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CHARACTERISTIC LIE RINGS AND INTEGRABLE MODELS IN MATHEMATICAL PHYSICS

A.V. ZHIBER, R.D. MURTAZINA, I.T. HABIBULLIN, A.B. SHABAT

Abstract. The survey is devoted to a systematic exposition of the algebraic approach based on the concept of the characteristic vector field to the study of nonlinear integrable partial differential equations and their discrete analogues. A special attention is paid to Darboux integrable equations and to soliton equations. The problem of constructing generalized symmetries for the equations as well as of their particular and general solutions is discussed. In particular, it is shown that a hyperbolic partial differential equation is integrated by quadrature if and only if its characteristic Lie rings in both directions are of finite dimension. For the hyperbolic type equations integrable by the inverse scattering method, the characteristic rings are of minimal growth. We suggest the ways of applying the concept of characteristic Lie rings to the systems of hyperbolic differential equations with more than two characteristic directions, to evolution equations, and to ordinary differential equations.

Keywords: characteristic vector field, symmetry, Darboux integrability

CONTENTS

A.V. Zhiber, R.D. Murtazina, I.T. Habibullin, A.B. Shabat, Characteristic Lie rings and integrable models in mathematical physics.

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1. Intorduction

The basic ideas of studying of the problem on the integration of hyperbolic partial differential equations go back to classical works by Laplace, Liouville, Lie, Darboux, Goursat, Vessiot, et al. And the meaning of the integration as obtaining an explicit formula for the general solution was almost immediately supplanted by others less exacting definitions. For instance, the Darboux method of integration of a hyperbolic equation consists in finding the integrals in each characteristic direction and the consequent reducing it to two ordinary differential equations. It is clear that in a general case it is quite complicated to end up with explicit formulas expressing a simultaneous solution of these equations.

For finding the integrals (as well as for identification the integrability of a given equation) Darboux employed the Laplace cascade method. In later studies (see [49, 60, 61]), an algebraic approach using characteristic vector fields became the main tool of finding integrals (exactly in the framework of such approach the first lists of the equations possessing the integrals in both directions were likely obtained [49]). Another approach to the integrating of nonlinear equation is related with one-parametric transformation groups, i.e., with symmetries. The notion of symmetry introduced more than one hundred years ago in the works of S. Lie and E. Noether, serves as the base for the modern integrability theory. The discover of the inverse scattering method and appearance of the class of soliton equations gave a powerful incentive to the developing of the symmetry approach in the integrability theory. It became clear that the equations integrable by the method of inverse scattering problem possess an infinite hierarchy of generalized symmetries.

During last three decades in the framework of symmetry approach effective algorithms for solving classification problems were created and the complete lists for very important classes of nonlinear partial differential equations and their discrete analogues were made up ([2, 10, 11, 33, 34, 38–40, 46–48, 59]). In so doing, the greatest successes were related with the classification of the evolution equations. However, in certain cases like the classification of the integrable equations of dimension $1+2$ and higher, and also the classification of hyperbolic equations with two independent variables and their discrete analogues the symmetry approach is not so effective. In the last years new methods for classification of integrable equations appeared, like Painlevé test, the method of algebraic entropy [55], the 3D compatibility condition [45], etc. The monograph [24] devoted to a detailed description of some aspects of the theory of integration for partial differential equations is also of interest for the experts.

In the present paper we consider an alternative approach to the problem on the classification of integrable equations going back to the classical works of Goursat. Am important milestone in forming of this approach was the work [44], where the system of hyperbolic equations

$$u^i_{xy} = \exp(a_{i1}u^1 + a_{i2}u^2 + \ldots + a_{in}u^n), \quad i = 1, 2, \ldots, n, \tag{1.1}$$

was studied. In this work the notion of characteristic Lie algebra of vector fields was introduced and it was shown that the characteristic Lie algebra for the system (1.1) has a finite dimension if and only if the matrix $A = (a_{ij})$ is the Cartan matrix of a simple Lie algebra. Then in the work [30] for a system of hyperbolic equations of a more general form

$$u^i_{xy} = F^i(u^1, u^2, \ldots, u^n), \quad i = 1, 2, \ldots, n, \tag{1.2}$$

it was shown that the condition of integrability by quadrature is the finite dimension of its characteristic Lie algebra.

The characteristic algebra for the hyperbolic systems

$$u^i_x = c^i_{jk}u^j v^k + c^i_k u^k, \quad v^k_y = d^k_{jl}u^j v^l + d^k_j u^j, \quad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, n \tag{1.3}$$

were studied in the work [13]. In particular, there was given a complete description for the basis of the characteristic algebra for the equation $u_{xy} = \sin u$.

Below in the first, third, and fourth section we shall give the definition and detailed description of the notion of the characteristic Lie ring for hyperbolic partial differential equations (and the system of equations) and their discrete analogues. Here we just briefly dwell on main aspects of the content. For a scalar hyperbolic equation (both continuous and discrete) the characteristic Lie ring on each characteristic direction is generated by two operators; denote them by X_1 and X_2. We indicate by V_j the linear space over the field of locally analytic functions spanned on X_1, X_2, and all multiple commutators of the operators X_1 and X_2 of order less or equal to j, so that

$$V_0 = \{X_1, X_2\}, \quad V_1 = \{X_1, X_2, [X_1, X_2]\}, \quad \ldots.$$

We introduce the function $\Delta(k) = \dim V_{k+1} - \dim V_k$.

A deep connection between the properties of the characteristic Lie ring and the integrability property of an equation was realized in the work [18]. In this work it was found that the spaces of multiple commutators forming characteristic rings for such integrable equations like Sine-Gordon equations, Tzitzeica equation, etc. grow very slowly in the first steps, saying more precisely, $\Delta(1) = \Delta(2) = \Delta(3) = \Delta(4) = 1$. It was conjectured that such behavior of the function $\Delta(k)$ is intrinsic for all integrable equations. Later the idea was specified and justified by numerous examples of integrable continuous and discrete models (see [35, 53]). Then in the works [42, 51] it was formulated the following

Conjecture 1.1. *(algebraic test). Each integrable scalar (continuous or discrete) hyperbolic equation satisfies the condition that there exists a sequence of natural numbers $\{t_k\}_{k=1}^{\infty}$ for which $\Delta(t_k) \leqslant 1$.*

Definition 1.1. *The characteristic Lie ring for which there exists such sequence of natural numbers is called a ring of minimal growth.*

The property of minimal growth of a ring began to be considered as a classification criterion for the integrable equations. For special classes of equations a series of model classification problem was solved ([18,42,51]). These results convince that the property of minimal growth of Lie ring is as universal property of integrable equation as the existence of an infinite hierarchy of generalized symmetries.

The paper presents a survey of the authors' results devoted to applications of the algebraic method based on the notion of characteristic vector field to nonlinear integrable models.

The paper is organized as follows. In the second section we make the classification of scalar hyperbolic equations of special form with infinite-dimensional characteristic Lie ring of minimal growth. It is shown that the system of equations $u_x = f(u,v)$, $v_y = \varphi(u,v)$, for which first three D and $\overline{D}$-conditions of the existence of generalized symmetries are satisfied, possesses x- and y-characteristic rings of minimal growth. We describe the classes of the equations with finite-dimensional Lie ring. By using the generators of the characteristic Lie rings we construct generalized symmetries of Liouville, Sine-Gordon, Tzitzeica, and modified Sine-Gordon equations.

In the third section we provide a short review of the authors' results (see [13,44,56]) devoted to the classification of integrable hyperbolic systems of equations basing on the notions of characteristic Lie rings and algebras.

In the forth section we introduce characteristic rings for a differential-difference equation. We illustrate the application of characteristic vector fields in the classification problem of Liouville type equations. We provide classification results. We study in details the characteristic ring of an differential-difference analogue of Sine-Gordon equation. It is notable that in this case the ring has a minimal growth.

In the fifth section we consider fully discrete equations. We give a general definition of the integral, introduce the notion of characteristic Lie ring, and discuss possible ways of applying these notions in the problems of classification integrable discrete equations.

