{12}\vec{\varphi}d\mu$. It results from the Theorem 1 that

$$C_{21}\vec{x}^* = \frac{1}{2\pi i} \oint\limits_{|\mu-\mu_*|=\delta} C_{21}(C_{11} - \mu I)^{-1}C_{12}\vec{\varphi}d\mu = 0,$$

and it contradicts the condition $C_{21}\vec{x}^* \neq 0$. Consequently, $(C_{11} - \mu I_1)^{-1}C_{12}$ is bounded in the neighbourhood of the point $\mu = \mu^*$.

Let us introduce the matrix

$$D(\mu) = \begin{pmatrix} I & -(C_{11} - \mu I)^{-1}C_{12} \\ 0 & I \end{pmatrix}, \tag{19}$$

into consideration. Due to the boundedness of $(C_{11} - \mu I)^{-1}C_{12}$ the matrix $D(\mu)$ is determined and degenerated in the point $\mu = \mu^*$ and its neighbourhood.

Let us determine the matrix $G(\mu) = (C_0(\mu))^2 D(\mu)$. It is obvious that in the point $\mu = \mu^*$ and in some its neighbourhood $rankG(\mu) = rank(C_0(\mu))^2$. According to the statement of the Theorem 1 with all μ, when $det(C_{11} - \mu I) \neq 0$, the following identity holds: $C_{21}(C_{11} - \mu I)^{-1}C_{12} \equiv 0$. Taking this into account it is easy to obtain that

$$G(\mu) = \begin{pmatrix} (C_{11} - \mu I)^2 + C_{12}C_{21} & 0 \\ C_{21}(C_{11} - \mu I) & 0 \end{pmatrix}. \tag{20}$$

Since $(C_{11}^* - \bar{\mu}_* I)\vec{y_*} = 0$, $C_{12}^*\vec{y} = 0$ then

$$[(C_{11} - \mu^* I)^2 + C_{12}C_{21}]^*\vec{y_*} = 0,$$

consequently, $rank((C_{11} - \mu I)^2 + C_{12}C_{21}) \leqslant m - 1$.

The latest denotes that there is a vector $\vec{\varphi}^* \neq 0$ such that $(C_{11} - \mu I)^2\vec{\varphi}^* = -C_{12}C_{21}\vec{\varphi}^*$. Taking into account the boundedness of $(C_{11} - \mu I)^{-1}C_{12}$ in the point $\mu = \mu_*$, we have $(C_{11} - \mu^* I)\vec{\varphi}^* = -(C_{11} - \mu^* I)^{-1}C_{12}C_{21}\vec{\varphi}^*$. Hence we obtain

$$C_{21}(C_{11} - \mu^* I)\vec{\varphi}^* = -C_{21}(C_{11} - \mu^* I)^{-1}C_{12}C_{21}\vec{\varphi}^* = 0.$$

Therefore, the vectors

$$\begin{pmatrix} \vec{\varphi}^* \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \vec{e}_k \end{pmatrix}, \quad k = \overline{1, n-m},$$

where $\vec{e}_k$, $k = \overline{1, n-m}$ are basis vectors of the space V_2, and are contained in $kerG(\mu^*)$.

Consequently, $rank[C_0(\mu)]^2 = m$ when $\mu \neq \mu^*$ and $rank[C_0(\mu)]^2 \leqslant m - 1$ when $\mu = \mu^*$. It follows that $\mu - \mu^*$ is a divisor of all the minors of the m-th order of the matrix $[C_0(\mu)]^2$.

The third possible case resulting from $\|C_{21}\vec{x}_*\| \cdot \|C_{12}^*\vec{y}_*\| = 0$, namely $C_{12}^*\vec{y}_* \neq 0$, $C_{21}\vec{x}_* = 0$, is investigated similarly to the case $C_{21}\vec{x}_* \neq 0$, and $C_{12}^*\vec{y}_* = 0$.

Thus, we have shown that for the arbitrary eigenvalue μ_* of the matrix C_{11} all the minors of the m-th order of the pencil $(C_0(\mu))^2$ are divided into $\mu - \mu_*$. Since all the eigenvalues C_{11} are simple and μ_* is an arbitrary eigenvalue, then $det(C_{11} - \mu I_1)$ is a divisor of all the minors of the m-th order of the matrix $(C_0(\mu))^2$. The Theorem has been proved.

