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EXTENSION OF THE CONIC FLOWS

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Abstract. All partial invariant solutions of gas dynamic equations, that are constructed on the conic subalgebra admitted by the model are found. The conic subalgebra consists of operators of rotation, translation by time and expansion. A submodel comprises a system of ordinary differential equations. Solutions form a series of submodels. In the basis of these submodels lies conic submodel with respect to the invariant variable depending on independent variables and constants of the submodels depending on an invariant function. To determine this dependence, various additional overdetermined equations are obtained. Moreover, two submodels, expanding the conic submodel, are derived from the system of partial differential equations. All formulas mapping the solutions to physical space are defined for these two submodels.

Keywords: conic flows, partial invariant solutions, gas dynamics.

Introduction

Canonic flows are invariant solutions constructed on the three-dimensional subalgebra with basis operators of rotation, translation by time and expansion (conic subalgebra). The submodel of conic flows is composed by a quasilinear non-autonomous system of ordinary differential equations. This submodel without rotation is the topic of the books [1, 2, 3]. Exact solutions with rotation were obtained in [3, 4]. The extension of conical flows can be obtained by two methods. The first extension presents partial invariant solutions of the conic algebra. In this paper we consider an irregular partial invariant submodel of rank 2 defect 1. The second extension consists in consideration of group solutions of the overalgebra of the conic algebra. In further papers we plan to consider a differential-invariant submodel of the rank $1+3$ of the five-dimensional subalgebra [3], where we add two operators of translation noncommuting with rotation to the operators of conic algebra. This submodel coincides with the partial invariant submodel of rank 1 defect 2 [5]. In both cases we have obtained partial solutions of equation of gas dynamics different from solutions of a conic submodel. Almost all the considered solutions are reduced to conic flows. It suggests some partial stability of conic flows with respect to perturbations from the considered partial invariant submodels.

1. The irregular partially invariant submodel of rank 2 defect 1 of the conic algebra

The basis of the conic subalgebra is set by basis operators in the cylindrical system of coordinates: ∂_t is the translation by time, ∂_θ is the rotation, $t\partial_t + x\partial_x + r\partial_r$ is the extension.The invariants of the subalgebra are: the quantity $s = xr^{-1}$, the coordinates of the velocity U, V, W, the density ρ and the entropy S. The pressure is determined by the equation of state $p = f(\rho, S)$. The representation of the partial invariant solution of rank 2 defect 1 of the equation of continuum mechanics consists in the fact that all the required functions depend on

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s and $\alpha = \alpha(t, x, r, \theta)$ [5]. The function α of the general form can be any invariant. In the case of equations of gas dynamics substitution of the representation provides the equalities

$$\begin{gathered}
r^{-1}(U - sV)S_s + S_\alpha(\alpha_t + \vec{u}\cdot\nabla\alpha) = 0,\\
r^{-1}\left[(U - sV)\rho_s + \rho(U_s - sV_s + V)\right] + \rho_\alpha(\alpha_t + \vec{u}\cdot\nabla\alpha) + \rho\vec{u}_\alpha\cdot\nabla\alpha = 0,\\
r^{-1}\left[(U - sV)U_s + \rho^{-1}p_s\right] + U_\alpha(\alpha_t + \vec{u}\cdot\nabla\alpha) + \rho^{-1}p_\alpha\alpha_x = 0,\\
r^{-1}\left[(U - sV)V_s - s\rho^{-1}p_s - W^2\right] + V_\alpha(\alpha_t + \vec{u}\cdot\nabla\alpha) + \rho^{-1}p_\alpha\alpha_r = 0,\\
r^{-1}\left[(U - sV)W_s + VW\right] + W_\alpha(\alpha_t + \vec{u}\cdot\nabla\alpha) + \rho^{-1}\rho^{-1}p_\alpha\alpha_\theta = 0,
\end{gathered} \tag{1.1}$$

where $\vec{u}$ is a vector with the coordinates U, V, W; ∇ is a vector gradient with the coordinates ∂_x, ∂_r, $r^{-1}\partial_\theta$. If gas dynamic functions do not depend on α, then we obtain the conic submodel [3].

