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REDUCTIONS OF STATIONARY BOUNDARY LAYER EQUATION

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Abstract. The equation describing a steady laminar boundary layer with a pressure gradient is considered in the paper. All reductions to the ordinary differential equations are obtained. It has been shown that this equation has a reduction which can be obtained neither by classical nor by nonclassical symmetry methods.

Keywords: similarity reduction, symmetry operator, invariant solution, boundary layer.

1. Introduction

The most important problem for the given differential equation with partial derivatives with two independent variables is the problem of reductions of this equation, i.e. construction of such ansatzes (types of solutions), which finding is reduced to the solution of an ordinary differential equation (ODE). The reduction makes it possible to reduce the solution of the equation with partial derivatives to the solution of ODE. Reductions are widely used in applications. The mostly used are self-similar solutions [1]. Self-similar solutions have the form

$$u = x^{\alpha}\varphi(\zeta)\,, \qquad \zeta = y/x^{\beta}\,, \tag{1.1}$$

where u is a dependent variable; x, y are independent variables; α, β are constants. These solutions are obtained from one another by transformation of similarity. Self-similar solutions are a particular case of invariant solutions (or symmetry reductions), obtained with the help of symmetries [2]. Symmetry reductions are obtained by standard methods of group analysis. Similarly, the reductions of the type of progressive waves [3] are used. The solutions of the progressive waves type have the form

$$u = \varphi(\zeta) + u_0(y)\,, \qquad \zeta = x + V(y)\,.$$

These solutions are obtained, in case of a fixed value of the variable y, one from the other by means of a transformation of the shift. In the paper [4] reductions of the type

$$u = P(x) + A(x)\varphi(\zeta)\,, \qquad \zeta = y/B(x) + Q(x)\,, \tag{1.2}$$

are considered that generalize reductions of the type (1.1).

The paper [5] contains the method of finding reductions of equations in partial derivatives with two independent variables. In this paper there were all the reductions of the form

$$u = U(x, y, w(z))\,, \tag{1.3}$$

obtained for the Boussinesq's equations

$$u_{yy} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0$$

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where $z = z(x, y)$ and the function $w(z)$ is the solutions of an ODE. It was shown that there are reductions different from the reductions obtained by means of symmetries. The cited paper also contains all the reductions of the Boussinesq's equation

$$u_y + uu_x = u_{xx}\,,$$

of the Korteweg–de Vries equation

$$u_y + uu_x = u_{xxx}$$

and the modified Korteweg–de Vries equation

$$u_y + u^2 u_x = u_{xxx}\,.$$

It is shown for these equations that the obtained reductions coincide with symmetry reductions. It was also demonstrated that the reductions (1.3) have for these equations the form

$$u = \alpha(x, y) + \beta(x, y)w(z)\,. \tag{1.4}$$

In the present paper we consider the equation

$$u_{yyy} - u_y u_{xy} + u_x u_{yy} + P(x) = 0\,. \tag{1.5}$$

The equation (1.5) describes the movement of the viscous incompressible liquid in a laminar stationary flat boundary layer with a gradient of pressure [6]. The equation is written in dimensionless variables, u where is the function of the current, $P(x) = -\partial p/\partial x$ is a given function, p is the pressure. Self-similar solutions of the equation (1.5) were considered in the monographs [1, 6, 7]. Symmetry reductions of the equation (1.5) can be obtained on the basis of the results, presented in [2]. The paper [4] contains reductions of the type (1.2). In the paper [8] we obtained new reductions of the equation (1.5) of the form (1.4) based on the use of the method of non-classical symmetries [9] and consideration of its generalization.

In the present paper we obtain all the reductions of the equation (1.5) of the form (1.3). It is shown that the considered equation has reductions, which are not obtained by means of symmetries.

