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SYMMETRIES AND EXACT SOLUTIONS OF THE MODEL OF DYNAMIC CONVECTION OF THE SEA.

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Abstract. Equations of the mathematical model of dynamic convection of the sea are considered. The model describes incompressible flows of shallow water with variable density under the action of the Coriolis force. This approximation is widely used for modeling mit-latitude oceanic and atmospheric flows.

This model is exceptional from the group-theoretical point of view to its infinite-dimensional group of transformations that involves five arbitrary functions of time. The goal of the paper is to demonstrate the physical meaning of the symmetry transformations, to construct the optimal system of small dimensional subalgebras, and to represent new exact solutions, constructed on the base of the symmetry analysis.

Keywords: equations of dynamic convection of the sea, optimal system of subalgebras, partially invariant solution.

1. Formulation of the problem

We consider the system of equations of dynamic convection of the sea [1, 2] describing motions of shallow water with variable density under the action of Coriolis force:

$$
\begin{aligned}
&u_t + uu_x + vu_y + wu_z - fv = -\rho^{-1}p_x,\\
&v_t + uv_x + vv_y + wv_z + fu = -\rho^{-1}p_y,\\
&0 = -\rho^{-1}p_z - g,\\
&\rho_t + u\rho_x + v\rho_y + w\rho_z = 0,\\
&u_x + v_y + w_z = 0.
\end{aligned}
\tag{1}
$$

The aim of the paper is to investigate symmetries of the system (1), and to construct some new partially-invariant exact solutions.

The system of coordinates is chosen as shown in Figure 1. N and S denote North and South poles, θ is the geographical latitude of the point on the sphere. The axis Ox is directed to the East, Oy is directed to the North, gravitation acts in the negative direction of the axis Oz. The velocity components (u, v, w) are determined in the respective way. The pressure and density are denoted by letters p and ρ; g is the gravitation acceleration.

The equations of model are derived under the following assumption [2]:

$$H/R \ll 1, \tag{2}$$

$$(L/R)^2 \ll 1, \tag{3}$$

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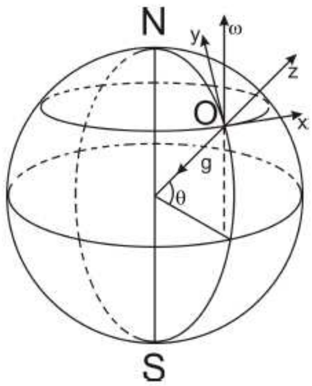

FIGURE 1. The system of coordinates

$$\tan\varphi_0(L/R) \ll 1, \tag{4}$$

here R is Earth's radius, L, H are characteristic horizontal and vertical motion scales, φ_0 is the latitude; $f = 2\Omega\cos\varphi_0 = 1$ is the Coriolis parameter. The approximation (2) means that we consider a thin liquid layer neglecting the radial distortions in the course of motion from one value z to another. This relationship matches with real relationships of the ocean depth and Earth's radius (numerical estimates are presented in [2]). According to the approximation (3) horizontal motion scales are significantly less then the Earth radius. The last approximation (4) is the strictest. It holds for middle or low latitudes, for which $\tan\varphi_0 \leqslant 1$. It excludes application of the model to the analysis of flows in high latitudes, that is in polar seas.

From the group theoretical point of view the infinite group G admitted by the equations of the model [1] is of interest. This group is generated by the following infinitesimal operators:

$$\begin{aligned}
&X_1 = x\partial_x + y\partial_y + 2z\partial_z + u\partial_u + v\partial_v + 2w\partial_w + 2p\partial_p;\\
&X_2 = -y\partial_x + x\partial_y - v\partial_u + u\partial_v;\\
&X_3 = p\partial_p + \rho\partial_\rho;\\
&\langle\tau\rangle_4 = 2\tau\partial_t + (\tau' x + \tau y)\partial_x - (\tau x - \tau' y)\partial_y - [(\tau' + \tau''')\tfrac{x^2+y^2}{2} + 2\tau' z]\partial_z +\\
&\quad +(-\tau' u + \tau v + \tau'' x + \tau' y)\partial_u - (\tau u + \tau' v + \tau' x - \tau'' y)\partial_v -\\
&\quad -[(\tau''' + \tau')(xu + yv) + 4\tau' w + (\tau'''' + \tau'')\tfrac{x^2+y^2}{2} + 2\tau'' z]\partial_w - 2\tau' p\partial_p;\\
&\langle\alpha\rangle_5 = \alpha\partial_x - (\alpha'' x + \alpha' y)\partial_z + \alpha'\partial_u - (\alpha'' u + \alpha' v + \alpha''' x + \alpha'' y)\partial_w;\\
&\langle\beta\rangle_6 = \beta\partial_y + (\beta' x - \beta'' y)\partial_z + \beta'\partial_v + (\beta' u - \beta'' v + \beta'' x - \beta''' y)\partial_w;\\
&\langle\gamma\rangle_7 = \gamma\partial_z + \gamma'\partial_w;\\
&\langle\delta\rangle_8 = \delta\partial_p.
\end{aligned} \tag{5}$$

The operators of the algebra L contain five arbitrary smooth functions of time $\tau(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$. The primes denote derivatives of these functions by t.

