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FRACTIONAL DIFFERENTIAL EQUATIONS: CHANGE OF VARIABLES AND NONLOCAL SYMMETRIES

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Abstract. In this paper point transformations of variables in fractional integrals and derivatives of different types are considered. In the general case, fractional integro-differentiation of a function with respect to another function arises after such substitution. The problem of applying a point transformations group to this type of operators is considered, the corresponding prolongation formulae for infinitesimal operator of the group are constructed. Usage of prolongation formulae for finding nonlocal symmetries of equations and checking their admittance is demonstrated on a simple example of an ordinary fractional differential equation.

Keywords: fractional derivative, prolongation formulae, nonlocal symmetry.

Introduction

Differential equations with fractional derivatives have a wide range of applications as mathematical models of various processes with anomalous kinetics [1, 2]. Investigation of symmetry properties of such equations is an important problem. Meanwhile, unlike the classical integer-order derivatives, there exists a number of differerent definitions of fractional derivatives [3, 4, 5, 6, 7]. These definition differences leads to the fractional differential equations (FDEs) having similar form but significantly different properties.

The Riemann-Liouville left-hand side fractional derivative [3] defined as

$$\left({}_cD_x^\alpha y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_c^x \frac{y(t)}{(x-t)^{\alpha-n+1}}dt \tag{1}$$

and the Caputo left-hand side fractional derivative [4] defined as

$$\left({}_c^CD_x^\alpha y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\int_c^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt \tag{2}$$

are used most often in practice (here $n=[\alpha]+1$, $\Gamma(x)$ is the gamma-function).

In general case, a solution of a differential equation with the derivative (1) may contain an integrable singularity (of the order not higher than $1-\alpha$) at the point $x=c$, while the existence of the derivative (2) implies that the solution is bounded at this point. It is known (see, for example, [4]) that if a finite limit $\lim_{x\to c+} y(x) = y(c)$ exists, then the derivatives (1) and (2) are connected by the relationship

$$\left({}_cD_x^\alpha y\right)(x) = \left({}_c^CD_x^\alpha y\right)(x) + \frac{1}{\Gamma(1-\alpha)}\frac{y(c)}{(x-c)^\alpha}. \tag{3}$$

Methods of constructing the point transformation groups admitted by differential equations were developed in the papers [8, 9, 10, 11] for equations with fractional derivatives of the form (1) and (2). Prolongation of the group infinitesimal operator to the fractional integrals and derivatives was constructed. Algorithms of finding the admitted group for equations containing

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these derivatives were developed, and some problems of group classification of ordinary differential equations and fractional-order partial differential equations were solved. However, it was found that the class of variables substitutions preserving the form of fractional derivatives is very narrow. The general form of such a point substitution for the Riemann-Liouville-type derivatives (1) is defined by the expression

$$\bar{x} = \frac{cc_1 + (x - c)}{c_1 + c_2(x - c)}, \quad \bar{y} = \psi_0(x) + y\psi_1(x),$$

where c, c_1, c_2 are constants, $\psi_0(x)$, $\psi_1(x)$ are some functions with specific form being determined by the equation under consideration.

Nevertheless, the derivatives (1) and (2) are only the particular forms of fractional derivatives, though they are used most frequently. More general definition is a fractional derivative of a function with respect to another function. This type of derivative arise when general point transformation of variables is applied to any fractional derivative of a definite type. FDEs with a fractional derivative of a function with respect to another function are considered, in particular, in the process of constructing invariant solutions of fractional partial differential equations. For example, when constructing scale-invariant solution of anomalous transport equations, one obtains the ordinary differential equations with the Erdélyi-Kober fractional derivatives [8, 12]. The existing methods of sovling such equations are very complicated and work only for a limited class of equations.

Using fractional derivatives of a function with respect to another function makes it possible to expand the class of allowed variables substitutions. These variable changes can be considered as a new type of equivalence transformations (brief discussion of this problem can be found in [13]). This approach provides new possibilities for reduction of the variables number, and, in particular, for constructing invariant solutions.

In this paper, application of the Lie group analysis methods to the class of FDEs containing fractional derivatives of a function with respect to another function is considered. The first necessary step is the construction of prolongation formulae to extend an infinitesimal operator of the group to the fractional integrals and derivatives of a function with respect to another function. Section 1 of this paper is devoted to this construction.

Since fractional derivatives are represented by integro-differential operators (i.e. they are nonlocal), it seems natural that the equations with such derivatives should have nonlocal symmetries. One of the ways to construct such symmetries is to use non-point transformation of variables (containing fractional derivatives and integrals). Then one can determine the form of infinitesimal operators in the space extended to the corresponding nonlocal variables.

Using the prolongation formulae for construction and verification of nonlocal symmetries is illustrated by a simple example. Working with fractional derivatives, verification of an operator admittance is often a non-trivial problem as demonstrated in the section 2 of the present paper.

1. A fractional derivative of a function with respect to another function. The prolongation formula

In the general case, an arbitrary change of variables $\bar{x} = \varphi(x, y)$, $\bar{y} = \psi(x, y)$ does not preserve the form of the fractional differential operator. In particular, this substitution transforms the Riemann-Liouville fractional derivative (1) of the order $\alpha \in (0, 1)$ to the left-hand side derivative of the function $\psi(x, y)$ with respect to the function $\varphi(x, y)$:

$$\left({}_cD^{\alpha}_{\varphi[x]}\psi\right)[x] = \frac{1}{\Gamma(1-\alpha)}\frac{1}{D_x\varphi[x]}\frac{d}{dx}\int_{\bar{c}}^{x}\frac{\psi[t]D_t\varphi[t]}{(\varphi[x]-\varphi[t])^{\alpha}}dt, \qquad \text{where } \bar{c} : \varphi(x, y(x))\big|_{x=\bar{c}} = c.$$

For the sake of brevity the notation $f[x] \equiv f(x, y(x))$ is introduced here. The definition and general properties of derivatives of a function with respect to another function are presented, for example, in [3].

Let us give some examples of transforming the Riemann-Liouville operator into other known forms of fractional differentiation operators by the change of variables.

1) Translation with respect to x

$$\bar{x} = x + a,\ \bar{y} = y$$

conserves the type of the operator but changes the lower limit of integration:

$$\left({}_cD_x^\alpha y\right)(x) = \left({}_{\bar{c}}D_{\bar{x}}^\alpha \bar{y}\right)(\bar{x}), \quad \bar{c} = c + a.$$

2) After substitution of variables

$$\bar{x} = x^a,\ \bar{y} = y,$$

the Riemann-Liouville operator is replaced by the Erdélyi-Kober operator [3]:

$$\left({}_cD_x^\alpha y\right)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\bar{x}^{b-1}}\frac{d}{d\bar{x}}\int_{\bar{c}}^{\bar{x}}\frac{\bar{y}(\bar{t})\bar{t}^{b-1}}{(\bar{x}^b - \bar{t}^b)^\alpha}d\bar{t}, \quad b = 1/a,\ \bar{c} = c^a.$$

Such change of variables is often performed to find invariant solutions of the anomalous transport equations with respect to the group of scaling transformations [8].