Theorem 4. *Let the conditions of the Theorem 3 hold. Then if $\mu - \mu_0$ is a general divisor of all the minors of the m-th order of the pencil $(C_0(\mu))^2$, then $q(\mu_0) = det(C_{11} - \mu_0 I) = 0$.*

Proof. Assume that $q(\mu_0) = det(C_{11} - \mu_0 I) \neq 0$. Similarly to the proof of the Theorem (3) let us consider the matrix $D(\mu)$ set in (19) and the matrix $G(\mu) = (C_0(\mu))^2 D(\mu)$. According to (19) and (20) we have

$$G(\mu) = \begin{pmatrix} (C_{11} - \mu I)^2 + C_{12}C_{21} & 0 \\ C_{21}(C_{11} - \mu I) & 0 \end{pmatrix}.$$

Since μ_0 is zero of all the minors of the order m, there is $\vec{x}_0 \neq 0$ such that $((C_{11} - \mu_0 I)^2 + C_{12}C_{21})\vec{x}_0 = 0$.

It results from the supposition $q(\mu_0) \neq 0$ that $\vec{x}_0 = -(C_{11} - \mu_0 I)^{-2}C_{12}C_{21}$. Since it results from Theorem 1 that $C_{21}(C_{11} - \mu I)^{-2}C_{12} \equiv 0$, then $C_{21}\vec{x}_0 = 0$. It means that $(C_{11} - \mu_0 I)^2\vec{x}_0 = 0$. That is μ_0 is an eigenvalue of C_{11}. The resulting contradiction proves the Theorem.

Therefore, $(C_0(\mu))^2$ has "real" eigenvalues in the following sense. Namely, if μ_* is an eigenvalue of the matrix C_{11} and $\mu \notin \sigma(C_{11})$, then

$$rank(C_0(\mu)) = rank(C_0(\mu))^2 = m,$$

and at the same time $rank(C_0(\mu_*))^2 \leqslant m - 1$. In other words, all the eigenvalues of the matrix C_{11} are regular eigenvalues for the pencil $[C_0(\mu)]^2$.

As an example we consider the pencil

$$C_0(\mu) = \begin{pmatrix} \mu_1-\mu & 0 & 0 & 0 & 0 \\ 0 & \mu_2-\mu & 0 & 1 & 0 \\ 0 & 0 & \mu_3-\mu & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to calculate with all $\mu \in \bar{\mathbb{C}}$ that $rankC_0(\mu) \equiv 3$. Let us write the square of the pencil $C_0(\mu)$,

$$C_0^2(\mu) = \begin{pmatrix} (\mu_1-\mu)^2 & 0 & 0 & 0 & 0 \\ 1 & (\mu_2-\mu)^2 & 0 & \mu_2-\mu & 0 \\ 0 & 0 & (\mu_3-\mu)^2 & 0 & \mu_3-\mu \\ \mu_1-\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now $rankC_0^2(\mu) = 3$ with all $\mu \neq \mu_k$, but $rankC_0^2(\mu_k) = 2$ for $k = 1, 2, 3$.

Let us note that the highest general divisor of all the minors of the m-th order of the matrix $C_0(\mu)$ is 1, and the corresponding minors of the matrix $C_0^2(\mu)$ are divided into $(\mu_1-\mu)(\mu_2-\mu)(\mu_3-\mu) = detC_0(\mu)$.

3. A perturbed irregular pencil.

Let us now proceed to consideration of the perturbed pencil

$$L(\mu, \varepsilon) = A_0 - \mu B_0 + \varepsilon(A_1 - \mu B_1) : E^n \to E^n,$$

where

$$rankB_0 = \max_{\mu\in\mathbb{C}} rank(A_0 - \mu B_0) = m < n$$

and

$$\max_{\mu\varepsilon\in\mathbb{C}} rankL(\mu, \varepsilon) = n.$$

Due to regularity of the pencil $L(\mu, \varepsilon)$ the eigenvalues of this pencil $\mu_1(\varepsilon), \ldots, \mu_n(\varepsilon)$ are zeros of the characteristic equation

$$detL(\mu, \varepsilon) = 0 \tag{21}$$

when $\varepsilon \neq 0$. As it was shown above (16) there are degenerated matrices S_1 and S_2, which do not depend on μ and ε such that

