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THE PROBLEM OF NUMERICAL REALIZATION OF INTEGRAL OPERATORS OF AXISYMMETRIC BOUNDARY VALUE PROBLEMS (ALGORITHMS WITHOUT SATURATION)

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Abstract. In the paper a fundamentally new, *unsaturated,* method of numerical implementation integral operators of C^∞-smooth axisymmetric boundary value problems is described. The method allows to take into account the specifics of the axisymmetric problems automatically. This specifics is an obstacle to any numerical methods with the principal term of error.

The method was tested on the problem of precise evaluation of the Gauss integral of the theory of harmonic potential in highly prolonged ellipsoids of evolution.

Keywords: unsaturated numerical method, Gauss integral, axisymmetric region, quadrature formula without saturation.

1. Preamble.

Extended opportunities of modern computers are basically determined by success in microelectronics, whereas the value of the obtained numerical results significantly depends on quality of the applied calculation algorithms. In this connection the search of new principles of construction of numerical algorithms becomes an indispensable condition of development and existence of computational mathematics as a science in the whole.

With the development of computational practice and appearance of new classes of applied problems [1] in the scientific world, the role of smoothness reserve of their solutions is considered differently. It is understood as existence of set of continuous derivatives in the solutions. This set exceeds rather definite threshold. But such extraordinary smoothness reserves are usually potential and are not realized if the method error has a dominant term, since with the extension of approximate opportunities of the compact of solutions determined by smoothness of its elements the error of the method does not decrease [2]. The specified defect significantly decreases practical value of numerical methods having the dominant term of the error. It is the factor that explains the tendency why calculators usually strive for the use of high order methods, in which the mentioned defect decreases though does not vanish completely.

Modern computational science (both fundamental and applied) experience great demand in appearance of numerical methods without the specified defect - existence of the dominant term of the error, and called by K.I. Babenko *unsaturated* [2]. Genetically these methods join methods of variable orders, with the only difference which corresponds an increased consistency with the class of the problem correctness: unsaturated numerical methods are self-improved with the increase of smoothness reserve. It means they automatically obtain the increment of their practical effectiveness directly from the differential nature of the searched solutions of

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the problems (*nonsaturation phenomenon* [3]). Meanwhile the peak of effectiveness, that is the *exponential convergence* is achieved for the classes of C^∞-smooth solutions.

There are classes of problems, in which the information on extraordinary smoothness of solutions can be applied with the profit for calculations. Thus, the quality discretions in the elliptical problems differ by their ability to inherit characteristic properties, that is Shaudier smoothness estimates [4]. Earlier this property was not demanded by practice, but it becomes acute due to the known context, striving for the numerical answer of the guaranteed quality. All this in the aggregate with good stipulation of discretization is able to result in the actual *terra incognita* of computer calculations, that is their evidentiary [2,5]. Good stipulation of discretization is necessary in this case, though mathematical essentials, which are on the stage of creating a numerical method very often become an ideal field for possible speculations not considering such realities of computer calculations as rounding errors.

The classical method of representation of solutions of boundary-value problems for the Laplace equation is based on the notion of harmonic potential and admits an equivalent formulation in terms of solutions of boundary integral equations. In this connection in mathematical folklore there is a supposition that if a problem is reduced to a "good" integral equation, then its effective numerical solution is determined in advance. Meanwhile it is naively supposed that finding the solution is a simple and sometimes standard activity. Indeed, the problem of value or precision of the computer answer under construction, which is the same, completely depends on the method of approximate realization of an integral operator of the problem and directly depends on the properties of the applied cubature formulae.

Although, the use of standard cubature formulae, i.e. the ones having the dominant term of the error faces only a formal statement of the existing difficulties, i.e. ("curses") of the finite mathematics (dimension of the space of independent variables [2,6] and order of integral [7]) without specifying possible ways to avoid them. As a result we can only hope for the chance and not for a guaranteed success. It is the reason why the problem of finding the guaranteed quality of the numerical solution of the Laplace equation in smooth three-dimensional domains of a sufficiently arbitrary form is still understood as a very difficult computational problem. Its solution, as theory [2] stipulates, is referred to the sphere of a higher intellectual degree: they highlight the problem of construction of unsaturated cubature formulae. Existence of difficulties on this way is also stipulated by the absence of the common sense highlights, which could satisfy real demands of practice. Though, in the recent paper [8] an interesting attempt to consider it was made.

The present point of view on the quality of computer calculations seriously influenced the choice of data for the paper. We confine ourselves only to the consideration of the axisymmetric boundary-value problems by advancing a new *nonsaturated* method of dicretization of the integral operators of these problems. It is considered in general terms in [5]. Here we describe the general source of difficulties computational these problems by the example of a precise calculation of the Gauss integral: great increase of integrants close to the symmetry axis of the domain, which is called a "boundary layer", and which is a "stumbling block" for any standard numerical methods. In this paper we specify an effective mechanism of numerical neutralization of these difficulties due to a "high" smoothness of the problems solutions.

The limit of the article size does not allow to present detailed results and proofs, making us exclude the considerations, including complicated manipulations with special functions. Therefore, some topics here remain untouched. But we demonstrate all the key notions of the new method in detail.

2. The Gauss integral on the axisymmetric surface

The new numerical methods are constructed under the influence and with the objective of solution of some definite applied problem. In the problem under consideration this role is

played by the Gauss integral of the theory of a harmonic potential [9]. This potential value does not depend on the surface upon which it is calculated. At the same time its analytical nature is sufficiently complicated to be a model for the general problem of smoothness in the axisymmetric domain. This specifies its uniqueness as a test example.

Calculation of the Gauss integral is a visual demonstration of all the aspects of the precise numerical realization of integral operators of the axisymmetric boundary-value problems. The general attention is paid to the process of a numerical algorithm construction, general computational difficulties and methods of their effective overcoming.

Assume that $\boldsymbol{x} \in \mathbb{R}^3,\ \boldsymbol{x} = (x, y, z)$; $\omega \subset \mathbb{R}^3$ is a domain with the axis of symmetry z, bounded by the C^∞-smooth closed surface of rotation $\partial\omega$; the values $r = \sqrt{x^2 + y^2}$ and z are invariants of the group of rotation of the domain ω with respect to the axis z. The meridian section $\partial\omega$ is a parameter curve $\gamma : [\,0, 1] \to \{r(s), z(s)\}$, $r \geq 0$, $dz/ds \geq 0$, $\gamma(s) \in C^\infty[0, 1]$. The points $\gamma(0)$ and $\gamma(1)$ are poles $\partial\omega$, and $r(s)$ and $z(s)$ have C^∞-smooth 2-periodical extensions with $[\,0, 1]$ on $[\,0, 2\,]$ (odd, even respectively).