3) Change of variables

$$\bar{x} = e^x,\ \bar{y} = y$$

results in the transforming the Riemann-Liouville operator to the fractional derivative operator [3] of the Hadamard type:

$$\left({}_cD_x^\alpha y\right)(x) = \frac{\bar{x}}{\Gamma(1-\alpha)}\frac{d}{d\bar{x}}\int_{e^{\bar{c}}}^{\bar{x}}\frac{\bar{y}(\bar{t})}{\left(\ln\frac{\bar{x}}{\bar{t}}\right)^\alpha}\frac{d\bar{t}}{\bar{t}}.$$

Together with the fractional derivative of a function with respect to another function, the definition of the fractional integral of order $\beta > 0$ of a function with respect to another function [3] is also used:

$$\left({}_cI_{g(x)}^\beta y\right)(x) = \frac{1}{\Gamma(\beta)}\int_c^x \frac{y(t)g'(t)}{(g(x)-g(t))^{1-\beta}}dt. \tag{4}$$

Here it is assumed [3] that $g(x) > 0$ within the interval (c, d) and the function $g(x)$ has a continuous derivative $g'(x)$ which is strictly positive or negative ($g'(x) > 0$ or $g'(x) < 0$ for all x). The function $y(x)$ is considered to be Lebesgue integrable within the interval (c, d), i.e. $y \in L_1(c, d)$.

In what follows, for the sake of simplicity, we consider the left-hand side fractional derivative of the order $\alpha \in (0, 1)$ of the function $y(x)$ with respect to the function $g(x)$:

$$\left({}_cD_{g(x)}^\alpha y\right)(x) \equiv \frac{1}{g'(x)}\frac{d}{dx}\left({}_cI_{g(x)}^\alpha y\right)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{1}{g'(x)}\frac{d}{dx}\int_c^x \frac{y(t)g'(t)}{(g(x)-g(t))^\alpha}dt. \tag{5}$$

The fractional derivative (1) for $\alpha \in (0, 1)$ is a particular case of (5) when $g(x) = x$.

The fractional derivative (5) has two properties which are used below to deduce the prolongation formula.

Property 1. *The following relationship holds:*

$${}_cD_{g(x)}^\alpha\left(g(x)y(x)\right) = g(x)\,{}_cD_{g(x)}^\alpha y(x) + \alpha\,{}_cI_{g(x)}^{1-\alpha}y(x). \tag{6}$$

Proof.

$$\begin{aligned}
{}_cD_{g(x)}^\alpha\left(g(x)y(x)\right) &\equiv \frac{1}{\Gamma(1-\alpha)}\frac{1}{g'(x)}\frac{d}{dx}\int_c^x \frac{g(t)y(t)g'(t)}{[g(x)-g(t)]^\alpha}dt = \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{1}{g'(x)}\frac{d}{dx}\left[\int_c^x \frac{g(x)y(t)g'(t)}{[g(x)-g(t)]^\alpha}dt - \int_c^x [g(x)-g(t)]^{1-\alpha}y(t)g'(t)dt\right] = \\
&= \frac{1}{\Gamma(1-\alpha)}\int_c^x \frac{y(t)g'(t)}{[g(x)-g(t)]^\alpha}dt + \frac{1}{\Gamma(1-\alpha)}\frac{g(x)}{g'(x)}\frac{d}{dx}\int_c^x \frac{y(t)g'(t)}{[g(x)-g(t)]^\alpha}dt - \\
&\quad - \frac{1-\alpha}{\Gamma(1-\alpha)}\int_c^x \frac{y(t)g'(t)}{[g(x)-g(t)]^\alpha}dt \equiv \alpha\,{}_cI_{g(x)}^{1-\alpha}y(x) + g(x)\,{}_cD_{g(x)}^\alpha y(x).
\end{aligned}$$

□

Property 2. *If* $\lim_{t\to c+} y(t)\,(g(t)-g(c)) = 0$, *then the following equality holds:*

$$ {}_cD^{\alpha}_{g(x)}\left(y(x)(g(x)-g(c))\right) = \frac{1}{\Gamma(1-\alpha)}\int_c^x \frac{D_t\left[y(t)(g(t)-g(c))\right]}{[g(x)-g(t)]^{\alpha}}dt. \quad (7)$$

Proof. The proof consists of integration by parts and differentiation of the resulting integral with a variable upper limit:

$$ {}_cD^{\alpha}_{g(x)}[y(x)(g(x)-g(c))] = \frac{1}{\Gamma(1-\alpha)}\frac{1}{g'(x)}\frac{d}{dx}\int_c^x \frac{y(t)(g(t)-g(c))}{(g(x)-g(t))^{-\alpha}}g'(t)dt = $$
$$ = \frac{1}{\Gamma(1-\alpha)}\frac{1}{g'(x)}\frac{d}{dx}\left[-y(t)[g(t)-g(c)]\frac{(g(x)-g(t))^{1-\alpha}}{1-\alpha}\bigg|_c^x + \right. $$
$$ \left. + \int_c^x D_t[y(t)(g(t)-g(c))]\frac{(g(x)-g(t))^{1-\alpha}}{1-\alpha}dt\right] = $$
$$ = \frac{1}{(1-\alpha)\Gamma(1-\alpha)}\frac{1}{g'(x)}\int_c^x D_t[y(t)(g(t)-g(c))]\frac{d}{dx}(g(x)-g(t))^{1-\alpha}dt = $$
$$ = \frac{1}{\Gamma(1-\alpha)}\int_c^x \frac{D_t\left[y(t)(g(t)-g(c))\right]}{[g(x)-g(t)]^{\alpha}}dt. $$

□

Proposition 1. *Let us consider a one-parameter group of point transformations in the infinitesimal form:*

$$ \bar{x} = x + a\xi[x] + o(a), \quad \bar{y}(\bar{x}) = y(x) + a\eta[x] + o(a). $$

Assume that the function $y(x)\in L_1(c,d)$ *has a continuous derivative* $y'(x)$ *for* $x\in(c,d)$, *the functions* $\xi[x]=\xi(x,y(x))$ *and* $\eta[x]=\eta(x,y(x))$ *are sufficiently smooth at every point* $x\in(c,d)$, $g(x)$ *is a monotonous positive twice differentiable function.*

Then the infinitesimal transformation of the fractional integral (4) for $\beta = 1-\alpha$ *can be presented in the form*

$$ \left({}_cI^{1-\alpha}_{g(\bar{x})}\bar{y}\right)(x) = \left({}_cI^{1-\alpha}_{g(x)}y\right)(x) + a\zeta_{\alpha-1}[x] + o(a), $$

where $\zeta_{\alpha-1}$ *is determined by the prolongation formula*

$$ \zeta_{\alpha-1}[x] = {}_cI^{1-\alpha}_{g(x)}(\eta-\xi y')(x) + \xi[x]g'(x)\left({}_cD^{\alpha}_{g(x)}y\right)(x). \quad (8)$$

