ISSN 2304-0122 Ufa Mathematical Journal. Vol. 4. No 4 (2012). P. 105-115.

UDC 517.5+517.9

ON UNIFORM APPROXIMABILITY BY SOLUTIONS OF ELLIPTIC EQUATIONS OF ORDER HIGHER THAN TWO

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Abstract. We consider uniform approximation problems on compact subsets of $\mathbb{R}^d$, $d > 2$ by solutions of homogeneous constant coefficients elliptic equations of order $n > 2$. We construct an example showing that in the general case for compact sets with nonempty interior there is no uniform approximability criteria analogous to the well-known Vitushkin's criterion for analytic functions in $\mathbb{C}$. On the contrary, for nowhere dense compact sets the situation is the same as for analytic and harmonic functions including instability of the corresponding capacities.

Keywords: elliptic equations, capacities, instability of capacities, uniform approximation, Vitushkin's scheme.

1. Introduction

Let us assume that $L(x)$ is a homogeneous elliptic polynomial with complex coefficients (where $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, $d \geqslant 2$, $L(x) = 0 \Leftrightarrow x = 0$), $L = L(\nabla)$ is the corresponding differential operator; further we consider only such operators L. Let us denote the order of the operator L by n. Recall [1, Theorem 7.1.20] that L has a fundamental solution of the form

$$E(x) = E_0(x) - E_1(x) \log |x|, \tag{1.1}$$

where E_0 is a real-analytical function in $\mathbb{R}^d \setminus \{0\}$, which is homogeneous and has the degree $n-d$, E_1 is a homogeneous polynomial of degree $n - d$ (if $n < d$, then $E_1 \equiv 0$).

Let $X \subset \mathbb{R}^d$ be a compact, X^o be the set of all inner points of X,

$$h(X, L) = C(X) \cap \{Lf = 0 \text{ in } X^o\},$$

$H(X, L)$ be the closure in $C(X)$ of the set of functions

$$\{f|_X : Lf = 0 \text{ in a certain neighbourhood } X\}$$

(the neighbourhood depends on the function f).

Since L is an elliptic operator, then $H(X, L) \subset h(X, L)$. Criteria of identity of the classes $H(X, L) = h(X, L)$ were obtained in the case of analytic functions ($d = 2$, L is the Cauchy-Riemann operator) by A.G. Vitushkin [2], and in the case of harmonic functions ($d > 2$, L is the Laplace operator) by J.Deny [3] and M.V. Keldysh [4] independently. There is the following statement (we simplify formulation to some extent):

$$H(X, L) = h(X, L) \Longleftrightarrow \mathrm{Cap}_L(B \setminus X^o) \leqslant A\mathrm{Cap}_L(kB \setminus X), \tag{1.2}$$

where $\mathrm{Cap}_L(\cdot)$ is considerable analytic or harmonic capacity, respectively, B is an arbitrary open ball (disk with $d = 2$), $A > 0$ and $k \geqslant 1$ are fixed constants.

The work was supported by the grant of RFFR 12-01-00434 and the grant NSh-3476.2010.1 of the program of support of the leading scientific schools.

Submitted on October 1, 2011.

While studying removable singularities of continuous solutions of the equation $Lf = 0$ R. Harvey and G. Polking introduced capacities [5], which naturally generalize analytic and harmonic capacities. In the present paper we consider only the case $n < d$. Following [5, definition 1.1], we term

$$\sup_g\{|\langle Lg|1\rangle| : \|g\|_{\mathrm{L}_\infty} \leqslant 1,\, g \in C(\mathbb{R}^d),\, \lim_{x\to\infty} g(x) = 0,\, \mathrm{Spt}(Lg) \subset U\} \tag{1.3}$$

as *the capacity* of the bounded set U and denote it by $\mathrm{Cap}_\mathrm{L}(U)$ (here and in what follows $\|\cdot\|_{\mathrm{L}_\infty} = \|\cdot\|_{\mathrm{L}_\infty}(\mathbb{R}^d)$). In the formula (1.3), the brackets indicate the action of a distribution with a compact support to an infinitely smooth function, $\mathrm{Spt}(\cdot)$ is the closure of the distribution support. Namely,

$$\langle Lg|1\rangle = (-1)^n \int g(x)L\varphi(x)dm_x, \tag{1.4}$$

where φ is an arbitrary function from $C_0^\infty(\mathbb{R}^d)$ such that $\varphi(x) \equiv 1$ in some neighbourhood of $\mathrm{Spt}(Lg)$, integration is carried out by the Lebesgue measure in $\mathbb{R}^d$. The function $g \in C(\mathbb{R}^d)$ such that $\mathrm{Spt}(Lg) \subset U$ and $\lim_{x\to\infty} g(x) = 0$ is said to be *an admitted for* U.

Since the capacity $\mathrm{Cap}_L(\cdot)$ characterises "solidity" of sets of unremovable singularities of continuous solutions of the equation $Lf = 0$ [5, Theorem 1.4], the inequality in the right-hand side (1.2) has the following natural meaning: supplement to the compact is not "less massive", then its boundary locally.

We have stated the following in the present paper.

1. We have shown that when $d > 2$ and $n < d$ for every corresponding L the inequality in the right-hand side (1.2) is necessary for the equality $H(X, L) = h(X, L)$ (see corollary 1 of Lemma 4), but it is not sufficient (see example 1 in §4).

Note that in the case of approximation in the Lipschitz spaces of the nonintegral order and BMO spaces (namely when we apply the capacity which is unlike (1.3), comparable with the corresponding Hausdorff content) the similar examples are constructed in [6, §4]; the construction of the example 1 in the present paper is significantly more simple than that one in [6].

