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THE NON-AUTONOMOUS DYNAMICAL SYSTEMS AND EXACT SOLUTIONS WITH SUPERPOSITION PRINCIPLE FOR EVOLUTIONARY PDEs

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Abstract. In the present article we introduce a new application of S. Lie's non-autonomous dynamical systems with the generalized separation of variables in right hand-sides. We consider non-autonomous dynamical equations as some sort of external action on a given evolution equation, which transforms a subset of solutions into itself. The goal of our approach is to find a subset of solutions of an evolution equation with a superposition principle. This leads to an integration of ordinary differential equations in a process of constructing exact solutions of PDEs. In this paper we consider the application of the most simple one-dimensional case of the Lie theorem.

Key words: evolutionary equations, exact solutions, superposition of solutions.

Introduction

The concept of linear superposition of solutions in classical theory of linear ordinary differential equations

$$x^i_t(t) = \phi_{i1}(t)x^1 + ... + \phi_{in}(t)x^n, \quad i = 1, 2, ..., n,$$

was generalized by Sophus Lie [1] for nonlinear dynamical systems with *the generalized separation of variables* in right hand-sides. Namely, S.Lie proved the following theorem.
Theorem. The equations

$$x^i_t(t) = f^i(t, x), \quad i = 1, 2, ..., n,$$

possess a *fundamental set of solutions*, i.e. its general solution can be represented by finite number m of particular solutions $x^1_1, ..., x^n_1; ..., x^1_m, ..., x^n_m$ and n number of arbitrary constants $C_1, ..., C_n$, if and only if they have the following form

$$\frac{\partial x^i}{\partial t}(t) = \phi_1(t){\xi^i}_1(x) + ... + \phi_r(t){\xi^i}_r(x), \quad i = 1, 2, ..., n, \tag{1}$$

where the coefficients ${\xi^i}_\alpha(x)$ satisfy the condition that the operators

$$X_\alpha = {\xi^i}_\alpha(x)\frac{\partial}{\partial x^i}, \quad \alpha = 1, 2, ..., r, \tag{2}$$

span a Lie Algebra L_r of a finite dimension r. The number m of necessary particular solutions is estimated by inequality $mn \geq r$. The superposition formulae for a general solution

$$x^i = S^i(x^1, ..., x^n; x^1_1, ..., x^n_1; ..., x^1_m, ..., x^n_m; C_1, ..., C_n), \quad i = 1, 2, ..., n, \quad \det \| \frac{\partial S^i}{\partial C_k} \| \neq 0.$$

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are defined implicitly by n equations

$$J_i(x^1,...,x^n;x_1^1,...,x_1^n;...,x_m^1,...,x_m^n) = C_i, \quad i=1,2,...,n, \quad \det \| \frac{\partial J_i}{\partial x^k} \| \neq 0,$$

where J_i are functionally independent with respect to x^i invariants of the operators (2) prolonged to the $(n+mn)$-dimensional space

$$\bar{X}_\alpha = \xi^i{}_\alpha(x)\frac{\partial}{\partial x^i} + \xi^i{}_\alpha(x_1)\frac{\partial}{\partial x_1^i} + ... + \xi^i{}_\alpha(x_m)\frac{\partial}{\partial x_m^i}, \quad \alpha = 1,2,...,r.$$

The non-autonomous dynamical system (1) will be referred as *the Lie non-autonomous dynamical system* (NADS) or *the Lie system.* Notice, that the statement of the Lie theorem has various aspects. The first one states that *the superposition* (which is nonlinear in general) of finite number particular solutions is again a solution. The second one is that its *general solution* can be represented by finite number m of particular solutions. Thirdly, the Theorem based on *an invariant object* which is Lie Algebra of operators X_α.

Notice, that if all functions $\phi_i = const, \quad i=1,2,...,n,$ then system (1) simply represents a one-parameter Lie group of point transformations, which generated by linear combinations (with constant coefficients ϕ_i) of operators (2). For variable coefficients $\phi_i(t)$ the system (1) is sufficient different from a one-parameter Lie group of point transformations.