Proof. Let us write the fractional integration operator in the new variables $\bar{x},\bar{y}$:

$$ \left({}_cI^{1-\alpha}_{g(\bar{x})}\bar{y}\right)(\bar{x}) \equiv \frac{1}{\Gamma(1-\alpha)}\int_c^{\bar{x}}\frac{\bar{y}(\bar{\tau})g'(\bar{\tau})d\bar{\tau}}{[g(\bar{x})-g(\bar{\tau})]^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_c^{x+a\xi[x]}\frac{\bar{y}(\bar{\tau})g'(\bar{\tau})d\bar{\tau}}{[g(x+a\xi[x])-g(\bar{\tau})]^{\alpha}} + o(a). \quad (9)$$

To make the substitution of the function $\bar{y}(\bar{\tau})$, a certain substitution of the integration variable $\bar{\tau}$ is necessary. The most natural type of the substitution $\bar{\tau}=\tau+a\xi[\tau]$ allows one to turn easily from $\bar{y}(\bar{\tau})$ to $y(\tau)$ ($\bar{y}(\bar{\tau}) = y(\tau)+a\eta[\tau]+o(a)$). However, after this substitution the lower limit of integration becomes a function of the parameter a, which complicates further transformations significantly and requires imposing additional restrictions on the type of the function $\xi[x]$ [9].

The substitution of variables

$$ \bar{\tau} = t + ah(x,t), $$

where

$$ h(x,t) = \xi[x]\frac{g'(x)}{g(x)-g(c)}\cdot\frac{g(t)-g(c)}{g'(t)} \quad (10)$$

is more optimal. Here t is a new variable of integration. Such a substitution preserves the integration limits because $t = c$ transforms to $\bar{\tau} = c$, and $t = x$ transforms to $\bar{\tau} = x + a\xi[x]$.

Carrying out this substitution in (9) we obtain

$$\left({}_cI^{1-\alpha}_{g(\bar{x})}\bar{y}\right)(x) = \frac{1}{\Gamma(1-\alpha)}\int_c^x \left(\bar{y}(\bar{\tau})\cdot[g(x+a\xi[x]) - g(\bar{\tau})]^{-\alpha}\cdot g'(\bar{\tau})\frac{d\bar{\tau}}{dt}\right)\Big|_{\bar{\tau}=t+ah(x,t)} dt + o(a). \tag{11}$$

Let us consider transformation of every subintegral factor in detail.

To express $\bar{y}(\bar{\tau})$ via t it is necessary to substitute in $y(\tau)+a\eta[\tau]+o(a)$ the argument τ, which is transformed exactly into $\bar{\tau}$ during the substitution. It is known that the inverse infinitesimal substitution has the form $\tau = \bar{\tau} - a\xi[\bar{\tau}] + o(a)$. Expressing $\bar{\tau}$ via t determined earlier, we obtain

$$\tau = t + ah(x,t) - a\xi[t] + o(a).$$

As a result we have

$$\begin{aligned}\bar{y}(\bar{\tau})|_{\bar{\tau}=t+ah(x,t)} &= (y(\tau) + a\eta[\tau] + o(a))|_{\tau=t+ah(x,t)-a\xi[t]+o(a)} = \\ &= y(t+ah(x,t) - a\xi[t]) + a\eta[t] + o(a) = y(t) + ay'(t)(h(x,t) - \xi[t]) + a\eta[t] + o(a) \overset{(10)}{=} \\ &\overset{(10)}{=} y(t) + a\eta[t] - a\xi[t]y'(t) + a\frac{\xi[x]g'(x)}{g(x)-g(c)}\cdot\frac{g(t)-g(c)}{g'(t)}y'(t) + o(a).\end{aligned} \tag{12}$$

Then,

$$\begin{aligned}(g(x+a\xi[x]) - g(\bar{\tau}))|_{\bar{\tau}=t+ah(x,t)} &= g(x) + a\xi[x]g'(x) - g(t) - ah(x,t)g'(t) + o(a) = \\ &= g(x) - g(t) + a\left(\xi[x]g'(x) - \xi[x]\frac{g(t)-g(c)}{g(x)-g(c)}\frac{g'(x)}{g'(t)}g'(t)\right) + o(a) = \\ &= (g(x)-g(t))\left(1 + a\frac{\xi[x]g'(x)}{g(x)-g(c)}\right) + o(a),\end{aligned}$$

whence

$$[g(x+a\xi[x]) - g(\bar{\tau})]^{-\alpha}\big|_{\bar{\tau}=t+ah(x,t)} = (g(x)-g(t))^{-\alpha}\left(1 - \alpha a\frac{\xi[x]g'(x)}{g(x)-g(c)}\right) + o(a). \tag{13}$$

Finally,

$$\begin{aligned}\left(g'(\bar{\tau})\frac{d\bar{\tau}}{dt}\right)\Big|_{\bar{\tau}=t+ah(x,t)} &= (g'(t) + ag''(t)h(x,t))(1 + ah_t(x,t)) + o(a) \overset{(10)}{=} \\ &\overset{(10)}{=} g'(t) + a\frac{\xi[x]g'(x)}{g(x)-g(c)}\left(g''(t)\frac{g(t)-g(c)}{g'(t)} + g'(t)\frac{d}{dt}\frac{g(t)-g(c)}{g'(t)}\right) + o(a) = \\ &= g'(t)\left(1 + a\frac{\xi[x]g'(x)}{g(x)-g(c)}\right) + o(a).\end{aligned} \tag{14}$$

Substituting (12),(13),(14) into (11), we obtain

$$\begin{aligned}\left({}_cI^{1-\alpha}_{g(\bar{x})}\bar{y}\right)(\bar{x}) &= \frac{1}{\Gamma(1-\alpha)}\int_c^x \left(\bar{y}(\bar{\tau})\,[g(x+a\xi[x]) - g(\bar{\tau})]^{-\alpha}\,g'(\bar{\tau})\frac{d\bar{\tau}}{dt}\right)\Big|_{\bar{\tau}=t+ah(x,t)} dt + o(a) = \\ &= \frac{1}{\Gamma(1-\alpha)}\int_c^x \left(y(t) + a\eta[t] - a\xi[t]y'(t) + a\frac{\xi[x]g'(x)}{g(x)-g(c)}\,\frac{g(t)-g(c)}{g'(t)}y'(t)\right)\times \\ &\times (g(x)-g(t))^{-\alpha}\left(1 - \alpha a\frac{\xi[x]g'(x)}{g(x)-g(c)}\right)g'(t)\left(1 + a\frac{\xi[x]g'(x)}{g(x)-g(c)}\right)dt + o(a) =\end{aligned}$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_c^x \frac{y(t)g'(t)dt}{(g(x)-g(t))^\alpha} + \frac{a}{\Gamma(1-\alpha)} \int_c^x \frac{g'(t)dt}{(g(x)-g(t))^\alpha} \times$$

$$\times \left(\eta[t] - \xi[t]y'(t) + \frac{\xi[x]g'(x)}{g(x)-g(c)} \left[\frac{g(t)-g(c)}{g'(t)} y'(t) + (1-\alpha)y(t) \right] \right) + o(a) =$$

$$= \left({}_cI^{1-\alpha}_{g(x)} y \right)(x) + a {}_cI^{1-\alpha}_{g(x)} \left(\eta - \xi y' \right)(x) +$$

$$+ \frac{a\xi[x]g'(x)}{\Gamma(1-\alpha)(g(x)-g(c))} \int_c^x \frac{((g(t)-g(c))y'(t) + (1-\alpha)y(t)g'(t)}{(g(x)-g(t))^\alpha} dt + o(a).$$