We consider the curve $\gamma(s)$ convex, so that the points close to the axis z are always concentrated near the poles $\gamma(0)$ and $\gamma(1)$ of the surface $\partial\omega$. The general (non-convex) case is investigated by the corresponding division of $\gamma(s)$ into parts.

The position of the points $\boldsymbol{x}$ and $\boldsymbol{\xi}$ on $\partial\omega$ is determined by the cylindrical coordinates (r, z, v) and (ρ, ζ, φ) respectively: $\boldsymbol{x} \equiv \boldsymbol{x}(s, v) = (r\cos v, r\sin v, z)$, $\boldsymbol{\xi} \equiv \boldsymbol{x}(\sigma, \varphi) = (\rho\cos\varphi, \rho\sin\varphi, \zeta)$. We use the notation here: $r = r(s), z = z(s)$, $\rho = r(\sigma), \zeta = z(\sigma)$ and $0 \leq s, \sigma \leq 1,\ 0 \leq v, \varphi < 2\pi$.

Let us consider the point $\boldsymbol{x} \in \partial\omega$ fixed, and the point $\boldsymbol{\xi} \in \partial\omega$ to be variable.

The normal $\boldsymbol{n}$ and the element of the area $d\,\omega_\xi$ in the point $\boldsymbol{\xi} \in \partial\omega$ are denoted by:

$$\boldsymbol{n} = \Big(-\delta^{-1}\,\zeta'\cos\,\varphi,\ -\delta^{-1}\,\zeta'\sin\,\varphi,\ \delta^{-1}\,\rho' \Big), \quad d\,\omega_\xi = \rho\,\delta\,d\sigma\,d\varphi,$$

respectively. Here $\zeta' \equiv d\,\zeta/d\,\sigma$, $\rho' \equiv d\,\rho/d\,\sigma,\ \delta \equiv \delta(\sigma) = \sqrt{\rho'^{\,2} + \zeta'^{\,2}}$.

The direct value of the normal derivative $\nabla_{\boldsymbol{x}} f < \boldsymbol{n} >$ of the function $f(\boldsymbol{\xi}, \boldsymbol{x})$ on the surface of rotation $\partial\omega$ is determined by the equality

$$\nabla_{\boldsymbol{x}} f < \boldsymbol{n} >= -\delta^{-1} \left(\zeta' \frac{\partial f}{\partial \rho} - \rho' \frac{\partial f}{\partial \zeta} \right) \Bigg|_{\boldsymbol{\xi},\, \boldsymbol{x} \in \partial\omega}.$$

Let us consider the Gauss integral with respect to the closed rotation surface $\partial\omega$:

$$\pi \equiv \Gamma(\boldsymbol{x}) = \frac{1}{2} \int\limits_{\partial\omega} \nabla_{\boldsymbol{x}} \frac{1}{P} < \boldsymbol{n} > \, d\,\omega_\xi, \quad \boldsymbol{x} \in \partial\omega, \quad \boldsymbol{\xi} \in \partial\omega. \tag{1}$$

Here $P = |\,\boldsymbol{\xi} - \boldsymbol{x}|$ is the distance between the points $\boldsymbol{\xi}, \boldsymbol{x} \in \partial\omega$, and $P^{\,2} = r^2 + \rho^2 + (\zeta - z)^2 - 2\,\rho\,r\cos(\varphi - v)$, and the integral is understood as a principal value.

Let us introduce the notation

$$h^2 \equiv h^2(\sigma, s) = (\rho - r)^2 + (\zeta - z)^2, \quad h_*^2 \equiv h_*^2(\sigma, s) = (\rho + r)^2 + (\zeta - z)^2,$$

$$q \equiv q(\sigma, s) = 4\,\rho\,r\,h_*^{-2}, \quad u \equiv 1 - q = h^2\,h_*^{-2},$$

$$K(\beta) = \int\limits_0^{\pi/2} (1 - \beta\,\sin^2\theta)^{-1/2}\,d\theta,\ E(\beta) = \int\limits_0^{\pi/2} (1 - \beta\,\sin^2\theta)^{1/2}\,d\theta,\ D(\beta) = K(\beta) - E(\beta).$$

Here $0 \leq \beta < 1$ is a *module* of complete elliptical integrals $K(\beta)$, $E(\beta)$, $D(\beta)$.

The axis symmetry of the surface $\partial\omega$ allows one to integrate once in (1), reducing the calculations into the meridian plane $v = const = 0$ on the basis of the following factors.

LEMMA 1. *The following representation holds on the smooth closed surface of the rotation* $\partial\omega$*:*

$$\nabla_{\boldsymbol{x}} \frac{1}{P} = \Omega(\sigma, s)\, \frac{h^2}{P^3} - r\, \frac{\zeta'}{\delta}\, \frac{1-\cos\varphi}{P^3},$$

where

$$\Omega \equiv \Omega(\sigma, s) = \begin{cases} (\sigma - s)^{-1}\ \boldsymbol{n} \cdot \boldsymbol{r}[\,\sigma, s\,] \,/\, |\,\boldsymbol{r}[\,\sigma, s\,]|^2, & \sigma \neq s, \\ 0.5\,(z'r'' - r'z'')\,/\,d^3, & \sigma = s, \end{cases}$$

$$\boldsymbol{r}[\,\sigma, s\,] = \left[\frac{\rho - r}{\sigma - s}\,\cos\varphi,\ \frac{\rho - r}{\sigma - s}\,\sin\varphi,\ \frac{\zeta - z}{\sigma - s}\right], \quad d \equiv d(s) = \sqrt{r'^2 + z'^2}.$$

Here r', z' *and* r'', z'' *are the first and the second derivatives of the function* $r = r(s)$, $z = z(s)$ *with respect to the local coordinate* s*, respectively.*

LEMMA 2. *One has*

$$\int_0^{2\pi} P^{-3}\, d\varphi = 4\, h^2\, h_*^{-1}\, E(q), \qquad \int_0^{2\pi} (1 - \cos\varphi) P^{-3}\, d\varphi = 2\, r^{-1}\, \rho^{-1}\, h_*^{-1}\, D(q)$$

on the smooth closed surface of the rotation $\partial\omega$.

Due to Lemmas 1,2 the Gauss integral (1) is transformed to the form

$$\pi \equiv \Gamma(s) = \int_0^1 \Big(2\,\rho\,\delta\ \Omega(\sigma, s)\, E(q) - \zeta' D(q)\Big)\, h_*^{-1}\, d\,\sigma, \quad \forall s \in [\,0, 1].$$

The existence of complete elliptical integrals here requires special methods of approximation $E(q)$ and $D(q)$ due to the fact that the module $q \equiv q(\sigma, s)$ is a function of two points: variable σ and fixed s. Meanwhile, there is always $q(s, s) = 1$.