2. Reductions of a boundary layer equation

Substituting the expression of the form (1.3) into the equation (1.5) we obtain the following equation

$$\begin{aligned}
&U_w z_y^3 w''' + 3U_{ww} z_y^3 w' w'' + z_y(3U_{yw} z_y + U_x U_w z_y - U_y U_w z_x + 3U_w z_{yy})w'' + \\
&\quad + U_{www} z_y^3 (w')^3 + (3U_{yww} z_y^2 - U_w U_{xw} z_y^2 + U_w U_{yw} z_x z_y + U_x U_{ww} z_y^2 - \\
&\quad - U_y U_{ww} z_x z_y + U_w^2 z_x z_{yy} - U_w^2 z_y z_{xy} + 3U_{ww} z_y z_{yy})(w')^2 + (3U_{yyw} z_y - \\
&\quad - U_w U_{xy} z_y + U_w U_{yy} z_x - U_y U_{xw} z_y - U_y U_{yw} z_x + 2U_x U_{yw} z_y + U_x U_w z_{yy} - \\
&\quad - U_y U_w z_{xy} + 3U_{yw} z_{yy} + U_w z_{yyy})w' + U_{yyy} - U_y U_{xy} + U_x U_{yy} + P(x) = 0\,.
\end{aligned} \tag{2.1}$$

Let us divide both sides of the equation (2.1) by the coefficient of the higher derivative, i.e. by $U_w z_y^3$. The condition of the obtained equation to be an ODE is dependence of each coefficients of different powers of the derivatives of the function $w(z)$ only on the variables z and w. Let us consider the coefficient of the term, containing $w'w''$. It has the form

$$\frac{3U_{ww}}{U_w} = \Gamma_1(z, w)\,. \tag{2.2}$$

Integrating the equation (2.2) twice we obtain

$$U(x, y, w) = \beta(x, y)\Gamma(z, w) + \alpha(x, y).$$

If we take an arbitrary function of w and z as the function $w(z)$ then the reduction of the equation (1.5) can be looked for in the form (1.4).

In the process of deducing the equation (2.2) it was supposed that $z_y \neq 0$. The case $z_y = 0$ corresponds to the degenerated reduction. This case is not of special interest and its detailed consideration is not presented.

Substituting (1.4) into the equation (1.5) we obtain the relationship

$$\begin{aligned}
&\beta z_y^3 w''' + \beta z_y(\beta_x z_y - \beta_y z_x) w w'' + z_y(3\beta z_{yy} + 3\beta_y z_y + \alpha_x \beta z_y - \alpha_y \beta z_x) w'' + \\
&\quad + \beta(\beta z_x z_{yy} - \beta z_y z_{xy} + \beta_y z_x z_y - z_y^2 \beta_x)(w')^2 + \\
&\quad + (\beta_x \beta z_{yy} + \beta z_x \beta_{yy} - \beta_y^2 z_x + \beta_y \beta_x z_y - \beta_y \beta z_{xy} - \beta z_y \beta_{xy}) w w' + \\
&\quad + (\beta z_{yyy} - \alpha_y \beta_x z_y - \alpha_y \beta_y z_x - \alpha_y \beta z_{xy} - \beta z_y \alpha_{xy} + 2\alpha_x \beta_y z_y + \\
&\qquad\qquad + \alpha_x \beta z_{yy} + \beta z_x \alpha_{yy} + 3\beta_y z_{yy} + 3\beta_{yy} z_y) w' + \\
&\quad + (\beta_x \beta_{yy} - \beta_y \beta_{xy}) w^2 + (\beta_{yyy} + \alpha_x \beta_{yy} - \beta_y \alpha_{xy} - \alpha_y \beta_{xy} + \beta_x \alpha_{yy}) w + \\
&\quad + \alpha_{yyy} - \alpha_y \alpha_{xy} + \alpha_x \alpha_{yy} + P(x) = 0\,.
\end{aligned} \tag{2.3}$$