Let us use properties 1 and 2 to transform the last integral:

$$\frac{1}{\Gamma(1-\alpha)} \int_c^x \ldots = \frac{1}{\Gamma(1-\alpha)} \int_c^x \frac{D_t[y(t)(g(t)-g(c))] - \alpha y(t)g'(t)}{(g(x)-g(t))^\alpha} dt \overset{(7)}{=}$$

$$\overset{(7)}{=} {}_cD^\alpha_{g(x)}[y(x)(g(x)-g(c))] - \alpha {}_cI^{1-\alpha}_{g(x)} y(x) = {}_cD^\alpha_{g(x)}(g(x)y(x)) - g(c) {}_cD^\alpha_{g(x)} y(x) - \alpha {}_cI^{1-\alpha}_{g(x)} y(x) \overset{(6)}{=}$$

$$\overset{(6)}{=} g(x) {}_cD^\alpha_{g(x)} y(x) + \alpha {}_cI^{1-\alpha}_{g(x)} y(x) - g(c) {}_cD^\alpha_{g(x)} y(x) - \alpha {}_cI^{1-\alpha}_{g(x)} y(x) =$$

$$= (g(x)-g(c)) \left({}_cD^\alpha_{g(x)} y \right)(x).$$

This finally results in

$$\left({}_cI^{1-\alpha}_{g(\bar{x})} \bar{y} \right)(\bar{x}) = \left({}_cI^{1-\alpha}_{g(x)} y \right)(x) + a \left({}_cI^{1-\alpha}_{g(x)} \left(\eta - \xi y' \right)(x) + \xi[x]g'(x) \left({}_cD^\alpha_{g(x)} y \right)(x) \right) + o(a),$$

which proves the proposition. □

Proposition 2. *In the conditions of proposition 1 the infinitesimal transformation of the fractional derivative (5) of the order $\alpha \in (0,1)$ has the form*

$$\left({}_cD^\alpha_{g(\bar{x})} \bar{y} \right)(\bar{x}) = \left({}_cD^\alpha_{g(x)} y \right)(x) + a\zeta_\alpha[x] + o(a),$$

where

$$\zeta_\alpha[x] = {}_cD^\alpha_{g(x)}(\eta - \xi y')(x) + \xi[x]g'(x) \left({}_cD^{\alpha+1}_{g(x)} y \right)(x). \tag{15}$$

Proof. By the definition, we have

$$\left({}_cD^\alpha_{g(\bar{x})} \bar{y} \right)(\bar{x}) \equiv \frac{1}{g'(\bar{x})} \frac{d}{d\bar{x}} \left({}_cI^{1-\alpha}_{g(\bar{x})} \bar{y} \right)(\bar{x}).$$

Using the infinitesimal expansions

$$\frac{d}{d\bar{x}} = \left(\frac{d\bar{x}}{dx} \right)^{-1} \frac{d}{dx} = (1 - aD_x\xi[x] + o(a)) \frac{d}{dx},$$

$$\frac{1}{g'(\bar{x})} = \frac{1}{g'(x)} \left(1 - a\xi[x] \frac{g''(x)}{g'(x)} + o(a) \right),$$

$$\left({}_cI^{1-\alpha}_{g(\bar{x})} \bar{y} \right)(\bar{x}) = \left({}_cI^{1-\alpha}_{g(x)} y \right)(x) + a\zeta_{\alpha-1}[x] + o(a),$$

we obtain

$$\left({}_cD^\alpha_{g(\bar{x})}\bar{y}\right)(\bar{x}) = \frac{1}{g'}\left(1 - a\xi\frac{g''}{g'} + o(a)\right)(1 - aD_x\xi + o(a))\frac{d}{dx}\left({}_cI^{1-\alpha}_g y + a\zeta_{\alpha-1} + o(a)\right) =$$
$$= \frac{1}{g'}\left(1 - a\frac{D_x(\xi g')}{g'}\right)\left(D_x({}_cI^{1-\alpha}_g y) + aD_x\zeta_{\alpha-1}\right) + o(a) =$$
$$= \frac{1}{g'}D_x({}_cI^{1-\alpha}_g y) + \frac{a}{g'}\left(D_x\zeta_{\alpha-1} - \frac{D_x(\xi g')}{g'}D_x({}_cI^{1-\alpha}_g y)\right) + o(a) \overset{(8)}{=}$$
$$\overset{(8)}{=} {}_cD^\alpha_g y + \frac{a}{g'}\left(D_x({}_cI^{1-\alpha}_g(\eta - \xi y')) + D_x(\xi g' {}_cD^\alpha_g y) - D_x(\xi g')_c D^\alpha_g y\right) + o(a) =$$
$$= {}_cD^\alpha_g y + a\left(\frac{1}{g'}D_x({}_cI^{1-\alpha}_g(\eta - \xi y')) + \xi D_x({}_cD^\alpha_g y)\right) + o(a) =$$
$$= {}_cD^\alpha_g y + a\left({}_cD^\alpha_g(\eta - \xi y') + \xi g' {}_cD^{\alpha+1}_g y\right) + o(a).$$

Here we omit the argument x for all functions for the sake of brevity. □

Remark 1. When $g(x) = x$ (15) transforms into the prolongation formula for the Riemann-Liouville type derivative obtained earlier [9], and in case of integer α, it coincides with the known classical prolongation formulae for integer-order derivatives [14].

Remark 2. Unlike integer-order derivatives, it is impossible to expand brackets in the right-hand side of (15) in the general case, because the fractional derivative of separate summands η and $\xi y'$ may not exist. An example of an operator with such coefficients is X_1 from section 3.

Remark 3. One can show that the formulae (8) and (15) are valid for fractional integrals and derivatives of an arbitrary order, respectively.

2. Nonlocal symmetries

Nonlocal symmetries of differential equations with integer-order derivatives have been known for a long time [15] and provide possibilities to construct additional invariant solutions and conservation laws in a number of cases. Meanwhile, it should be mentioned that there exists no constructive algorithm of finding such symmetries. Several heuristic approaches are known that allow one to construct particular types of nonlocal symmetries. One of them is the introduction of nonlocal variables and extension of the transformation action to these variables. This approach can be successfully applied for equations with fractional-order derivatives. In this case the prolongation formulae (8), (15) constructed in the previous section can be used both for construction of nonlocal symmetries and for verification of their admittance by the equation.