In the equations (1.1) all gas dynamic functions depend on s and α. Therefore it is convenient to use the variables s, α, t, θ as new independent variables. The substitution is set by the equations $x = sr$, $r = r(t, \theta, \alpha, s)$, $r_\alpha \neq 0$. The identity $\alpha \equiv \alpha\left(t, sr, r(t, \theta, \alpha, s), \theta\right)$ holds. The derivatives of the functions r and α are connected by the equations

$$\alpha_t = -\frac{r_t}{r_\alpha}, \quad \alpha_\theta = -\frac{r_\theta}{r_\alpha}, \quad \alpha_x = -\frac{r_s}{r r_\alpha}, \quad \alpha_r = -\frac{1}{r_\alpha}\left(1 + \frac{s r_s}{r}\right).$$

The equations (1.1) are reduced to the form

$$\begin{gathered}
S_\alpha\left(r_t + (U - sV)r^{-1}r_s + Wr^{-1}r_\theta\right) - (U - sV)S_s r^{-1}r_\alpha = S_\alpha V,\\
\rho_\alpha r_t + \left(\rho(U - sV)\right)_\alpha r^{-1}r_s + (\rho W)_\alpha r^{-1}r_\theta -\\
-\left(\left(\rho(U - sV)\right)_s + 2\rho V\right) r^{-1}r_\alpha = (\rho V)_\alpha,\\
U_\alpha r_t + \left((U - sV)U_\alpha + \rho^{-1}p_\alpha\right) r^{-1}r_s + WU_\alpha r^{-1}r_\theta -\\
-\left((U - sV)U_s + \rho^{-1}p_s\right) r^{-1}r_\alpha = VU_\alpha,\\
V_\alpha r_t + \left((U - sV)V_\alpha - s\rho^{-1}p_\alpha\right) r^{-1}r_s + WV_\alpha r^{-1}r_\theta -\\
-\left((U - sV)V_s - s\rho^{-1}p_s - W^2\right) r^{-1}r_\alpha = VV_\alpha + \rho^{-1}p_\alpha,\\
W_\alpha r_t + (U - sV)W_\alpha r^{-1}r_s + (WW_\alpha + \rho^{-1}p_\alpha)r^{-1}r_\theta -\\
-\left((U - sV)W_s + VW\right)r^{-1}r_\alpha = VW_\alpha.
\end{gathered} \tag{1.2}$$

We have obtained five linear equations for the quantities r_t, $r^{-1}r_s$, $r^{-1}r_\theta$, $r^{-1}r_\alpha$.

Lemma 1. *If all the derivatives of the function r are determined by (1.2) as the functions s, α, then they are reduced to the invariant solution of the two-dimensional subalgebra* [5].

Proof.

Let us assume that $r_t = A(\alpha, s)$, $r^{-1}r_s = B(\alpha, s)$, $r^{-1}r_\theta = C(\alpha, s)$, $r^{-1}r_\alpha = \mathcal{D}(\alpha, s)$. The conditions of compatibility provide $C_s = C_\alpha = 0 \Rightarrow C = C_0$ is a constant; $A_s = BA$, $A_\alpha = \mathcal{D}A$, $AC_0 = 0$, $B_\alpha = \mathcal{D}_s \Rightarrow B = E_s$, $\mathcal{D} = E_\alpha$, $A = Ke^E$, K is a constant, $KC_0 = 0$.

If $C_0 = 0$, then $r = t\exp(E) \Rightarrow \alpha = \alpha(s, t^{-1}r)$ is an invariant of the subalgebra $\{\partial_\theta, t\partial_t + x\partial_x + r\partial_r\}$.

If $C_0 \neq 0$, then $K = 0$, $r = \exp(E + C_0\theta) \Rightarrow \alpha = \alpha(s, \ln r - C_0\theta)$ is an invariant of the subalgebra $\{\partial_\theta, t\partial_t + x\partial_x + r\partial_r + C_0^{-1}\partial_\theta\}$. This completes the proof.