The condition that the equation (2.3) is an ODE, means that the coefficients depending on the functions $\alpha(x,y)$, $\beta(x,y)$, $z(x,y)$ and their derivatives should be the functions of the variable z. Hence we obtain the following overdetermined system of equations:

$$\begin{aligned}
&\frac{\beta_x z_y - \beta_y z_x}{z_y^2} = \Gamma_1(z)\,, \\
&\frac{3\beta z_{yy} + 3\beta_y z_y + \alpha_x \beta z_y - \alpha_y \beta z_x}{\beta z_y^2} = \Gamma_2(z)\,, \\
&\frac{\beta(\beta z_x z_{yy} - \beta z_y z_{xy} + \beta_y z_x z_y - z_y^2 \beta_x)}{\beta z_y^3} = \Gamma_3(z)\,, \\
&\frac{\beta_x \beta z_{yy} + \beta z_x \beta_{yy} - \beta_y^2 z_x + \beta_y \beta_x z_y - \beta_y \beta z_{xy} - \beta z_y \beta_{xy}}{\beta z_y^3} = \Gamma_4(z)\,, \\
&\frac{\beta z_{yyy} - \alpha_y \beta_x z_y - \alpha_y \beta_y z_x - \alpha_y \beta z_{xy} - \beta z_y \alpha_{xy}}{\beta z_y^3} + \\
&\qquad + \frac{2\alpha_x \beta_y z_y + \alpha_x \beta z_{yy} + \beta z_x \alpha_{yy} + 3\beta_y z_{yy} + 3\beta_{yy} z_y}{\beta z_y^3} = \Gamma_5(z)\,, \\
&\frac{\beta_x \beta_{yy} - \beta_y \beta_{xy}}{\beta z_y^3} = G_6(z)\,, \\
&\frac{\beta_{yyy} + \alpha_x \beta_{yy} - \beta_y \alpha_{xy} - \alpha_y \beta_{xy} + \beta_x \alpha_{yy}}{\beta z_y^3} = \Gamma_7(z)\,, \\
&\frac{\alpha_{yyy} - \alpha_y \alpha_{xy} + \alpha_x \alpha_{yy} + P(x)}{\beta z_y^3} = \Gamma_8(z)\,.
\end{aligned} \tag{2.4}$$

Let us formulate the basic principles of the suggested method of construction of reductions:
1. Each equation from (2.4) is equivalent to the condition of vanishing the Jacobian of the left side of this equation and the function $z(x,y)$. As a result we can obtain the overdetermined system of equations ($\mathcal{A}$-system) for determining the functions $\alpha(x,y)$, $\beta(x,y)$, $z(x,y)$ (we do not present it because of its inconvenience).

2. We introduce auxiliary functions $\mu_1(x,y)$, $\mu_2(x,y)$, $\mu_3(x,y)$ determined from the equations

$$\begin{aligned} &z_x - \mu_1(x,y) z_y = 0\,, \\ &\beta_x - \mu_1(x,y)\beta_y - \mu_2(x,y)\beta = 0\,, \\ &\alpha_x - \mu_1(x,y)\alpha_y - \mu_2(x,y)\alpha - \mu_3(x,y) = 0\,. \end{aligned} \tag{2.5}$$

The introduction of auxiliary functions is crucial for the suggested method. As it was noted in the paper [5], the reductions of the form (1.4) admit the following transformations, mapping the reduction into the reduction

$$\begin{aligned} &z \to F_1(z)\,, \\ &\beta \to \frac{\beta}{F_2(z)}\,, \\ &\alpha \to \alpha + \frac{\beta}{F_3(z)}\,, \end{aligned} \tag{2.6}$$

where $F_1(z)$, $F_2(z)$, $F_3(z)$ are arbitrary functions. These transformations are connected with the arbitrary way of finding ODE for the function $w(z)$. It can be shown, that the introduced auxiliary functions are invariants of the transformations (2.6). We can also show that $\mathcal{A}$-system admits transformations (2.6).