Let us demonstrate it by a simple example. We consider the equation

$$ {}_0D^{\alpha+1}_x y = 0, \quad \alpha \in (0,1), \tag{16}$$

which has the well-known general solution $y = x^{\alpha-1}(c_1x + c_2)$ (c_1, c_2 are arbitrary constants). According to the definition of the fractional derivative, the equation (16) can be written in the form

$$D^2_x({}_0I^{1-\alpha}_x y) = 0,$$

where

$$\left({}_0I^{1-\alpha}_x y\right)(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{y(t)}{(x-t)^\alpha}dt$$

is the left sided integral of the fractional order $1 - \alpha$.

Upon the nonlocal substitution $z = {}_0I^{1-\alpha}_x y$, the equation (16) is written in the form

$$z'' = 0, \tag{17}$$

which admits the known eight-parameter group [9] determined by the infinitesimal operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial z}, \quad X_3 = x\frac{\partial}{\partial x}, \quad X_4 = z\frac{\partial}{\partial x},$$

$$X_5 = x\frac{\partial}{\partial z}, \quad X_6 = z\frac{\partial}{\partial z}, \quad X_7 = x^2\frac{\partial}{\partial x} + xz\frac{\partial}{\partial x}, \quad X_8 = xz\frac{\partial}{\partial x} + z^2\frac{\partial}{\partial z}.$$

By virtue of the identity ${}_0D_x^{1-\alpha}\,{}_0I_x^{1-\alpha}y = y$ it is possible to reverse the nonlocal substitution:

$$y = {}_0D_x^{1-\alpha}z.$$

Applying the prolongation formula (15) with $g(x) = x$, we can construct the prolongation of each operator to the fractional derivative ${}_0D_x^{1-\alpha}z$:

$$\zeta_{1-\alpha} = {}_0D_x^{1-\alpha}(\eta - \xi z') + \xi\,{}_0D_x^{2-\alpha}z. \tag{18}$$

We omit the lower indices 0 and x in the operators of fractional differentiation and integration for the sake of simplicity .

The known relationship between the Riemann-Liouville and the Caputo derivatives (3) can be written in this case as

$$D^\beta f \equiv DI^{1-\beta}f = I^{1-\beta}f' + \frac{f(0)x^{-\beta}}{\Gamma(1-\beta)}, \quad \beta \in (0,1). \tag{19}$$

Differentiating (19), we have the relation

$$D^{\beta+1}f = D^\beta f' + \frac{f(0)x^{-\beta-1}}{\Gamma(-\beta)}. \tag{20}$$

The relations (19) and (20) allow one to write

$$I^\alpha z' = D^{1-\alpha}z - \frac{z(0)x^{\alpha-1}}{\Gamma(\alpha)}, \quad D^{1-\alpha}z' = D^{2-\alpha}z - \frac{z(0)x^{\alpha-2}}{\Gamma(\alpha-1)} \tag{21}$$

when $\beta = 1-\alpha$. Since the value $z(0)$ exists, then the fractional derivative ${}_0D_x^{1-\alpha}z'$ also exists.

The Leibniz rule for the fractional differentiation of the product of two functions (see [3]) also appears useful for construction of the prolongations:

$$D^\beta(fg) = \sum_{k=0}^{\infty}\binom{\beta}{k}D^{\beta-k}f\,D^k g. \tag{22}$$

Here $\binom{\beta}{k}$ are binomial coefficients, $D^{\beta-k}f = I^{k-\beta}f$ when $k > \beta$. In particular,

$$D^\beta(xf) = xD^\beta(f) + \beta D^{\beta-1}(f), \tag{23}$$

$$D^\beta(x^2 f) = x^2D^\beta(f) + 2\beta xD^{\beta-1}(f) + \beta(\beta-1)D^{\beta-2}(f). \tag{24}$$

The fractional derivative of the power function has the form [3]

$$D^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}x^{\gamma-\alpha}, \quad \gamma > -1, \quad \alpha \in \mathbb{R}. \tag{25}$$

Prolongation of the operator X_1: Here $\xi = 1, \eta = 0$ and

$$\zeta_{1-\alpha} = -D^{1-\alpha}(z') + D^{2-\alpha}z \overset{(21)}{=} \frac{z(0)x^{\alpha-2}}{\Gamma(\alpha-1)}.$$

Prolongation of the operator X_2:

$$\zeta_{1-\alpha} = D^{1-\alpha}(1) \overset{(25)}{=} \frac{x^{\alpha-1}}{\Gamma(\alpha)}.$$

Prolongation of the operator X_5:

$$\zeta_{1-\alpha} = D^{1-\alpha}(x) \overset{(25)}{=} \frac{x^\alpha}{\Gamma(\alpha+1)}.$$

Prolongation of the operator X_6:

$$\zeta_{1-\alpha} = D^{1-\alpha}(z).$$

Prolongation of the operator X_3:

$$\zeta_{1-\alpha} = -D^{1-\alpha}(xz') + xD^{2-\alpha}(z).$$

The assumption that the finite value $z(0)$ exists entails that $(xz)|_{x=0} = 0$. Then, due to (19) we have $D^{1-\alpha}(xz)' = D^{2-\alpha}(xz)$ and substituting xz' as $(xz)' - z$, we obtain

$$\zeta_{1-\alpha} = -D^{2-\alpha}(xz) + D^{1-\alpha}z + xD^{2-\alpha}(z) \overset{(23)}{=} (\alpha-1)D^{1-\alpha}z.$$

Prolongation of the operator X_4:

$$\zeta_{1-\alpha} = -D^{1-\alpha}(zz') + zD^{2-\alpha}(z).$$

Applying the equation (17), the Leibniz rule (22) and the representations (21) it is possible to eliminate nonlinearity under the operator of fractional differentiation:

$$\zeta_{1-\alpha} \overset{(22)}{=} -\sum_{n=0}^{\infty}\binom{1-\alpha}{n}D^n(z)\,D^{1-\alpha-n}z' + zD^{2-\alpha}(z) \overset{(17)}{=} -zD^{1-\alpha}z' - (1-\alpha)z'I^{\alpha}z' + zD^{2-\alpha}(z) \overset{(21)}{=}$$

$$\overset{(21)}{=} -(1-\alpha)z'\left(D^{1-\alpha}z - \frac{z(0)x^{\alpha-1}}{\Gamma(\alpha)}\right) + \frac{zz(0)x^{\alpha-2}}{\Gamma(\alpha-1)} = (\alpha-1)z'D^{1-\alpha}z - \frac{z'z(0)x^{\alpha-1}}{\Gamma(\alpha-1)} + \frac{zz(0)x^{\alpha-2}}{\Gamma(\alpha-1)}.$$