For the uniform approximations it is necessary to note a special role of the dimension $d = 2$: in [7, Theorem 1] it has been proved that the equality $H(X, L) = h(X, L)$ holds for any compact X and the operator L when $d = 2$ in case of local boundedness of E from (1.1); in addition, when $d > 2$ for any of the considered operators L (including the locally bounded fundamental solution) there is a compact X such that $H(X, L) \neq h(X, L)$ (for example [8, Theorem 8.2]).

2. When $X^o = \emptyset$ (and respectively, $h(X, L) = C(X)$) the situation is significantly more simple than in the general case: there hold not only (1.2), but also an instability of the capacity $\mathrm{Cap}_L(\cdot)$, similar to the instability of the analytic and harmonic capacities (see Theorem 1 from [2, Ch. 6, §2], Theorem B from [9] and Theorem B from [10]). Namely, the following statement holds.

Theorem 1. *Assume that $X^o = \emptyset$.*

(1) If the equality $C(X) = H(X, L)$ holds, then for any open ball $B(x, r)$ (with the centre x of the radius r) the estimate

$$\mathrm{Cap}_L(B(x, r) \setminus X) \geqslant Ar^{d-n}, \tag{1.5}$$

where $A = A(L) > 0$, exists.

(2) Let the following estimate hold for almost all $x \in X$ (by the Lebesgue measure of the space $\mathbb{R}^d$)

$$\limsup_{r\to 0} \frac{\mathrm{Cap}_L(B(x, r) \setminus X)}{r^d} > 0, \tag{1.6}$$

then $C(X) = H(X, L)$.

Statement (1) of Theorem 1 results from the definition of the capacity (see corollary 2 of Lemma 4). The proof of statement (2) consists of two parts.

1. The proof (1.5)$\Rightarrow$ $C(X) = H(X, L)$ is given with the help of the improved Vitushkin's scheme [2] of separation of singularities and approximation of the function by parts and it results from the Lemmas 5 and 8.

2. In the proof (1.6)$\Rightarrow$(1.5) (see Lemma 9) there are arguments different from the ones used in the papers [2] by A.G. Vitushkin, [9] A.A. Gonchar and [10] Yu.A. Lysenko and B.M. Pisarevsky.

The problem of the (natural) criterion of the equality $H(X, L) = h(X, L)$ in the case $d > 2$, $n > 2$ and compacts X with not empty interior remains open. Let us recall, that N.N. Tarhanov in [11] proved the analogue of the A.G. Vitushkin theorem for solutions of elliptic systems. In a particular case of uniform approximations, the result is formulated in terms of the capacity corresponding to (1.3), and several capacities, generalizing (1.3), which appear "redundant" for the classes of analytic and harmonic functions Let us note in this connection that applying Theorem 2 from [7], in condition 3) of Lemma 3.8 from [11] (on approximation of functions by parts) we can substitute $|x|^n$ by $|x|^{n-1}$ to decrease the number of capacity applied for all the operators L.

2. Preparatory results.

Let us apply elementary properties of the capacities $\mathrm{Cap}_L(\cdot)$, resulting from (1.3).

(1) $\mathrm{Cap}_L(U) \leqslant \mathrm{Cap}_L(U')$ *when* $U \subset U'$.

(2) $\mathrm{Cap}_L(B(a, r)) = Ar^{d-n}$, *where* $A = A(L) > 0$.

(Let us remind that n is order of the operator L, d is dimension of the space).

Let us consider that every function from $h(X, L)$ can be extended from X to the entire space $\mathbb{R}^d$ as continuous and compactly supported (it can be carried out, e.g., by the Whitney Theorem [12, Ch. 6, s. 2.2]).

Let us fix a fundamental solution E from (1.1). Let the function f be continuous in $\mathbb{R}^d$, $\mathrm{Spt}(Lf)$ be compact and $\lim_{x\to\infty} f(x) = 0$. Then (for example, [7, Lemma 1.3]) the following equality holds

$$f = E * (Lf), \tag{2.1}$$

considered in the sense of distributions.

Let $\{\varphi_j\}$ be the finite family of non-negative functions $\varphi_j \in C_0^\infty(\mathbb{R}^d)$ such that $\sum_j \varphi_j(x) \equiv 1$ in some neighbourhood of $\mathrm{Spt}(Lf)$. Let $\{\varphi_j\}$ be *a partition of unit* on $\mathrm{Spt}(Lf)$. The function f is presented in the form of the sum of *localizations*:

$$f = \sum_j f_j, \quad \text{where} \quad f_j = E * (\varphi_j Lf), \tag{2.2}$$

and the corresponding operator V_φ:

$$V_\varphi \Psi = E * (\varphi L\Psi), \tag{2.3}$$

where $\varphi \in C_0(\mathbb{R}^d)$, $\Psi \in (C_0^\infty(\mathbb{R}^d))'$ is called an *operator of localization.*

Further $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ is a multi-index,

$$|\alpha| = \sum_{k=1}^{d} \alpha_k, \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}},$$

$$\alpha! = \alpha_1! \alpha_2! \dots \alpha_d!, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}.$$

Everywhere in the paper we denote closed cubes with edges parallel to the coordinates axes by *cubes*. For the cube $Q = Q(a, s)$ with the centre $a \in \mathbb{R}^d$ and the edge s we denote a cube with the same centre and edge λs via λQ. We term cubes of the form

$$Q = Q_k^{m_1,\dots,m_d} = [m_1 2^{-k}, (m_1 + 1)2^{-k}] \times \dots \times [m_d 2^{-k}, (m_d + 1)2^{-k}], \tag{2.4}$$

where k, m_1, m_2,..., $m_d \in \mathbb{Z}$ as *binary cubes.*

Considering covers of compacts sets by finite families of binary cubes, we always assume that the cubes are disjoint (do not have common inner points).