3. Finding from the relationships (2.5) derivatives α_x, β_x, z_x, ... and substituting them into the $\mathcal{A}$-system, we obtain the following overdetermined system for the auxiliary functions:

$$\begin{aligned} &\mu_{1y}\mu_2 + \mu_{1xy} - \mu_{1y}^2 - \mu_1\mu_{1yy} = 0\,, \\ &\mu_2^2 + \mu_{2x} - \mu_2\mu_{1y} - \mu_1\mu_{2y} = 0\,, \\ &\mu_2\mu_3 + 3\mu_{1yy} + 3\mu_{2y} + \mu_{3x} - \mu_1\mu_{3y} - \mu_{1y}\mu_3 = 0\,, \\ &\mu_{2y}(2\mu_{1y} - \mu_2) = 0\,, \\ &\mu_2\mu_{2yy} = \mu_{2y}^2\,, \\ &4\mu_{1yyy} + 6\mu_{2yy} - \mu_2\mu_{3y} + 2\mu_{2y}\mu_3 - 2\mu_{1y}\mu_{3y} = 0\,, \\ &\mu_{2yyy} + 2\mu_{2y}\mu_{3y} - \mu_{2yy}\mu_3 - \mu_2\mu_{3yy} = 0\,, \\ &\mu_{3yyy} - \mu_{3y}^2 + \mu_3\mu_{3yy} - 3P(x)\mu_{1y} - P(x)\mu_2 + P'(x) = 0\,. \end{aligned} \tag{2.7}$$

Therefore, $\mathcal{A}$-system is reduced to a more simple system of equations (2.7). The writing of the $\mathcal{A}$-system only via invariants of the transformations (2.6) results from its invariance with respect to these transformations.

4. By the obtained auxiliary functions we can find the functions $\alpha(x,y)$, $\beta(x,y)$, $z(x,y)$ from the corresponding linear equations (2.5). Below we construct the demanded reductions with the precision to the transformations (2.6) by the functions $\alpha(x,y)$, $\beta(x,y)$, $z(x,y)$.

As it is seen from the forth equation of the system (2.7), it is necessary for its solution to consider the following two cases: $\mu_{2y} = 0$ and $2\mu_{1y} - \mu_2 = 0$. Hence in the first of them it is needed to consider the cases $\mu_2 \neq 0$ and $\mu_2 = 0$.

Case I. $\boldsymbol{\mu_2 = \mu_2(x) \neq 0}$. In this case integration of the initial system is carried out in the elementary way and results in several possible cases.

Case I.1. $\boldsymbol{P(x) = \lambda(x+\kappa)^{\delta}}$**.** In this case the solution of the system (2.7) has the form

$$\mu_1 = y\frac{(\delta-1)}{4(x+\kappa)} + f(x)\,,$$
$$\mu_2 = \frac{(\delta+3)}{4(x+\kappa)}\,,$$
$$\mu_3 = \frac{c_3(\delta+3)}{4(x+\kappa)}\,, \qquad c_3 = const\,.$$

Here and below $f(x)$ is an arbitrary function of the variable x. Then we can assume that

$$\alpha(x,y) = const\,, \qquad \beta(x,y) = (x+\kappa)^{\frac{\delta+3}{4}}\,,$$
$$z(x,y) = y(x+\kappa)^{\frac{\delta-1}{4}} + \int f(x)(x+\kappa)^{\frac{\delta-1}{4}}dx\,.$$

Hence we obtain the following ODE:

$$4w''' + (\delta+3)ww'' - 2(\delta+1)w'^2 + 4\lambda = 0\,.$$

Case I.2. $\boldsymbol{P(x) = \lambda\exp(\delta x)}$**.** In this case the solution of the system (2.7) has the form

$$\mu_1 = \frac{\delta y}{4} + f(x), \qquad \mu_2 = \frac{\delta}{4}, \qquad \mu_3 = c_3\,.$$

Then we can assume that

$$\alpha(x,y) = const\,, \qquad \beta(x,y) = \exp\left(\frac{\delta x}{4}\right)\,,$$
$$z(x,y) = y\exp\left(\frac{\delta x}{4}\right) + \int f(x)\exp\left(\frac{\delta x}{4}\right)dx\,.$$