The presented form of the coefficient of the prolonged operator is not the only one possible. In particular, we can eliminate the variable z' by applying the representation of the fractional derivative $D^{1-\alpha}z$ to the equation (17):

$$D^{1-\alpha}z \overset{(22)}{=} \sum_{n=0}^{\infty}\binom{1-\alpha}{n}D^n(z)\,D^{1-\alpha-n}1 \overset{(17),(25)}{=} \frac{zx^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-\alpha)z'x^{\alpha}}{\Gamma(\alpha+1)},$$

whence, due to relation $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ we have

$$(1-\alpha)z' = \Gamma(\alpha+1)x^{-\alpha}D^{1-\alpha}z - \alpha\frac{z}{x}.$$

As a result we find

$$\zeta_{1-\alpha} = -\left(\Gamma(\alpha+1)x^{-\alpha}D^{1-\alpha}z - \alpha\frac{z}{x}\right)\left(D^{1-\alpha}z - \frac{z(0)x^{\alpha-1}}{\Gamma(\alpha)}\right) + \frac{zz(0)x^{\alpha-2}}{\Gamma(\alpha-1)} =$$

$$= -\Gamma(\alpha+1)\frac{(D^{1-\alpha}z)^2}{x^{\alpha}} + \frac{\alpha zD^{1-\alpha}z}{x} + \frac{z(0)\Gamma(\alpha+1)D^{1-\alpha}z}{x\Gamma(\alpha)} + \left(\frac{1}{\Gamma(\alpha-1)} - \frac{\alpha}{\Gamma(\alpha)}\right)zz(0)x^{\alpha-2} =$$

$$= -\Gamma(\alpha+1)\frac{(D^{1-\alpha}z)^2}{x^{\alpha}} + \frac{\alpha(z+z(0))D^{1-\alpha}z}{x} - \frac{zz(0)x^{\alpha-2}}{\Gamma(\alpha)}.$$

Prolongation of the operator X_7:

$$\zeta_{1-\alpha} = D^{1-\alpha}(xz - x^2z') + x^2D^{2-\alpha}z.$$

Acting similarly to the procedure of the operator X_3 prolongation, we find

$$\zeta_{1-\alpha} = D^{1-\alpha}(xz) - D^{1-\alpha}D(x^2z) + D^{1-\alpha}(2xz) + x^2D^{2-\alpha}z =$$

$$= 3D^{1-\alpha}(xz) - D^{2-\alpha}(x^2z) + x^2D^{2-\alpha}z \overset{(23),(24)}{=} 3xD^{1-\alpha}z + (3-3\alpha)I^{\alpha}z - (4-2\alpha)xD^{1-\alpha}z -$$

$$- (2-\alpha)(1-\alpha)I^{\alpha}z = (2\alpha-1)xD^{1-\alpha}z + (1-\alpha^2)D^{\alpha}z.$$

Prolongation of the operator X_8:

$$\zeta_{1-\alpha} = D^{1-\alpha}(z^2 - xzz') + xzD^{2-\alpha}z. \tag{26}$$

Applying the Leibniz rule (22), due to the equation (17) we find

$$D^{1-\alpha}(z^2) \overset{(22)}{=} \sum_{n=0}^{\infty}\binom{1-\alpha}{n}D^nz\,D^{1-\alpha-n}z \overset{(17)}{=} zD^{1-\alpha}z + (1-\alpha)z'I^{\alpha}z. \tag{27}$$

Similarly, applying the Leibniz rule for $xz \cdot z'$ and taking into account that due to the equation (17) $D^3(xz) = 0$, we have

$$-D^{1-\alpha}(xzz') \overset{(22),(17)}{=} -xzD^{1-\alpha}(z') - (1-\alpha)(z+xz')I^{\alpha}z' - (1-\alpha)(-\alpha)z'I^{\alpha+1}z' \overset{(21)}{=}$$

$$\overset{(21)}{=} -xzD^{2-\alpha}z + xz\frac{z(0)x^{\alpha-2}}{\Gamma(\alpha-1)} - (1-\alpha)(z+xz')\left[D^{1-\alpha}z - \frac{z(0)x^{\alpha-1}}{\Gamma(\alpha)}\right] + \alpha(1-\alpha)z'\left[I^{\alpha}z - \frac{z(0)x^{\alpha}}{\Gamma(\alpha+1)}\right] =$$

$$= -xzD^{2-\alpha}z + \frac{zz(0)x^{\alpha-1}}{\Gamma(\alpha-1)} - (1-\alpha)\left[zD^{1-\alpha}z - \frac{zz(0)x^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)} + xz'D^{1-\alpha}z - \frac{z'z(0)x^{\alpha}}{(\alpha-1)\Gamma(\alpha-1)}\right] +$$

$$+\alpha(1-\alpha)\left[z'I^{\alpha}z - \frac{z'z(0)x^{\alpha}}{\alpha(\alpha-1)\Gamma(\alpha-1)}\right] =$$

$$= -xzD^{2-\alpha}z - (1-\alpha)zD^{1-\alpha}z - (1-\alpha)xz'D^{1-\alpha}z + \alpha(1-\alpha)z'I^{\alpha}z. \quad (28)$$

Substituting (27) and (28) into (26) we obtain

$$\zeta_{1-\alpha} = \alpha zD^{1-\alpha}z + (1-\alpha^2)z'I^{\alpha}z - (1-\alpha)xz'D^{1-\alpha}z.$$

As a result, the prolonged operators take the form

$$\tilde{X}_1 = \frac{\partial}{\partial x} + \frac{z(0)x^{\alpha-2}}{\Gamma(\alpha-1)}\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_2 = \frac{\partial}{\partial z} + \frac{x^{\alpha-1}}{\Gamma(\alpha)}\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_3 = x\frac{\partial}{\partial x} + (\alpha-1)z^{(1-\alpha)}\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_4 = z\frac{\partial}{\partial x} + \left((\alpha-1)z'z^{(1-\alpha)} - \frac{z'z(0)x^{\alpha-1}}{\Gamma(\alpha-1)} + \frac{zz(0)x^{\alpha-2}}{\Gamma(\alpha-1)}\right)\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_5 = x\frac{\partial}{\partial z} + \frac{x^{\alpha}}{\Gamma(\alpha+1)}\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_6 = z\frac{\partial}{\partial z} + z^{(1-\alpha)}\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_7 = x^2\frac{\partial}{\partial x} + xz\frac{\partial}{\partial z} + \left[(2\alpha-1)xz^{(1-\alpha)} + (1-\alpha^2)z^{(-\alpha)}\right]\frac{\partial}{\partial z^{(1-\alpha)}},$$

$$\tilde{X}_8 = xz\frac{\partial}{\partial x} + z^2\frac{\partial}{\partial z} + \left[\alpha zz^{(1-\alpha)} - (1-\alpha)xz'z^{(1-\alpha)} + (1-\alpha^2)z'z^{(-\alpha)}\right]\frac{\partial}{\partial z^{(1-\alpha)}},$$

where $z^{(1-\alpha)} \equiv {}_0D_x^{1-\alpha}z$. Whence, after the reverse substitution of the variables $z = {}_0I_x^{1-\alpha}y$, we find symmetries of the equation (16):

$$X_1 = \frac{\partial}{\partial x} + \frac{y^{(\alpha-1)}(0)x^{\alpha-2}}{\Gamma(\alpha-1)}\frac{\partial}{\partial y},$$

$$X_2 = x^{\alpha-1}\frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + (\alpha-1)y\frac{\partial}{\partial y},$$

$$X_4 = y^{(\alpha-1)}\frac{\partial}{\partial x} + \left((\alpha-1)yy^{(\alpha)} - \frac{y^{(\alpha)}y^{(\alpha-1)}(0)x^{\alpha-1}}{\Gamma(\alpha-1)} + \frac{y^{(\alpha-1)}y^{(\alpha-1)}(0)x^{\alpha-2}}{\Gamma(\alpha-1)}\right)\frac{\partial}{\partial y},$$

$$X_5 = x^{\alpha}\frac{\partial}{\partial y}, \qquad X_6 = y\frac{\partial}{\partial y},$$

$$X_7 = x^2\frac{\partial}{\partial x} + [(2\alpha-1)xy + (1-\alpha^2)Iy]\frac{\partial}{\partial y},$$

$$X_8 = xy^{(\alpha-1)}\frac{\partial}{\partial x} + [\alpha yy^{(\alpha-1)} - (1-\alpha)xyy^{(\alpha)} + (1-\alpha^2)y^{(\alpha)}Iy]\frac{\partial}{\partial y}.$$

Here $y^{(\alpha-1)} \equiv {}_0I_x^{1-\alpha}y$, $Iy \equiv {}_0I_xy$.