Finite transformations for the operators $\langle\alpha\rangle_5, \langle\beta\rangle_6, \langle\gamma\rangle_7$ have the form:

$$\begin{aligned}
\langle\alpha\rangle_5 : \;&\bar{x} = x + \alpha(t),\; \bar{u} = u + \alpha'(t),\\
&\bar{z} = z + (\alpha''(t)x - \alpha'(t)y) + \tfrac{1}{2}\alpha(t)\alpha''(t),\\
&\bar{w} = w + (\alpha''(t)u - \alpha'(t)v + \alpha'''(t)x - \alpha''(t)y) + \tfrac{1}{2}(\alpha(t)\alpha''(t))';
\end{aligned}$$

$$\begin{aligned}
\langle\beta\rangle_6 : \;&\bar{y} = y + \beta(t),\; \bar{v} = v + \beta'(t),\\
&\bar{z} = z + (\beta'(t)x + \beta''(t)y) + \tfrac{1}{2}\alpha(t)\alpha''(t),\\
&\bar{w} = w + (\beta'(t)u + \beta''(t)v + \beta'(t)x + \beta'''(t)y) + \tfrac{1}{2}(\beta(t)\beta''(t))';
\end{aligned}$$

$$\langle\gamma\rangle_7 : \bar{z} = z + \gamma(t),\; \bar{w} = w + \gamma'(t).$$

We specify only nontrivial transformations, the remaining quantities transform identically. These transformations represent generalized Galilean translations in the directions of the axis Ox, Oy, Oz, respectively. Application of the present transformations to the solutions containing singularities (source, drain, vortex) appears to obtain the solution with the same singularities moving along a given trajectory.

The direct calculation of the finite transformation, corresponding to the operator $\langle\tau\rangle_4$ leads to the implicit form which is inconvenient for its further application. To find an explicit form of the transformation, the symmetry of infinitesimal operators of the algebra L with respect to the transformations of the admitted group G can be used. Indeed, since the transformations of the group G conserve the system of equations (1), their action upon any of the operators of the algebra L provides a linear combination of the same operators. Using this observation and the information on the form of the admitted transformation following from the implicit representation one can obtain the required symmetry transformation explicitly. For the convenience this transformation is written in the cylindrical coordinates (r, θ, z), $x = r\cos\theta$, $y = r\sin\theta$; U and V are radial and circular components of velocity in the plane Oxy, W is the component of the velocity along the axis Oz (Figure 1).

$$
\begin{aligned}
&\bar{t} = \varphi(t),\\
&\bar{r} = r\sqrt{\varphi'(t)},\\
&\bar{\theta} = \theta - \frac{1}{2}\left(\varphi(t) - t\right),\\
&\bar{z} = \frac{1}{\varphi'(t)}z + \frac{-\varphi'(t)^2 + \varphi'(t)^4 - 3\varphi''(t)^2 + 2\varphi'(t)\varphi^{(3)}(t)}{4\varphi'(t)^3}\frac{r^2}{2},\\
&\bar{u} = \frac{u_r}{\sqrt{\varphi'(t)}} + \frac{r}{2}\left(\frac{\varphi''(t)}{2\varphi'(t)^{3/2}}\right),\\
&\bar{v} = v_\theta\frac{1}{\sqrt{\varphi'(t)}} + \frac{r}{2}\frac{1-\varphi'(t)}{\varphi^{3/2}(t)},\\
&\bar{w} = \frac{1}{\varphi'(t)^2}w + \frac{1}{8(\varphi'(t))^5}\big(2ru_r\varphi'(t)(-\varphi'(t)^2 + \varphi'(t)^4 - 3\varphi''(t)^2) +\\
&\qquad + r^2\varphi''(t)(\varphi'(t)^2 + \varphi'(t)^4 + 9\varphi''(t)^2) +\\
&\qquad + 2r(2u_r\varphi'''(t) + r\varphi^{(IV)}(t)) - 10r^2\varphi'(t)\varphi''(t)\varphi'''(t)\big),\\
&\bar{p} = \frac{p}{\varphi'(t)}.
\end{aligned}
\tag{6}
$$

Here $\varphi(t)$ is an arbitrary function. The present transformation allows one generate nonstationary solutions from solutions describing stationary flows of the liquid.

2. Optimal system of subalgebras

For the systematic description of invariant and partially invariant solutions of the system (1) it is necessary to construct an optimal system of subalgebras for the algebra L. Nontriviality of the problem consists in the infinity of the algebra L. The first step to the construction of the optimal system of subalgebras is calculation of the group of the inner automorphisms $\mathrm{Aut}\,L$ [3]. Application of the standard algorithm [4] provides the following result:

$A_1 : \overline{\tau} = \tau, \overline{\alpha} = e^{-t_1}\alpha, \overline{\beta} = e^{-t_1}\beta, \overline{\gamma} = e^{-2t_1}\gamma, \overline{\delta} = e^{-2t_1}\delta;$

$A_2 : \overline{\alpha} = \alpha\cos t_2 - \beta\sin t_2, \overline{\beta} = \alpha\sin t_2 + \beta\cos t_2, \quad \overline{\tau} = \tau, \quad \overline{\gamma} = \gamma, \overline{\delta} = \delta;$

$A_3 : \overline{\tau} = \tau, \quad \overline{\alpha} = \alpha, \quad \overline{\beta} = \beta, \quad \overline{\gamma} = \gamma, \quad \overline{\delta} = e^{-t_3}\delta;$

$A_4 : \overline{\tau} = \frac{T(t_4+\chi(t))}{2\chi'(t)}$, where $\tau(t) = \frac{T(\chi(t))}{2\chi'(t)}$,

$\overline{\delta} = 2\chi'(t)D(t_4 + \chi(t))$, where $\delta(t) = 2\chi'(t)D(\chi(t))$,