In [10] we obtain the representations

$$E(q) = E_p^*(q) - e_p^*(q)\, \ln u, \quad D(q) = D_p^*(q) - d_p^*(q)\, \ln u.$$

Here the parameter $p \geq 0$ is an integer number, whose algorithm of choice will be determined below; $u = 1 - q$, and the functions $E_p^*(q),\ e_p^*(q),\ D_p^*(q),\ d_p^*(q)$ are set by the formulae:

$$E_p^*(q) = \begin{cases} 0.5\,\pi(1 - q\, E_0 < q >) + e_p^*(q)\, \ln(1-q), & \text{if} \quad 0 \leq q \leq 0.5, \\ -D_p < u >\ u^{p+1}\, \ln u + 1 + u\, G < u >, & \text{if} \quad 0.5 < q \leq 1, \end{cases}$$

$$D_p^*(q) = \begin{cases} \pi\, q\, D_0 < q > + d_p^*(q)\, \ln(1-q), & \text{if} \quad 0 \leq q \leq 0.5, \\ 0.5\, E_p < u >\ u^{p+1}\, \ln u + \ln(4/e) - u\, W < u >, & \text{if} \quad 0.5 < q \leq 1. \end{cases}$$

where

$$E_p < x > = \sum_{n=p+1}^{\infty} \frac{\gamma_n\, x^{\,n-p-1}}{2n-1}, \quad e_p^*(x) = \sum_{n=1}^{p} \frac{n\,\gamma_n\, x^n}{2n-1}, \quad e_0^*(x) = 0, \quad G < x > = \sum_{n=1}^{\infty} G_n,$$

$$D_p < x > = \sum_{n=p+1}^{\infty} \frac{n\,\gamma_n\, x^{\,n-p-1}}{2n-1}, \quad d_p^*(x) = -0.5 \sum_{n=1}^{p} \frac{\gamma_n\, x^n}{2n-1}, \quad d_0^*(x) = 1, \quad W < x > = \sum_{n=1}^{\infty} W_n,$$

$$G_n = \Big(\lambda_n + \lambda_{n-1}\Big)\, \frac{n\,\gamma_n\, x^{\,n-1}}{2n-1}, \quad W_n = \Big(\frac{1}{2n-1} + \lambda_n\Big)\, \frac{\gamma_n\, x^{\,n-1}}{2n-1},$$

$$\gamma_0 = 1,\ \gamma_n = \Big((1/2)_n\,/\,n!\Big)^2, \qquad \lambda_0 = \ln 4,\ \lambda_n = \psi(n+1) - \psi(n+1/2), \quad n > 0,$$

$$(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha), \quad \alpha > 0,\ n > 0, \quad \psi(z) = d\,\Gamma(z)/\,d\,z \quad (z > 0).$$

Here $\Gamma(z)$ and $\psi(z)$ are Γ- and ψ-functions of Euler, respectively (don't mix it with the integral $\Gamma(s)$). Algorithms of calculation of the finite totals $e_p^*(q)$ and $d_p^*(q)$ and fast-converging degree series $E_0 < q >,\ D_0 < q >,\ E_p < u >,\ D_p < u >,\ W < u >,\ G < u >$ are specified in [10].

Therefore, the Gauss integer (1) is reduced to a standard one for the axisymmetric domains due to

$$\pi \equiv \Gamma(s) = \int\limits_0^1 \left(2\,\rho\,\delta\ \Omega(\sigma,s)\ E_p^*(q) - \zeta'\ D_p^*(q) \right) h_*^{-1}\,d\,\sigma -$$

$$\int\limits_0^1 \left(2\,\rho\,\delta\ \Omega(\sigma,s)\ e_p^*(q)\ - \zeta'\ d_p^*(q)\ \right) h_*^{-1}\ \ln(1-q)\,d\,\sigma. \tag{2}$$

Let us note that the representation (2) is regular in the poles $\gamma(0)$ and $\gamma(1)$ of the smooth surface of the rotation $\partial\omega$, i.e. it does not contain logarithmic singularities.

3. The specific character of computational difficulties in axisymmetric problems

Far not every quadrature formula for the approximate realization (2) can be of practical interest: the structure of complication of the integrant in (2) is determined both by the properties of the module $q(\sigma,s)$, and by the behaviour of the function $h_*^{-1}(\sigma,s)$ in points close to the axis of symmetry z, i.e. near the poles $\gamma(0)$ and $\gamma(1)$. Indeed, the integrant in (2) has a "mobile" logarithmic singularity on the diagonal $\sigma = s$, and in the points s is close to the axis of symmetry z, it demonstrates a highly expressed increase, we call it a *boundary layer* [3,11]. In terms of mathematics this supposes existence in (2) of a parameter such that when it tends to zero we can observe the structure of increase of the integrant and its derivatives close to the axis of symmetry z. In (2) this parameter is presented by the weighting function $h_*^{-1}(\sigma,s)$. Thus, being a peculiar payment for the cylindrical symmetry of the problem, the boundary layer in (2) proves to be applicable to all the problems with the axis symmetry. It is clear that this influences the realization (2) by quadrature formulae.

The precise computational experiments show that for the effective computer realization of the representation (2) we need:

1) to apply properties of the module $q(\sigma,s)$ for the finding logarithmic singularity uniform in (σ,s), and the usual derivation according to the rule

$$\ln(1-q) = 2\ln|\,\sigma - s| + A(\,\sigma,s), \quad (\sigma,s) \in [\,0,1] \times (\,0,1)$$

is useless near the axis z, since the function $A(\sigma,s)$ is not uniformly continuous;

2) to ensure numerical neutralization of the boundary layer.

Mathematical investigation [3] formed the notation of the constructive terms convenient to describe the specified difficulties and resources which ensure their numerical neutralization.

Let us reduce (2) to the form convenient for further analysis. Let us introduce the functions, which are uniformly continuous in the square $K \times K$ (here $K \equiv [\,0,1]$):

$$R^2(\sigma,s) = \left(\frac{\rho - r}{\sin\frac{\pi(\sigma-s)}{2}}\right)^2 + \left(\frac{\zeta - z}{\sin\frac{\pi(\sigma-s)}{2}}\right)^2,\ R_*^2(\sigma,s) = \left(\frac{\rho + r}{\sin\frac{\pi(\sigma+s)}{2}}\right)^2 + \left(\frac{\zeta - z}{\sin\frac{\pi(\sigma+s)}{2}}\right)^2,$$

$$Q(\sigma,s) = 4\,\frac{(r/\sin\pi s)\,(\rho/\sin\pi\sigma)}{R_*^2(\sigma,s)}, \quad B(\sigma,s) = \frac{R^2(\sigma,s)}{R_*^2(\sigma,s)}. \tag{3}$$

The functions in (3) are uniformly continuous and belong to $C^\infty[K \times K]$ when $\gamma(s) \in C^\infty[K]$.