All the derivatives S_α, ρ_α, U_α, V_α, W_α are not equal to zero, otherwise reducing to the conic invariant submodel [3] may occur. It follows that $r_t = R(\alpha, s) \Rightarrow r = tR(\alpha, s) + r_1(\alpha, s\theta)$, $r_1 \neq 0$. Substitution of the expression for r into (1.2) and equating the coefficients with the free variable t to zero provides two systems. The first one is for r_1

$$
\begin{gathered}
S_\alpha(U - sV)r_{1s} + S_\alpha W r_{1\theta} - S_s(U - sV)r_{1\alpha} = S_\alpha(V - R)r_1,\\
\big(\rho(U - sV)\big)_\alpha r_{1s} + (\rho W)_\alpha r_{1\theta} -\\
-\big(2\rho V + \big(\rho(U - sV)\big)_s\big)\, r_{1\alpha} = \big((\rho V)_\alpha - \rho_\alpha R\big) r_1,\\
\left((U - sV)U_\alpha + \rho^{-1}p_\alpha\right) r_{1s} + WU_\alpha r_{1\theta} -\\
-\left((U - sV)U_s + \rho^{-1}p_s\right) r_{1\alpha} = U_\alpha(V - R)r_1,\\
\left((U - sV)V_\alpha - s\rho^{-1}p_\alpha\right) r_{1s} + WV_\alpha r_{1\theta} -\\
-\left((U - sV)V_s - s\rho^{-1}p_s - W^2\right) r_{1\alpha} = \left(V_\alpha(V - R) + \rho^{-1}p_\alpha\right) r_1,\\
(U - sV)W_\alpha r_{1s} + \left(WW_\alpha + \rho^{-1}p_\alpha\right) r_{1\theta} - \left((U - sV)W_s + VW\right) r_{1\alpha} =\\
= W_\alpha(V - R)r_1;
\end{gathered}
\tag{1.3}
$$

The second system is obtained from (1.3) by means of substitution of r_1 for R. If all the derivatives of the function r_1 are determined by (1.3), then the derivatives of the function R are determined the same way. Reducing to the invariant solution under the two-dimensional algebra occurs.

2. The case when the derivative $r_{1\theta}$ is not determined by (1.3)

Let $r_{1\theta}$ not be determined by (1.3):

$$
WS_\alpha = 0, \quad (\rho W)_\alpha = 0, \quad WU_\alpha = 0, \quad WV_\alpha = 0, \quad WW_\alpha + \rho^{-1}p_\alpha = 0. \tag{2.1}
$$

Let $W \neq 0$. It results from (2.1) that $S(s)$, $U(s)$, $V(s)$, $\rho W = C(s)$, $C(s)W + p = K(s)$. Substitution into the equation of state results in the equality

$$
K(s) - C^2(s)s^{-1} = f\left(\rho, S(s)\right),
$$

which holds only for the Chaplygin gas

$$
p = f(\rho, S) = N(S) - \rho^{-1}\mathcal{D}^2(S), \quad K(s) = N\left(S(s)\right), \quad C(s) = \mathcal{D}\left(S(s)\right).
$$

The following equations follow from (1.3)

$$
\begin{aligned}
&(U - sV)S' r_{1\alpha} = 0,\\
&\mathcal{D}W^{-2}W_\alpha(U - sV)r_{1s} + \left[2\mathcal{D}VW^{-1} + \left(\mathcal{D}W^{-1}(U - sV)\right)_s\right] r_{1\alpha} =\\
&= \mathcal{D}W^{-2}W_\alpha(V - R)r_1,\\
&WW_\alpha r_{1s} + \left[(U - sV)U' + \rho^{-1}p_s\right] r_{1\alpha} = 0,\\
&sWW_\alpha r_{1s} - \left[(U - sV)V' - s\rho^{-1}p_s - W^2\right] r_{1\alpha} = -WW_\alpha r_1,\\
&(U - sV)W_\alpha r_{1s} - \left[(U - sV)W_s + VW\right] r_{1\alpha} = W_\alpha(V - R)r_1.
\end{aligned}
\tag{2.2}
$$

Similar equations hold if we replace r_1 by R. If $(U-sV)S' \neq 0$, then $r_{1\alpha} = 0 = R_\alpha$, $r_\alpha = 0$ which is a contradiction. Therefore $(U-sV)S' = 0$ and the following alternative equation occurs: $U - sV = 0$ or $U - sV \neq 0$.