$\overline{\gamma} = 2\chi'(t)\Gamma(t_4 + \chi(t)), \gamma(t) = 2\chi'(t)\Gamma(\chi(t));$

$\overline{\alpha} = \frac{\alpha_0(t_4+\chi(t))\cos\frac{t}{2} - \beta_0(t_4+\chi(t))\sin\frac{t}{2}}{\sqrt{2\chi'(t)}}$, where $\alpha(t) = \frac{\alpha_0(\chi(t))\cos\frac{t}{2} - \beta_0(\chi(t))\sin\frac{t}{2}}{\sqrt{2\chi'(t)}}$

$\overline{\alpha} = \frac{\alpha_0(t_4+\chi(t))\sin\frac{t}{2} + \beta_0(t_4+\chi(t))\cos\frac{t}{2}}{\sqrt{2\chi'(t)}}$, where $\beta(t) = \frac{\alpha_0(\chi(t))\sin\frac{t}{2} + \beta_0(\chi(t))\cos\frac{t}{2}}{\sqrt{2\chi'(t)}}$

$A_5 : \overline{\tau} = \tau, \overline{\alpha} = \alpha + A_5 t_5, \overline{\beta} = \beta + B_5 t_5,$

$\overline{\gamma} = \gamma + \Gamma_{51}t_5 + \Gamma_{52}\frac{t_5{}^2}{2}, \overline{\delta} = \delta,$

$A_5 = [(x^1 + \tau')\sigma - 2\tau\sigma'], B_5 = (\tau - x^2)\sigma,$

$\Gamma_{51} = \sigma\alpha'' - \alpha\sigma'' - \sigma'\beta - \sigma\beta', \Gamma_{52} = \sigma A_5'' - \sigma'' A_5 - \sigma' B_5 - \sigma B_5';$

$A_6 : \overline{\tau} = \tau, \overline{\alpha} = \alpha + A_6 t_6, \overline{\beta} = \beta + B_6 t_6,$

$\overline{\gamma} = \gamma + \Gamma_{61}t_6 + \Gamma_{62}\frac{t_6{}^2}{2}, \overline{\delta} = \delta,$

$A_6 = (x^2 - \tau)\sigma, B_6 = [(x^1 + \tau')\sigma - 2\tau\sigma'],$

$\Gamma_{61} = \sigma\beta'' - \sigma''\beta + \alpha\sigma' + \sigma\alpha', \Gamma_{62} = \sigma B_6'' - \sigma'' B_6 + \sigma' A_6 + \sigma A_6';$

$A_7 : \overline{\tau} = \tau, \overline{\alpha} = \alpha, \overline{\beta} = \beta, \overline{\gamma} = \gamma + \Gamma_{71}t_7, \overline{\delta} = \delta,$

$\Gamma_{71} = 2x^1\sigma - (\tau\sigma)';$

$A_8 : \overline{\tau} = \tau, \overline{\alpha} = \alpha, \overline{\beta} = \beta, \overline{\gamma} = \gamma, \overline{\delta} = \delta + \Delta_{81}t_8,$

$\Delta_{81} = 2x^1\sigma + x^3\sigma - 2(\tau\sigma' + \sigma\tau').$

The above formulae specify transformed coordinates of the infinitesimal operator

$$X = aX_1 + bX_2 + cX_3 + \langle\tau\rangle_4 + \langle\alpha\rangle_5 + \langle\beta\rangle_6 + \langle\gamma\rangle_7 + \langle\delta\rangle_8 \quad (7)$$

under the action of every automorphism. Instead of the implicit representation of the inner automorphism A_4 it is convenient to use its explicit form obtained by the action of the transformation (6) on every operator of the algebra L written in the coordinate form (5).

The considered algebra L can be expanded into the semidirect sum $L = L_4 \dot{\oplus} L_\infty$, where $L_4 = \{X_1, X_2, X_3, \langle\tau\rangle_4\}$. We construct the optimal system of subalgebras using the two-steps algorithm [3]. First we construct an optimal system for the subalgebra L_4, which in its turn admits expansion into a direct sum of the finite and infinite parts. For $L_3 = \{X_1, X_2, X_3\}$ the optimal system is easily constructed since this subalgebra is Abelian. It consists of one three-dimensional representative $\{X_1, X_2, X_3\}$, three two-dimensional representatives $\{aX_1 + X_2, bX_2 + X_3\}$, $\{X_1, cX_2 + X_3\}$, $\{X_1, X_2\}$, three one-dimensional representatives $\{aX_1 + bX_2 + X_3\}$, $\{aX_1 + X_2\}, \{X_1\}$ and the zero subalgebra.

The one-dimensional subalgebras from L_4 having a nontrivial coordinate of $\langle\tau\rangle_4$ can be reduced to the form with $\tau = 1$ by the action of the automorphism A_4. Indeed, the infinitesimal operator $\langle\tau\rangle_4$ changes under the action of the substitution (6) in the following way:

$$\langle\tau\rangle_4 = \tau(t)\partial_t + \dots \quad \to \quad \langle\bar{\tau}\rangle_4 = \tau(t)\varphi'(t)\partial_{\bar{t}} + \dots .$$

The choice $\varphi = \int 1/\tau(t)dt$ reduces the operator $\langle\tau\rangle_4$ to the translation operator ∂_t. Therefore the optimal system includes the one-dimensional subalgebra $\{aX_1 + bX_2 + cX_3 + \langle 1\rangle_4\}$.