The integer (2), in which the variable σ is implicitly substituted by the new variable

$$t \equiv t_S(\sigma) = t(\sigma, s) = \sin\frac{\pi(\sigma - s)}{2} / \sin\frac{\pi(\sigma + s)}{2}, \tag{4}$$

where $s \in (0, 1)$ is fixed, and the function $\sigma = \sigma_S(t) \equiv \sigma(t, s)$ is inverse to $t \equiv t_S(\sigma) \equiv t(\sigma, s)$, has the following form

$$\varepsilon\,\Gamma(s) = \int\limits_{-1}^{1} \widetilde{\Gamma}_C(t)\,d\,t - \int\limits_{-1}^{1} \widetilde{\Gamma}_L(t)\,\ln|\,t|\,d\,t, \quad \varepsilon \equiv \varepsilon(s) = 0.5\,\pi\sin\pi\,s, \tag{5}$$

where the subintegral functions $\widetilde{\Gamma}_C(t)$ and $\widetilde{\Gamma}_L(t)$ are set by the formulae (see formula (2))

$$\widetilde{\Gamma}_C(t) = \Big(2\widetilde{\rho}\widetilde{\Omega}(\widetilde{E}_p^* - \widetilde{b}\,\widetilde{e_p^*}) - \widetilde{\delta}^{-1}\widetilde{\zeta}\,'(\widetilde{D}_p^* - \widetilde{b}\,\widetilde{d_p^*})\Big)\widetilde{a},\ \widetilde{\Gamma}_L(t) = 2\Big(2\widetilde{\rho}\widetilde{\Omega}\,\widetilde{e_p^*} - \widetilde{\delta}^{-1}\widetilde{\zeta}\,'\widetilde{d_p^*})\Big)\widetilde{a},$$

$$\widetilde{a}(t) \equiv \widetilde{\delta}\widetilde{R}_*^{-1}\sin\frac{\pi\,(\sigma + s)}{2}, \quad \widetilde{b} \equiv \ln\widetilde{B}. \tag{6}$$

For the uniformly continuous function $g(\sigma, s)$ on $K \times K$ according to the definition

$$\widetilde{g} \equiv \widetilde{g}(t) = g\big(\,\sigma_S(t), s\,\big), \quad t \in I \equiv [\,-1, 1], \quad s \in (\,0, 1)\,\text{fixed}. \tag{7}$$

If $g(\sigma, s) \in C^{\infty}[K \times K]$, then $\widetilde{g}(t) \in C^{\infty}[I]$ and the following equality holds

$$\left(\frac{d}{dt}\right)^k \widetilde{g}(t) = \varepsilon^{-k}\left(\sin^2\frac{(\sigma + s)}{2}\frac{d}{d\sigma}\right)^k g(\sigma, s), \quad \varepsilon \equiv \varepsilon(s) = 0.5\pi\sin\pi\,s, \quad \forall k \geq 0. \tag{8}$$

The embedding $\sigma : [-1, 1] \to [\,0, 1]$ plays the key role in reducing integral (2) to the form (5): on the one hand, it discovers the structure of "bad" functions of the problem:

$$\widetilde{q} = (1 - t)(1 + t)\,\widetilde{Q}(t),\ \widetilde{u} = 1 - \widetilde{q} = t^2\widetilde{B}(t),\ h_*^{-1}(\sigma, s)\,\delta\,d\,\sigma = \varepsilon^{-1}\,\widetilde{\delta}\,\widetilde{R}_*^{-1}(t)\sin\frac{\pi(\sigma + s)}{2}\,d\,t,$$

on the other hand, it transforms the "mobile" logarithmic singularity from (2) into the immobile one, i.e. the middle of the segment I in (5).

For the approximate realization of integrals in (5) we apply the quadrature formulae, which take into account the singularity of behaviour of a subintegral function. Their construction is connected with the method of approximation of the functions (6) by polynomials from the subspace $\mathcal{P}^n$ of the algebraic polynomials of the degree not higher than $n - 1$.

Let $C[I]$ and $C^k[I]$ ($k > 0$ is an integer) be the space of continuous and k times continuously differentiated on the segment $I = [-1, 1]$ functions f, respectively; $\|f\| = \max\limits_{t \in I}|\,f(t)|$ be the Chebyshev norm and $E_n(f) = \inf\limits_{H_n \in \mathcal{P}^n}\|f - H_n\| = \|f - R_n\|$ ($n > 0$ is an integer) be the best Chebyshev approximation of the function $f \in C[I]$ by the polynomial R_n from $\mathcal{P}^n$.

Let $t_1, t_2, \ldots, t_n$ be the splitting the segment I by distinct nodes. The interpolation polynomial $P_n(t;\, f)$ with these nodes and the functional of the error $\delta_n(t, f)$ has the form:

$$P_n(t;\, f) = \sum_{i=1}^{n} f(t_i)\,\omega_{n\,i}(t), \quad \delta_n(t, f) \equiv f(t) - P_n(t;\, f).$$

Here $\omega_{n\,i}(t)$ is the fundamental polynomial of the Lagrangian interpolation [2].

What precision of approximation can be obtained if we limit the degree n of the approximating polynomials and the method of their construction in advance? To answer this question we should note that for any polynomial $H_n(t) \in \mathcal{P}^n$ the identity

$$\delta_n(t, f) = f(t) - P_n(t;\, f) = f(t) - H_n(t) - P_n(t;\, f - H_n) holds.$$

From this relationship we obtain the estimate of the error applying the Lebesgue inequality [2]:

$$|\,\delta_n(t, f)| = |\,f(t) - P_n(t;\, f)| \leq (1 + \Lambda_n)\,E_n(f),$$

where Λ_n is the corresponding Lebesgue constant. According to the Lozinsky-Kharshiladze theorem [12] $\Lambda_n \geq \frac{\ln n}{8\sqrt{\pi}}$. In practice splitting of the segment with a minimal growth of the constant Λ_n is preferable. The zeros of the first type Chebyshev polynomial $T_n(t) = \cos(n \arccos t)$ $(n \geq 1)$ are explicitly calculated:

$$t_i = \cos \frac{\pi\,(2\,i-1)}{2\,n}, \quad t_i \in I,\ i = 1,\, 2, \ldots n, \quad n > 0.$$

The corresponding Lebesgue constant is majorized according to the Bernstein theorem [13]: $\Lambda_n \leq 8 + \frac{4}{\pi} \ln n$. Thus, the polynomial interpolating the function $f(t) \in C[I]$ has the form

$$P_n(t;\, f) = \sum_{i=1}^{n} f(t_i)\, \frac{T_n(t)}{T_n'(t_i)\,(t-t_i)}, \quad T_n'(t_i) = \frac{(-1)^{i-1}\, n}{\sqrt{(1-t_i)(1+t_i)}},$$

and the estimate of the error satisfies the inequality

$$|\,\delta_n(t,f)| = |\,f(t) - P_n(t;\, f)| \leq \left(9 + \frac{4}{\pi}\, \ln n\right)\, E_n(f). \tag{9}$$

Any other choice of splitting the segment I can result in the increase of the constant Λ_n and, generally speaking, in worsening the quality of the approximation of the function f, as it results from (9).