The sixth section is devoted to the discussion of open questions and perspectives of algebraic approach presented in the paper. For instance, we suggest the scheme for studying the characteristic ring of the system of hyperbolic equations with more than two characteristic directions. A typical example of such system is that of n-waves. We discuss briefly the possibility of extending the presented approach to other classes of nonlinear equations like of evolution type, ordinary differential equations (see [8]).

2. Scalar integrable equations

2.1. Definition of characteristic Lie ring. To study the integrability of the equations

$$u_{xy} = f(x,y,u,u_x,u_y) \tag{2.4}$$

we use an approach based on the notion of "characteristic ring".

On the space of locally analytic functions depending on a finite number of variables $x, y, \overline{u}_1, u, u_1, u_2, \ldots$, the operator of total differentiation w.r.t. y reads as

$$\overline{D} = \frac{\partial}{\partial y} + \overline{u}_2\frac{\partial}{\partial \overline{u}_1} + \overline{u}_1\frac{\partial}{\partial u} + f\frac{\partial}{\partial u_1} + D(f)\frac{\partial}{\partial u_2} + \ldots,$$

while the operator of total differentiation w.r.t. x is

$$D = \frac{\partial}{\partial x} + \overline{D}(f)\frac{\partial}{\partial \overline{u}_2} + f\frac{\partial}{\partial \overline{u}_1} + u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + \ldots,$$

where $u_1 = u_x, \overline{u}_1 = u_y, u_2 = u_{xx}, \overline{u}_2 = u_{yy}, \ldots$.

We represent

$$\overline{D} = \overline{u}_2 X_2 + X_1, \tag{2.5}$$

where

$$X_1 = \frac{\partial}{\partial y} + \overline{u}_1\frac{\partial}{\partial u} + f\frac{\partial}{\partial u_1} + D(f)\frac{\partial}{\partial u_2} + \ldots, \quad X_2 = \frac{\partial}{\partial \overline{u}_1}.$$

In accordance with (2.5), the characteristic equation

$$\overline{D}W(x, y, u, u_1, \ldots, u_m) = 0 \tag{2.6}$$

is equivalent to the system

$$X_1 W = 0, \quad X_2 W = 0. \tag{2.7}$$

We note that the solution to equation (2.6) is called a x-integral of equation (2.4).

In a natural way with equations (2.7) one associates a Lie ring generated by the vector fields X_1 and X_2. In a similar way, while considering the characteristic equation $D\overline{W}(x, y, u, \overline{u}_1, \ldots, \overline{u}_m) = 0$, one introduces the Lie ring generated by the elements Y_1 and Y_2.

Let L_n be the linear space of the commutators of generators of the length $n-1$, $n = 2, 3, \ldots$. For instance, L_2 is the linear span of the vector fields X_1, X_2, and L_3 is generated by the element X_3=$[X_1, X_2]$, L_4 is generated by the commutators X_4=$[X_2, X_3]$, X_5=$[X_1, X_3]$, etc. Then the x-characteristic Lie ring A can be represented as

$$A = \sum_{i=2}^{\infty} L_i,$$

and the y-characteristic Lie ring $\overline{A}$ of equation (2.4) is

$$\overline{A} = \sum_{i=2}^{\infty} \overline{L}_i.$$

We introduce the notation $\pounds_k = \sum_{i=2}^{k} L_i$.

The classification of integrable equations is based on the following statement.

Lemma 2.1. *Assume u is a solution to equation (2.4) and the vector fields Z and $\overline{Z}$ read as follows*

$$Z = \sum_{i=1}^{\infty} \alpha_i \frac{\partial}{\partial u_i}, \ \ \alpha_i = \alpha_i(u, \overline{u}_1, u_1, u_2, \ldots, u_{n_i}),$$
$$\overline{Z} = \sum_{i=1}^{\infty} \overline{\alpha}_i \frac{\partial}{\partial \overline{u}_i}, \ \ \overline{\alpha}_i = \overline{\alpha}_i(u, u_1, \overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{n_i}), \ \ i = 1, 2, \ldots.$$

If $[D, Z] = 0$, then $Z = 0$. In the same way, if $[\overline{D}, \overline{Z}] = 0$, then $\overline{Z} = 0$.

Proof. Since the operator of total differentation w.r.t. x on the set of locally analytic functions depending on a finite number of variables $\overline{u}_1, u, u_1, u_2, \ldots$

$$D = f\frac{\partial}{\partial \overline{u}_1} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \ldots,$$

then

$$[D, Z] = (D(\alpha_1)\tfrac{\partial}{\partial u_1} + D(\alpha_2)\tfrac{\partial}{\partial u_2} + D(\alpha_3)\tfrac{\partial}{\partial u_3} + \ldots) -$$
$$-(\alpha_1 f_{u_1} \tfrac{\partial}{\partial \overline{u}_1} + \alpha_1 \tfrac{\partial}{\partial u} + \alpha_2 \tfrac{\partial}{\partial u_1} + \alpha_3 \tfrac{\partial}{\partial u_2} + \ldots).$$

By the assumption $[D, Z] = 0$, and hence

$$\alpha_1 = 0, \quad D(\alpha_i) - \alpha_{i+1} = 0, \quad i = 1, 2, \ldots$$

and, therefore, $\alpha_i = 0$ as $i = 1, 2, 3, \ldots$. In the same way, if $[\overline{D}, \overline{Z}] = 0$ and

$$\overline{D} = f\frac{\partial}{\partial u_1} + \overline{u}_1 \frac{\partial}{\partial u} + \overline{u}_2 \frac{\partial}{\partial \overline{u}_1} + \ldots,$$

then $\overline{Z} = 0$. The lemma is proven.

2.2. Classification of integrable hyperbolic equation with an infinite-dimensional characteristic Lie ring. In the case of $f = f(u)$ on the set of locally analytic functions depending on the variables $u, u_1, u_2, \dots, u_m$

$$\overline{D} = \overline{u}_1\frac{\partial}{\partial u} + f\frac{\partial}{\partial u_1} + D(f)\frac{\partial}{\partial u_2} + \ldots = \overline{u}_1 X_2 + X_1.$$

2.2.1. Klein-Gordon equation. In this subsection we consider the equations (see [15,16])

$$u_{xy} = f(u). \tag{2.8}$$

We have

$$[D, X_1] = -fX_2, \quad [D, X_2] = 0. \tag{2.9}$$

We note that the operators X_1, X_2 are linearly independent as $f(u) \neq 0$.

Let $X_3 = [X_2, X_1]$. Employing Jacobi identity and (2.9), we get

$$[D, X_3] = -f_u X_2. \tag{2.10}$$

Lemma 2.2. *The dimension of the linear space* $\pounds_3 = \sum_{i=2}^{3} L_i$ *equals to two if and only if*

$$X_3 - cX_1 = 0.$$

And here the right hand side of equation (2.8) *becomes*

$$f(u) = \alpha e^{cu},$$

where α, c *are constants,* $\alpha \neq 0$.

Proof. Let $\dim \pounds_3 = 2$. Then due to

$$X_3 = f'\frac{\partial}{\partial u_1} + f'' u_1 \frac{\partial}{\partial u_2} + \ldots,$$

then $X_3 = c(u)X_1$, in accordance with Lemma 2.1 and formulas (2.9) and (2.10) we get

$$[D, X_3 - cX_1] = -f'X_2 - D(c)X_1 + cfX_2 = 0.$$

The last relation is equivalent to the following system of equations,

$$f' - cf = 0, \quad D(c) = 0.$$

Therefore, c – *const* and $f = \alpha e^{cu}$. The lemma is proven.

Thus, nonlinear equation (2.8) with a two-dimensional characteristic Lie algebra A is reduced to the Liouville equation

$$u_{xy} = e^u. \tag{2.11}$$

Let $X_4 = [X_2, X_3]$, $X_5 = [X_1, X_3]$. Employing Jacobi identity and relations (2.9), (2.10), we obtain

$$[D, X_4] = -f'' X_2, \quad [D, X_5] = f'X_3 - fX_4. \tag{2.12}$$

In what follows we assume that the dimension of the linear space $\pounds_3$ equals to three (X_1, X_2, X_3 are linear independent), and we shall show that the case $\dim \pounds_4 = 3$ is not realized.