The corresponding ODE has the form

$$4w''' + \delta ww'' - 2\delta w'^2 + 4\lambda = 0.$$

Case I.3. $\boldsymbol{P(x) = \lambda = const}$**.**
Case I.3.1. $\boldsymbol{\lambda \neq 0}$**.** In this case the solution of the system (2.7) has the form

$$\mu_1 = -y\frac{3}{4x+\kappa} + f(x)\,, \qquad \mu_2 = \frac{3}{4x+\kappa}\,, \qquad \mu_3 = \frac{c_3}{4x+\kappa}\,.$$

Then we can assume that

$$\alpha(x,y) = const\,, \qquad \beta(x,y) = (4x+\kappa)^{\frac{3}{4}}\,,$$
$$z(x,y) = \frac{y}{(4x+\kappa)^{\frac{1}{4}}} + \int \frac{f(x)dx}{(4x+\kappa)}\,.$$

The corresponding ODE has the form

$$w''' + 3ww'' - 2w'^2 + \lambda = 0.$$

Case I.3.2. $\boldsymbol{\lambda = 0}$**.** In this case the solution of the system (2.7) has the form

$$\mu_1 = y\frac{1-c_1}{c_1x+c_2} + f(x)\,, \qquad \mu_2 = \frac{1}{c_1x+c_2}\,, \qquad \mu_3 = \frac{c_3}{c_1x+c_2}\,.$$

Then we can assume that

$$\alpha(x,y) = const\,, \qquad \beta(x,y) = (c_1x+c_2)^{1/c_1}\,,$$
$$z(x,y) = y(c_1x+c_2)^{1/c_1-1} + \int f(x)(c_1x+c_2)^{1-1/c_1}dx\,.$$

The corresponding ODE has the form

$$w''' + ww'' - 2w'^2 + (c_1 - 2)w^2 = 0\,.$$

Case I.4. $\boldsymbol{P(x) = -\lambda_1^2/(3x+\kappa)^{5/3} + \lambda_2/(3x+\kappa)^{1/3}}$**.** In this case the solution of the system (2.7) has the form

$$\mu_1 = -\frac{y}{3x+\kappa} + f(x)\,, \qquad \mu_2 = \frac{2}{3x+\kappa}\,,$$
$$\mu_3 = \frac{\lambda_1 y}{(3x+\kappa)^{4/3}} + \frac{c}{3x+\kappa} + \frac{\lambda_1 \int \frac{f(x)dx}{(3x+\kappa)^{1/3}}}{(3x+\kappa)}\,.$$

In this case we can assume that

$$\alpha(x,y) = \frac{\lambda_1 y}{(3x+\kappa)^{1/3}} - \frac{c}{2} - \lambda_1 \int \frac{f(x)dx}{(3x+\kappa)^{1/3}}\,,$$
$$\beta(x,y) = (3x+\kappa)^{2/3}\,,$$
$$z(x,y) = \frac{y}{(3x+\kappa)^{1/3}} + \int \frac{f(x)dx}{(3x+\kappa)^{1/3}}\,.$$

The corresponding ODE has the form

$$w''' + 2ww'' - w'^2 + \lambda_2 = 0\,.$$

Case II. $\boldsymbol{\mu_2 = 0}$**.** In this case the system (2.7) is simplified and takes the form

$$\begin{aligned}
&\mu_{1xy} - \mu_{1y}^2 - \mu_1\mu_{1yy} = 0\,,\\
&3\mu_{1yy} + \mu_{3x} - \mu_3\mu_{1y} - \mu_{3y}\mu_1 = 0\,,\\
&2\mu_{1yyy} = \mu_{3y}\mu_{1y}\,,\\
&\frac{dP(x)}{dx} - \mu_{3yyy} + \mu_{3y}^2 - \mu_3\mu_{3yy} - 3P(x)\mu_{1y} - P(x)\mu_2 = 0\,.
\end{aligned} \tag{2.8}$$