Let us proceed to the construction of the one-dimensional representatives of an optimal system for the algebra L. The general from the operator is (7). First we consider the case $\tau \neq 0$. As it was shown above, the action of the automorphism A_4 provides $\tau = 1$. Application of the automorphisms A_5, A_6 allows making make $\alpha = 0$, $\beta = 0$. The automorphisms A_7, A_8 allow one to annull the coordinates γ and δ. Therefore the optimal system contains the subalgebra $\{aX_1 + bX_2 + cX_3 + \langle 1\rangle_4\}$.

If $\tau = 0$, then in the case $a \neq 0$ we still make $\alpha = 0$, $\beta = 0$ with the help of A_5, A_6. We make $\gamma = 0$, $\delta = 0$ with the help of A_7, A_8 and include the subalgebra $\{X_1 + bX_2 + cX_3\}$ into the optimal system. If $a = 0$ and $b \neq 0$, it is impossible to annull γ with the help of A_7. In this case we obtain one more one-dimensional representative $\{bX_2 + X_3 + \langle\gamma\rangle_7\}$.

In the case $a = 0$, $b = 0$ the automorphism A_4 allows making $\alpha = 1$, $\beta = 0$. The coordinate γ will remain arbitrary. Acting by the operator A_8, we get $\delta = 0$. Then we include $\{X_3 + \langle 1\rangle_5 + \langle\gamma\rangle_7\}$ in the optimal system.

It remains to consider the case $c = 0$, that is X_3 is not included in the basis operator of the subalgebra. If $a = 0$ and $b = 0$, then we obtain the subalgebra $\{\langle 1\rangle_5 + \langle\gamma\rangle_7 + \langle\delta\rangle_8\}$. If $a \neq 0$, we deal with the case considered above. When $b \neq 0$ we obtain the one-dimensional subalgebra $\{bX_2 + \langle\gamma\rangle_7 + \langle\delta\rangle_8\}$.

Therefore, the optimal system contains the following one-dimensional subalgebras:

$$\begin{gathered}\{aX_1 + bX_2 + cX_3 + \langle 1\rangle_4\}, \quad \{X_1 + bX_2 + cX_3\},\\ \{bX_2 + X_3 + \langle\gamma\rangle_7\}, \quad \{\langle 1\rangle_5 + \langle\gamma\rangle_7 + \langle\delta\rangle_8\},\\ \{bX_2 + \langle\gamma\rangle_7 + \langle\delta\rangle_8\}, \quad \{X_3 + \langle 1\rangle_5 + \langle\gamma\rangle_7\}, \{aX_1 - 2aX_3 + \langle\delta\rangle_8\}.\end{gathered}$$

Similarly we construct an optimal system of two-dimensional subalgebras. The condition of the subalgebra (the commutator of the basis operators of the subalgebra is their linear combination) allows to additionally specify the form of the functions included in the basis operators. Like for one the one-dimensional subalgebras, the most bulky operator $\langle\tau\rangle_4$ can always be reduced to the operator of time translation $\langle 1\rangle_4 = \partial_t$ by the action of the transformation (6). Finally, the optimal system of subalgebras has the following form (the functions $\alpha_{1,2}(t)$, $\beta_{1,2}(t)$, $\gamma_{1,2}(t)$, $\delta_{1,2}(t)$ are arbitrary if there is no other indication):

$$\begin{aligned}&\{X_1 + \langle 1\rangle_4, X_2 + \langle C\rangle_4 + \langle C_1 e^{t/2}\cos(\tfrac{t}{2}) - C_2 e^{t/2}\sin(\tfrac{t}{2})\rangle_5 +\\ &+\langle C_2 e^{t/2}\cos(\tfrac{t}{2}) + C_1 e^{t/2}\sin(\tfrac{t}{2})\rangle_6 + \langle C_3 e^{2t}\rangle_7 + \langle C_4 e^{t}\rangle_8\},\end{aligned}$$

$$\begin{aligned}&\{X_1 + \langle 1\rangle_4, cX_2 + X_3 + \langle C\rangle_4 + \langle C_1 e^{t/2}\cos(\tfrac{t}{2}) - C_2 e^{t/2}\sin(\tfrac{t}{2})\rangle_5 +\\ &+\langle C_2 e^{t/2}\cos(\tfrac{t}{2}) + C_1 e^{t/2}\sin(\tfrac{t}{2})\rangle_6 + \langle C_3 e^{2t}\rangle_7 + \langle C_4 e^{t}\rangle_8\},\end{aligned}$$

$$\{dX_1 + X_2 + \langle 1\rangle_4, fX_2 + X_3 + \langle C\rangle_4 + \langle C_1 e^{\frac{d}{2}t}\rangle_5 + \langle C_2 e^{\frac{d}{2}t}\rangle_6 + \langle C_3 e^{2dt}\rangle_7 + \langle C_4 e^{dt}\rangle_8\},$$

$$\{aX_1 + bX_2 + cX_3 + \langle 1\rangle_4, \langle C\rangle_4 + \langle\alpha_2\rangle_5 + \langle\beta_2\rangle_6 + \langle C_1 e^{2at}\rangle_7 + \langle C_2 e^{(2a+C)t}\rangle_8\},$$

where the functions α_2, β_2 are the solution of the following system:

$$\begin{pmatrix}\alpha_2'\\ \beta_2'\end{pmatrix} = \begin{pmatrix}-a & b+1\\ b-1 & -a\end{pmatrix}\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix},$$

$\{\langle 1\rangle_4, \langle C\rangle_4 + \langle\alpha_2\rangle_5 + \langle\beta_2\rangle_6 + \langle\gamma_2\rangle_7 + \langle\delta_2\rangle_8\}$,