Indeed, if $\dim \pounds_4 = 3$, then

$$X_4 = c_1 X_1 + c_2 X_3 \quad \text{and} \quad X_5 = \overline{c}_1 X_1 + \overline{c}_2 X_3, \tag{2.13}$$

where $c_i = c_i(u, u_1, u_2, \dots, u_{n_i}), \overline{c}_i = \overline{c}_i(u, u_1, u_2, \dots, u_{\overline{n}_i}), i = 1, 2$.

In accordance with Lemma 2.1 and the formulas (2.9)–(2.12), the first relation in (2.13) is equivalent to the system

$$D(c_1) = 0, \quad c_1 f - f'' + c_2 f' = 0, \quad D(c_2) = 0.$$

This is why c_1, c_2 are constants and

$$f'' - c_2 f' - c_1 f = 0.$$

The second relation in (2.13) is equivalent to the system

$$D(\overline{c}_1) + c_1 f = 0, \quad \overline{c}_1 f + \overline{c}_2 f' = 0, \quad D(\overline{c}_2) + c_2 f - f' = 0.$$

The last equation implies that $\overline{c}_2$ is constant and $f' = c_2 f$. Then, as it was shown above, $\dim \pounds_3 = 2$.

Lemma 2.3. *The dimension of the space $\pounds_4$ generated by the operators X_1, X_2, X_3, X_4, and X_5 equals 4 if and only if the function f satisfies the equation*

$$f'' - pf' - qf = 0, \tag{2.14}$$

where p, q are constants and $f' \neq \beta f$. At that $X_4 = pX_3 + qX_1$.

Proof. Employing Lemma 2.1 and formulas (2.9)–(2.12), we obtain that either

$$X_4 = c_1 X_1 + c_2 X_3 + c_3 X_5,$$

and, therefore,

$$\begin{aligned} D(c_1) - c_1 c_3 f = 0, \quad f'' - c_1 f - c_2 f' = 0, \\ D(c_2) + c_3 f' - c_2 c_3 f = 0, \end{aligned} \tag{2.15}$$

or

$$X_5 = \overline{c}_1 X_1 + \overline{c}_2 X_3 + \overline{c}_3 X_4,$$

and then

$$\begin{aligned} D(\overline{c}_1) = 0, \quad \overline{c}_1 f + \overline{c}_2 f' + \overline{c}_3 f'' = 0, \\ D(\overline{c}_2) - f' = 0, \quad D(\overline{c}_3) + f = 0. \end{aligned} \tag{2.16}$$

According to the first and third equations in (2.15), c_1, c_2 are constants, $c_3 = 0$ (otherwise $f' = c_2 f$ and then $\dim \pounds_3 = 2$), and the function f satisfies equation (2.14). If (2.16) holds true, then $f = 0$.

Vice-versa, if the function f satisfies equation (2.14), then

$$[D, X_4] = -(pf' + qf)X_2 = p[D, X_3] + q[D, X_1] = [D, pX_3 + qX_1].$$

Thus, $X_4 = pX_3 + qX_1$ and $\dim \pounds_4 = 4$. The lemma is proven.

Remark 2.1. *If $X_4 = 0$, then $p = q = 0$, and equation* (2.8) *is reduced to the equation $u_{xy} = u$.*

In what follows we assume that the assumption of Lemma 2.3 holds. We introduce the operators of length 4,

$$X_6 = [X_2, X_5] \quad \text{and} \quad X_7 = [X_1, X_5].$$

Employing Jacobi identity

$$[X_2, [X_1, X_3]] + [X_3, [X_2, X_1]] + [X_1, [X_3, X_2]] = 0,$$

it is easy to show that $X_6 = pX_5$. This is why $\dim \pounds_5 \leqslant 5$.

Remark 2.2. *If $X_6 = 0$, then $p = 0$, and identity* (2.14) *becomes*

$$f'' - qf = 0.$$

Then equation (2.8) *is reduced to the Sine-Gordon equation*

$$u_{xy} = e^u + e^{-u}. \tag{2.17}$$

By formulas (2.9)–(2.12) we obtain that

$$[D, X_7] = (f' - 2pf)X_5. \tag{2.18}$$

Let us check that $\dim \pounds_5 = 5$. Suppose the opposite, $\dim \pounds_5 = 4$ and $X_7 = c_1X_1 + c_2X_3 + c_3X_5$. Then

$$\begin{aligned} D(\overline{c}_1) - c_3qf &= 0, \\ \overline{c}_1 f + \overline{c}_2 f' &= 0, \\ D(\overline{c}_2) + c_3f' - c_3pf &= 0, \\ D(\overline{c}_3) + 2pf - f' &= 0. \end{aligned} \tag{2.19}$$

It is clear that c_i are constants, $i = 1, 2, 3$. From the last equation in (2.19) one can see that $f' = 2pf$, i.e., $X_3 = 2pX_1$ (Lemma 2.2).

We introduce now the operators of length 5,

$$X_8 = [X_2, X_7], \quad X_9 = [X_1, X_7], \quad [X_3, X_5].$$

It is easy to check that $[X_3, X_5] = -pX_7 + X_8$, and thus $\dim \pounds_6 \leqslant 7$.

Employing (2.9)–(2.12), (2.18), we obtain that

$$[D, X_8] = (q - 2p^2)fX_5, \quad [D, X_9] = -fX_8 + (f' - 2pf)X_7. \tag{2.20}$$

If $\dim \pounds_6 = 5$, the following relations

$$X_8 = \overline{c}_1X_1 + \overline{c}_2X_3 + \overline{c}_3X_5 + \overline{c}_4X_7,$$

$$X_9 = c_1X_1 + c_2X_3 + c_3X_5 + c_4X_7$$

hold true. We rewrite the former in accordance with Lemma 2.1 and the formulas (2.10)–(2.12), (2.18), (2.20) as

$$D(c_1) - qc_3f = 0, \quad c_1f + c_2f' = 0, \quad D(c_3) + c_3f' - pc_3f = 0,$$

$$D(c_3) + c_4f' - 2pc_4f = 0, \quad D(c_4) - f' + 2pf = 0.$$

It follows from the last equation that $c_4 = 0$ and $f' = 2pf$. Hence, $X_3 = 2pX_1$, then $\dim \pounds_3 = 2$. Thus, $\dim \pounds_6 \geq 6$.

The following statement holds true.

Lemma 2.4. *Let* $\dim \pounds_i = i$, $i = 3, 4, 5$. *Then the dimension of the space* $\pounds_6$ *equals 6 if and only if*

$$X_8 = 0.$$

Proof. Let $\dim \pounds_6 = 6$. Then either

$$X_9 = c_1X_1 + c_2X_3 + c_3X_5 + c_4X_7 + c_5X_8$$

and therefore

$$\begin{aligned} D(c_1) - qc_3f = 0, \quad c_1f + c_2f' = 0, \quad D(c_2) + c_3f' - pc_3f &= 0, \\ D(c_3) + c_4f' - 2pc_4f + c_5f'' - c_5pf' - 2c_5p^2f &= 0, \\ D(c_4) - f' + 2pf = 0, \quad D(c_5) + f &= 0, \end{aligned} \tag{2.21}$$

or

$$X_8 = \overline{c}_1X_1 + \overline{c}_2X_3 + \overline{c}_3X_5 + \overline{c}_4X_7 + \overline{c}_5X_9,$$

and then

$$\begin{aligned} D(\overline{c}_1) - \overline{c}_3qf - \overline{c}_1\overline{c}_5f = 0, \quad \overline{c}_1f + \overline{c}_2f' &= 0, \\ D(\overline{c}_2) + \overline{c}_3f' - \overline{c}_3pf - \overline{c}_2\overline{c}_5f &= 0, \\ D(\overline{c}_3) - (q - 2p^2)f + \overline{c}_4(f' - 2pf) - \overline{c}_3\overline{c}_5f &= 0, \\ D(\overline{c}_4) - \overline{c}_4\overline{c}_5f = 0, \quad D(\overline{c}_5) - \overline{c}_5^2f &= 0. \end{aligned} \tag{2.22}$$

One can see that the last equation in (2.21) yields $f = 0$. We rewrite the system (2.22) as

$$\overline{c}_3 q = 0, \quad \overline{c}_1 f + \overline{c}_2 f' = 0, \quad \overline{c}_3(f' - pf) = 0,$$
$$-(q - 2p^2)f + \overline{c}_4(f' - 2pf) = 0,$$

where $\overline{c}_1$, $\overline{c}_2$, $\overline{c}_3$, $\overline{c}_4$ – *const*, $\overline{c}_5$=0.