Positive constants, which can depend only on L (in particular, on n or d) are denoted by A, A_0, A_1, ... Values of each of these constants in various relationships can be different. We will use partitions of unity by R. Harvey and G. Polking (see [13, Lemma 3.1], [7, Lemma 1.1]).

Lemma 1. *Let $\{Q_j\}$ be a finite family of disjoint binary cubes. Then there exists $\{\varphi_j\}$ a partition of unity on $\bigcup_j Q_j$ such that*

(1) $\mathrm{Spt}\varphi_j \subset (3/2)Q_j$*;*

(2) $\|\partial^\alpha \varphi_j\|_{\mathrm{L}_\infty} \leqslant As(Q_j)^{-|\alpha|}$ *when* $|\alpha| \leqslant n$*.*

Further we consider the localizations (2.3) only with respect to the functions φ, satisfying conditions of Lemma 1. The following Lemma in proved in a standard way (for example, [7, Lemma 1.2], [14, Lemma 14.10]).

Lemma 2. *Let $f \in h(X, L)$, $\omega_f(s)$ be the module of continuity f in $\mathbb{R}^d$, $Q = Q(a, s)$ be the cube (not necessarily binary), the function $\varphi \in C_0^\infty(\mathbb{R}^d)$ satisfy the conditions (1)–(2) of Lemma 1 with respect to the cube Q, $V_\varphi f$ from (2.3). Then:*

(1) $V_\varphi f \in C(\mathbb{R}^d)$ *and* $\lim_{x\to\infty} V_\varphi f(x) = 0$*;*

(2) $\mathrm{Spt}(L(V_\varphi f)) \subset (\mathrm{Spt}\varphi \bigcap \mathrm{Spt}Lf)$*;*

(3) $\|V_\varphi f\|_{\mathrm{L}_\infty} \leqslant A\omega_f(s)$*.*

Meanwhile everywhere outside the cube $A_1 Q$ the function $V_\varphi f$ is expanded into the Laurent series converging in C^∞ (for example, [7, §1] [14, §7, s. 2^o], [15, p. 163]):

$$V_\varphi f = \sum_{|\alpha|\geqslant 0} c_\alpha \partial^\alpha E(x - a), \tag{2.5}$$

where

$$c_\alpha = c_\alpha(V_\varphi f, a) = \frac{(-1)^{|\alpha|}}{\alpha!}\langle \varphi(y)Lf(y)|(y - a)^\alpha\rangle \tag{2.6}$$

are Laurent coefficients. In particular,

$$c_0(V_\varphi f) = \langle \varphi Lf|1\rangle. \tag{2.7}$$

Estimates of the Laurent coefficients of the localizations result from Lemma 2 and (1.3)-(1.4). Since the function $V_\varphi f$ is admissible for $(3/2)Q \setminus X^o$, we have:

$$|c_0(V_\varphi f)| \leqslant A\omega_f(s)\mathrm{Cap}_L((3/2)Q \setminus X^o) \tag{2.8}$$

due to the definition of the capacity (1.3)

To estimate the Laurent coefficients c_α, $|\alpha| > 0$, we use the same line of reasoning as in the proof of Lemma 3.3 from [16]: it is evident that condition (2) of Lemma 1 holds with sufficiently small $A_1 = A_1(n) > 0$ for the function $\psi(y) = A_1(2s)^{-|\alpha|}(y - a)^\alpha \varphi(y)$; applying Lemma 2 to the localization $V_\psi f$ and taking into account

$$c_0(V_\psi f) = \langle \psi Lf|1\rangle =$$

$$= A_1(2s)^{-|\alpha|}\langle \varphi(y)Lf(y)|(y - a)^\alpha\rangle = A_1(2s)^{-|\alpha|}\alpha!(-1)^{-|\alpha|}c_\alpha(V_\varphi f, a),$$

we obtain

$$|c_\alpha(V_\varphi f, a)| \leqslant \frac{A_2}{\alpha!}\omega_f(s)(2s)^{|\alpha|}\mathrm{Cap}_L((3/2)Q \setminus X^o). \tag{2.9}$$

In view of the inequalities

$$|\partial^\alpha E(x)| \leqslant \frac{\alpha! A_3^{|\alpha|}}{|x|^{d-n+|\alpha|}}$$

(for example, [14, §7, Lemma 7.3]), carrying out the summation of the geometric progression, we obtain from (2.9) that outside the sufficiently big cube A_4Q the following estimate holds:

$$|V_\varphi f(x)| \leqslant A\omega_f(s)\frac{\mathrm{Cap}_L((3/2)Q\setminus X^o)}{|x-a|^{d-n}}. \tag{2.10}$$

Remark 2.1. Applying supplementary partition of unity on $(3/2)Q$, one can easily demonstrate (see, for example, [17, Lemma 1.5]) that the estimate (2.10) holds for any $\lambda>0$ everywhere outside the cube $(3/2+\lambda)Q$ with (an increased) constant $A=A(\lambda)$. Likewise, let $B=B(a,r)$ be a ball, $\mathrm{Spt}\varphi\subset B$ and $\|\partial^\alpha\varphi\|_{\mathrm{L}_\infty}\leqslant r^{-|\alpha|}$ when $|\alpha|\leqslant n$, $V_\varphi f$ be a localization. Then, if $\lambda>1$, everywhere outside the ball λB the following estimate holds:

$$|V_\varphi f(x)| \leqslant A(\lambda)\omega_f(r)\frac{\mathrm{Cap}_L(B\setminus X^o)}{|x-a|^{d-n}}. \tag{2.11}$$

Due to Lemma 2, the equalities $\mathrm{Cap}_L(B(a,r))=A(L)r^{d-n}$ and the monotony of the capacity, we obtain the following statement, which is a simple corollary (2.9).