$\{X_1 + \langle C_1\rangle_5 + \langle C_2\rangle_6 + \langle C_3\rangle_7 + \langle C_4\rangle_8, X_2 + \langle 1\rangle_4\}$,

$\{X_1 + \langle C_1 \cos\left(\frac{1}{2}(c-1)t\right) + C_2 \sin\left(\frac{c-1}{2}t\right)\rangle_5 +$
$+\langle C_2 \cos\left(\frac{c-1}{2}t\right) - C_1 \sin\left(\frac{c-1}{2}t\right)\rangle_6 + \langle C_3\rangle_7 + \langle C_4 e^{t/2}\rangle_8, cX_2 + X_3 + \langle 1\rangle_4\}$,

$\{dX_1 + X_2 + \langle C_1 \cos\left(\frac{1}{2}(c-1)t\right) + C_2 \sin\left(\frac{c-1}{2}t\right)\rangle_5 +$
$+\langle C_2 \cos\left(\frac{c-1}{2}t\right) - C_1 \sin\left(\frac{c-1}{2}t\right)\rangle_6 + \langle C_3\rangle_7 + \langle C_4 e^{t/2}\rangle_8, fX_2 + X_3 + \langle 1\rangle_4\}$,

$\{aX_1 + bX_2 + cX_3 + \langle C_1 \cos\left(\frac{t}{2}\right) - C_2 \sin\left(\frac{t}{2}\right)\rangle_5 + \langle C_2 \cos\left(\frac{t}{2}\right) + C_1 \sin\left(\frac{t}{2}\right)\rangle_6 + \langle C_3\rangle_7 + \langle C_4\rangle_8, \langle 1\rangle_4\}$,

$\{X_1, X_2\}$,

$\{X_1, cX_2 + X_3\}$,

$\{dX_1 + X_2, fX_2 + X_3\}$,

$\{bX_2 + X_3 + \langle\gamma_1\rangle_7, \langle\gamma_2\rangle_7\}$,

$\{\langle 1\rangle_5 + \langle\gamma_1\rangle_7 + \langle\delta_1\rangle_8, \langle\alpha_2\rangle_5 + \langle C + \alpha_2'\rangle_6 + \langle\gamma_2\rangle_7 + \langle\delta_2\rangle_8\}$,

$\{bX_2 + \langle\gamma_1\rangle_7 + \langle\delta_1\rangle_8, \langle\gamma_2\rangle_7 + \langle\delta_2\rangle_8\}$,

$\{X_3 + \langle 1\rangle_5 + \langle\gamma\rangle_7, \langle\alpha_2\rangle_5 + \langle C + \alpha_2'\rangle_6 + \langle\gamma_2\rangle_7\}$.

3. The partially invariant solution of the system

For convenience of the further investigation we write the equations of dynamic convection of the sea in cylindrical coordinates:

$$\begin{array}{l} U_t + UU_r + r^{-1}VU_\theta + WU_z - V + \rho^{-1}p_r = r^{-1}V^2, \\ V_t + UV_r + r^{-1}VV_\theta + WV_z + U + (r\rho)^{-1}p_\theta = -r^{-1}UV, \\ \rho_t + U\rho_r + r^{-1}V\rho_\theta + W\rho_z = 0, \\ U_r + r^{-1}U + r^{-1}V_\theta + W_z = 0, \\ \rho = -p_z. \end{array} \tag{8}$$

Let us consider the following five-dimensional subalgebra of the symmetry Lie algebra:

$$L_5 = \langle \partial_z, t\partial_z + \partial_W, \partial_\theta, p\partial_p + \rho\partial_\rho, \partial_p\rangle. \tag{9}$$

The basis operators of the algebra (9) correspond, in the succession order, to the operators $\langle 1\rangle_7, \langle t\rangle_7, X_2, X_3, \langle 1\rangle_8$ of the Lie algebra.

The algebra (9) generates the partially invariant solution of rank 2 and defect 3 having the representation:

$$\begin{array}{c} U = U(t,r),\ V = V(t,r),\ W = W(t,r,\theta,z), \\ \rho = \rho(t,r,\theta,z),\ p = p(t,r,\theta,z), \end{array} \tag{10}$$

where $\rho > 0$, $p > 0$ due to non-negativity of the density and the pressure.

Substituting the representation (10) in the original system we obtain the following submodel:

$$\begin{array}{l} U_t + UU_r - V + \rho^{-1}p_r = r^{-1}V^2, \\ V_t + UV_r + U + (r\rho)^{-1}p_\theta = -r^{-1}UV, \\ \rho_t + U\rho_r + r^{-1}V\rho_\theta + W\rho_z = 0, \\ U_r + r^{-1}U + W_z = 0, \\ \rho = -p_z. \end{array} \tag{11}$$

Let us solve the obtained equations with respect to the derivatives of the function p:

$$\begin{gathered} p_r = r^{-1}\rho\left(rV + V^2 - r\left(UU_r + U_t\right)\right), \\ p_\theta = -\rho\left(U\left(rV_r\right) + rV_t\right), \\ p_z = -\rho. \end{gathered} \tag{12}$$

The remained equations of the system take the following form:

$$W_z = -\frac{1}{r}\frac{\partial}{\partial r}\left(rU\right), \tag{13}$$

$$\rho_t + U\rho_r + r^{-1}V\rho_\theta + W\rho_z = 0. \tag{14}$$

The equation (13) determines the function $W(t, r, \theta, z)$:

$$W(t, r, \theta, z) = -\frac{z}{r}\frac{\partial}{\partial r}\left(rU\right) + W_0(t, r, \theta). \tag{15}$$