If $\overline{c}_3 \neq 0$, the function f satisfies the equation $f' = pf$, then $\dim \pounds_3 = 2$. If $\overline{c}_3 = 0$, then $\overline{c}_4 = 0$ (otherwise $\dim \pounds_3 = 2$), and the fourth equation implies $q = 2p^2$. Hence, $X_8 = 0$. Thus, the necessary condition is proven.

Let us prove the sufficient condition. Suppose $X_8 = 0$, since $[X_3, X_5] = -pX_7$, then $\dim \pounds_6 \leqslant 6$. If $\dim \pounds_6 = 5$, then the operator X_9 should be expressed as a linear combination of the operators X_1, X_3, X_5, and X_7, but as it is shown above in this case $\dim \pounds_3 = 2$. The lemma is proven.

Remark 2.3. *Thus, if $X_8 = 0$, then $q = 2p^2$, and equation* (2.8), (2.14) *is reduced to the Tzitzeica equation*

$$u_{xy} = e^u + e^{-2u}. \tag{2.23}$$

2.2.2. Hyperbolic equations $u_{xy} = f(u, u_x, u_y)$. We consider a nonlinear equation

$$u_{xy} = f(u, u_x, u_y). \tag{2.24}$$

In this subsection we obtain the conditions for the right hand side of the equation (2.24) (see [16, 18, 35]), for which

$$\dim \pounds_i = i, \quad i = 2, 3, 4, 5, 6.$$

We exclude the equations (2.24) which are linear w.r.t. the variable u_x or u_y.

We let

$$X_1 = \overline{u}_1 \frac{\partial}{\partial u} + f\frac{\partial}{\partial u_1} + \ldots + D^{n-1}(f)\frac{\partial}{\partial u_n} + \ldots, \qquad X_2 = \frac{\partial}{\partial \overline{u}_1},$$

then

$$\overline{D} = X_1 + \overline{u}_2 X_2. \tag{2.25}$$

We have

$$[D, X_1] = -(\overline{u}_1 f_u + f f_{u_1})X_2, \qquad [D, X_2] = -f_{\overline{u}_1} X_2. \tag{2.26}$$

Employing Jacobi identity

$$[D, X_3] = [D, [X_2, X_1]] = -[X_1, [D, X_2]] - [X_2, [X_1, D]]$$

and the relations (2.26), we get

$$[D, X_3] = -(f_u + f_{u_1} f_{\overline{u}_1})X_2 - f_{\overline{u}_1} X_3. \tag{2.27}$$

The operators X_4, X_5 satisfy the relations

$$\begin{gathered}[D, X_4] = -f_{u_1} f_{\overline{u}_1\overline{u}_1} X_2 - f_{\overline{u}_1\overline{u}_1} X_3 - 2f_{\overline{u}_1} X_4,\\ [D, X_5] = (f_u + f_{u_1} f_{\overline{u}_1} - \overline{u}_1 f_{u\overline{u}_1} - f f_{u_1\overline{u}_1})(f_{u_1} X_2 + X_3)-\\ -(\overline{u}_1 f_u + f f_{u_1})X_4 - f_{\overline{u}_1} X_5.\end{gathered} \tag{2.28}$$

Theorem 2.1. *Suppose the dimension of the space $\pounds_4$ generated by the operators of length* 1, 2, *and* 3 *equals four. Then*

$$X_4 + c_1(X_1 - \overline{u}_1 X_3) + c_2 X_5 = 0,$$

and we have one of the following relations for the right hand side of the equation (2.24), *either*

$$\begin{gathered}f = \overline{c}\left(u_1 \int \tfrac{\overline{c}_u}{\overline{c}^2} d\overline{u}_1 + B\right), \quad \overline{c}_{\overline{u}_1} + \tfrac{\delta \overline{u}_1}{\overline{c}} = \lambda,\\ B = B(u, u_1), \quad \overline{c} = \overline{c}(u, \overline{u}_1),\end{gathered} \tag{2.29}$$

$$\textit{where} \quad c_1 = \frac{1}{\overline{c}^2}, \quad c_2 = 0, \quad \delta, \lambda - \textit{const};$$

or the function f satisfies the relations

$$\begin{gathered} f_u + f_{u_1} f_{\overline{u}_1} - \overline{u}_1 f_{u\overline{u}_1} - f f_{u_1\overline{u}_1} - c f_{\overline{u}_1\overline{u}_1} = 0, \\ D(c) - c f_{\overline{u}_1} - (\overline{u}_1 f_u + f f_{u_1}) = 0, \quad c = c(u, \overline{u}_1), \end{gathered} \tag{2.30}$$

$$\text{where} \quad c_1 = 0, \quad c_2 = \frac{1}{c}, \quad c_2 \neq 0.$$

Considering the y-characteristic ring, we obtain a "symmetric" version of Theorem 2.1.

Theorem 2.2. *If the dimension of the space $\overline{\mathcal{L}}_4$ equals four, then*

$$Y_4 + \overline{c}_1 (Y_1 - u_1 Y_3) + \overline{c}_2 Y_5 = 0,$$

and we have one of the following relations for the right hand side of the equation (2.24)*,*

either

$$\begin{gathered} f = c \left(\overline{u}_1 \int \tfrac{c_u}{c^2} du_1 + \overline{B} \right), \quad c_{u_1} + \tfrac{\overline{\delta} u_1}{c} = \overline{\lambda}, \\ \overline{B} = \overline{B}(u, \overline{u}_1), \quad c = c(u, u_1), \end{gathered} \tag{2.31}$$

$$\text{where} \quad \overline{c}_1 = \frac{1}{c^2}, \quad \overline{c}_2 = 0, \quad \overline{\delta}, \overline{\lambda} - const;$$

or the function f satisfies the relations

$$\begin{gathered} f_u + f_{u_1} f_{\overline{u}_1} - u_1 f_{uu_1} - f f_{u_1\overline{u}_1} - \overline{c} f_{u_1 u_1} = 0, \\ \overline{D}(\overline{c}) - \overline{c} f_{u_1} - (u_1 f_u + f f_{\overline{u}_1}) = 0, \quad \overline{c} = \overline{c}(u, u_1), \end{gathered} \tag{2.32}$$

$$\text{where} \quad \overline{c}_1 = 0, \quad \overline{c}_2 = \frac{1}{\overline{c}}, \quad \overline{c}_2 \neq 0.$$

We observe that the relations (2.31), (2.32) can be obtained from the equations (2.29), (2.30) by replacing u_1 by $\overline{u}_1$ and $\overline{u}_1$ by u_1.

Lemma 2.5. *Let the right hand side of the equation* (2.24) *satisfies the identities* (2.29)*,* (2.31)*. Then*

$$\begin{gathered} u_{xy} = K(u) L(u_x) \overline{B}(u_y), \quad L' + \eta \left(\tfrac{u_x}{L} \right) = \widetilde{\lambda}, \quad \overline{B}' + \delta \left(\tfrac{u_y}{\overline{B}} \right) = \lambda, \\ \widetilde{\lambda}, \lambda, \eta, \delta - const. \end{gathered} \tag{2.33}$$

We note that for the equation (2.33) the operators X_4 and Y_4 read as

$$X_4 + \frac{\delta}{\overline{B}^2} (X_1 - u_y X_3) = 0, \quad Y_4 + \frac{\eta}{L^2} (Y_1 - u_x Y_3) = 0.$$

We introduce the operators of length 4,

$$X_6 = [X_2, X_5], \quad X_7 = [X_1, X_5].$$

It is easy to show that $X_6 = \frac{\delta \overline{u}_1}{\overline{B}^2} X_5$.