Lemma 3. *The following estimates hold:*

$$|c_\alpha(V_\varphi f,a)| \leqslant \frac{A}{\alpha!}\omega_f(s)(2s)^{d-n+|\alpha|}. \tag{2.12}$$

Assume that $m\in\mathbb{Z}_+$, $m\leqslant n+1$; if $c_\alpha(V_\varphi f,a)=0$ for $|\alpha|<m$, then

$$|V_\varphi f(x)| \leqslant A\omega_f(s)\min\left(1,\frac{s^{d-n+m}}{|x-a|^{d-n+m}}\right). \tag{2.13}$$

The fact that the estimate $\mathrm{Cap}_L(B\setminus X^o)\leqslant A\mathrm{Cap}_L(kB\setminus X)$ is necessary for the equality $h(X,L)=H(X,L)$ results from the following statement.

Lemma 4. *Assume that $f\in H(X,L)$. Then for any localization $V_\varphi f$ satisfying conditions of Lemma 2 the following estimate holds:*

$$|c_0(V_\varphi f)| \leqslant A\omega_f(s)\mathrm{Cap}_L((3/2)Q\setminus X). \tag{2.14}$$

Proof. Assume that $f\in H(X,L)$. Then for any $\varepsilon>0$ there is a function $F\in C(\mathbb{R}^d)$ such that in some neighbourhood X the conditions $LF=0$ and $|f(x)-F(x)|<\varepsilon\leqslant\omega_f(s)$ hold (if we extend the difference $f-F$ by the Whitney theorem [12, Ch. 6, s. 2.2], we consider that this inequality holds everywhere in $\mathbb{R}^d$).

Due to (1.4), (2.7), $|f(x)-F(x)|<\varepsilon$ and the arbitrariness of ε, for the proof of (2.14) it is sufficient to state that

$$|c_0(V_\varphi F)| \leqslant A\omega_f(s)\mathrm{Cap}_L((3/2)Q\setminus X).$$

But the last inequality results from the definition of the capacity. Actually, having estimated localizations $V_\varphi f=E*(\varphi Lf)$ and $V_\varphi(f-F)=E*(\varphi L(f-F))$ with the help of Lemma 2, we obtain that $\|V_\varphi F\|_{\mathrm{L}_\infty}\leqslant A\omega_f(s)$, and $V_\varphi F$ is admitted for $(3/2)Q\setminus X$. The Lemma has been proved.

Let us consider a corollary of Lemma 4.

Corollary 1. *Let the equality $H(X,L)=h(X,L)$ hold. Then for an arbitrary cube Q there is an estimate $\mathrm{Cap}_L(Q\setminus X^o)\leqslant A\mathrm{Cap}_L((3/2)Q\setminus X)$, which is equivalent to the right-hand side (1.2).*

Proof. Obviously, we may consider that $\mathrm{Cap}_L(Q\setminus X^o)>0$. Due to the definition of the capacity there is a function g, admitted for $Q\setminus X^o$ (and, consequently, by the condition $g\in H(X,L)$) such that $\|g\|_{\mathrm{L}_\infty}\leqslant 2$ and $\langle Lg|1\rangle=\mathrm{Cap}_L(Q\setminus X^o)$. Due to (2.1) and Lemma 1 there is the equality $g=V_\varphi g$, where φ satisfies the conditions (1)-(2) of Lemma 1, and $\varphi\equiv 1$ in some neighbourhood of Q. Since $\langle Lg|1\rangle=c_0(V_\varphi g)=\mathrm{Cap}_L(Q\setminus X^o)$, it remains to apply (2.14). The corollary has been proved.

Corollary 2. *Assume that* $X^o = \emptyset$ *and the equality* $C(X) = H(X, L)$ *holds. Then for an arbitrary cube* Q *there is the estimate* $\mathrm{Cap}_L(Q \setminus X) \geqslant A_1 s^{d-n}$ *which is equivalent to (1.5).*

The proof follows from Corollary 1 and the equality $\mathrm{Cap}_L(Q) = A_2 s^{d-n}$.

Let us now consider the problem of the necessary precision of approximation of localization. The following statement holds (see [7, Theorem 2]).

Lemma 5. *Let us assume that for any binary cube* $Q = Q(a, s)$ *and the corresponding localization* $V_\varphi f$*, satisfying conditions of the Lemma 2, there is a function* F_Q *such that:*

(1) $\mathrm{Spt}(LF_Q) \subset (10Q \setminus X)$*;*

(2) the estimate

$$|V_\varphi f(x) - F_Q(x)| \leqslant A\omega_f(s) \min\left(1, \frac{s^d}{|x-a|^d}\right) \tag{2.15}$$

holds. Then $f \in H(X, L)$*.*

Due to (2.13) all the coefficients of the Laurent expansions of the functions $V_\varphi f$ and F_Q should coincide when $|\alpha| \leqslant n-1$ for the estimate (2.15) to hold. Note that repeating reasoning of Lemma 1 from [2, Ch. 2, §4] would require satisfying a stricter condition, i.e. substitution of the degree d by $d+1$ in (2.15).