In 1980-th there was the renewed interest to this Lie's theorem, and several important applications were found [2,3,4,5]. The discussion of Lie's theorem and several examples of applications one can find in [6].

In this paper we consider the new application the above theorem. We do not investigate system of type (1) itself, but consider non-autonomous equations as some sort of *external action* on some given evolution equation. The goal of our approach is to find a subset of solutions of evolution equation which possess the superposition principle. Solutions of a non-autonomous equation will be considered as some generalization of symmetry transformations, which act on an evolution equation and transform a subset of solutions into itself. This leads to an integration of ordinary differential equations in a process of developing exact solutions of PDEs. We supply the theory with examples.

The article is organized as following. In Section 1 we formulate the most simple one-dimensional case of the Lie theorem. Section 2 devoted to the technique for constructing special solutions with linear superposition principle, what demonstrated on examples in Section 3. In Section 4 we generalize the approach for evolutionary PDEs in two space dimensions. Sections 5 and 6 devoted to solutions with the Bernoulli and the Riccati types superpositions. In final Section 7 we develop the necessary and sufficient conditions for evolutionary PDEs to possess subset of solutions with linear superpositions principle.

1. One-dimensional case of the Lie's theorem

For $n = 1$ the most general Lie's non-autonomous dynamical equation is the Riccati equation

$$u_t(t) = \phi_1(t) + \phi_2(t)u + \phi_3(t)u^2, \tag{3}$$

where $\phi_i, i = 1,2,3$ are some smooth functions of t.

The equation (3) possesses the fundamental set of solutions as far it is associated with the Lie Algebra L_3

$$X_1 = \frac{\partial}{\partial u}, \quad X_2 = u\frac{\partial}{\partial u}, \quad X_2 = u^2\frac{\partial}{\partial u},$$

of the projective group. For the equation (3) $n = 1, r = 3, m \geq 3$. In fact three particular solutions casts the minimum number of solutions to develop a general solution of Riccati equation.

The subalgebras of (3) are casted by 1-dimensional subalgebras spanned by each operators individually and by two-dimensional subalgebras $(X_1; X_2)$ and $(X_2; X_3)$. The equation (3) possesses the well-known non-linear superposition principle for their solutions

$$\frac{(u_4 - u_3)(u_2 - u_1)}{(u_4 - u_2)(u_3 - u_1)} = C,$$

where C is an arbitrary constant, and its general solution can be expressed by means of three particular solutions

$$u = \frac{u_3(u_2 - u_1) - Cu_2(u_3 - u_1)}{u_2 - u_1 - C(u_3 - u_1)}.$$

For the 1-dimensional subalgebras spanned by each operators $X_1; X_2; X_3$ individually there exists point transformations, which change corresponding dynamical equations into classical Lie group equations. Thus, nontrivial cases start from two-dimensional subalgebras $(X_1; X_2)$ and $(X_2; X_3)$.

2. The subset of solutions with linear superposition

We firstly consider the subalgebra L_2 spanned by the operators $X_1 = \dfrac{\partial}{\partial u}$, $X_2 = u\dfrac{\partial}{\partial u}$. The corresponding non-autonomous evolution is a linear equation

$$u_t(t) = \phi_1(t) + \phi_2(t)u. \tag{4}$$

For equation (3) $n = 1, r = 2, m \geq 2$. In fact two particular solutions casts the minimum number of solutions to develop a general solution of the equation (4). Thus, (4) has a fundamental set of special solutions u_1 and u_2 with superposition

$$\frac{u - u_1}{u_2 - u_1} = C,$$

which yield the general solution as

$$u = (1 - C)u_1 + Cu_2, \quad C = const. \tag{5}$$

In this paper we consider non-autonomous equations as some sort of *external action* on a given evolution equation. Following the idea of classical Lie group analysis of differential equations [7,8] we involve *the prolongation of non-autonomous dynamical system for spatial derivatives*.