Having integrated the first equation of the system (2.8) in y we obtain

$$\mu_{1x} - \mu_1\mu_{1y} = f'(x)\,, \tag{2.9}$$

where $f'(x)$ is an arbitrary function. We can write the general solution of the equation (2.9) in the form

$$F(I_1, I_2) = 0,$$

where F is an arbitrary function, $I_1 = y + x(\mu_1 - f(x)) + \int f(x)dx$, $I_2 = \mu_1 - f(x)$. Therefore, one of the first integrals is a function of the other, i.e. $I_1 = G(I_2)$ or $I_2 = G(I_1)$. Let us dwell on the latter case. We have

$$\mu_1 - f(x) = G(y + x(\mu_1 - f(x)) + \int f(x)dx)\,. \tag{2.10}$$

Then, as it is easy to see that the general solution of the second equation of the system (2.8) is

$$\mu_3 = -3\frac{\mu_{1yy}}{\mu_{1y}} + \mu_{1y}H(\mu_1 - f(x)), \tag{2.11}$$

where H is an arbitrary function of its argument. Substituting (2.10), (2.11) into the last two equations of the system (2.8) results in the relationships in which the functions G and H and their derivatives are included polynomially, where the coefficients of monomials are powers of the variable x. Since $H = H(\mu_1 - f(x))$, then H also depends on G. Therefore, the functions G, H depend only on one complex argument and we should carry out splitting by the variable x. This results an overdetermined system of ODE for the functions G and H. Further calculations are carried out in the elementary way and provide as a result the following cases.

Case II.1. $\boldsymbol{P(x) = \lambda(x+\kappa)^{-3}}$**.** In this case

$$\mu_1 = -\frac{y}{x+\kappa} + f(x)\,, \qquad \mu_2 = 0\,, \qquad \mu_3 = \frac{c}{x+\kappa}\,.$$

Then we can assume that

$$\alpha(x,y) = c\ln(x+\kappa)\,, \qquad \beta(x,y) = 1\,, \qquad z(x,y) = \frac{y}{x+\kappa} + \int \frac{f(x)dx}{x+\kappa}\,.$$

The corresponding ODE has the form

$$w''' + cw'' + w'^2 + \lambda = 0\,.$$

Case II.2. $\boldsymbol{P(x) = \lambda x + \kappa}$**.** In this case

$$\mu_1 = f(x)\,, \qquad \mu_2 = 0,$$

and μ_3 satisfies the equation

$$\mu_{3yyy} + \mu_3\mu_{3yy} - \mu_{3y}^2 - \frac{dP(x)}{dx} = 0.$$

Then the function $\mu_3 = H(y + \int f(x)dx)$ should be the solution of the equation

$$H''' + HH'' - H'^2 - \lambda = 0\,.$$

Then we can assume that

$$\alpha(x,y) = xH(y + \int f(x)dx)\,, \qquad \beta(x,y) = 1\,, \qquad z(x,y) = y + \int f(x)dx\,.$$

The corresponding ODE has the form

$$w''' + Hw'' - H'w' + \kappa = 0\,.$$

Let us note that in the case $\mu_2 - 2\mu_{1y} = 0$, as it is easy to verify, we obtain the considered above reductions. Therefore, the consideration of this case is not presented.

3. Reductions obtained by means of symmetries

Let us find symmetries of the equations (1.5). The operator of the symmetry is searched in the form

$$X = \xi^1(x,y,u)\frac{\partial}{\partial x} + \xi^2(x,y,u)\frac{\partial}{\partial y} + \eta(x,y,u)\frac{\partial}{\partial u}\,.$$

The system of the defining equations for the coefficients of the symmetry operator lead to the following equations:

$$\begin{aligned}
&\xi^1_y = 0, \qquad \xi^1_u = 0, \qquad \xi^2_u = 0, \qquad \xi^2_{xy} = 0,\\
&\eta_x = 0, \qquad \eta_y = 0, \qquad \xi^1_x - \xi^2_y - \eta_u = 0,\\
&\xi^1 P'(x) + (3\xi^2_y - \eta_u)P(x) = 0.
\end{aligned}$$

Symmetries of the equation (1.5) depend on the form of the given function $P(x)$. The result of the group classification is as follows.