In the first case $U = sV$ it results from (2.1) that $V = U = R = 0$, $N = N_0$ is a constant, $p = N_0 - \mathcal{D}^2(S)\rho^{-1}$ is the equation of state, $W = m(\theta)r^{-1}\left[\mathcal{D}\left(S(s)\right)\right]^{-1}$, $m(\theta)$ is an arbitrary function. This solution describes arbitrary circular motion of particles.

In the second case the motion is isentropic $S = S_0$, $\rho = \mathcal{D}_0 W^{-1}$, $p = N_0 - \mathcal{D}_0 W$. The following equations result from (2.1)

$$R = 0, \quad U^2 + V^2 = C_0^2, \quad r = m(\theta)\left|2V(U-sV)^2(V+sU)^{-1} - W^2\right|^{-1/2},$$

where C_0 is a constant. Furthermore the following equations hold:

$$\frac{ds}{dV} = \frac{s}{2V} + \frac{1}{2\sqrt{C_0^2 - V^2}},$$
$$V\left[-4V(V+sU)^2 + U(U-sV)(V+sU) - 2V(U-sV)^2\right] = 0. \tag{2.3}$$

If $V \neq 0$, then it results from the last equation that s is irrationally expressed via V, and it results from (2.3) that s is a transcendental function V. Therefore, $V = 0$, $U = C_0$, $W = m(\theta)/r$. We obtain an arbitrary spiral motion of particles by cylinders, which is set by the invariant solution of the equations of gas dynamics for the two-dimensional subalgebra $\{\partial_t, \partial_x\}$.

The case $W = 0 \Rightarrow p(s)$, $r_{1\theta}$ is arbitrary. The following equations result from: (1.3)

$$\begin{gathered}
(U-sV)(S_\alpha r_{1s} - S_s r_{1\alpha}) = S_\alpha (V-R) r_1, \\
\big(\rho(U-sV)\big)_\alpha r_{1s} - \big(2\rho V + \big(\rho(U-sV)\big)_s\big) r_{1\alpha} = \big(\rho_\alpha (V-R) + \rho V_\alpha\big) r_1, \\
(U-sV)U_\alpha r_{1s} - \left((U-sV)U_s + \rho^{-1}p'\right) r_{1\alpha} = U_\alpha (V-R) r_1, \\
(U-sV)V_\alpha r_{1s} - \left((U-sV)V_s - s\rho^{-1}p'\right) r_{1\alpha} = V_\alpha (V-R) r_1,
\end{gathered} \tag{2.4}$$

and also equations where R stands for r_1.

Let all the coefficients in r_{1s} in (2.4) be equal to zero (r_{1s} is not determined). Then $U = sV$ (otherwise reducing to a conic model occurs), and (2.4) gives the constant solution $U = V = W = 0$, $p = p_0$.

Let $r_1^{-1}r_{1s} = R^{-1}R_\alpha$ and $R \neq 0 \Rightarrow r_1 = Rk(\alpha,\theta) \neq 0$ be determined from (2.4). Then the following equations result from (2.4)

$$\begin{gathered}
S_s(U-sV)k_\alpha = 0, \quad \left[2\rho V + \left(\rho(U-sV)\right)_s\right] k_\alpha = 0, \\
\left[(U-sV)U_s + \rho^{-1}p'\right] k_\alpha = 0, \quad \left[(U-sV)V_s - s\rho^{-1}p'\right] k_\alpha = 0.
\end{gathered}$$

If $k_\alpha \neq 0$, then $U = sV$ results in rest: $U = V = W = 0$, $p = p_0$; and $U \neq sV$ results in the equation of a conic model without rotation

$$\begin{gathered}
V_s + sU_s = 0, \quad \rho(U-sV)U_s + p' = 0, \\
(U-sV)\rho_s + \rho\left(V + (1+s^2)U_s\right) = 0, \\
p(s) = f\big(\rho(\alpha,s), S(\alpha)\big).
\end{gathered}$$

The equations (2.4) for $r_1 = R$ take the form

$$(U-sV)R_sR^{-1} = V - R, \quad R(U_\alpha - sV_\alpha) = VU_\alpha - UV_\alpha.$$

Hence it follows that the integral $U_s = (U-sV)d(s)$. The equations of the conic model result in $V = n(\alpha)V_1(s)$, $U = n(\alpha)U_1(s) \Rightarrow R = 0$ which is a contradiction.