To argue for the record of extraordinary reserves of the smoothness of the function f, we develop not only advantages but also the analytical nature of adaptation of the procedure of approximation to the reserves of smoothness.

It is well known [2,14] that any continuous function f has the numerical characteristics $E_n(f)$ and according to the Weierstrass theorem, $\lim\limits_{n\to\infty} E_n(f) = 0$. The fundamental value has the sufficient refinement of the Weierstrass theorem in the form of the so-called Jackson inequality: if $f \in C^{\,k}[I]$ $(k > 0)$, then

$$E_n(f) \leq \frac{\pi}{2}\, \min_{0\leq k\leq n}\, \frac{a^{\,k}\|f^{\,(k)}\|}{n^{\,k}}, \qquad a > 1 \ \text{ is an } \quad \text{absolute constant} \tag{10}$$

(strengthened[1] Jackson inequality [2, p. 307]).

Reducing to zero with the increase of the parameter n of the approximation of the function follows from (9) and (10). If in this case the structure of the function f is sufficiently good (for example, f has many derivatives), then the interpolation polynomial in the nodes of Chebyshev presents f with precision depending on the structure of the function f itself providing approximation almost as good as the best R_n. In other words, the method of interpolation in the nodes of Chebyshev by the function of high smoothness does not differ much from the method of approximation by its polynomial R_n of the best Chebyshev approximation.

There are definite advantages in the consideration of infinitely differentiated functions. This case is probably easier for consideration, but we also expect more from it. Indeed, when $f \in C^{\,\infty}[I]$, $f \notin \mathcal{P}^{\,n}$, $\|f\| = G(0) \neq 0$, $\|f^{\,(k)}\| \leq G(k)$, $\overline{\lim\limits_{k\to\infty}} \sqrt[k]{G(k)} = \infty$ the following functions of the argument $r \in [\,0, \infty)$ are determined:

$$\lambda(r) = \begin{cases} G(0) & \text{when} \quad 0 \leq r < 1, \\ \min\limits_{0\leq k\leq r}\, \dfrac{G(k)}{r^{\,k}} & \text{when} \quad r \geq 1, \end{cases}$$

$$\theta(r) = \begin{cases} 0 & \text{when} \quad 0 \leq r < 1, \\ \max\left\{\, k \mid 1 \leq k \leq r \quad \text{and} \quad \lambda(r) = \dfrac{G(k)}{r^{\,k}} \right\} & \text{when} \quad r \geq 1, \end{cases}$$

[1] In the paper [3] this inequality contains a misprint. The author is thankful to Professor R.M. Trigub, who paid his attention to this circumstances.

and

$$\lambda(r) = \min_{0\le k\le r} \frac{G(k)}{r^{\,k}} = \frac{G[\,\theta(r)\,]}{x^{\,\theta(r)}} \quad \text{and} \quad \|f - R_n\| = E_n(f) \le \frac{\pi}{2}\,\lambda\left(\frac{n}{a}\right) \tag{11}$$

holds.

Theorem 1 [3]. *When $r \ge 1$ the function $\theta(r)$ is integer and non-negative, non-decreasing, continuous from the right-hand side and tends to infinity together with r; the function $\lambda(r)$ decreases monotonously, it is continuous from the right-hand side and tends to zero when $r \to \infty$. The function $\lambda(r)$ experiences splits from the left-hand side only in the points of splitting of the function $\theta(r)$. Meanwhile for any $\xi \ge 0$ the following equality holds:*

$$\lambda(r) = \lambda(\xi)\, e^{\,-\int\limits_{\xi}^{r} \frac{\theta(t)}{t}\,dt \;-\sum\limits_{\xi<r_i\le x} |\,\sigma_i|}, \quad r \ge \xi.$$

Here $\sigma_0 = 0$ and $\sigma_i = \ln\lambda(r_i - 0) - \ln\lambda(r_i)$ for every $i > 0$.

Corollary. *The equality* $\lim_{r\to\infty} r^p\,\lambda(r) = 0$ holds for any $p \ge 0$.

It results from the inequality (11) that existence of information about the "great" reserve of smoothness of the function f obtains, a definite and tactile value in practice due to Theorem 1. The Chebyshev approximation process self-improves with the increase of n, i.e. it gets its practical effectiveness in the differential nature of f automatically, being fixed by the factual smoothness f for the maximum for the given n convergence order $\theta(n)$ (*phenomenon of unsaturation* [3]). Such a property of the method (9) intuitively means that the velocity of decreasing the approximation method to zero increases with the increasing "reserve" of smoothness of the function f. Potential opportunities of the development of the method depend only on the velocity of the increase of the function $\theta(n)$ in case of increase of the parameter n. The method becomes of the maximum practical effectiveness, i.e.*the exponential convergence* in the class of C^∞-smooth functions. For $f \in C^\infty[I]$ under the condition $\overline{\lim}_{k\to\infty} \sqrt[k]{G(k)}/k^{\,\alpha} < \infty$, where $\alpha > 0$, we have $|\,E_n(f)| \le C\,e^{-\varrho\,\sqrt[\alpha]{n}}$, here C, ϱ are positive constants.

In other words, the construction of the method (9) is initially such that it is able to contain, figuratively speaking, opportunities of the infinite set of numerical methods. Developing the opportunity of the "flexible" participating reserves of the smoothness of the function f in the estimate of the error, Theorem 1 also provides that the approximation method, whose error is estimated in terms of characteristics $E_n(f)$, is *unsaturated* [2,3]. In particular, such is one method (9).

The unsaturation property expands the sphere of practical application of the numerical method to the class of C^∞-smooth functions with the following property.