Thus, we involve x as one more independent variable and let u be dependent on two variables: $u = u(t, x)$. We rewrite (4) in the form

$$u_t(t, x) = \bar{\phi}(t) + \psi'(t)u(t, x),$$

(we write $\psi'(t)$ for convenience) and prolong it for evolution of partial derivative $u_x, u_{xx}, \ldots$:

$$u_t(t, x) = \bar{\phi}(t) + \psi'(t)u(t, x), \tag{6}$$

$$u_{xt}(t, x) = \psi'(t)u_x(t, x), \quad u_{xxt}(t, x) = \psi'(t)u_{xx}(t, x), \ldots.$$

The above evolutionary system can be associated with the Lie algebra L_2 spanned by the following operators:

$$X_1 = \frac{\partial}{\partial u}, \quad X_2 = u\frac{\partial}{\partial u} + u_x\frac{\partial}{\partial u_x} + u_{xx}\frac{\partial}{\partial u_{xx}}, \ldots.$$

Notice, that for the system (6) we still have $n = 1, \quad r = 2$, as far as all differential sequences of the first evolution equations do not produce new dependent variables. One can write down a solution of the linear equations (6) in the following form

$$u(t, x) = \phi(t) + e^{\psi(t)}V(x),$$

where $V(x)$ describes dependence of u on space variable x, and $\phi(t)$ is a special solution of inhomogeneous equation

$$\phi' = \bar{\phi}(t) + \psi'(t)\phi.$$

Notice, that the superposition formula (5) yields the superposition for $V(x)$ as well:

$$V = (1 - C)V_1 + CV_2,$$

where $C = const.$

Thus, the set of solutions (V_1, V_2) span a linear space.

Now we consider an evolutionary equation

$$u_t = F(u, u_x, u_{xx}), \tag{7}$$

which, in general, is nonlinear. We consider the compatibility condition for evolution (6) and evolutionary eq. (7), that gives ODEs for unknown functions $\phi(t), \psi(t)$.

Now we consider a row of examples of evolutionary equations, when such a compatibility exists. Most of the following examples of equations were taken from [9,10,11,12,13] and yield subspaces of solutions with various dimensions. In all cases we have linear superposition and two functions $\phi(t), \psi(t)$ to describe the corresponding subset independently of space dimension.

3. Examples of evolutionary equations with a linear subspace of solutions

Example 1. We consider the nonlinear equation

$$u_t = u_{xx} + {u_x}^2 - u^2,$$

and looking for the special solution in the form

$$u(t,x) = \phi(t) + e^{\psi(t)}V(x), \quad u_x(t,x) = e^{\psi(t)}V_x(x), \quad u_{xx}(t,x) = e^{\psi(t)}V_{xx}(x), \tag{8}$$

which yields the following equation

$$\phi' + \psi' e^{\psi} V = e^{\psi} V'' + e^{2\psi} V'^2 - \phi^2 - 2\phi e^{\psi} V - e^{2\psi} V^2.$$

Splitting the last equation gives

$$\phi' = -\phi^2 + e^{2\psi}((V')^2 - V^2), \quad \psi' V = V'' - 2\phi V.$$

Then we have the following overdetermined system

$$(V')^2 - V^2 = k = const, \quad V'' = mV, \quad m = const.$$

From the last relations in the case $k \neq 0$ we have

$$m = 1, \quad V(x) = Ae^x + Be^{-x}, \quad k = -4AB.$$

So, we can write down the family of solution as

$$u(t,x) = \phi(t) + e^{\psi(t)}(Ae^x + Be^{-x}),$$

where functions $\phi(t), \psi(t)$ satisfy the following dynamical system:

$$\phi' = -\phi^2 - 4ABe^{2\psi}, \quad \psi' = 1 - 2\phi.$$

In the case $k = 0$ one can integrate the corresponding dynamical system and express the solution explicitly

$$u(t,x) = \frac{1}{t + C_1} + \frac{C_2 e^t}{(t + C_1)^2}(Ae^x + Be^{-x}),$$

where C_1, C_2, A, B are arbitrary constants, while $AB = 0$.