The condition of compatibility of the equations (12) yield the expressions for the derivatives of the density:

$$\rho_r = -r^{-1}\big(rV + V^2 - r\left(UU_r + U_t\right)\big)\,\rho_z, \tag{16}$$

$$\rho_\theta = \left(U\left(r + V + rV_r\right) + rV_t\right)\rho_z. \tag{17}$$

Integration of the corresponding compatibility conditions of the system (12) taking into account to the obtained representations (16), (17) allows to determine V_t:

$$rV_t = -rV_rU - UV - rU - h(t),$$

where $h(t)$ is an arbitrary function of time, which appears after integration. Substitution of the obtained expression into the second equation of the system (11) provides the following:

$$p_\theta = h(t)\rho.$$

Let us denote $f(t, r) = -U_t - UU_r + V + \dfrac{V^2}{r}$. Then the first equation of the system (12) takes the form:

$$p_r = f(t, r)\rho.$$

Finally the system (12) is rewritten in the following form:

$$\left\{\begin{array}{l} p_\theta = h(t)\rho, \\ p_r = f(t, r)\rho, \\ p_z = -\rho. \end{array}\right. \tag{18}$$

The density ρ is obtained by integrating equations (16), (17):

$$\rho(t, r, \theta, z) = R\left(t, \lambda\right), \quad \lambda = h(t)\theta - z + g(t, r), \tag{19}$$

where $R > 0$ is some arbitrary function, $g(t, r) = \displaystyle\int f(t, r)dr$. Substituting the obtained result in (18) we find the function p:

$$p = \int R\left(t, \lambda\right) d\lambda + p_0(t). \tag{20}$$

In the relationship (19) the dependence of the variable λ on the polar angle θ is linear, therefore the function R should be periodic for the continuity of the density over the whole space. Due to the same reason the pressure should be a periodic function. But due to non-negativity of ρ it results from the formula (20) that the pressure is a strictly increasing function of the variable λ. Hence, the continuity of the solution can be achieved only for $h(t) = 0$.

Substitution of the representation (19) into the equation (14) provides the following:

$$R_t + R_\lambda \left(g_t(t,r) + U g_r(t,r) + \frac{z}{r}\frac{\partial}{\partial r}(rU) + W_0(t,r,\theta) \right) = 0.$$

The factor for R_λ should depend only on λ. This can be achieved by the choice of the functions W_0 and U:

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) = f(t), \quad W_0(t,r,\theta) = g_t(t,r) + U g_r(t,r) + \frac{g}{r}\frac{\partial}{\partial r}(rU),$$
$$U(t,r) = f(t)\frac{r}{2} + \frac{a(t)}{r},$$

where $f(t)$, $a(t)$ are arbitrary functions of time.

As a result we have the equation for the function $R(t,\lambda)$:

$$R_t - f(t)\lambda R_\lambda = 0. \tag{21}$$

Its solution is written as follows:

$$R(t,\lambda) = R\left(\lambda \exp(\int f(t)dt)\right).$$

Now we can find function V from the second equation of the system (11):

$$V(t,r) = -\frac{r}{2} + \frac{C(\xi)}{r}, \tag{22}$$

where ξ is the Lagrangian coordinate given as follows:

$$\begin{cases} \dfrac{dr}{dt} = U(t,r), \\ r(0) = \xi. \end{cases} \tag{23}$$

Then the first equation of the system (11) determines function $g(t,r)$:

$$g(t,r) = -\frac{r^2}{4}\left(f'(t) + \frac{1}{2}f(t)^2 + \frac{1}{2} \right) - \frac{a(t)^2}{2r^2} - a'(t)\ln r + \int \left(\frac{C(\xi)^2}{r^3} \right) dr. \tag{24}$$

Finally, we obtain the following exact solution:

$$\begin{aligned}
U &= \frac{rf(t)}{2} + \frac{q(t)}{2\pi r},\\
V &= -\frac{r}{2} + \frac{\Gamma(\xi)}{2\pi r},\\
W &= -zf(t) + W_0(t,r),\\
R &= R\Big(\lambda \exp(\int f(t)dt)\Big),\\
P &= \int R\Big(\lambda \exp(\int f(t)dt)\Big)d\lambda + p_0(t)\\
g &= -\frac{r^2}{4}\left(f'(t) + \frac{1}{2}f(t)^2 + \frac{1}{2} \right) - \frac{q(t)^2}{4\pi^2 r^2} - \frac{q'(t)}{2\pi}\ln r + \frac{1}{4\pi^2}\int \left(\frac{\Gamma(\xi)^2}{r^3} \right) dr.
\end{aligned}$$

Here

$$a(t) = \frac{q(t)}{2\pi}, \quad C(\xi) = \frac{\Gamma(\xi)}{2\pi}.$$

The arbitrary functions involved in the solution have a clear physical interpretation. The function $q(t)$ gives the rate of the source distributed over the axis Oz. The one function $f(t)$ determines velocity of the radial dispersion of particles. Function $\Gamma(\xi)$ allows one prescribe arbitrarily the circulation of the velocity vector along the circle $r = \text{const}$ in some cylindrical layer.