Proposition 1. *In case of an arbitrary function $P(x)$ the basis of the symmetry operators of the equation* (1.5) *is*

$$X_1 = b(x)\frac{\partial}{\partial y}\,, \qquad X_2 = \frac{\partial}{\partial u}\,,$$

where $b(x)$ is an arbitrary function.
When $P(x) = \lambda(x+\kappa)^\delta$, $\lambda \neq 0$, $\delta \neq 0$, the basis of the symmetry operators consists of the operators

$$X_1\,, \quad X_2\,, \quad Y_1 = 4(x+\kappa)\frac{\partial}{\partial x} + (1-\delta)y\frac{\partial}{\partial y} + (\delta+3)u\frac{\partial}{\partial u}.$$

When $P(x) = \lambda e^{\delta x}$, $\lambda \neq 0$, $\delta \neq 0$, the basis of the symmetry operators consists of the operators

$$X_1\,, \quad X_2\,, \quad Y_2 = 4\frac{\partial}{\partial x} - \delta y\frac{\partial}{\partial y} + \delta u\frac{\partial}{\partial u}.$$

When $P(x) = a$, $a \neq 0$ the basis of the symmetry operators consists of the operators

$$X_1\,, \quad X_2\,, \quad Z_1 = \frac{\partial}{\partial x}\,, \quad Z_2 = 4x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 3u\frac{\partial}{\partial u}.$$

When $P(x) = 0$ the basis of the symmetry operators consists of the operators

$$X_1\,, \quad X_2\,, \quad Z_1\,, \quad Z_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,, \quad Z_4 = x\frac{\partial}{\partial a} + u\frac{\partial}{\partial u}\,.$$

The results of the group classification of the equation (1.5) correspond to the results of the group classification of the system of equations describing a laminar stationary flat boundary layer with the gradient of pressure [2].

In all the stated cases of the extension of the symmetries it is easy to determine the forms of functions $\mu_1(x,y)$, $\mu_2(x,y)$, $\mu_3(x,y)$ which correspond to invariant solutions. Let us formulate the corresponding formulae.

Case 1. $P(x) = \lambda(x+\kappa)^\delta$. The symmetry operator of the general form is

$$X = 4c_1(x+\kappa)\frac{\partial}{\partial x} + (c_1(1-\delta)y + b(x))\frac{\partial}{\partial y} + (c_1(\delta+3)u + c_2)\frac{\partial}{\partial u}\,.$$

The functions $\mu_1(x,y)$, $\mu_2(x,y)$, $\mu_3(x,y)$ corresponding to any invariant solution have the form

$$\mu_1 = \frac{y(\delta-1)}{4(x+\kappa)} - \frac{b(x)}{4c_1(x+\kappa)}\,,$$
$$\mu_2 = \frac{(\delta+3)}{4(x+\kappa)}\,,$$
$$\mu_3 = -\frac{c_2}{4c_1(x+\kappa)}\,.$$

By analogy we can also find the functions $\mu_1(x,y)$, $\mu_2(x,y)$, $\mu_3(x,y)$ in other cases.