Example 2. Consider the nonlinear heat equation

$$u_t = (uu_x)_x + u^2 = uu_{xx} + {u_x}^2 + u^2.$$

The special solution in the form (8) yields the following equation

$$\phi' + \psi' e^{\psi} V = (\phi + e^{\psi} V) e^{\psi} V'' + e^{2\psi} V'^2 + \phi^2 + 2e^{\psi} V + e^{2\psi} V^2.$$

Splitting the last equation yields

$$\phi' = \phi^2 + e^{2\psi}(VV'' + V'^2 + V^2), \quad \psi' V = \phi V'' + 2V.$$

Then we have the following overdetermined system:

$$VV'' + V'^2 + V^2 = k = const, \quad V'' = mV, \quad m = const.$$

From the last relations we have

$$V(x) = \sqrt{2} C \cos \frac{x}{\sqrt{2}}, \quad C = const.$$

So, we can write down the family of solution as

$$u(t,x) = \phi(t) + e^{\psi(t)} \sqrt{2} C \cos \frac{x}{\sqrt{2}},$$

where functions $\phi(t), \psi(t)$ satisfy the following dynamical system:

$$\phi' = \phi^2 + C^2 e^{2\psi}, \quad \psi' = 2 - \frac{\phi}{2}.$$

4. Some generalizations within linear superposition

1. The linear superposition evolution can easily be generalized for higher order time-derivatives. Indeed, one can prolong evolutionary system (6) for partial derivative $u_{tt}, u_{ttt}, \ldots$:

$$u_{tt}(t,x) = \bar{\phi}'(t) + \bar{\phi}(t)\psi'(t) + (\psi'(t) + \psi''(t))u(t,x), \ldots .$$

In that case the solution has evidently the same form

$$u(t,x) = \phi(t) + e^{\psi(t)} V(x). \tag{9}$$

To determine unknown functions $\phi(t), \psi(t), V(x)$ one should substitute (9) into corresponding evolutionary equation.

2. The linear superposition evolution can be generalized for 2-D space-dimensional solutions $u(t,x,y)$. For that case we prolong the evolution system (4) for partial derivative $u_x, u_y, u_{xx}, u_{yy}, \ldots$:

$$u_t(t,x,y) = \bar{\phi}(t) + \psi'(t) u(t,x,y), \tag{10}$$
$$u_{xt}(t,x,y) = \psi'(t) u_x(t,x,y), \quad u_{yt}(t,x,y) = \psi'(t) u_y(t,x,y),$$
$$u_{xxt}(t,x,y) = \psi'(t) u_{xx}(t,x,y), \quad u_{yyt}(t,x,y) = \psi'(t) u_{yy}(t,x,y), \ldots .$$

In 2-D case the solution has similar to 1-D case form

$$u(t,x) = \phi(t) + e^{\psi(t)} V(x,y),$$

and the same linear superposition formula.

Below we consider an example of evolutionary equation of the second order

$$u_t = F(u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}),$$

which is compatible with the evolution (10).

Example3. Let us consider the nonlinear equation

$$u_t = u(u_{xx} + u_{yy}) + 2u_{xx}u_{yy} - {u_x}^2 - {u_y}^2. \tag{11}$$

We substitute a solution

$$u(t,x) = \phi(t) + e^{\psi(t)} V(x,y),$$

and obtain

$$\phi' + \psi' e^{\psi} V = (\phi + V e^{\psi})(V_{xx} + V_{yy}) e^{\psi} + 2e^{2\psi} V_{xx} V_{yy} - e^{2\psi}(V_x^2 + V_y^2).$$

Then we split the last equality

$$V(V_{xx}+V_{yy})+2V_{xx}V_{yy}-(V_x^2+V_y^2)=k, \quad k=const,$$

$$mV=V_{xx}+V_{yy}, \quad m=const, \quad \phi'=ke^{2\psi}, \quad m\psi'=\phi.$$

For the special case $m=-1, k=-2$ one can find the special solution

$$V(x,y)=\cos x+\cos y,$$

and then write the solution for equation (11) in the form

$$u(t,x,y)=\phi(t)+e^{\psi(t)}(\cos x+\cos y),$$

while $\phi(t),\psi(t)$ should obey the following dynamical system

$$\phi'=-2e^{2\psi}, \quad \psi'=-\phi,$$

which can be integrated completely.