The surfaces of a constant density $\rho = \rho_0$ are characteristic for the solution:

$$z = g(t, r) + C \exp\left(\int f(t)dt\right). \tag{25}$$

Let us determine the function $q(t)$ in the form:

$$\begin{aligned} &q(t) = 2\pi(1 - \cos t) \text{ for } 0 \leqslant t \leqslant 2\pi, \\ &q(t) = 0 \text{ for } t > 2\pi, \quad t < 0. \end{aligned} \tag{26}$$

The remaining arbitrary functions are takes to be equal to zero: $f(t) = \Gamma(\xi) = 0$. The generatrices of surfaces of constant density are shown in Figure 2,*a*. At the initial moment of time the surfaces of an equal density are paraboloids. When $t > 0$ and when the function $q(t)$ takes nonzero values, a singularity appears at $r = 0$. Due to the source on the axis. The surfaces of the equal density move away from the axis $r = 0$. When the action of the source stops the surfaces again become paraboloids.

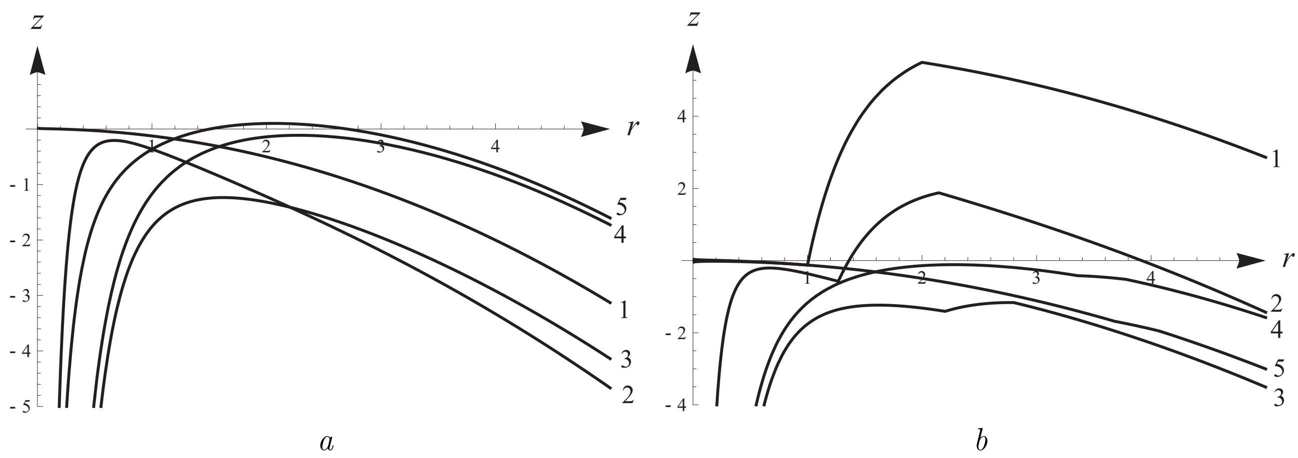

FIGURE 2. The surfaces of the constant density (25) for equal periods of time $t \in (0, 2\pi)$ are enumerated in the increasing order of t when *a* - $\Gamma(\xi) = 0$, *b* - $\Gamma(\xi) \neq 0$

Let us consider the case when besides the source on the axis there is a finite twist of the flow in some cylindrical layer. For this purpose we set the function $\Gamma(\xi)$ in the following way:

$$\Gamma(\xi) = 0 \text{ when } \xi < 1, \xi > 2,$$

$$\Gamma(\xi) = 8\pi \text{ when } 1 \leqslant \xi \leqslant 2.$$

We leave the function $q(t)$ defined by (26) as above and $f(t)$ identically equal to zero. In this case we obtain surfaces of constant density illustrated in Figure 2, *b*. In the domain, where $\Gamma(\xi)$ has a non-vanishing value we observe a "buckling" of surfaces of constant density, which conserves in time but moves away from the axis Oz.

4. Self-similar solution

In this part of the paper we consider construction of a partially invariant solution with respect to the following four-dimensional subalgebra of the Lie algebra of the symmetry of the system:

$$L_4 = \langle \partial_z, t\partial_z + \partial_W, p\partial_p + \rho\partial_\rho, r\partial_r + 2z\partial_z + U\partial_U + V\partial_V + 2W\partial_W + p\partial_p \rangle, \tag{27}$$

where the basis operators correspond to the operators of the algebra $\langle 1\rangle_7$, $\langle t\rangle_7$, X_3, X_1.

The subalgebra (27) has the invariants $t, \theta, U/r, V/r$ and generates a partially invariant solution of rank 2 and defect 3 having the representation:

$$\begin{gathered} U = ru(t,\theta),\ V = rv(t,\theta),\ W = w(t,r,\theta,z), \\ \rho = \rho(t,r,\theta,z),\ p = p(t,r,\theta,z). \end{gathered} \tag{28}$$

An attempt to investigate such a solution when $\rho = r^a R(t,\theta)$, $a = \text{const}$ was made in [5].

After substitution of the representation of the solution (28) we obtain a factor-system in the following form:

$$rv\,(u_\theta - 1) + ru^2 - rv^2 + \frac{1}{\rho}p_r + ru_t = 0, \tag{29}$$

$$rvv_\theta + u(2rv + r) + \frac{1}{r\rho}p_\theta + rv_t = 0, \tag{30}$$

$$\rho_z w + ru\rho_r + v\rho_\theta + \rho_t = 0, \tag{31}$$

$$2u + v_\theta + w_z = 0, \tag{32}$$

$$\rho = -p_z. \tag{33}$$

We obtain from equation (32) a representation for the function $w(t,r,\theta,z)$:

$$w(t,r,\theta,z) = -(2u+v)z + w_0(t,r,\theta). \tag{34}$$

The equations (29), (30) and (33) provide an explicit expression for the derivatives of the function $p(t,r,\theta,z)$. Calculation of mixed second derivatives of p provides two conditions of compatibility being equations for the function $\rho(t,r,\theta,z)$. They are integrated in the form

$$\rho(t,r,\theta,z) = R(t,\lambda), \quad \lambda = \frac{r^2}{2}f(t,\theta) - z, \tag{35}$$

where the following notation is introduced:

$$f(t,\theta) = v\,(u_\theta - 1) + u^2 - v^2 + u_t. \tag{36}$$

After substitution of (35) into the initial equations we obtain the equation for the function $R(t,\lambda)$:

$$R_{\lambda\lambda}\left(\frac{r^2}{2}\left(f_t + f\,(4u + v_\theta) + vf_\theta\right) + \lambda\,(-2u - v_\theta) - w_0\right) + R_{t\lambda} = 0. \tag{37}$$