We let

$$\begin{gathered} \alpha = -(\overline{u}_1 f_u + f f_{u_1}), \quad \beta = -f_{\overline{u}_1}, \quad \gamma = -(f_u + f_{u_1} f_{\overline{u}_1}), \quad p = -f_{\overline{u}_1\overline{u}_1}, \\ q = f_{u_1} p, \quad r = f_u + f_{u_1} f_{\overline{u}_1} - \overline{u}_1 f_{u\overline{u}_1} - f f_{u_1\overline{u}_1}, \quad s = f_{u_1} r. \end{gathered}$$

Employing Jacobi identity and the relations (2.26), (2.27) (2.28), we have

$$\begin{gathered} [D, X_7] = -\tfrac{\delta}{\overline{B}^2} (\overline{u}_1 \alpha_u + f \alpha_{u_1})(X_1 - \overline{u}_1 X_3) + (\overline{u}_1 r_u + f r_{u_1} - \\ - s)(f_{u_1} X_2 + X_3) + (2\alpha \tfrac{\delta \overline{u}_1}{\overline{B}^2} + \overline{u}_1 \beta_u + f \beta_{u_1} + r) X_5 + \beta X_7. \end{gathered} \tag{2.34}$$

One can see that the dimension of the space $\mathcal{L}_5$ increases at most by one, i.e., $\dim \mathcal{L}_5 \leqslant 5$.

Let for the equation (2.33) the dimension of the space $\mathcal{L}_5$ equals five, i.e., the operators X_1, X_2, X_3, X_5, X_7 are linear independent. Then we introduce the operators of length 5, $X_8 = [X_2, X_7]$, $X_9 = [X_1, X_7]$, $[X_3, X_5]$. Employing Jacobi identity, we have

$$[X_3, X_5] = -\frac{\delta \overline{u}_1}{\overline{B}^2} X_7 + X_8,$$

i.e., $\dim \mathcal{L}_6 \leqslant 7$.

According to (2.26), (2.27), (2.28), and (2.34), we obtain

$$\begin{gathered}[D, X_8] = (\overline{u}_1 r_u + f r_{u_1} - s)_{\overline{u}_1}(f_{u_1} X_2 + X_3) - (\tfrac{\delta}{\overline{B}^2}(\overline{u}_1 r_u + f r_{u_1} - s) + \\ + \left(\tfrac{\delta}{\overline{B}^2}(\overline{u}_1 \alpha_u + f\alpha_{u_1})\right)_{\overline{u}_1} + \tfrac{\delta^2}{\overline{B}^4}\overline{u}_1(\overline{u}_1\alpha_u + f\alpha_{u_1}))(X_1 - \overline{u}_1 X_3) + \\ + \Big(\left(2\delta\alpha \tfrac{\overline{u}_1}{\overline{B}^2} + \overline{u}_1 \beta_u + f\beta_{u_1} + r\right)_{\overline{u}_1} + \delta \tfrac{\overline{u}_1}{\overline{B}^2}(2\delta\alpha\tfrac{\overline{u}_1}{\overline{B}^2} + \\ + \overline{u}_1\beta_u + f\beta_{u_1} + r)\Big) X_5 + \beta_{\overline{u}_1} X_7 + 2\beta X_8,\end{gathered} \tag{2.35}$$

$$\begin{gathered}[D, X_9] = (\overline{u}_1(\overline{u}_1 r_u + f r_{u_1} - s)_u + f(\overline{u}_1 r_u + f r_{u_1} - s)_{u_1} - \\ - f_{u_1}(\overline{u}_1 r_u + f r_{u_1} - s))(f_{u_1} X_2 + X_3) + (2\overline{u}_1 r_u + 2 f r_{u_1} - s + \\ + \overline{u}_1(3\delta\tfrac{\overline{u}_1}{\overline{B}^2}\alpha_u + \overline{u}_1\beta_{uu} + f_u \beta_{u_1}) + 2\overline{u}_1 f \beta_{uu_1} + \\ + f\left(3\delta\tfrac{\overline{u}_1}{\overline{B}^2}\alpha_{u_1} + f_{u_1}\beta_{u_1} + f\beta_{u_1u_1}\right))X_5 + (2\delta\tfrac{\overline{u}_1}{\overline{B}^2}\alpha + 2\overline{u}_1\beta_u + \\ + 2f\beta_{u_1} + r)X_7 + \alpha X_8 + \beta X_9 - \tfrac{\delta}{\overline{B}^2}(2\overline{u}_1 f \alpha_{uu_1} + \\ + \overline{u}_1(\overline{u}_1\alpha_{uu} + f_u\alpha_{u_1}) + f(f_{u_1}\alpha_{u_1} + f\alpha_{u_1u_1}))(X_1 - \overline{u}_1 X_3).\end{gathered} \tag{2.36}$$

Lemma 2.6. *If the dimension of the space $\mathcal{L}_6$ for the equation (2.33) equals six, then the functions $K(u)$, $L(u_x)$, and $\overline{B}(u_y)$ satisfy the relations*

$$K'' = 4\lambda^2 k_2^2 K^3 + 2k_2\lambda K K', \quad L' = k_2(1 + 2k_2\frac{u_x}{L}), \quad \overline{B}' = \lambda(1 + 2\lambda\frac{u_y}{\overline{B}}). \tag{2.37}$$

At that

$$X_8 + d_1(X_1 - u_y X_3) + d_3 X_7 = 0,$$

where

$$d_1 = 2\lambda k_2 (1 + \lambda \frac{u_y}{\overline{B}})(2\lambda k_2 K^2 + K'), \quad d_3 = 2\lambda\frac{1}{\overline{B}}(1 + \lambda\frac{u_y}{\overline{B}}).$$

Remark 2.4. *For the equation (2.33), (2.37) the constant λ is non-zero, otherwise $\overline{B}' = 0$.*

Remark 2.5. *The equation (2.33), (2.37) by the point change*

$$K = \frac{1}{\lambda k_3}\widetilde{K}, \quad L = k_2 \widetilde{L}, \quad \overline{B} = \lambda\widetilde{B}$$

is reduced to the equation

$$u_{xy} = \widetilde{K}\widetilde{L}\widetilde{B}, \quad \widetilde{K}'' = 4\widetilde{K}^3 + 2\widetilde{K}\widetilde{K}', \quad \widetilde{L}' = 1 + 2\frac{u_x}{\widetilde{L}}, \quad \widetilde{B}' = 1 + 2\frac{u_y}{\widetilde{B}},$$

which is related with the Tzitzeica equation $v_{xy} = e^v + e^{-2v}$ by the differential substitution (see [4, 20])

$$v = -\frac{1}{2}\ln(u_x - \widetilde{L}) - \frac{1}{2}(u_y - \widetilde{B}) + P(u),$$

where the function P is determined by the ordinary differential equation

$$P'^2 - 2\widetilde{K}P' - 3\widetilde{K}' - 2\widetilde{K}^2 = 0.$$

For the equation (2.33) ($\lambda = \widetilde{\lambda} = 0$)

$$u_{xy} = K(u)\sqrt{1-u_1^2}\sqrt{1-\overline{u}_1^2} \tag{2.38}$$

the dimension of the linear space L_6 equals 2, i.e., the space $\pounds_6$ is generated by the elements $X_1, X_2, X_3, X_5, X_7, X_8, X_9$ ($\dim \pounds_6 = 7$).

We introduce the operators of length 6,

$$X_{10} = [X_2, X_8] = \tfrac{3\overline{u}_1}{1-\overline{u}_1^2} X_8, \quad X_{11} = [X_1, X_8],$$
$$X_{12} = [X_2, X_9], \quad X_{13} = [X_1, X_9].$$

Theorem 2.3. *Let the dimension of the space $\pounds_7$ for equation* (2.38) *equals nine. Then*

$$X_{11} = -3KK'\overline{u}_1(X_1 - \overline{u}_1 X_3) + (3K^2 + \mu)\overline{u}_1 X_5 + \frac{\overline{u}_1}{1-\overline{u}_1^2} X_9.$$

And the function K satisfies the relation

$$K'' - 2K^3 - \mu K = 0, \quad \mu - const. \tag{2.39}$$

The equation (2.38), (2.39) in a much more cumbersome form appeared for the first time in the work [3]. The latter by the change (see [20])

$$v = \arcsin u_x + \arcsin u_y + P(u), \quad P'^2 = 2K' - 2K^2 - \lambda,$$

is reduced to Sine-Gordon equation $v_{xy} = e^v + e^{-v}$.