Definition [3]. *The function $f \in C^\infty[I]$ has a boundary layer of thickness ε_0, $0 < \varepsilon_0 < 1$ on the segment $I \equiv [-1, 1\,]$, if there exists such a low positive number $\tau = \tau(\varepsilon_0)$ and such a positive function $F(k)$ independent of ε_0 that with any integral $k \ge 0$ the following inequality holds:*

$$|f^{\,(k)}(t)| \le \begin{cases} F(k), & \text{if} \quad t \in I_\tau \equiv [-1+\tau,\, 1-\tau], \\ {\varepsilon_0}^{-k} F(k), & \text{if} \quad t \in I \setminus I_\tau, \end{cases} \tag{12}$$

where $\varepsilon_0 \equiv \varepsilon_0(\tau) > 0$ is the thickness of the boundary layer, and $F(k)$ is the function of the parameter k.

Peculiarities of the behaviour of the function $f \in C^\infty[I]$ near the ends of the segment I reflect on the convergence to zero with the increase of n of its Chebyshev characteristics $E_n(f)$. It follows from the following statement.

Theorem (V.K. Dzyadyk [14]). *For the k-th derivative of a function f to satisfy the Hölder condition with the index $0 < \alpha < 1$ on the segment $I \equiv [-1, 1\,]$ it is necessary and sufficient to*

have with any integral $n \geq k$ such an algebraic polynomial $Q_n \in \mathcal{P}^n$ of the degree $n-1$, that for all $t \in I$ the following estimates hold:

$$|\, f(t) - Q_n(t)| \leq \frac{A_k}{n^{\,k+\alpha}} \left(\sqrt{1-t^2} + \frac{1}{n} \right)^{k+\alpha},$$

where A_k is a constant independent of t and n.

The presented result strengthens the Jackson inequality (10), since it establishes an opportunity of approximation near the ends of I with the error $O(n^{\,-2(\,k+\alpha)}\,)$ in case of the uniform estimate $E_n(f) \leq M n^{\,-(\,k+\alpha)}$. Let us formulate the last observation strictly connecting it with the definition of the boundary layer.

Theorem 2 [3]. *If the function $f \in C^{\,\infty}[I]$ and the condition* (12) *holds, then*

$$E_n(f) \leq \frac{\pi}{2} \min_{0 \leq k \leq n} \frac{{\varepsilon_0}^{\,-k/2}\, F_k}{n^{\,k}}. \tag{13}$$

Here the coefficients F_k are effectively calculated according to the values $F(k)$ from (12)(see [3]).

The applied meaning of estimate (13) is that due to redistribution of the boundary layer along all the segment I, it is possible to increase its thickness to the value $\sqrt{\varepsilon_0}$. Thus the smoothness of the function $f \in C^{\,k}[I]$ is characterized by the value $\varepsilon_0^{\,-k/2}\, F_k$, whose increase is compensated according to Theorem 1 and the choice of the parameter n.

Indeed, in case of the fixed n among the inequalities (13), satisfying different $0 \leq k \leq n$, we have the best: its number $k_0 = \theta(n)$ is an order of that (maximal) derivative, which participates in constructing estimate (13). Derivatives of higher orders $k > k_0$ can influence the value of the estimate (13) only in the case $n > n_{\min}$ starting with some threshold value $n_{\min} = n_{\min}(\varepsilon_0)$. Therefore, the procedure of the neutralization of the boundary layer is carried out due to Theorem 1, according to the choice of the parameter $n > n_{\min}$: $E_n(f) \leq \pi/2 \, \min_{0 \leq k \leq n} {\varepsilon_0}^{\,-k/2} F_k / n^{\,k} \leq \epsilon, \;\; 0 < \epsilon \leq \varepsilon_0$.

Certainly, in a real situation there is not always any opportunity to find $k_0 = \theta(n)$ in the explicit form. Fortunately, the precise value of k_0 is not of great importance, and there is a reason for it. With the increase of $n > n_{\min}$ the choice of the parameter $\theta(n)$ is carried out automatically, due to the definite situation. In this connection for the neutralization of the boundary layer $f \in C^{\,\infty}[I]$ is not necessary: it is possible to do without the finite $\theta(n)$ reserve of derivatives. Their number is estimated as $\theta(n) = O(\varepsilon_0^{\,-1/2})$ when $n \to \infty$.

4. Quadrature formula without saturation.

Taking into account that

$$\widetilde{\Gamma}_C(t) = P_n(t;\,\widetilde{\Gamma}_C) + \delta_n(t;\Gamma_C), \quad \widetilde{\Gamma}_L(t) = P_n(t;\,\widetilde{\Gamma}_L) + \delta_n(t;\Gamma_L)$$

and substituting them in (5), we obtain the equality

$$\Gamma(s) = \varepsilon^{-1} \sum_{i=1}^{n} \frac{\widetilde{\Gamma}_C(t_i)}{T_n'(t_i)} \int_{-1}^{1} \frac{T_n(t)}{t-t_i}\, d\,t - \varepsilon^{-1} \sum_{i=1}^{n} \frac{\widetilde{\Gamma}_L(t_i)}{T_n'(t_i)} \int_{-1}^{1} \frac{T_n(t)}{t-t_i} \ln|\,t|\, d\,t + \varrho_n(s,\Gamma), \tag{14}$$

where $\varrho_n(s,\Gamma)$ is the functional of the error. Special integrals in (14) are calculated explicitly and we construct the following quadrature formulae on their basis in [15]:

$$\int_{-1}^{+1} f(t)\, dt = \sum_{i=1}^{n} c_i\, f(t_i) + \wp_n^c(f), \quad -\int_{-1}^{+1} f(t)\, \ln|\,t\,|\, dt = \sum_{i=1}^{n} l_i\, f(t_i) + \wp_n^l(f). \tag{15}$$

Here $\wp_n^c$ and $\wp_n^l$ are functionals of the error, and the weights c_k and l_k are set by the formulae

$$c_i = 2\,\frac{C(n,t_i)}{T_n'(t_i)}, \quad l_i = -\,2\,\frac{L(n,t_i)}{T_n'(t_i)}, \quad T_n'(t_i) = \frac{(-1)^{i-1}\,n}{\sqrt{(1-t_i)(1+t_i)}},$$

which are realized with the help of the relationships

$$\left\{\begin{array}{c} C(n,t) \\ L(n,t) \end{array}\right\} = \sum_{m=1}^{[\frac{n+1}{2}]} \left\{\begin{array}{c} C_m \\ -\,L_m \end{array}\right\} T_{n+1-2m}(t) - \frac{1-(-1)^n}{4} \left\{\begin{array}{c} C_{[\frac{n+1}{2}]} \\ -\,L_{[\frac{n+1}{2}]} \end{array}\right\} \quad (\,n>0).$$