$$C(\mu, \varepsilon) := S_1 L(\mu, \varepsilon) S_2 = C_0(\mu) + \varepsilon C_1(\mu) \tag{22}$$

$$C_1(\mu) = S_1 A_1 S_2 - \mu S_1 B_1 S_2 = C_1^{(1)} - \mu C_1^{(2)}, \tag{23}$$

and $C_0(\mu)$, has the form:

$$C_0(\mu) = \begin{pmatrix} C_{11} - \mu I_1 & C_{12} \\ C_{21} & 0 \end{pmatrix} : \mathbb{E}^n \to \mathbb{E}^n$$

as it was above in (15) and (16). Since the pencils $C(\mu, \varepsilon)$ and $L(\mu, \varepsilon)$ have similar eigenvalues $\mu_k(\varepsilon)$,we proceed to research of eigenvalues of the pencil $C(\mu, \varepsilon)$.

Theorem 5. *Let the spectre of the matrix C_{11} consist of simple eigenvalues $\mu_1^*, \ldots, \mu_m^*$. Then the pencil $C(\mu, \varepsilon)$ has exactly m eigenvalues $\mu_{j_1}(\varepsilon), \ldots, \mu_{j_m}(\varepsilon)$ (in case of the corresponding enumeration) such that for all $k = \overline{1, m}$*

$$\mu_{j_k}(0) = \mu_k^*.$$

Proof. Let $G_1(\mu)$, $G_2(\mu)$ be degenerated (to be more exact, $detG_1(\mu) \equiv 1$, $detG_2(\mu) \equiv 1$) matrices reducing the pencil $(C_0(\mu))^2$ to the normal Smith form (see [3]).

Since $rank(C_0(\mu))^2 = m$, then the matrices $G_1(\mu)$ and $G_2(\mu)$ can be chosen so that

$$G_1(\mu)C_0^2(\mu)G_2(\mu) = \begin{pmatrix} \Lambda(\mu) & 0 \\ 0 & 0 \end{pmatrix}, \tag{24}$$

where $\Lambda(\mu)$ is a diagonal matrix of the size $m \times m$. It results from the Theorems 3 and 4 that $det[\Lambda(\mu)] = 0$ if and only if $det(C_{11} - \mu I) = 0$.

Assume that

$$D(\mu,\varepsilon) = G_1(\mu)C_0^2(\mu,\varepsilon)G_2(\mu) = D_0(\mu) + \varepsilon D_1(\mu,\varepsilon). \tag{25}$$

Then, taking into account the representation (24), we obtain

$$D(\mu,\varepsilon) = \begin{pmatrix} \Lambda(\mu) + \varepsilon D_{11}(\mu,\varepsilon) & \varepsilon D_{12}(\mu,\varepsilon) \\ \varepsilon D_{21}(\mu,\varepsilon) & \varepsilon D_{22}(\mu,\varepsilon) \end{pmatrix}. \tag{26}$$

Note that it results from (25) that

$$g(\mu,\varepsilon) = detD(\mu,\varepsilon) = [detC_0(\mu,\varepsilon)]^2. \tag{27}$$

Since $detD(\mu,0) \equiv 0$, then $g(\mu,\varepsilon)$ being a polynomial from μ and ε can be presented in the form

$$g(\mu,\varepsilon) = \varepsilon^{\alpha} g_0(\mu,\varepsilon), \tag{28}$$

where $\alpha \in \mathbb{N}$, $g_0(\mu,\varepsilon)$ is a polynomial from ε and μ, and $g_0(\mu,0) \not\equiv 0$.

Let us now show that $g_0(\mu,\varepsilon)$ is divided into $det\Lambda(\mu)$.

For this purpose we calculate the determinant of the matrix $D(\mu,\varepsilon)$ as a product of zeros $\sigma_k(\mu,\varepsilon)$ of the polynomial $det(D(\mu,\varepsilon) - \sigma I) = 0$.