2.3. System of equations $u_x = f(u, v)$, $v_y = \varphi(u, v)$. In this subsection we consider the system of the equations

$$u_x = f(u, v), \quad v_y = \varphi(u, v). \tag{2.40}$$

In work [21] for classification of integrable equations the symmetry approach was employed and it was shown that if first three D and $\overline{D}$-conditions of generalized symmetries existence hold true, then system (2.40) is reduced to one of the following

$$\begin{gathered} u_x = v, \quad v_y = \sin u, \\ u_x = v, \quad v_y = e^u + e^{-2u}, \\ u_x = \sin v, \quad v_y = \sin u, \\ u_x = \alpha(v), \quad v_y = e^u, \\ u_x = \tfrac{1}{v}, \quad v_y = uv + 1, \\ u_x = v, \quad v_y = e^u v + e^{2u}, \\ u_x = uv + 1, \quad v_y = uv + 1. \end{gathered} \tag{2.41}$$

On the set of locally-analytic functions depending on a finite number of the variables $u, v, \overline{u}_1, v_1, \overline{u}_2, v_2, \overline{u}_3, v_3, \ldots$, the operator of total differentation w.r.t. x reads as

$$D = v_1 \frac{\partial}{\partial v} + f\frac{\partial}{\partial u} + \overline{D} f \frac{\partial}{\partial \overline{u}_1} + \overline{D}^2 f \frac{\partial}{\partial \overline{u}_2} + \ldots,$$

where $\overline{u}_1 = u_y, v_1 = v_x, \overline{u}_2 = u_{yy}, v_2 = v_{xx}, \ldots$.

Then

$$D = v_1 X_1 + X_2, \tag{2.42}$$

where

$$X_1 = \frac{\partial}{\partial v}, \quad X_2 = f\frac{\partial}{\partial u} + \overline{D} f \frac{\partial}{\partial \overline{u}_1} + \overline{D}^2 f \frac{\partial}{\partial \overline{u}_2} + \ldots.$$

Thus, in a natural way one can associate the Lie ring generated by the vector fields X_1 and X_2 with the system of equations (2.40).

Let $\dim \mathcal{L}_3 \leqslant 3$, $\dim \mathcal{L}_4 \leqslant 4$. Then one of the following relations

$$\begin{array}{l} (i) \quad X_3 = c_1X_1 + c_2X_2, \quad c_1 = c_1(v),\, c_2 = c_2(v); \\ (ii) \quad X_4 = c_1X_1 + c_2X_2 + c_3X_3,\, X_5 = \widetilde{c}_1X_1 + \widetilde{c}_2X_2 + \widetilde{c}_3X_3, \\ \qquad c_i = c_i(v),\, \widetilde{c}_i = \widetilde{c}_i(v),\, i = 1, 2, 3; \\ (iii) \quad X_4 = c_1X_1 + c_2X_2 + c_3X_3 + c_5X_5, \\ \qquad c_i = c_i(v),\, i = 1, 2, 3, 5; \\ (iv) \quad X_5 = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4, \\ \qquad c_i = c_i(v),\, i = 1, 2, 3, 4, \end{array} \tag{2.43}$$

hold true.

The operator $\overline{D} = \varphi\frac{\partial}{\partial v} + \overline{u}_1\frac{\partial}{\partial u} + \overline{u}_2\frac{\partial}{\partial \overline{u}_1} + \ldots$ coincides with that of total differentiation w.r.t. y on the set of the functions depending on the variables $v, u, \overline{u}_1, \overline{u}_2, \overline{u}_3, \ldots$.

Lemma 2.7. *Let u and v be solutions to the system of equations* (2.40) *($\varphi'_u \neq 0$) and a vector field Z reads as*

$$Z = \alpha_0(u, v)\frac{\partial}{\partial u} + \alpha_1(u, v, \overline{u}_1)\frac{\partial}{\partial \overline{u}_1} + \alpha_2(u, v, \overline{u}_1, \overline{u}_2)\frac{\partial}{\partial \overline{u}_2} + \ldots.$$

If $[\overline{D}, Z] = 0$, then $Z = 0$.

We have

$$[\overline{D}, X_1] = -\varphi_v X_1, \quad [\overline{D}, X_2] = -f\varphi_u X_1. \tag{2.44}$$

Employing Jacobi identity and relations (2.44), we also obtain

$$\begin{gathered} [\overline{D}, X_3] = [\overline{D}, [X_1, X_2]] = -[X_2, [\overline{D}, X_1]] - [X_1, [X_2, \overline{D}]] = \\ = [X_2, \varphi_v X_1] - [X_1, f\varphi_u X_1] = -f_v\varphi_u X_1 - \varphi_v X_3, \\ [\overline{D}, X_4] = [\overline{D}, [X_1, X_3]] = -f_{vv}\varphi_u X_1 - \varphi_{vv} X_3 - 2\varphi_v X_4, \\ [\overline{D}, X_5] = [\overline{D}, [X_2, X_3]] = \varphi_u\left(f_u f_v - f f_{uv}\right) X_1 + \\ + \left(f_v\varphi_u - f\varphi_{uv}\right) X_3 - f\varphi_u X_4 - \varphi_v X_5. \end{gathered} \tag{2.45}$$

Let the dimension of the characteristic ring equal two. Then $X_3 = c_1X_1 + c_2X_2$. In according with Lemma 2.7 and relations (2.44) and (2.45) we have

$$c_1 = 0, \quad (f_v - c_2 f)\varphi_u = 0, \quad \overline{D}(c_2) + c_2\varphi_v = 0. \tag{2.46}$$

If the operators X_1, X_2, and X_3 are linearly independent and the dimension of the characteristic ring equals three, then $X_4 = c_1X_1 + c_2X_2 + c_3X_3$, $X_5 = \widetilde{c}_1X_1 + \widetilde{c}_2X_2 + \widetilde{c}_3X_3$. Employing Lemma 2.7 and relations (2.44) and (2.45), we rewrite the last identities in an equivalent form,

$$\begin{gathered} c_1 = \widetilde{c}_1 = 0, \quad \left(c_2 f + c_3 f_v - f_{vv}\right)\varphi_u = 0, \\ c_{2v}\varphi + 2c_2\varphi_v = 0, \quad c_{3v}\varphi + \varphi_{vv} + c_3\varphi_v = 0, \\ \widetilde{c}_2 f + \widetilde{c}_3 f_v + f_u f_v - f f_{uv} = 0, \quad \widetilde{c}_{2v}\varphi + \widetilde{c}_2\varphi_v + c_2 f\varphi_u = 0, \\ \widetilde{c}_{3v}\varphi + c_3 f\varphi_u - f_v\varphi_u + f\varphi_{uv} = 0. \end{gathered} \tag{2.47}$$

Let us consider now the cases when the characteristic ring is of minimal growth, i.e., $\dim \mathcal{L}_4 = 4$. If the operators X_1, X_2, X_3, and X_4 are linearly independent, and $X_5 = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4$, then

$$\begin{gathered} c_1 = \widetilde{c}_1 = 0, \quad \left(c_2 f + c_3 f_v + c_4 f_{vv} + f_u f_v - f f_{uv}\right)\varphi_u = 0, \\ c_{2v}\varphi + c_2\varphi_v = 0, \quad c_{3v}\varphi - c_4\varphi_{vv} - f_v\varphi_u + f\varphi_{uv} = 0, \\ c_{4v}\varphi - c_4\varphi_v + f\varphi_u = 0. \end{gathered} \tag{2.48}$$

If the operators X_1, X_2, X_3, and X_5 are linearly independent, and $X_4 = c_1X_1 + c_2X_2 + c_3X_3 + c_5X_5$, then

$$\begin{gathered} c_1 = \widetilde{c}_1 = 0, \quad (c_2 f + c_3 f_v - f_{vv} - c_5(f_u f_v - f f_{uv}))\,\varphi_u = 0, \\ c_{2v}\varphi + 2c_2\varphi_v - c_2c_5 f\varphi_u = 0, \\ c_{3v}\varphi + c_3\varphi_v + \varphi_{vv} + c_5(f_v\varphi_u - f\varphi_{uv}) - c_3c_5 f\varphi_u = 0, \\ c_{5v}\varphi + c_5\varphi_v - c_5^2 f\varphi_u = 0. \end{gathered} \tag{2.49}$$