3. Proof of Theorem 1

Recall that statement (1) of Theorem 1 is established due to corollary 2 of Lemma 4. Let us prove the following statement.

Lemma 6. *Let the estimate (1.5) hold for the compact set* X *with* $X^o = \emptyset$*; then the following equality* $C(X) = H(X, L)$ *holds.*

Proof. Let us demonstrate that if (1.5) is satisfied, we can not only obtain the estimate (2.15), whence Lemma 6 follows, but also obtain the estimate (3.2) with any given in advance $m \in \mathbb{N}$. The following statement is elementary.

Lemma 7. *Assume that* $\mathbf{Q} = [0,1]^d$*. Then for any multi-index* α*,* $|\alpha| \geqslant 0$ *there exists a function*

$$F^\alpha = \sum_j \lambda_j E(x - a_j), \tag{3.1}$$

where the sum is finite, the number of indices j *does not exceed* $A(\alpha)$*,* $a_j \in \mathbf{Q}$*,* $|\lambda_j| \leqslant A(\alpha)$*,* $\min_{j,j'} |a_j - a_{j'}| > A_1(\alpha) > 0$*, and there exists the asymptotics*

$$F^\alpha(x) = \partial^\alpha E(x) + o(|x|^{n-d-|\alpha|}).$$

To prove Lemma 7 it is sufficient to note that the functions F^α are obtained from the standard formulae of numerical differentiation and they are easily constructed by induction: if the vector a is directed along the axis x_k, then

$$F^\alpha(x-a) - F^\alpha(x) = |a| \frac{\partial}{\partial x_k}(\partial^\alpha E(x)) + o(|x|^{n-d-|\alpha|-1}).$$

Corollary of Lemma 7. *Let* $Q = Q(a, s)$ *be a cube,* $m \in \mathbb{Z}_+$*, and with* $|\alpha| \leqslant m$ *arbitrary numbers* $b_\alpha \in \mathbb{C}$*,* $|b_\alpha| \leqslant s^{d-n+|\alpha|}$ *are given. Then there exists a function* F_m *such that:*

(1) $F_m = \sum_j \lambda'_j E(x - a_j)$*, where* $a_j \in Q$*,* $|\lambda'_j| \leqslant A(m)s^{d-n}$*, the number of indices* j *does not exceed* $A(m)$*,* $\min_{j,j'} |a_j - a_{j'}| > A_1(m)s$*, where* $A_1(m) > 0$*;*

(2) when $|\alpha| \leqslant m$ *the equality* $c_\alpha(F_m, a) = b_\alpha$ holds.

(The corollary is obvious: for $Q = [0,1]^d$ the function F_m is a consistent linear combination of functions F^α from Lemma 7, and its coefficients are obtained from the system of linear equations with a triangle matrix; the general case is obtained by variation of the scale).

Let us return to the proof of Lemma 6. Considering that (1.5) holds, we modify a little functions F_α from (3.1).

Let for the fixed $c > 0$ and some set K we have $\mathrm{Cap}_L(B(x,r) \cap K) \geqslant cr^{d-n}$ uniformly on all $x \in \mathbf{Q}$ and $r \leqslant 1$. For $r \in (0,1]$ in the sum in the right-hand side (3.1) we substitute every function $E(x-a_j)$ by $r^{n-d}g_j$, where $\|g_j\|_{L_\infty} \leqslant 2c^{-1}$, $\mathrm{Spt}(Lg_j) \subset (B(a_j,r) \cap K)$, $\lim_{x\to\infty} g_j(x) = 0$ and $c_0(g_j) = r^{d-n}$. Due to (2.12) we obtain (independently of K) $|c_\alpha(r^{n-d}g_j, a_j)| \leqslant Ac^{-1}r^{|\alpha|}$ when $|\alpha| \geqslant 0$, at that $c_0(r^{n-d}g_j) = c_0(E(x-a_j)) = 1$.

Whence and from (2.6) it follows that for any $m \in \mathbb{Z}_+$ and all β, $|\beta| \leqslant m$ the following estimate holds $|c_\beta(E(x-a_j),0) - c_\beta(r^{n-d}g_j,0)| \leqslant A(m,L)r$. Consequently, for any $\epsilon > 0$ and $m \in \mathbb{Z}_+$ there exists $r_0 = r_0(\epsilon, m, L)$ such that the inequality

$$\sum_{\{\beta:|\beta|\leqslant m\}} |c_\beta(\widetilde{F}_\alpha, 0) - c_\beta(F_\alpha, 0)| < \epsilon$$

holds for all $r \leqslant r_0$ where $\widetilde{F}_\alpha$ indicates the sum obtained from (3.1) by the substitution of $E(x-a_j)$ for $r^{n-d}g_j$.

Similarly to the corollary in Lemma 7, due to (2.12) there exists the function $\widetilde{F}_m$ which is a consistent linear combination of the functions $\widetilde{F}_\alpha$ such that there holds the following statement (the matrix of the system of linear equations from which we obtain the coefficients, corresponding to $\widetilde{F}_\alpha$, under small r/s is close to be triangular).

Lemma 8. *Let $Q = Q(a,s)$ be a cube, $f \in h(X,L)$, $V_\varphi f$ be a localization from Lemma 2. Let $c > 0$ be such that for all $r \leqslant s/10$ and $x \in Q'$, where Q' is a cube, $Q' \subset 10Q$, $s(Q') \geqslant (1/10)s$, the inequality $\mathrm{Cap}_L(B(x,r) \setminus X) \geqslant cr^{d-n}$ holds. Then for any $m \geqslant 0$ there is a function $\widetilde{F}_m$ such that $\mathrm{Spt}(L\widetilde{F}_m) \subset (10Q \setminus X)$, and there exists the estimate*

$$|V_\varphi f(x) - \widetilde{F}_m(x)| \leqslant A(c,m)\omega_f(s) \min\left(1, \frac{s^{d-n+m}}{|x-a|^{d-n+m}}\right). \tag{3.2}$$

Due to Lemmas 5 and 8 Lemma 6 has been proved.