The symmetries X_2, X_3, X_5, X_6 are local, other symmetries are nonlocal. Let us note that the initial value $y^{(\alpha-1)}(0)$ contained in the operators X_1 and X_4 is a natural initial condition for the formulation of the Cauchy problem for fractional differential equations.

Let us show that the coefficients of the operators $X_1 \dots X_8$ satisfy the definite equation

$$\zeta_{\alpha+1}|_{D^{\alpha+1}y=0} = 0,$$

which takes the form

$$D^{\alpha+1}(\eta - \xi y')\big|_{D^{\alpha+1}y=0} = 0$$

for the equation (16).

Operators X_2,X_5,X_6. The verification is trivial. For X_6, we have

$$D^{\alpha+1}(y)\big|_{D^{\alpha+1}y=0} = 0.$$

For X_2, we have $\eta - \xi y' = x^\alpha$. By virtue of (25) we obtain

$$D^{\alpha+1}x^\alpha = 0,$$

because the gamma-function has poles of the first order at the points $x = 0,\ x = -n, n \in \mathbb{R}$. Likewise for X_5: $D^{\alpha+1}x^{\alpha-1} = 0$.

Operator X_1*:*

$$\zeta_{\alpha+1} = D^{\alpha+1}\left(\frac{y^{(\alpha-1)}(0)x^{\alpha-2}}{\Gamma(\alpha-1)} - y'\right).$$

Note that derivatives $D^{\alpha+1}y'$ and $D^{\alpha+1}x^{\alpha-2}$ do not exist, therefore it is impossible to apply the operator $D^{\alpha+1}$ to the separate summands in this case.

The relationship (19) with $f = I^{1-\alpha}y$ allows one to write the following representation of y:

$$y = D^{1-\alpha}I^{1-\alpha}y = I^\alpha DI^{1-\alpha}y + \frac{(I^{1-\alpha}y)(0)\cdot x^{\alpha-1}}{\Gamma(\alpha)} = I^\alpha D^\alpha y + \frac{y^{(\alpha-1)}(0)x^{\alpha-1}}{\Gamma(\alpha)}, \tag{29}$$

whence

$$y' = D^{1-\alpha}D^\alpha y + \frac{y^{(\alpha-1)}(0)x^{\alpha-2}}{\Gamma(\alpha-1)} \tag{30}$$

due to $(\alpha-1)\Gamma(\alpha-1) = \Gamma(\alpha)$. Then

$$\zeta_{\alpha+1} = -D_x^{\alpha+1}(D_x^{1-\alpha}D_x^\alpha y).$$

Due to the relationship (19) and the equation (16) we have

$$D^{1-\alpha}D^\alpha y = DI^\alpha D^\alpha y \overset{(19)}{=} I^\alpha D^{\alpha+1}y + \frac{(D^\alpha y)(0)x^{\alpha-1}}{\Gamma(1-\beta)} \overset{(16)}{=} \frac{(D^\alpha y)(0)x^{\alpha-1}}{\Gamma(1-\beta)} \tag{31}$$

(the existence of $(D^\alpha y)(0)$ follows from the formulation of the Cauchy problem for the original equation or from the existence of $z'(0)$).

Due to (25) the $\alpha+1$ fractional derivative of the expression (31) is equal to zero, whence

$$\zeta_{\alpha+1}|_{D^{\alpha+1}y=0} = 0.$$

Operator of the dilation group X_3*:*

$$\zeta_{\alpha+1} = D^{\alpha+1}((\alpha-1)y - xy').$$

Applying the representation $xy' = (xy)' - y$ and the relationship $D^{\alpha+1}(xy)' = D^{\alpha+2}(xy)$ (which holds due to $(xy)|_{x=0} = 0$), we obtain

$$\zeta_{\alpha+1} = D^{\alpha+1}(\alpha y - (xy)') = \alpha D^{\alpha+1}y - D^{\alpha+2}(xy) \overset{(23)}{=} -xD^{\alpha+2}(y) - 2D^{\alpha+1}(y),$$

and

$$\zeta_{\alpha+1}|_{D^{\alpha+1}y=0} = 0.$$

Operator X_4:

$$\zeta_{\alpha+1} = D_x^{\alpha+1}\left((\alpha-1)yy^{(\alpha)} - \frac{y^{(\alpha)}y^{(\alpha-1)}(0)x^{\alpha-1}}{\Gamma(\alpha-1)} + \frac{y^{(\alpha-1)}y^{(\alpha-1)}(0)x^{\alpha-2}}{\Gamma(\alpha-1)} - y^{(\alpha-1)}y'\right).$$

Applying the Leibniz rule (22) it is easy to see that due to the equation (16) the fractional derivatives of the first and the second summands vanish:

$$D_x^{\alpha+1}(yy^{(\alpha)})\big|_{D^{\alpha+1}y=0} \overset{(22)}{=} \sum_{n=0}^{\infty}\binom{\alpha+1}{n}D^{\alpha+1-n}y\cdot D^{\alpha+n}y\Big|_{D^{\alpha+1}y=0} = 0,$$

$$D_x^{\alpha+1}(x^{\alpha-1}y^{(\alpha)})\big|_{D^{\alpha+1}y=0} \overset{(22)}{=} \sum_{n=0}^{\infty}\binom{\alpha+1}{n}D^{\alpha+1-n}x^{\alpha-1}\cdot D^{\alpha+n}y\Big|_{D^{\alpha+1}y=0} \overset{(25)}{=} 0.$$

To simplify the remaining parts of the expression we use the representation y' (30):

$$D_x^{\alpha+1}\left(\frac{y^{(\alpha-1)}(0)y^{(\alpha-1)}x^{\alpha-2}}{\Gamma(\alpha-1)} - y^{(\alpha-1)}y'\right) \overset{(30)}{=} -D_x^{\alpha+1}\left(I^{1-\alpha}y\cdot D^{1-\alpha}D^{\alpha}y\right) \overset{(22)}{=}$$

$$\overset{(22)}{=} -\sum_{n=0}^{\infty}\binom{\alpha+1}{n}D^{\alpha+1-n}(D^{1-\alpha}D^{\alpha}y)\cdot D^{n}(I^{1-\alpha}y).$$