3. The linear superposition evolution can be extended for four-dimensional Lie Algebra, involving variable x as non-evolutionary parameter.

Indeed, let us consider the following extended evolution system (4):

$$u_t(t,x)=\phi_1(t)+\phi_2(t)u+\phi_3(t)x+\phi_4(t)x^2, \tag{12}$$

$$u_{xt}(t,x)=\phi_2(t)u_x+\phi_3(t)+2\phi_4(t)x, \quad u_{xxt}(t,x)=\phi_2(t)u_{xx}+2\phi_4(t),\dots .$$

which still has linear superposition for its solutions. Thus, formally we have two dependent variables u,x, while the second one does not evolute in time: $\dfrac{dx}{dt}=0$, since $x=const$. The corresponding Lie Algebra is the following four-dimensional algebra L_4, which we prolong for high-order derivatives:

$$X_1=\frac{\partial}{\partial u}, \quad X_2=u\frac{\partial}{\partial u}+u_x\frac{\partial}{\partial u_x}+u_{xx}\frac{\partial}{\partial u_{xx}}, \quad X_3=x\frac{\partial}{\partial u}+\frac{\partial}{\partial u_x}, \quad X_4=x^2\frac{\partial}{\partial u}+2x\frac{\partial}{\partial u_x}+2\frac{\partial}{\partial u_{xx}}.$$

In accordance with the Lie theorem we have two dependent variables u,x since $n=2$, and four-dimensional algebra L_4, consequently $r=4$. The Lie theorem yields $m\geq 2$. In fact two particular solutions cast the minimum number of solutions and allow to write down a general solution of equations (12). Thus, (12) has the fundamental set of special solutions u_1 and u_2 with superposition

$$\frac{u-u_1}{u_2-u_1}=C,$$

and a general solution is $u=(1-C)u_1+Cu_2, \quad C=const$. In accordance with the evolution (12) a special solution has the following form:

$$u(t,x)=\psi_1(t)+\psi_2(t)V(x)+\psi_3(t)x+\psi_4(t)x^2, \tag{13}$$

where functions $\psi_i(t)$ are connected with the functions $\phi_j(t)$ by means of the following dynamical system

$$\psi_1'=\phi_1+\phi_2\psi_1, \quad \psi_2'=\phi_2\psi_2, \quad \psi_3'=\phi_2\psi_3+\phi_3, \quad \psi_4'=\phi_2\psi_4+\phi_4.$$

Thus, we got the new representation of solution, which possesses two additional functions of t. Now we consider the example of such extension.

Example 4. We consider the nonlinear equation

$$u_t=u_{xx}^2 \tag{14}$$

and seek the special solution of equation (14) in the form (13). It yields the following equation

$$\psi_1'+\psi_2'V+\psi_3'x+\psi_4'x^2={\psi_2}^2(V'')^2V+4{\psi_4}^2+4\phi_4\phi_2V''.$$

Splitting the last equality and integrating corresponding equations for $V(x)$ and $\psi_i(t)$ yields

$$u(t,x) = C_5 - \frac{C_3^2}{12}(C_1 - 144t)^{1/3} + C_4 x + \frac{C_3}{(C_1-144t)^{1/3}}(x^2 + \frac{C_2}{2}x + \frac{C_2^2}{16}) +$$

$$+ \frac{1}{C_1 - 144t}\left(x^4 + C_2 x^3 + \frac{3C_2^2}{8}x^2 + \frac{C_2^3}{16}x + \frac{C_2^4}{256}\right),$$

where C_1, C_2, C_3, C_4, C_5 are arbitrary constants.

Thus, we developed the 4-order in x polynomial expression for the solution, which contains three independent functions of t.