To conclude the proof of Theorem 1 it remains to prove the following statement of instability of the capacity.

Lemma 9. *Let X be a compact set with $X^o = \emptyset$. If the estimate (1.6) holds for almost all $x \in X$, then for any ball $B(x,r)$ with the centre $x \in \mathbb{R}^d$ the estimate (1.5) holds.*

Proof. Lemma 9 results from the following three Lemmas.

Lemma 10. *Let us assume that K is a subset of the ball $B = B(a,r)$, $a_0 \in B$, and the estimate $\mathrm{Cap}_L(B(a_0,\delta) \cap K) \leqslant c\delta^d$ holds for some $c > 0$ and any ball $B(a_0,\delta)$ with $\delta \leqslant 2r$. Let $g \in C(\mathbb{R}^d)$ be a function such that $\mathrm{Spt}(Lg) \subset K$, $\|g\|_{L_\infty} \leqslant 1$ and $\lim_{x\to\infty} g(x) = 0$. Then the estimate $|g(a_0)| \leqslant Acr^n$ holds.*

Proof of Lemma 10. Due to (2.1) we have $g = E*(Lg)$. Assume that $B_0 = B(a_0, 2r)$, and for $m \in \mathbb{N}$ we suppose $B_m = B(a_0, 2r/2^m)$; it is clear that the rings $D_m = (3/2)B_m \setminus (1/4)B_m$ cover B_0. Having expanded the space $\mathbb{R}^d$ into binary cubes whose edges lengths are almost "equivalent" to the distances to a_0, and having applied Lemma 1 for the arbitrary $m_0 \in \mathbb{N}$ we represent g in the form of a sum of localizations:

$$g = \sum_{m=0}^{m_0} E*(\varphi_m Lg) + E*(\psi Lg),$$

where $\varphi_m = \sum_{j=1}^{p_m} \varphi_{m,j}$, and $\mathrm{Spt}\varphi_{m,j} \subset D_m$, $\varphi_{m,j}$ satisfies the conditions of Lemma 1 for the corresponding cubes, contained in D_m, $p_m \leqslant A(d)$, $\mathrm{Spt}\psi \subset (3/2)B_{m_0}$, and ψ satisfies conditions of Lemma 1 for the cube comparable with B_{m_0}.

Having applied Lemma 2 and (2.11), we obtain:

$$|g(a_0)| \leqslant A\left(\sum_{m=0}^{m_0} \frac{\mathrm{Cap}_L((3/2)B_m \cap K)}{(r(B_m))^{d-n}} + \omega_g(r(B_{m_0}))\right).$$

If by condition we apply the estimate $\mathrm{Cap}_L((3/2)B_m \cap K) \leqslant A_1 c(r(B_m))^d$ and tending m_0 to infinity, we obtain the estimate $|g(a_0)| \leqslant Acr^n$. Lemma 10 has been proved.

Lemma 11. *Let $t_1 > 0$, $t_2 \geqslant t_1$, $K_0 = K_0(t_1, t_2)$ be a set of points x such that $\limsup\limits_{r\to 0} \dfrac{\mathrm{Cap}_L(B(x,r)\cap K_0)}{r^d} \geqslant t_1$ and $\mathrm{Cap}_L(B(x,r)\cap K_0) \leqslant t_2 r^d$ for all r, $r \leqslant r_0$. Then $mes(K_0) = 0$,where $mes(\cdot)$ is the Lebesgue measure in $\mathbb{R}^d$.*

Proof of Lemma 11. Lemma 11 is proved similarly to Lemma 1.7 from [10]. Reasoning by contradiction, we assume that $mes(K_0) > 0$; let x_0 be a density point of K_0, then there is $r_1 < r_0$ such that the inequality $mes(B(x_0, r)\cap K_0) > (1/2)mes(B(x_0, r))$ holds in case of all $r \leqslant r_1$. Let us prove that it results in

$$\liminf_{r\to 0} \frac{\mathrm{Cap}_L(B(x_0, r)\cap K_0)}{r^{d-n}} > 0, \tag{3.3}$$

consequently, due to the definition K_0, we have $x_0 \notin K_0$ and the obtained contradiction proves the lemma.

Let us prove estimate (3.3). Let us fix the ball $B = B(x_0, r)$, $r \leqslant r_1$. Due to the covering Lemma [9, §3], [12, Ch. 1, §1.6] there exists a finite family of balls $B_j = B(a_j, \delta_j)$ contained in B with $a_j \in K_0$ such that the following conditions hold (1)–(3):
(1) $\mathrm{Cap}_L(B(a_j, \delta_j)\cap K_0) \geqslant (1/2)t_1(\delta_j)^d$;
(2) $\sum_j (\delta_j)^d \geqslant Ar^d$;
(3) balls $2B_j$ do not intersect pairwise.

Due to condition (1) we take for each ball B_j the function g_j, admitted for $B(a_j, \delta_j)\cap K_0$, such that $\|g_j\|_{\mathrm{L}_\infty} \leqslant 1$ and $c_0(g_j) = (1/4)t_1(\delta_j)^d$. Assume that $g = \sum_j g_j$; it is clear, that $\mathrm{Spt}(Lg) \subset (B(x_0, r)\cap K_0)$, and due to condition (2) we have $c_0(g) \geqslant At_1 r^d$.