In the same we introduce the y-characteristic Lie ring of system of equations (2.40). The condition of "slow" growth yields that one of the following relations

$$\begin{aligned} &(i') \quad Y_3 = \beta_1 Y_1 + \beta_2 Y_2, \quad \beta_1 = \beta_1(u),\ \beta_2 = \beta_2(u); \\ &(ii') \quad Y_4 = \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 Y_3,\ Y_5 = \widetilde{\beta}_1 Y_1 + \widetilde{\beta}_2 Y_2 + \widetilde{\beta}_3 Y_3, \\ &\qquad \beta_i = \beta_i(u),\ \widetilde{\beta}_i = \widetilde{\beta}_i(u),\ i = 1,2,3; \\ &(iii') \quad Y_4 = \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 Y_3 + \beta_5 Y_5, \\ &\qquad \beta_i = \beta_i(u),\ i = 1,2,3,5; \\ &(iv') \quad Y_5 = \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 Y_3 + \beta_4 Y_4, \\ &\qquad \beta_i = \beta_i(u),\ i = 1,2,3,4, \end{aligned} \tag{2.50}$$

hold true.

For system of equations (2.41) one of conditions (2.43) and (2.50) holds true. Namely, for the first, second, sixth, and seventh system (2.41) $X_4 = 0$. And also for the first system we have $Y_4 = -Y_2$, for the second $Y_4 = 2Y_2 - Y_3$, for the sixth $Y_4 = -2Y_2 + 3Y_3$, for the seventh $Y_4 = 0$.

For the third system $X_4 = -X_2$ and $Y_4 = -Y_2$.

The y-characteristic Lie ring of the forth system of equations $u_x = \alpha(v)$, $v_y = e^u$ is three-dimensional ($Y_3 = Y_2$), and the x-characteristic ring for each of the cases (i)–(iv) determines the function $\alpha(v)$ as follows. As $X_3 = c_2X_2$, it reads $\alpha = \gamma e^{c_2 v}$ (c_2, γ are constants); in the case (ii) the function α satisfies the relations

$$\begin{gathered} c_2\alpha + c_3\alpha' - \alpha'' = 0, \quad \widetilde{c}_2\alpha + \widetilde{c}_3\alpha' = 0, \\ \widetilde{c}_2' + c_2\alpha = 0, \quad \widetilde{c}_3' + c_3\alpha - \alpha' = 0, \quad c_1 = \widetilde{c}_1 = 0, \end{gathered}$$

where c_2, c_3 are constants, $\widetilde{c}_2 = \widetilde{c}_2(v)$, $\widetilde{c}_3 = \widetilde{c}_3(v)$.

In the case (iii) the function α satisfies the relations

$$\begin{gathered} c_2\alpha + c_3\alpha' - \alpha'' = 0, \quad c_2' - c_2c_5\alpha = 0, \\ c_3' + c_5\alpha' - c_3c_5\alpha = 0, \quad c_5' - c_5^2\alpha = 0, \quad c_i = c_i(v),\ i = 2,3,5. \end{gathered}$$

At that $X_4 = c_2X_2 + c_3X_3 + c_5X_5$.

In the case (iv) $X_5 = c_2X_2 + c_3X_3 + c_4X_4$ the function α is so that

$$\begin{gathered} c_2\alpha + c_3\alpha' + c_4\alpha'' = 0, \quad c_3' = \alpha', \\ c_4' = -\alpha, \quad c_2 - const, \quad c_i = c_i(v),\ i = 3,4. \end{gathered}$$

Remark 2.6. *Nontrivial symmetries exist only as* $\frac{\alpha''}{\alpha} = const$ *(see [21]).*

The fifth system of equations $u_x = \frac{1}{v}$, $v_y = uv + 1$ ($Y_4 = 0$) by the change $v = e^{-w}$ is reduced to

$$u_x = e^w, \quad w_y = u + e^w.$$

For the last system of equations $X_4 = 0$.

Thus, it is shown that systems of equations (2.41) have the rings of minimal growth, i.e., they satisfy (2.43) and (2.50).

2.4. Nonlinear integrable equations with a finite dimensional characteristic ring. In this subsection we consider the equations (2.24) with the characteristic ring A of the dimension 2 and 3 (see [16, 18, 35]) and the equation (2.33) as $\dim A = 4$.

One can see that the operators X_1 and X_2 are linearly independent, i.e., $\dim L_2 = 2$.

The following statement holds true.

Lemma 2.8. *The dimension of the characteristic ring A equals two if and only if the right hand side f of the equation* (2.24) *has the form*

$$f = A(u, u_x)u_y.$$

At that $X_3 = \frac{1}{u_y}X_1$.

Let the dimension of the ring A equals three. In this case the relations

$$X_4 + c(X_1 - \overline{u}_1 X_3) = 0, \quad X_5 + \overline{c}(X_1 - \overline{u}_1 X_3) = 0,$$
$$c = c(u, \overline{u}_1, u_1, u_2, \ldots), \quad \overline{c} = \overline{c}(u, \overline{u}_1, u_1, u_2, \ldots)$$

hold true. Then

$$[D, X_4 + c(X_1 - \overline{u}_1 X_3)] = 0, \quad [D, X_5 + \overline{c}(X_1 - \overline{u}_1 X_3)] = 0.$$

In accordance with (2.26), (2.27), and (2.28), the last relations are equivalent to the following system of equations

$$\begin{gathered} D(c) + 2cf_{\overline{u}_1} = 0, f_{\overline{u}_1\overline{u}_1} + c(f - \overline{u}_1 f_{\overline{u}_1}) = 0, \\ D(\overline{c}) + c(\overline{u}_1 f_u + f f_{u_1}) + \overline{c} f_{\overline{u}_1} = 0, \\ f_u + f_{u_1} f_{\overline{u}_1} - \overline{u}_1 f_{u\overline{u}_1} - f f_{u_1\overline{u}_1} - \overline{c}(f - \overline{u}_1 f_{\overline{u}_1}) = 0. \end{gathered} \tag{2.51}$$

It is clear that $c = c(u, \overline{u}_1)$, $\overline{c} = \overline{c}(u, \overline{u}_1)$.

The statement holds true.

Lemma 2.9. *the equations* (2.24) *with the characteristic Lie ring A of the dimension* 3 *by the point change is reduced to one of the following*

$$u_{xy} = -\frac{1}{B_{u_x}}(B_u u_y + 1), \quad B = B(u, u_x), \quad c = \overline{c} = 0;$$

or

$$u_{xy} = e^u \Psi(u_x), \quad c = 0, \quad \overline{c} = \overline{c}(u, u_y);$$

or

$$u_{xy} = \frac{1}{u} p(u_x)\overline{r}(u_y), \quad \overline{r}' + \frac{u_y}{\overline{r}} = \lambda, \quad p' + \frac{u_x}{p} = \lambda,$$
$$\textit{where} \quad \lambda - \textit{const}, \quad \lambda \neq 0, \quad c = \frac{1}{\overline{r}^2}, \quad \overline{c} = -\frac{1}{u};$$

or

$$u_{xy} = q(u)p(u_x)\overline{r}(u_y), \quad (\ln q)'' = q^2, \quad \overline{r}' + \frac{u_y}{\overline{r}} = 0, \quad p' + \frac{u_x}{p} = 0,$$
$$\textit{where} \quad c = \frac{1}{\overline{r}^2}, \quad \overline{c} = \frac{q'}{q};$$

or

$$u_{xy} = \overline{F}(u, u_y)u_x,$$
$$\textit{where} \quad c = \frac{1}{u_y}(\ln(\overline{F} - u_y \overline{F}_{u_y}))'_{u_y}, \quad \overline{c} = (\ln(\overline{F} - u_y \overline{F}_{u_y}))'_u,$$

the function $\overline{F}$ satisfies the relation

$$u_y e^{-\varphi} + (\overline{F} - \varphi' u_y) \int e^{-\varphi} du = \Phi(\overline{F} - \varphi' u_y), \quad \varphi = \varphi(u).$$

Here B, Ψ, Φ are arbitrary functions of their arguments. And

$$X_4 = -c(X_1 - u_y X_3), \quad X_5 = -\overline{c}(X_1 - u_y X_3).$$

Let us consider now the equation (2.33) for which $\dim \pounds_4 = 4$.