Let us denote the diagonal elements of the matrix $\Lambda(\mu)+\varepsilon D_{11}(\mu,\varepsilon)$ via $\lambda_1(\mu,\varepsilon),\dots,\lambda_m(\mu,\varepsilon)$, and the diagonal elements of the matrix $D_{22}(\mu,\varepsilon)$ via $d_1(\mu,\varepsilon),\dots,d_{n-m}(\mu,\varepsilon)$. Expanding $detD(\mu,\varepsilon)$ by lines we obtain

$$det(D(\mu,\varepsilon) - \sigma I) = (\lambda_1(\mu,\varepsilon) - \sigma)\cdots(\lambda_m(\mu,\varepsilon) - \sigma)(\varepsilon d_1(\mu,\varepsilon) - \sigma)\cdots$$
$$\cdots(\varepsilon d_{n-m}(\mu,\varepsilon) - \sigma) + \varepsilon h(\mu,\varepsilon,\sigma),$$

where $h(\mu,\varepsilon,\sigma)$ is some polynomial from σ, μ and ε.

If we apply the Rouche Theorem for zeros $\sigma_k(\mu,\varepsilon)$ of this polynomial $det(D(\mu,\varepsilon) - \sigma I)$ to the variable σ, when $\varepsilon \to 0$ and $|\mu| < R_0 < \infty$, we obtain $\sigma_k(\mu,\varepsilon) = \lambda_k(\mu,0) + \varepsilon\alpha_k(\mu,\varepsilon)$ for all $k = \overline{1,m}$, where $\alpha_k(\mu,\varepsilon)$ are bounded functions and $\sigma_k(\mu,\varepsilon) = O(\varepsilon)$ for all $k = \overline{m+1,n}$.

Since the determinant of the matrix $g(\mu,\varepsilon) = D(\mu,\varepsilon)$ is equal to the product of all its eigenvalues $\sigma_k(\mu,\varepsilon)$, then we obtain

$$g(\mu,\varepsilon) = \sigma_1(\mu,\varepsilon)\cdots\sigma_m(\mu,\varepsilon)\sigma_{m+1}(\mu,\varepsilon)\cdots\sigma_n(\mu,\varepsilon).$$

It follows from the above that $\sigma_1(\mu,\varepsilon)\cdots\sigma_m(\mu,\varepsilon) = det[\Lambda(\mu)] + O(\varepsilon)$ and $\sigma_{m+1}(\mu,\varepsilon)\cdots\sigma_n(\mu,\varepsilon) = \varepsilon^{\beta} d(\mu,\varepsilon)$, where $d(\mu,\varepsilon)$ is some algebraic function and $d(\mu,0) \not\equiv 0$. Hence we obtain

$$g(\mu,\varepsilon) = \varepsilon^{\beta}(det\Lambda(\mu) + O(\varepsilon))d(\mu,\varepsilon). \tag{29}$$

Now comparing (28) with (29) we come to the conclusion that

$$g_0(\mu,0) = det\Lambda(\mu)d(\mu,0), \tag{30}$$

where $d(\mu,0)$ is a polynomial from μ.

It follows from the representation of the polynomial $det(D(\mu,\varepsilon))$ in the form (30) subject to (27) and (28) that every zero of the polynomial $det[C_{11} - \mu I_1]$ is analytically (as an algebraic function) expanded on ε. This proves the theorem.

It follows from the latest statement that the pencil of the form

$$C(\mu,\varepsilon) = \begin{pmatrix} C_{11} - \mu I_1 & C_{12} \\ C_{21} & 0 \end{pmatrix} + \varepsilon(C_1 - \mu C_2) : \mathbb{E}^n \to \mathbb{E}^n$$

is equal to m eigenvalues $\mu_{j_1}(\varepsilon), \dots, \mu_{j_m}(\varepsilon)$ when $\varepsilon \to 0$ have limits equivalent to the eigenvalues $\mu_1, \dots, \mu_m$ of the matrix C_{11} independently of the matrices C_1 and C_2. At the same time it is easy to show that the limits of other eigenvalues of the pencil $C(\mu, \varepsilon)$ depend on C_1 and C_2.

Therefore, according to Definition 1, the quasi-spectre of the pencil $C_0(\mu)$ consists of the eigenvalues $\mu_1, \dots, \mu_m$ of the matrix C_{11}.

Now returning to an irregular pencil of the general form (1)–(3) we note the following. To calculate a quasi-regular spectre of the pencil $L_0(\mu) = A_0 - \mu B_0$, we first reduce it to the form satisfying the condition (7).

Then with the help of this strict equivalent transformation (16) we obtain a pencil of the form $C(\mu, \varepsilon)$ and find the spectre of the matrix C_{11}.

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