5. The subset of solutions with nonlinear superposition principle

We now consider the second subalgebra L_2 spanned by the operators:

$$X_1 = u\frac{\partial}{\partial u}, \quad X_2 = u^2\frac{\partial}{\partial u}, \quad [X_1, X_2] = X_2. \tag{15}$$

The prolongation of the L_2 for spatial derivatives

$$X_1 = u\frac{\partial}{\partial u} + u_x\frac{\partial}{\partial u_x} + u_{xx}\frac{\partial}{\partial u_{xx}}, \quad X_2 = u^2\frac{\partial}{\partial u} + 2uu_x\frac{\partial}{\partial u_x} + 2(uu_{xx} + {u_x}^2)\frac{\partial}{\partial u_{xx}},$$

corresponds the following evolution (the Bernoulli equation):

$$u_t = \bar{\phi}(t)u + \psi(t)u^2, \tag{16}$$

$$u_{xt} = \bar{\phi}(t)u_x + \psi(t)2uu_x, \quad u_{xxt} = \bar{\phi}(t)u_{xx} + \psi(t)2(uu_{xx} + {u^2}_x).$$

Let apply the point change of variable

$$u(t,x) = -\frac{1}{v(t,x)},$$

then the subalgebra (15) becomes $Y_1 = -v\frac{\partial}{\partial v}$, $Y_2 = \frac{\partial}{\partial v}$; $[Y_1, Y_2] = Y_2$. The corresponding non-autonomous evolution (16) is transforming into the linear one

$$v_t = -\phi'(t)v + \bar{\psi}(t).$$

Thus, now we have a linear superposition situation, which defines the family of solutions in the form

$$v(t,x) = \psi(t) + e^{-\phi(t)}\alpha(x).$$

Coming back to $u(t,x)$, we have the solutions in the form

$$u(t,x) = \frac{-1}{\psi(t) + e^{-\phi(t)}\alpha(x)},$$

which possess the following nonlinear superposition principle for its solutions:

$$u(t,x) = \frac{u_1 u_2}{Cu_2 + (1-C)u_1}. \tag{17}$$

One can apply that approach to the Example 1 and obtain the following nonlinear equations with nonlinear superposition (17):

$$u_t = u_{xx} - 2\frac{{u_x}^2}{u} + \frac{{u_x}^2}{u^2} - 1.$$

6. The subset of solutions with the Riccati-type nonlinear superposition

We now consider the complete Riccati equation, prolonged to evolution of space derivatives

$$u_t = \alpha(t) + \beta(t)u + \gamma(t)u^2, \tag{18}$$

$$u_{xt} = \beta(t)u_x + 2\gamma(t)uu_x, \quad u_{xxt} = \beta(t)u_{xx} + 2\gamma(t)(uu_{xx} + u_x^2).$$

Now we will construct a solution of the Riccati equation. Let u_1 be a particular solution of the Riccati equation (18). Then one can change variables as follows:

$$u(t,x) = u_1 + w(t,x),$$

and get the corresponding Bernoulli's equation and its differential sequences

$$w_t = (\beta + 2\gamma u_1)w + \gamma w^2, \quad w_{xt} = (\beta + 2\gamma u_1)w_x + 2\gamma w w_x,$$

$$w_{xxt} = (\beta + 2\gamma u_1)w_{xx} + 2\gamma(ww_{xx} + w_x^2).$$

The change

$$w(t,x) = -\frac{1}{v(t,x)}$$

yields the following linear equation

$$v_t = \gamma(t) + \psi'(t)v, \quad \psi' = -(\beta + 2\gamma u_1).$$

The last equation possesses a linear superposition principle and has general solution

$$v(t,x) = \phi(t) + e^{\psi}C(x),$$

where $\phi(t)$ is a particular solution of a linear equation:

$$\phi' = \gamma + \psi'\phi.$$

The corresponding solution of the Bernoulli equation

$$w(t,x) = -\frac{1}{\phi(t) + e^{\psi(t)}C(x)},$$

yields the solution of the Riccati system

$$u(t,x) = u_1 - \frac{1}{\phi(t) + e^{\psi(t)}C(x)},$$

which possesses the nonlinear superposition of the particular solutions.

In the following example we develop an equation which possesses solutions with the Riccati-type superposition.