Thus we apply coefficients which are preliminary calculated: $C_1 = 2$, $L_1 = 2$,

$$C_m = 1/(m-0.5), \;\; L_m = -(m-1.5)\,C_m\; L_{m-1} - C_m^2\,/(m-1.5)/\,2, \;\; 2 \le m \le \Big[\frac{n+1}{2}\Big].$$

The construction of the formulae (15) is simple though the motivation of its practical application is not obvious. It would be untimely to express a stipulated meaning of its effectiveness without considering the influence of rounding errors, a defining component of modern computer calculations. Therefore the following becomes of a special importance

Theorem 3 [15]. *If $n \ge 1$ and is odd, then the weights c_i and l_i in the formulae* (15) *are strictly positive and their functinals of the errors $\wp_n^c$, $\wp_n^l$ satisfy the inequalities*

$$|\,\wp_n^c(f)\,| \le 4E_n(f), \quad |\,\wp_n^l(f)\,| \le 4E_n(f). \tag{16}$$

It results from (16), that the quadrature formulae (15) are unsaturated; positiveness of the coefficients c_i, l_i provides their resistance to the rounding errors.

The specified properties of the quadrature effectively solve the problem of numerical neutralization of the boundary layer in (5). The formal side of the solution of this problem can be described by the presented above scheme due to the estimates (16). Therefore all the conclusions made before with respect to the properties right-hand side of the inequality (13) also hold true for the subintegral functions from (5). For these functions, the definition (12) holds in advance if we assume that

$$\varepsilon_0 \le \varepsilon(s) = 0.5\,\pi|\sin\pi\,s|\,, \quad F(k) = \varepsilon^{-k}(s) \max_{(\sigma,\,s)\,\in K\times K} \left|\left(\sin^2\frac{\pi(\sigma+s)}{2}\frac{d}{d\sigma}\right)^k f(\sigma,s)\right|,$$

and for the function $f(\sigma,s)$ we take either $\Gamma_C\,(\sigma,s)$, or $\Gamma_L\,(\sigma,s)$ due to the transformation (4).

As a result the quadrature formulae (15) have a real opportunity to hold the inequalities $|\,\wp_n^c\,(\widetilde{\Gamma}_C)| \le \epsilon,\; |\,\wp_n^l\,(\widetilde{\Gamma}_L)| \le \epsilon,\; 0 < \epsilon \le \varepsilon_0$, which is carried out according to Theorem 1 due to the choice of the parameter $n > n_{\min}(\varepsilon_0)$ in the estimates (16). This results in neutralization

of the boundary layer in (5) due to the required number $k_0 = \theta(n)$ of the derivatives of the functions $\widetilde{\Gamma}_C(t)$, $\widetilde{\Gamma}_D(t) \in C^{2p+1}[I]$. The choice of the parameter p is subject to neutralization of the boundary layer, i.e. to the inequality $p \geq [\frac{\theta(n)-1}{2}]$.

In the precise algorithms of the numerical solution of the axisymmetric boundary-value problems we should use "flexible" methods of calculation of complete elliptical integrals (for example, the ones specified in [10]), since they most adequately correspond to the nature of the considered problems and successfully complete the construction of the unsaturated algorithm of numerical approximation of the Gauss integral (1).

5. The algorithm of calculation. Numerical parameters

Computer realization of the Gauss integral (1) is carried out according to following formulae:

$$\pi \equiv \Gamma(s_j),\ \Gamma(s) = \varepsilon^{-1}\ \left(\sum_{i=1}^{n} c_i\,\widetilde{\Gamma}_C(t_i) + \sum_{i=1}^{n} l_i\,\widetilde{\Gamma}_L(t_i)\right) + \varrho_n(s,\Gamma),\ \varepsilon = 0.5\,\pi\,\sin\,\pi s. \qquad (17)$$

The functions $\widetilde{\Gamma}_C(t), \widetilde{\Gamma}_L(t)$ are set by the formulae (6), and the weights c_i, l_i are given by the formulae (15). The values $\sigma_i \equiv \sigma_S(t_i, s)$ $(1 \leq i \leq n)$ are recovered due to the fixed value of $s \in (\,0,1)$ and a set ordered array of the numbers $\{t_i = \cos\,\frac{\pi\,(2\,i-1)}{2\,n}\}$:

$$t_i = \sin\frac{(\sigma_i - s)}{2} / \sin\frac{(\sigma_i + s)}{2}, \quad 1 \leq i \leq n.$$

Recovering of σ_i is carried out by the Newton method by means of the following algorithm:

$$\overset{k+1}{y} = \overset{k}{y} - f(\overset{k}{y})/f'(\overset{k}{y}), \quad k = 0,\,1,\,2,\ldots is the \text{iteration number},$$

$$f(x) = x - s - \frac{2}{\pi}\arcsin\left(t_i\,\sin\frac{\pi(x+s)}{2}\right),\ \ f'(x) = 1 - \frac{t_i\,\cos\frac{\pi(x+s)}{2}}{\sqrt{\left(1 - t_i^2\sin^2\frac{\pi(x+s)}{2}\right)}}.$$

The problem of the choice of the initial approximation $\overset{0}{y} = \overset{0}{\sigma}_i$ $(1 \leq i \leq n)$ is very important in the Newton method. In the considered case it is solved very simply:

$$\overset{0}{\sigma}_1 = 1 - \pi^{-1}\left(1 - t_1\,\cot\,\frac{\pi\,s}{2}\right), \quad \overset{0}{\sigma}_{i+1} = \sigma_i + 2\,\frac{(t_{i+1} - t_i)}{\pi\sin\pi s}\,\sin^2\frac{\pi(\sigma_i + s)}{2},\ 1 \leq i \leq n-1.$$

The specified iteration process has the quadratic convergence: already $2-3$ iterations ensure not less than 10 precise decimal ranks for $s \in (\,0.01, 0.99\,)$. It is easy to specify the fast iteration process for $s \in (\,0.001, 0.999\,)$: it is necessary only to somehow modify the choice of the values $\overset{0}{y} = \overset{0}{\sigma}_i$ $(1 \leq i \leq n)$.

Let us now give test examples illustrating the property of unsaturation of the constructed numerical method. Unfortunately, the size of the paper does not allow us to illustrate out the extensive numerical data, however, we hope that it is still sufficient to give the definite meaning to the statement on effectiveness of the constructed unsaturated numerical method.

The admitted domains which are usually applied to verify methods of the numerical solution of the elliptical boundary-value problems are ellipsoids of rotation. These domains are convenient because the computational difficulties of the problems are characterised in them by one numerical parameter $\varkappa$ which denotes prolongation equivalent to the relationship of semiaxes. We can easily vary the range of the computational difficulties of the problem by varying the parameter $\varkappa$.