It is easy to see that it results from the estimate $\mathrm{Cap}_L(B(x, r)\cap K_0) \leqslant t_2 r^d$ for $x \in K_0$, that $\mathrm{Cap}_L(B(x, r/2)\cap K_0) \leqslant t_2 r^d$ for $x \in \mathbb{R}^d$. Due to Lemma 10, conditions (3) for $\{B_j\}$ and (2.11), we obtain

$$|g(x)| \leqslant At_2\left(r^n + \int_B \frac{dm_y}{|x-y|^{d-n}}\right) \leqslant A_1 t_2 r^n$$

for $x \in \mathbb{R}^d$.

We have obtained from the definition of capacity that $\mathrm{Cap}_L(B(x_0, r)\cap K_0) \geqslant A(t_1/t_2)r^{d-n}$, and, consequently, we have obtained (3.3). Lemma 11 has been proved.

The following statement is obvious.

Corollary of Lemma 11. Let the estimate (1.6) hold for almost all $x \in X$. Then for almost all $x \in X$:

$$\limsup_{r\to 0} \frac{\mathrm{Cap}_L(B(x, r)\setminus X)}{r^d} = \infty. \tag{3.4}$$

To complete proof of Lemma 9 and Theorem 1 it is remains to establish the following statement.

Lemma 12. *Let the estimate (3.4) hold for almost all $x \in X$, then for any ball $B(x, r)$ with the centre $x \in \mathbb{R}^d$ the estimate (1.5) holds.*

Proof of Lemma 12. Lemma 12 is proved by analogy with Lemma 11 and Theorem 1 from [9]. Let us note that the estimate (3.4) obviously holds for all $x \notin X$. Let us fix an arbitrary ball $B(x_0, r)$. According to the Vitali Covering Lemma (see [9, §3]) there exists a finite number of balls $B_j = B(a_j, \delta_j)$ contained in B such that the following conditions hold (1)-(3):

(1) $\mathrm{Cap}_L(B(a_j, \delta_j) \setminus X) \geqslant (\delta_j)^d / r^n$;

(2) $\sum_j (\delta_j)^d \geqslant A r^d$;

(3) balls $2B_j$ do not cross pairwise.

Due to condition (1) we take the functions g_j admitted for $B(a_j, \delta_j) \setminus X$, such that $\|g_j\|_{\mathrm{L}_\infty} \leqslant \dfrac{(\delta_j)^d}{r^n \mathrm{Cap}_L(B(a_j, \delta_j) \setminus X)} \leqslant 1$ and $c_0(g_j) = (1/2)(\delta_j)^d / r^n$. Assume that $g = \sum_j g_j$, then due to the condition (2) we have $c_0(g) \geqslant A r^{d-n}$. Due to Lemma 2, condition (3) and (2.11) we obtain for $x \in \mathbb{R}^d$:

$$|g(x)| \leqslant A + r^{-n} \int_B \frac{dm_y}{|x-y|^{d-n}} \leqslant A_1.$$

Therefore $c_0(g)/\|g\|_{\mathrm{L}_\infty} \geqslant A_2 r^{d-n}$. According to the definition of capacity we have obtained that $\mathrm{Cap}_L(B(x_0, r) \setminus X) \geqslant A_2 r^{d-n}$. Lemma 12 has been proved. The proof of Theorem 1 has been completed.

4. Construction of Example 1

Example 1. *Assume that $d > 2$ and $2 < n < d$ (where d is dimension of the space, n is order of the operator L). Then there is a compact set X, such that for any cube Q the estimate $\mathrm{Cap}_L(Q \setminus X^o) \leqslant A \mathrm{Cap}_L(2Q \setminus X)$ holds and the function $f \in h(X, L)$ is such that $f \notin H(X, L)$.*

Construction of the Example. Let us start with an auxiliary construction. For $N \in \mathbb{N}$ we consider a $d-1$-dimensional cube D_N: $D_N = [0,1]^d \cap \{x_d = 10^{-N}\}$ and the set of open balls B_j with centres $a_j \in D_N$, though the coordinates x_m of the points a_j have the form $k10^{-N}$ when $m = 1, \dots, d-1$, where $k = 0, 1, \dots, 10^N$ and $r(B_j) = 10^{-N(d-1)/(d-n)}$.

It is clear that the balls $2B_j$ do not intersect pairwise; let us take the functions g_j such that g_j is admitted for B_j, $\|g_j\|_{\mathrm{L}_\infty} \leqslant 2$ and $c_0(g_j) = \mathrm{Cap}_L(B_j) = A10^{-N(d-1)}$. Let $Q = Q(a, s)$ be an arbitrary cube such that $a \in D_N$ and $10^{-N+1} \leqslant s \leqslant 1$. Then, as it is easy to see, due to "uniformity" of location of the points a_j on D_N the following properties of the functions g_j occur.

(1) In summing the cubes $B_j \subset Q$ by all the indexes j, the sum $c_0(g_j)$ is not less then $A_1 s^{d-1}$, where $A_1 > 0$.

(2) For the arbitrary $x \in \mathbb{R}^d$ the sum $|g_j(x)|$ over all indices j such that $B_j \subset Q$ and $|a_j - x| \geqslant 10^{-N}$ does not exceed $A_2 \displaystyle\int_{Q \cap D_N} \frac{dx_1 \dots dx_{d-1}}{|x-y|^{d-n}} \leqslant A_3 s^{n-1}$.