Example 5. Let consider a special integrable case of the Riccati equation $u_t = \dfrac{2}{t^2} - u^2$, which has a general solution

$$u(x,t) = \frac{3t^2}{t^3 - C(x)} - \frac{1}{t}. \tag{19}$$

Applying a shift by special solution $u_0 = -1/t$ we transform the Ruccati equation into the Bernoulli equation

$$v_t = \frac{2}{t}v - v^2, \quad u = v - \frac{1}{t}.$$

We linearize the Bernoulli equation by change $w = -\dfrac{1}{v}$

$$w_t = -\frac{2}{t}w - 1, \tag{20}$$

and find the solution

$$w(x,t) = \frac{C(x)}{t^2} - \frac{t}{3}.$$

The evolution (20) is compatible with the following equation

$$w_t = \sqrt{w_x} w_{xx} - \frac{1}{3}.$$

The substitution of solution in form (19) yields the equation

$$(C')^{5/2} + \frac{5}{2}C^2 = c_1, \quad c_1 = const;$$

which can be solved in quadrature for $C(x)$:

$$x = \int \frac{dC}{(c_1 - \frac{5}{2}C^2)^{2/5}} + c_2, \quad c_2 = const.$$

Applying backward transformation we obtain the following equation for $v(x,t)$:

$$v_t = \sqrt{v_x}\left(\frac{v_{xx}}{v} - 2\frac{v_x^2}{v^2}\right) - \frac{v^2}{3},$$

and then the evolutionary equation for $u(x,t)$:

$$u_t = \sqrt{u_x}\left(\frac{tu_{xx}}{1+tu} - 2\frac{u_x^2 t^2}{(1+tu)^2}\right) - \frac{tu^2 + 2u}{3t}.$$

The last equation has solution (19) and Riccati-type superposition for particular solutions.

7. How to separate evolutionary equations which possess a linear subspace of solutions?

We now consider the compatibility conditions of the evolution (6) with a PDE

$$u_t = F(u, u_x, u_{xx}) \tag{21}$$

as a compatibility with certain differential constraint. There is a lot of different approaches to constructing exact solutions for PDEs based on differential constraints (see, for example, [14] and references therein). We restrict ourselves here with such constraint, which leads to solutions with linear superposition principle.

Excluding two functions of t by differentiation we rewrite linear evolution equations (6) as differential constrain

$$u_{xxt}u_x = u_{xt}u_{xx}. \tag{22}$$

Thus, now we can repose the problem of compatibility as a compatibility of (21) and differential constrain (22). We substitute time derivatives from (21) into (22) and obtain ODE

$$D_x\left(\frac{D_x(F)}{u_x}\right) = 0.$$

In the same way one can write down the constrain of k-th order:

$$D_x\left(\frac{D_x^k(F)}{u_k}\right) = 0, \quad k = 1, 2, \dots . \tag{23}$$

The operator of total differentiation with respect to t along trajectories of (21)

$$X^* = F(u, u_x, u_{xx})\frac{\partial}{\partial u} + D_x^2(F)\frac{\partial}{\partial u_{xx}} + \dots ,$$

casts a higher order symmetry operator. Then the criterion of an invariance of manifold (23) reads

$$X^*\left(D_x\left(\frac{D_x(F)}{u_x}\right)\right)\Bigg|_{(23)} = 0. \tag{24}$$

Being solved for F equation (24) yields evolution equation which potentially has linear subspace of solutions. Existence and particular form of solutions can be obtained by substituting solution into evolution equation and applying splitting procedure, which was demonstrated by examples.

In a similar way one can write down the compatibility condition for evolutionary PDEs which possesses a subset of solutions with Riccati-type superposition.

Concluding remarks

Thus, we considered the Lie non-autonomous dynamical equations (NADS) as some sort of external action on a given evolutionary equation. It makes it possible to find a subset of special solutions of evolutionary equation, which possesses a superposition principle which acts within subset of solutions. This leads to integration of ordinary differential equations in a process of constructing exact solutions of PDEs. In this paper we considered the application of the most simple one-dimensional case of the Lie theorem. It also was shown that the NADS approach can be generalized for 1+2 D equations as well. Feather generalizations will be published elsewhere.

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