Case 2. $P(x) = \lambda\exp(\delta x)$. The symmetry operator has the form

$$X = 4c_1\frac{\partial}{\partial x} + (b(x) - c_1\delta y)\frac{\partial}{\partial y} + (c_1\delta u + c_2)\frac{\partial}{\partial u}\,.$$

Therefore,

$$\mu_1 = \frac{\delta y}{4} - \frac{b(x)}{4c_1}\,, \qquad \mu_2 = \frac{\delta u}{4}\,, \qquad \mu_3 = -\frac{c_2}{4c_1}\,.$$

Case 3. $P(x) = const \neq 0$. The symmetry operator has the form

$$X = (4c_1x + c_2)\frac{\partial}{\partial x} + (c_1y + b(x))\frac{\partial}{\partial y} + (3c_1u + c_3)\frac{\partial}{\partial u}\,.$$

Therefore,

$$\mu_1 = -\frac{c_1y}{4c_1x + c_2} - \frac{b(x)}{4c_1x + c_2}\,,$$
$$\mu_2 = -\frac{3c_1}{4c_1x + c_2}\,,$$
$$\mu_3 = -\frac{c_3}{4c_1x + c_2}\,.$$

Case 4. $\boldsymbol{P(x) = 0}$**.** The symmetry operator has the form

$$X = ((c_1 + c_2)x + c_3)\frac{\partial}{\partial x} + (c_2 y + b(x))\frac{\partial}{\partial y} + (c_1 u + c_4)\frac{\partial}{\partial u}\,.$$

Therefore,

$$\mu_1 = -\frac{c_2 y}{(c_1 + c_2)x + c_3} - \frac{b(x)}{(c_1 + c_2)x + c_3}\,,$$
$$\mu_2 = -\frac{c_1}{(c_1 + c_2)x + c_3}\,,$$
$$\mu_3 = -\frac{c_4}{(c_1 + c_2)x + c_3}\,.$$

Comparing the enumerated expressions for the functions $\mu_1(x, y)$, $\mu_2(x, y)$, $\mu_3(x, y)$, $P(x)$ in case of invariant solutions and in case of reductions obtained by a direct method it can be concluded that the reductions of the cases I.4 and II.2 cannot be obtained with the help of the classical method of finding invariant solutions, but, probably, can be obtained as differential-invariant solutions or differential-invariant substitutions.

4. Reductions obtained with the help of non-classical symmetries

The so-called method of non-classical symmetries [9] generalizes the classical Lie method of finding reductions. In accordance with this method one searches symmetry operators of the system of equations consisting of the initial equation $E(u_x, u_y, u_{xx}, u_{xy}, ...) = 0$ and the supplementary condition (condition of invariance)

$$\xi^1(x, y, u)u_x + \xi^2(x, y, y)u_y - \eta(x, y, u) = 0\,.$$

In this case, as opposed to the classical Lie method, the system of defining equations for the components $\xi^1(x, y, u)$, $\xi^2(x, y, u)$, $\eta(x, y, u)$ of the symmetry operator

$$X = \xi^1(x, y, u)\frac{\partial}{\partial x} + \xi^2(x, y, y)\frac{\partial}{\partial y} + \eta(x, y, u)\frac{\partial}{\partial u}$$

is non-linear. In the present article the calculations of non-classical symmetries and corresponding solutions are not presented (all the necessary results can be found in the paper [8]). That comparison with the results of application of the non-classical method shows that all the obtained above reductions can be obtained with the help of the non-classical method. Therefore, the method of non-classical symmetries and the direct approach suggested in the present paper are equivalent and provide a similar result.

5. Conclusion

The general result of the paper is the suggested method of construction of reductions based on the idea of invariance. The key difference of the suggested in the paper approach from the Clarkson-Kruskal approach is the transformation from the non-linear inhomogeneous system of equations for the functions α, β, z to the homogeneous system of equations for the auxiliary functions μ_1, μ_2, μ_3. The obtained system of equations is more convenient for integrating than the system of equations for the functions α, β, z, containing indefinite functions $G_1(z)$, $\ldots$, $G_8(z)$. The suggested method due to its invariant nature is more simple in application than the Clarkson-Kruskal method. With the help of the suggested method we have found all the reductions of the equation (1.5) of the form (1.3). It was shown that the equation under consideration has reductions which are not obtained with the help of symmetries.

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