The factor for $R_{\lambda\lambda}$ should depend only on t and λ. This implies that

$$\begin{gathered} w_0 = \frac{r^2}{2}\left(f_t + f\,(4u + v_\theta) + vf_\theta\right) + \frac{h'(t)}{k(t)}, \\ u = \frac{1}{2}\left(-v_\theta + \frac{k'(t)}{k(t)}\right). \end{gathered} \tag{38}$$

Here $h(t)$, $k(t)$ are arbitrary functions of time.

Substitution of the relationships (38) into (37) provides a final expression for the density. For the sake convenience we write it in the form of a derivative from some function P:

$$\rho = P'(k(t)\lambda + h(t)). \tag{39}$$

Then the pressure has the following form:

$$p = \frac{1}{k(t)}P(\lambda k(t) + h(t)). \tag{40}$$

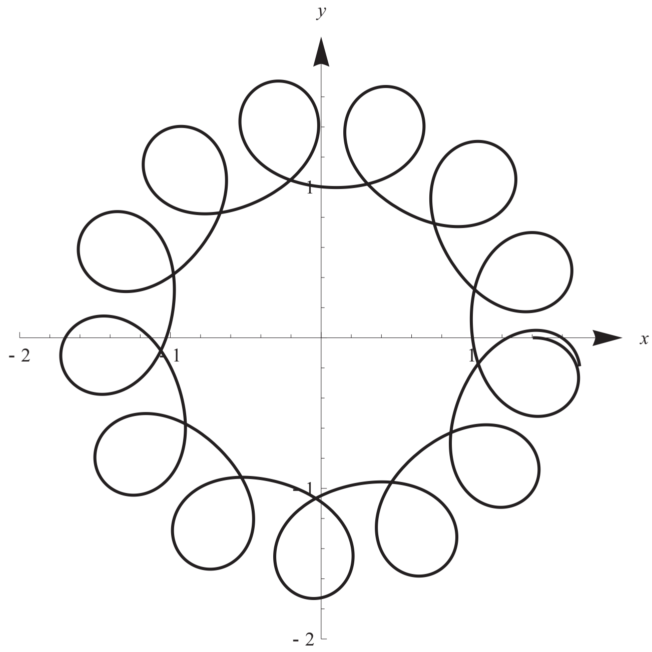

FIGURE 3. Trajectory of the self-similar solution

Substitution of the representations (39), (40) into the initial system of equations (29)-(33) results in the following system of equations for the functions $f(t,\theta), u(t,\theta), v(t,\theta)$:

$$\begin{cases} f_\theta = -2v_t + \left(-2u - 2v\dfrac{k'(t)}{k(t)}\right), \\ u_\theta = -\dfrac{1}{v}u_t + \left(1 - \dfrac{f}{v} - \dfrac{u^2}{v} + v\right), \\ v_\theta = -2u + \dfrac{k'(t)}{k(t)}. \end{cases} \tag{41}$$

Let us search for an axial-symmetric solution of the system (41):

$$u = u(t), v = v(t), f = f(t).$$

We obtain the following solution:

$$u = \frac{k'(t)}{k(t)}, \quad v = \frac{A}{k(t)} - \frac{1}{2},$$
$$f = \frac{1}{4}\left(\frac{k'(t)}{k(t)}\right)^2 - \frac{k''(t)}{k(t)} - \frac{1}{4} + \left(\frac{A}{k(t)}\right)^2.$$

Here $k(t)$ is an arbitrary function, A is an arbitrary constant. Let us construct a trajectory of motion of particles in the plane Oxy. For this purpose we choose

$$k(t) = 2 + \sin t, \quad A = 1.$$

The present solution is periodical in time. Figure 3 illustrates a characteristic trajectory of a particle in the liquid flow determined by the obtained solution.

CONCLUSION

We have constructed an optimal system of one-dimensional and two-dimensional subalgebras for the infinite group admitted by the equations of dynamic convection of the sea. In the course of the investigation we have calculated in an explicit form a finite transformation for the operator of the admitted group, which transforms the time t into an arbitrary function of t. The representation of transformation was crucial for construction of an optimal system.

We have also constructed new exact solutions of the equations of dynamic convection of the sea. The solutions are partially invariant of rank 2 and defect 3. The first one describes a three-dimensional vortex flow generated by the interaction of a source of an arbitrary power, distributed over a vertical straight line with an arbitrary rotation in the cylindrical layer enclosing the source. The second solution corresponds to a vortex periodical in the time axisymmetrical flow around the origin of coordinates. Applying the admissible symmetry transformations we can generate equivalent solutions, in which the vortex moves along an arbitrary trajectory in the horizontal plane and rotates rigidly with an arbitrary angular velocity. For the first solution we specify physical characteristics of surfaces of the constant density of the flow, for the second solution we have constructed trajectories of particles.

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