Therefore, $k_\alpha = 0 \Rightarrow r = R(s,\alpha)\big(t - k(\theta)\big) \Rightarrow \alpha = A\left(s, r\big(t-k(\theta)\big)^{-1}\right)$. Since all the gas dynamic functions depend on s and α, then we can let $A_s = 0$ or $R_s = 0$, $R = \alpha$. In this case the system (2.4) for R takes the form

$$
\begin{gathered}
S_s(U - sV) + S_\alpha \alpha(V - \alpha) = 0,\\
2\rho V + \big(\rho(U - sV)\big)_s + \alpha\left[\rho_\alpha(V-\alpha) + \rho V_\alpha\right] = 0,\\
(U - sV)U_s + \rho^{-1}p' + U_\alpha \alpha(V - \alpha) = 0,\\
(U - sV)V_s - s\rho^{-1}p' + V_\alpha \alpha(V - \alpha) = 0.
\end{gathered}
\tag{2.5}
$$

The equation of state can be written in the form $S = G\big(p(s), \rho\big)$. If the function G is given, then the system (2.5) consists of four equations for four functions U, V, ρ, p, but the pressure p depends only on s. The system (2.5) is overdetermined. If $p(s)$ is given then we find the function S and the equation of state from (2.5).

For example, let $p = p_0$ be a constant. Then $U(s)$, $V(\rho)$, $\alpha(U - sV) = F(\rho)(V - \alpha)$ is an integral, $(U' - sV')\rho_s + \alpha V'\rho_\alpha + V = 0$. The compatibility of the equations for ρ provides

$$
V\left[2\alpha(U' - sV') - 2FV' - (V-\alpha)F'\right] = 0.
$$

Only the following solution is possible: $V = 0$; $U(\alpha)$, $\rho(\alpha)$ are arbitrary functions $\alpha = r\big(t - k(\theta)\big)^{-1}$.

In the case $W = 0$ there is one more opportunity: r_{1s} is determined by (2.4) but $R = 0$. Then we can assume that $r = r_1(\alpha,\theta)$ and the system (2.4) takes the form

$$
\begin{gathered}
S_s(U - sV)r_\alpha + S_\alpha V r = 0,\\
\left[2\rho V + \big(\rho(U - sV)\big)_s\right] r_\alpha + (V\rho)_\alpha r = 0,\\
\left[(U - sV)U_s + \rho^{-1}p'\right] r_\alpha + VU_\alpha r = 0,\\
\left[(U - sV)V_s - s\rho^{-1}p'\right] r_\alpha + VV_\alpha r = 0.
\end{gathered}
\tag{2.6}
$$

If all the coefficients are equal to zero when r_α, then the solution is reduced to the invariant model of rank 2 or 1. Otherwise we can consider that $r = \alpha k(\theta)$ and the system (2.6) is simplified

$$
\begin{gathered}
S_s(U - sV) + \alpha V S_\alpha = 0,\\
2\rho V + \big(\rho(U - sV)\big)_s + \alpha(V\rho)_\alpha = 0,\\
(U - sV)U_s + \rho^{-1}p' + \alpha V U_\alpha = 0,\\
(U - sV)V_s - s\rho^{-1}p' + \alpha V V_\alpha = 0.
\end{gathered}
\tag{2.7}
$$

One and the same remarks hold both for the system (2.7) and the system (2.5). Isobaric motions are reduced to flat steady flows.