Lemma 13. *Let $\mathbf{B}$ be the union of balls B_j, constructed for all D_N, where $N = 1, 2, \dots$; $Q_0 = Q_0(a, s)$ is a cube with the centre $a \in D_0$, where $D_0 = [0,1]^d \cap \{x_d = 0\}$ and $s \leqslant 1$ (let us recall that we consider cubes with edges parallel to the axes of coordinates). Then the estimate $\mathrm{Cap}(Q_0 \cap \mathbf{B}) \geqslant A s^{d-n}$ occurs.*

Proof. The cube Q_0 intersects all D_N, starting with some D_{N_0}. Let g^N be a sum of functions g_j such that B_j is contained in Q_0, and the centres B_j belong to D_N. Due to properties (1) and (2) of the functions g_j, for all sufficiently large m the function

$$g = \frac{1}{m} \sum_{N=N_0}^{N_0+m-1} g^N$$

has the following two properties:

(1) $c_0(g) \geqslant A_1 s^{d-1}$; (2) $\|g\|_{\mathrm{L}_\infty} \leqslant A_3 s^{n-1}$. By definition of capacity this entails the statement of the lemma. The lemma has been proved.

Let us return to the construction of the Example 1. Let us take $X = Q(0,10)\setminus \mathbf{B}$. It is clear that the inner boundary of X coincides with D_0.

Let the cube Q not intersect D_0, then the estimate $\mathrm{Cap}_L(Q\setminus X^o) \leqslant A\mathrm{Cap}_L(2Q\setminus X)$ results from the possibility of uniform approximation with any degree of precision of the function h admitted for $Q\setminus X^o$ by admitted functions for $2Q\setminus X$ (this fact is standard as Q intersects only a finite number of balls B_j: h is presented in the form of the sum of localizations of A.G. Vitushkin [2, Ch. 2, §1] within the scale $\min r(B_j)$, and Lemmas 5 and 8) are applied.

Let Q intersect D_0; if Q intersects the boundary $Q(0,10)$ as well, then, it is obvious that $\mathrm{Cap}_L(2Q\setminus X) \geqslant A_1(s(Q))^{d-n}$ and, consequently, $\mathrm{Cap}_L(Q\setminus X^o) \leqslant A\mathrm{Cap}_L(2Q\setminus X)$.

Finally, when Q intersects D_0 and is contained inside $Q(0,10)$, the estimate $\mathrm{Cap}_L(2Q\setminus X) \geqslant A_1(s(Q))^{d-n}$ easily results from Lemma 13.

Therefore in the general case we obtain $\mathrm{Cap}_L(Q\setminus X^o) \leqslant A\mathrm{Cap}_L(2Q\setminus X)$.

Let us take $f = \dfrac{\partial E}{\partial x_d} * \chi_{D_0}$ for E from (1.1). Here $\chi_{(\cdot)}$ is a characteristic function. Since n (order of the operator L) is higher than two, then $f\in C(\mathbb{R}^d)$; it is clear that $f\in h(X,L)$.

It remains to show that $f\notin H(X,L)$. Let us take a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\mathrm{Spt}\varphi\subset Q(0,5)$, $\varphi\equiv 1$ on $Q(0,2)$ and $\|\partial^\alpha\varphi\|_{\mathrm{L}_\infty} \leqslant A$ when $|\alpha|\leqslant n$. Similarly, we take a function $\varphi_j\in C_0^\infty(\mathbb{R}^d)$, $\mathrm{Spt}\varphi_j\subset 2B_j$, $\varphi\equiv 1$ for every ball B_j on $(3/2)B_j$ and $\|\partial^\alpha\varphi\|_{\mathrm{L}_\infty} \leqslant A(r(B_j))^{-|\alpha|}$ when $|\alpha|\leqslant n$.

Let us consider the function $\mu = L(x_d\varphi) - \sum_{\{j:B_j\subset\mathbf{B}\}} L(x_d\varphi_j)$. It is clear that

$$\int_{Q(0,5)} |\mu(x)|dm_x \leqslant A_1 + A_2\sum_{N=1}^{\infty} 10^{-N} \sum_{\{j:a_j\in D_N\}} (r(B_j))^{d-n} \leqslant A_1 + A_3\sum_{N=1}^{\infty} 10^{-N} < \infty.$$

Due to the construction we obtain the equality $\mu\equiv 0$ on the union of $(3/2)B_j$ and everywhere outside $Q(0,5)$. Therefore, for any function $F\in H(X,L)$ the equality $\displaystyle\int_{Q(0,5)} F(x)\mu(x)dm_x = 0$ holds. To make sure that $f\notin H(X,L)$, it is left to show that $\displaystyle\int_{Q(0,5)} f(x)\mu(x)dm_x \neq 0$.

Since $f = \dfrac{\partial E}{\partial x_d} * \chi_{D_0}$, then for all j we have:

$$\int_{Q(0,5)} f(x)L(x_d\varphi_j(x))dm_x = (-1)^n\langle Lf|x_d\varphi_j\rangle = 0,$$

consequently, $\displaystyle\int_{Q(0,5)} f(x)\mu(x)dm_x = (-1)^n\langle Lf|x_d\varphi\rangle$. Since $\varphi\equiv 1$ on $Q(0,2)$, due to (2.5) and (2.6) we have: $-\langle Lf|x_d\varphi\rangle = \displaystyle\int_{D_0}\chi_{D_0}(x)dx_1\ldots dx_{d-1} = 1$.

Therefore $\left|\displaystyle\int_{Q(0,5)} f(x)\mu(x)dm_x\right| = 1$. Construction of Example 1 has been completed.

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