Let us keep to consideration of rotation ellipsoids with the meridian section:

$$\gamma(s) = \{a\sin\pi s,\ b\cos\pi s\}, \quad \varkappa = b/a, \quad s \in [\,0,1],$$

having given the preference to the ellipsoids extended along the axis of symmetry z ($b \geq a$), as the most "difficult" ones from the point of view of carrying out numerical calculations for the domains.

The integral $\Gamma(s)$ is calculated in the points $s_j = j/100$ ($1 \leq j \leq 100$) by the formulae (17).

In Tables 1-5 we specify the difference $\pi - \Gamma(s_j)$, which is an absolute error, in the first ten points near the pole γ_0: they are mostly subject to the influence of the boundary layer, though its presence appears in every considered point s_j ($1 \leq j \leq 100$), and the higher $\varkappa$, the more it appears. Thus the considered range of prolongations $1 \leq \varkappa \leq 100$ demonstrates real opportunities of the unsaturated numerical method.

The idea of the results presented in Tables 1–5, consists in the following: if subintegral functions in (5) are smoother when $p > p_0$ than when $p = p_0$, then under other equivalent conditions the method calculates $\Gamma(s_j)$ with the precision higher that in the case $p = p_0$. The case $p = 4$ corresponds to the known [16] method of approximation of complete elliptical integrals. Note that even a substitution of the parameter $p = 5$ by $p = 10$ in Table 5 increases precision up to 8 decimal positions. Other examples of application of the constructed method can be found, for example, in the paper [11].

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TABLE 1. **The Gauss integral Γ(s)**

$\mathbf{a = b = 1/\pi,\ s_j = j/100,\ n = 40}$

p	4	10	20	40
j	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$
1	0.00074996015	0.00085952336	0. 00094117975	0. 00096930719
2	0.00003620611	0.00005709499	0. 00005785123	0. 00005327380
3	0.00002540592	0.00000476493	0. 00000401157	0. 00000431754
4	0.00002168817	0.00000057541	0. 00000041157	0. 00000041578
5	0.00002993194	0.00000009031	0. 00000021862	0. 00000023572
6	0.00003338771	0.00000014374	0. 00000000286	0. 00000001958
7	0.00003562241	0.00000017582	0. 00000001728	0. 00000000286
8	0.00003693635	0.00000016574	0. 00000000166	0. 00000001053
9	0.00003752035	0.00000016399	0. 00000000158	0. 00000001187
10	0.00003751165	0.00000016414	0. 00000000009	0. 00000000887

TABLE 2. **The Gauss integral $\Gamma(s)$**

$$\mathbf{a = b = 1/\pi,\ s_j = j/100,\ n = 80}$$

p	4	10	40	80
j	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$
1	0.00000791729	0.00000030422	0.00000036784	00000034329
2	0.00001499472	0.00000008073	0.00000000447	00000000455
3	0.00002152174	0.00000011099	0.00000000197	00000000200
4	0.00002704800	0.00000013839	0.00000000216	00000000217
5	0.00003145025	0.00000015930	0.00000000203	00000000203
6	0.00003471950	0.00000017262	0.00000000045	00000000045
7	0.00003693750	0.00000018305	0.00000000148	00000000148
8	0.00003823076	0.00000018725	0.00000000105	00000000107
9	0.00003875586	0.00000018790	0.00000000094	00000000093
10	0.00003866926	0.00000018571	0.00000000101	00000000080

TABLE 3. **The Gauss integral $\Gamma(s)$**

$$\mathbf{a = b = 1/\pi,\ s_j = j/100,\ n = 100}$$

p	4	10	50	100
j	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$
1	0.00000775635	0.00000005011	00000000742	00000000831
2	0.00001508046	0.00000008207	00000000443	00000000443
3	0.00002164322	0.00000011590	00000000492	00000000490
4	0.00002718896	0.00000013937	00000000080	00000000079
5	0.00003160697	0.00000016184	00000000192	00000000190
6	0.00003488173	0.00000017458	00000000041	00000000041
7	0.00003710023	0.00000018590	00000000148	00000000147
8	0.00003838814	0.00000018960	00000000059	00000000061
9	0.00003890500	0.00000018984	00000000018	00000000015
10	0.00003880950	0.00000018815	00000000080	00000000080

TABLE 4. **The Gauss integral $\Gamma(s)$**

$\mathbf{a = b = 1/\pi,\ s_j = j/100,\ n = 100}$

p	5	25	55	95
j	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$	$\pi - \Gamma(s_j)$	
1	0.00000309766	0.00000000860	0.00000000742	0.00000000827
2	0.00000600681	0.00000000445	0.00000000443	0.00000000443
3	0.00000861302	0.00000000489	0.00000000491	0.00000000490
4	0.00001080007	0.00000000080	0.00000000082	0.00000000080
5	0.00001253228	0.00000000191	0.00000000192	0.00000000192
6	0.00001379995	0.00000000039	0.00000000040	0.00000000042
7	0.00001464497	0.00000000144	0.00000000148	0.00000000147
8	0.00001511599	0.00000000061	0.00000000061	0.00000000061
9	0.00001528089	0.00000000018	0.00000000017	0.00000000016
10	0.00001520526	0.00000000080	0.00000000081	0.00000000080

TABLE 5. **The Gauss integral Γ(s)**

$\mathbf{a = 1},\ \mathbf{b = 100},\ \mathbf{s_j = j/100},\ \mathbf{n = 2000},\ \mathbf{p = 5}$

j	$\pi - \Gamma(s_j)$	j	$\pi - \Gamma(s_j)$
1	0.00000019329	26	0.00000269393
2	0.00000123943	27	0.00000225247
3	0.00000186769	28	0.00000165033
4	0.00000199312	29	0.00000201233
5	0.00000196848	30	0.00000241428
6	0.00000207644	31	0.00000285311
7	0.00000201963	32	0.00000218227
8	0.00000238756	33	0.00000135977
9	0.00000224457	34	0.00000157173
10	0.00000206489	35	0.00000179447
11	0.00000189213	36	0.00000202522
12	0.00000234309	37	0.00000226069
13	0.00000216461	38	0.00000249735
14	0.00000224396	39	0.00000273245
15	0.00000215630	40	0.00000296148
16	0.00000191624	41	0.00000318214
17	0.00000214350	42	0.00000338972
18	0.00000234187	43	0.00000358196
19	0.00000171960	44	0.00000111728
20	0.00000243399	45	0.00000116935
21	0.00000236408	46	0.00000121328
22	0.00000217242	47	0.00000124865
23	0.00000287342	48	0.00000127462
24	0.00000165453	49	0.00000129018
25	0.00000213453	50	0.00000129555

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