By virtue of the equation $D^{\alpha+n}=0$ and all summands with $n>1$ vanish. The first two summands are also equal to zero due to the relationship (31), holding true for the equation:

$$-D^{\alpha+1}(D^{1-\alpha}D^{\alpha}y)\cdot I^{1-\alpha}y - (\alpha+1)D^{\alpha}(D^{1-\alpha}D^{\alpha}y)\cdot D^{\alpha}y \overset{(31)}{=} 0.$$

Remark. It is clear from the proof that an operator of a simpler form than X_4 is admitted:

$$\hat{X}_4 = y^{(\alpha-1)}\frac{\partial}{\partial x} + \frac{y^{(\alpha-1)}y^{(\alpha-1)}(0)x^{\alpha-2}}{\Gamma(\alpha-1)}\frac{\partial}{\partial y}.$$

Operator X_7:

$$\zeta_{\alpha+1} = D^{\alpha+1}((2\alpha-1)xy + (1-\alpha^2)Iy - x^2y').$$

The following equalities hold:

$$D^{\alpha+1}Iy = D^2I^{1-\alpha}Iy = D^2I(I^{1-\alpha}y) = DI^{1-\alpha}y = D^{\alpha}y,$$
$$D^{\alpha}(xy') = D^{\alpha}(xy)' - D^{\alpha}y = D^{\alpha+1}(xy) - D^{\alpha}y = xD^{\alpha+1}y + \alpha D^{\alpha}y,$$
$$D^{\alpha+1}(xy') = DD^{\alpha}(xy') = (\alpha+1)D^{\alpha+1}y + xD^{\alpha+2}y.$$

Thus,

$$D^{\alpha+1}Iy = D^{\alpha}y, \quad D^{\alpha}(xy')\big|_{D^{\alpha+1}y=0} = \alpha D^{\alpha}y, \quad D^{\alpha+1}(xy')\big|_{D^{\alpha+1}y=0} = 0. \tag{32}$$

Then,

$$\zeta_{\alpha+1} = (2\alpha-1)D^{\alpha+1}(xy) + (1-\alpha^2)D^{\alpha+1}Iy - D^{\alpha+1}(x\cdot xy') =$$
$$= (2\alpha-1)xD^{\alpha+1}(y) + (2\alpha-1)(\alpha+1)D^{\alpha}y + (1-\alpha^2)D^{\alpha}y - xD^{\alpha+1}(xy') - (\alpha+1)D^{\alpha}(xy').$$

After substituting the equation (16) and using the relationships (32), we obtain

$$\zeta_{\alpha+1}\big|_{D^{\alpha+1}y=0} = (2\alpha^2+\alpha-1+1-\alpha^2)D^{\alpha}y - 0 - \alpha(\alpha+1)D^{\alpha}y = 0.$$

Operator X_8:

$$\zeta_{\alpha+1} = D^{\alpha+1}[\alpha yy^{(\alpha-1)} - (1-\alpha)xyy^{(\alpha)} + (1-\alpha^2)y^{(\alpha)}Iy - xy^{(\alpha-1)}y'].$$

Let us use the Leibniz rule (22) for representation of every summand taking into account that $D^{\alpha+n}y=0$ due to the equation (16) when $n>0$.

$$D^{\alpha+1}[\alpha yI^{1-\alpha}y] \overset{(22)}{=} \alpha\sum_{n=0}^{\infty}\binom{\alpha+1}{n}D^{\alpha+1-n}y\,D^{n+\alpha-1}y \overset{(16)}{=} \alpha(\alpha+1)(D^{\alpha}y)^2,$$

$$D^{\alpha+1}[(\alpha-1)xyD^{\alpha}y] \overset{(22)}{=} (\alpha-1)\sum_{n=0}^{\infty}\binom{\alpha+1}{n}D^{\alpha+1-n}(xy)\,D^{n+\alpha}y \overset{(16)}{=} (\alpha-1)D^{\alpha+1}(xy) \overset{(23)}{=}$$
$$\overset{(23)}{=} (\alpha-1)xD^{\alpha+1}(y)+(\alpha-1)(\alpha+1)(D^{\alpha}y)^2 \overset{(16)}{=} (\alpha^2-1)(D^{\alpha}y)^2,$$

$$D^{\alpha+1}[(1-\alpha^2)D^{\alpha}y\,Iy] \overset{(22)}{=} (1-\alpha^2)\sum_{n=0}^{\infty}\binom{\alpha+1}{n}D^{\alpha+1-n}(Iy)\,D^{n+\alpha}y \overset{(16)}{=}$$
$$\overset{(16)}{=} (1-\alpha^2)D^{\alpha+1}(Iy)\,D^{\alpha}y \overset{(32)}{=} (1-\alpha^2)(D^{\alpha}y)^2,$$

$$D^{\alpha+1}[-xy'I^{1-\alpha}y] \overset{(22)}{=} -\sum_{n=0}^{\infty}\binom{\alpha+1}{n}D^{\alpha+1-n}(xy')\,D^{n+\alpha-1}y \overset{(16)}{=}$$
$$\overset{(16)}{=} -D^{\alpha+1}(xy')I^{1-\alpha}y-(\alpha+1)D^{\alpha}(xy')D^{\alpha}y \overset{(32)}{=} -\alpha(\alpha+1)(D^{\alpha}y)^2.$$

It is easy to see that the sum of right-hand sides turns to zero, which was to be proved. Note that simpler operators are also admitted:

$$\hat{X}_8 = xy^{(\alpha-1)}\frac{\partial}{\partial x}+\alpha yy^{(\alpha-1)}\frac{\partial}{\partial y}, \qquad \bar{X}_8 = \left((\alpha-1)xyy^{(\alpha)}+(1-\alpha^2)y^{(\alpha)}Iy\right)\frac{\partial}{\partial y}.$$

Remark. In our previous paper [9] five local symmetries, including the projective operator

$$X_9 = x^2\frac{\partial}{\partial x}+\alpha xy\frac{\partial}{\partial y}$$

were obtained from the principle of invariance of the equation (16). The present operator cannot be combined from $X_1,\ldots,X_8$, but the closest to it is X_7 (obtained from the projective operator for the equation $z''=0$). It is easy to verify that the nonlocal operator

$$X_{10} \equiv X_7 - X_9 = [(\alpha-1)xy+(1-\alpha^2)Iy]\frac{\partial}{\partial y}$$

is admitted by the equation (16). Meanwhile, in the boundary case $\alpha=1$ the operator X_{10} turns to zero, i.e. X_7 matches X_9.

Conclusion

The prolongation formulae obtained in the paper give an opportunity to investigate symmetry properties of a new class of differential equations, containing fractional derivatives of a function with respect to another function. Meanwhile, the following important problem to be solved is the development of a method that allows to solve the resulting determining equations. The main difficulty in this case is caused by splitting rules for the determining equation.

Another direction of the further research is systematization of results considering nonlocal symmetries of fractional differential equations and developing new algorithms of their construction. The problem of determining classification rules for nonlocal symmetries of such equations also seems to be important.

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