3. The case when the derivative $r_{1\theta}$ is determined

Let us assume that $r_1^{-1}r_{1\theta} = E(\alpha, s) \Rightarrow r_1 = r_2(s,\alpha)\exp\big(\theta E(s,\alpha)\big)$. The system (1.3) after substitution of r_1 and splitting by θ is reduced to two systems. The first homogeneous system for E is:

$$(U - sV)(E_s S_\alpha - E_\alpha S_s) = 0,$$

$$\big(\rho(U - sV)\big)_\alpha E_s - \left[2\rho V + \big(\rho(U - sV)\big)_s\right] E_\alpha = 0,$$

$$[(U - sV)U_\alpha + \rho^{-1}p_\alpha]\, E_s - [(U - sV)V_s - s\rho^{-1}p_s - W^2]\, E_\alpha = 0, \tag{3.1}$$

$$[(U - sV)V_\alpha - s\rho^{-1}p_\alpha]\, E_s - [(U - sV)V_s - s\rho^{-1}p_s - W^2]\, E_\alpha = 0,$$

$$(U - sV)W_\alpha E_s - [(U - sV)W_s + VW]\, E_\alpha = 0.$$

The second system for r_2 is:

$$(U - sV)\,[S_\alpha(\ln r_2)_s - S_s(\ln r_2)_\alpha] = S_\alpha(V - R - WE),$$

$$\big(\rho(U - sV)\big)_\alpha(\ln r_2)_s - \left[2\rho V + \big(\rho(U - sV)\big)_s\right](\ln r_2)_\alpha =$$

$$= \rho_\alpha(V - R) + \rho V_\alpha - (\rho W)_\alpha E,$$

$$[(U - sV)U_\alpha + \rho^{-1}p_\alpha]\,(\ln r_2)_s - [(U - sV)U_s + \rho^{-1}p_s]\,(\ln r_2)_\alpha =$$

$$= U_\alpha(V - R - WE), \tag{3.2}$$

$$[(U - sV)V_\alpha + \rho^{-1}p_\alpha]\,(\ln r_2)_s - [(U - sV)V_s - s\rho^{-1}p_s - W^2]\,(\ln r_2)_\alpha =$$

$$= V_\alpha(V - R - WE) + \rho^{-1}p_\alpha,$$

$$(U - sV)W_\alpha(\ln r_2)_s - [(U - sV)W_s + VW]\,(\ln r_2)_\alpha =$$

$$= W_\alpha(V - R - WE) - \rho^{-1}p_\alpha E.$$

Apart from these systems there is one more system (1.3) for R. If we assume that $E = 0$, then the system (3.2) coincides with (1.3) for R. We can show that in the case $U = sV$ there are no new solutions. Further we assume that $U \neq sV$. Due to Lemma 1 either a) $(\ln r_2)_s$ or b) $(\ln r_2)_\alpha$ is not determined by the system (3.2).

The case a). The coefficients of $(\ln r_2)_s$ in the system (3.2) are equal to zero. In this case

$$S = S(s), \quad W = W(s), \quad V + sU = c(s), \quad \rho(U - sV) = b(s), \quad p = l(s) - b(s)U.$$

There is only one function $U(s, \alpha)$, which significantly depends on α. All the other V, ρ, p are determined by it. The equation of compatibility redefines the solution. The solution holds only for the Chaplygin gas. It results from the system (3.1) that $E = E(s)$, and from the system (1.3) for R that $R = 0$. The contradiction $U_\alpha = 0$ results from the system (3.2).

The case b). The coefficients of $(\ln r_2)_\alpha$ in the system (3.2) are equal to zero. It results in the equation of the conic submodel [4], in which all the constants of the integrals and the general solution depend on the parameter α:

$$S = S(\alpha), \quad W^2 = \rho(U - sV)\mathcal{D}(\alpha),$$

$$U^2 + V^2 + W^2 + 2\textstyle\int \rho^{-1}dp = B^2(\alpha), \tag{3.3}$$

$$V_s + sU_s = \rho\mathcal{D}(\alpha), \quad U_s\left((U - sV)^2 - f_\rho\right) + sV_s = V,$$

where $p = f(\rho, S)$ is the equation of state.

Equations (3.1) yield that $E_s = 0$ (otherwise the reduction occurs) and the system (3.1) is satisfied identically. The equations (1.3) for R take the form

$$\begin{gathered} QS_\alpha = 0, \quad QW_\alpha = 0, \quad QV_\alpha = \rho^{-1}p_\alpha(R + sR_s), \\ QU_\alpha = -\rho^{-1}p_\alpha R_s, \quad Q(\ln\rho)_\alpha = RV_\alpha - (U_\alpha - sV_\alpha)R_s, \end{gathered} \tag{3.4}$$

where $Q = (U - sV)R_s - R(V - R)$.

If $Q \neq 0$, then $S = S_0$ is a constant, $W = W(s)$, and it results from (3.2) that $E = 0$, $r_2 = RK(\alpha)$. The compatibility of (3.4) and (3.3) show that $W = 0$, $R = BB'(V_\alpha + sV_\alpha)^{-1}$. The conditions of the overdetermined submodel have the form:

$$\begin{gathered} f_\rho(\ln\rho)^2_\alpha = U^2_\alpha + V^2_\alpha, \quad U^2 + V^2 + 2\int f_\rho d\ln\rho = B^2, \\ V_s + sU_s = 0, \quad U_s\left[(U - sV)^2 - s^2 - f_\rho\right] = V. \end{gathered} \tag{3.5}$$

Let us assume that $Q = 0$. The equations (3.4) give

$$Rp_\alpha = 0, \quad RV_\alpha = R_s(U_\alpha - sV_\alpha), \quad (U - sV)R_s = R(V - R). \tag{3.6}$$

If $p_\alpha \neq 0$, then $R = 0$, and the equations (3.6) hold. We obtain from the equations (3.2) that $(\ln r_2)_s = -U_\alpha(V_\alpha + sU_\alpha)^{-1}$, $E = -W_\alpha(V_\alpha + sU_\alpha)^{-1}$, where B, S are constants,

$$f_\rho(\ln\rho)^2_\alpha = U^2_\alpha + V^2_\alpha + W^2_\alpha. \tag{3.7}$$

The equations (3.4), (3.7) set a overdetermined submodel which extends the conic submodel.

If $p_\alpha = 0$, then two equations

$$R_s(U_\alpha - sV_\alpha) = RV_\alpha, \quad (U - sV)R_s = R(V - R) \tag{3.8}$$

remained from (3.4). The system (3.2) for irreducible solutions is reduced to two eqialities

$$(U - sV)(\ln r_2)_s = V - R - WE(\alpha), \quad (U_\alpha - sV_\alpha)(\ln r_2)_s = V_\alpha - EW_\alpha. \tag{3.9}$$

Hence when $R \neq 0$ we have

$$E\big(W_\alpha(U - sV) - W(U_\alpha - sV_\alpha)\big) = 0, \quad R = \frac{VU_\alpha - UV_\alpha}{U_\alpha - sV_\alpha}. \tag{3.10}$$

If $E \neq 0$, then the integral $W = n(s)(U - sV)$ holds and $r_2 = R\exp\left(-E(\alpha)n(s)\right)$. It follows from (3.8) that

$$(V_\alpha + sU_\alpha)\left[U_{s\alpha}(U - sV) - U_s(U_\alpha - sV_\alpha)\right] = 0.$$

If the first factor is equal to zero, then the equations (3.3) show that all the functions do not depend on α (reduction to a conic submodel). Equating the second factor to zero provides the integral $U(s) = m(s)(U - sV)$. Studying the compatibility with the system (3.3) provides $R = 0$ which is a contradiction.

Let us assume that $E = 0$, $R \neq 0$. Then $r_2 = RK(\alpha)$, the function R is determined by the formula (3.10). We obtain the additional equation for the system (3.3) from (3.8):

$$\frac{U_\alpha - sV_\alpha}{U - sV} = \frac{U_\alpha V_{\alpha s} - V_\alpha U_{\alpha s}}{U_\alpha V_s - V_\alpha U_s}. \tag{3.11}$$

The overdetermined system (3.3), (3.11) gives a model extending the conic submodel.

The last case $R = 0$, $E \neq 0$ gives the submodel consisting of the system (3.3) and the equation

$$\frac{U_\alpha - sV_\alpha}{U - sV} = \frac{V_\alpha - EW_\alpha}{V - EW}$$

with an arbitrary function $E(\alpha)$.

Thus, all the possible solutions of the overdetermined system (1.3) have been considered. They are reduced to the solution of a conic submodel with respect to the variable s with constants depending on α. To determine the dependence on α there are different additional redefining equations. Moreover, there are also two submodels (2.5) and (2.7), composed by the overdetermined systems of partial differential equations, which extend the conic submodel.

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