Lemma 2.10. *Let the dimension of the space pounds$_4$ equals for. For the equation* (2.33) $\dim \pounds_5 = 4$ *if and only if the function K satisfies the relation*

$$\left(\frac{K'}{K}\right)' = \kappa K^2, \quad \kappa - const. \tag{2.52}$$

We observe that the dimensions of the x- and y-characteristic Lie rings A and $\overline{A}$ of the equation (2.33), (2.52) equal four (see [20]). This is a Liouville type equation.

2.5. Equation $u_{xy} = f(u, u_x, u_y)$ with second order x- and y-integrals. In the work [36] the method for classification nonlinear hyperbolic equations (2.24) with second order x- and y-integrals based on studying characteristic Lie rings was suggested. The characteristic rings of such equations are three-dimensional.

Theorem 2.4. *Let the characteristic rings A and $\overline{A}$ of the equation* (2.24) *be three-dimensional. The the following relations*

$$A_{u_1} = 0, \quad A_u u_1 + A_{\overline{u}_1} f = -2 f_{\overline{u}_1} A, \tag{2.53}$$

$$B_{u_1} = 0, \quad B_u u_1 + B_{\overline{u}_1} f = -(f_u \overline{u}_1 + f f_{u_1}) A - f_{\overline{u}_1} B, \tag{2.54}$$

hold true, where $A = \frac{f_{\overline{u}_1 \overline{u}_1}}{f - \overline{u}_1 f_{\overline{u}_1}}, \quad B = \frac{\overline{u}_1 f_{u \overline{u}_1} + f f_{u_1 \overline{u}_1} - f_u - f_{u_1} f_{\overline{u}_1}}{f - \overline{u}_1 f_{\overline{u}_1}}$, *and*

$$\overline{A}_{\overline{u}_1} = 0, \quad \overline{A}_u \overline{u}_1 + \overline{A}_{u_1} f = -2 f_{u_1} \overline{A}, \tag{2.55}$$

$$\overline{B}_{\overline{u}_1} = 0, \quad \overline{B}_u \overline{u}_1 + \overline{B}_{u_1} f = -(f_u u_1 + f f_{\overline{u}_1}) \overline{A} - f_{u_1} \overline{B}, \tag{2.56}$$

where $\overline{A} = \frac{f_{u_1 u_1}}{f - u_1 f_{u_1}}, \quad \overline{B} = \frac{u_1 f_{u u_1} + f f_{u_1 \overline{u}_1} - f_u - f_{u_1} f_{\overline{u}_1}}{f - u_1 f_{u_1}}$.

Relations (2.53)–(2.56) allow one to make the complete list of the equations with second order integrals (see, for instance, [6]).

2.6. Linearized equation. For classification of nonlinear integrable equations instead of the Lie ring one can uses the characteristic ring of its linearization.

Consider the linearization

$$(D\overline{D} - f_{u_x} D - f_{u_y} \overline{D} - f_u) v = 0 \tag{2.57}$$

of equation (2.4). For this equation we can define the sequence of Laplace invariants (see [20]).

Definition 2.1. *Equation* (2.4) *is called Darboux integrable if there exist functions ω, $\overline{\omega}$ depending on a finite number of the variables*

$$x, y, u, u_1, u_2, u_3, \ldots, \overline{u}_1, \overline{u}_2, \overline{u}_3, \ldots \tag{2.58}$$

such that on the solutions of the equation (2.4) *the function ω is independent on the variable y, and the function $\overline{\omega}$ is independent of x.*

Let us adduce the criterion for Darboux integrability (see [19, 32, 41, 49]).

Theorem 2.5. *The nonlinear equation* (2.4) *is Darboux integrable if and only if the sequence of the Laplace invariants for the linearized equation* (2.57) *breaks on both sides.*

Employing the notion of Characteristic Lie ring, in the works [14, 17] it was shown that the sequence the Laplace invariants for the linearized equation (2.57) breaks on both sides only in the case when the characteristic Lie rings are finite-dimensional.

2.7. Generalized symmetries of integrable equations. In the present section we provide the description of generalized symmetries for integrable equations on the basis of the generators of the characteristic Lie ring (see [10, 32, 37]).

The right hand side of the nonlinear equation $u_{xy} = f(u)$ possessing nontrivial Lie-Bäcklund group is reduced to one of the forms $e^u, e^u + e^{-u}, e^u + e^{-2u}$.

2.7.1. Symmetries of Liouville equation. The x-characteristic Lie ring is generated by the operators

$$X_1 = e^u \frac{\partial}{\partial u_1} + D(e^u)\frac{\partial}{\partial u_2} + \dots, \quad X_2 = \frac{\partial}{\partial u}.$$

Let

$$X = e^{-u} \sum_{k=1}^{\infty} D^{k-1}(e^u)\frac{\partial}{\partial u_k} = e^{-u} X_1,$$

and obtain the operator $\overline{X}$ by the change $u_k \leftrightarrow \overline{u}_k$, $D \leftrightarrow \overline{D}$.

It is known (see [22]) that a symmetry can be represented as

$$F = \varphi(u_1, u_2, \dots, u_n) + \overline{\varphi}(\overline{u}_1, \overline{u}_2, \dots, \overline{u}_m),$$

where φ, $\overline{\varphi}$ are symmetries.

Now the generating equation

$$D\overline{D}\varphi = e^u \varphi$$

becomes

$$(D + u_1)X\varphi = \varphi. \tag{2.59}$$

Applying the operator operator X to the equation (2.59), we obtain

$$(D + u_1)X^2\varphi = 0.$$

Therefore, $h = X\varphi \in Ker\overline{D}$, in the same way $\overline{h} = \overline{X}\overline{\varphi} \in KerD$, and it follows from the formula (2.59) that each symmetry of the Liouville equation can be represented as

$$f = (D + u_1)h + (\overline{D} + \overline{u}_1)\overline{h}, \tag{2.60}$$

where $h(\overline{h})$ is an arbitrary element of $Ker\overline{D}(D)$. Thus, the following statement holds true.

Theorem 2.6. *The symmetries of Liouville equation are calculated by the formula*

$$f = (D + u_1)h(w, w_1, \dots) + (\overline{D} + \overline{u}_1)\overline{h}(\overline{w}, \overline{w}_1, \dots),$$

where $w = u_2 - \frac{u_1^2}{2}$ $(\overline{w} = \overline{u}_2 - \frac{\overline{u}_1^2}{2})$, $h(\overline{h})$ *is an arbitrary function of its arguments.*

2.7.2. The symmetries of Sine-Gordon equation. The vector field of the x-characteristic ring for the Sine-Gordon equation

$$X_1 = (e^u + e^{-u})\frac{\partial}{\partial u_1} + D(e^u + e^{-u})\frac{\partial}{\partial u_2} + \dots$$

can be represented as (see [11])

$$X_1 = e^u X + e^{-u} Y.$$

Then the generating equation $D\overline{D}F = (e^u - e^{-u})F$ is equivalent to the system

$$(D + u_1)XF = F, \quad (D - u_1)YF = F. \tag{2.61}$$

Since the commutator $[D, \overline{D}] = 0$, then the relations

$$(D + u_1)X = XD, \quad (D - u_1)Y = YD \tag{2.62}$$

hold true. Applying the operator of differentiation X and Y to the equations (2.61) and employing (2.62), we arrive at the formulas

$$\begin{gathered} DYXF = (Y-X)F, \quad (D+u_1)XYXF = YXF, \\ (D-u_1)Y^2XF = -YXF. \end{gathered} \tag{2.63}$$

It follows from (2.61) – (2.63) that if F is a symmetry of order n, then YXF is a symmetry of order $n-2$. Indeed, since

$$\begin{gathered} (Y-X)F = -2\left(u_1\tfrac{\partial}{\partial u_2} + u_2\tfrac{\partial}{\partial u_3} + \ldots + (u_{n-1}+\ldots)\tfrac{\partial}{\partial u_n}\right)(u_n+ \\ +cu_{n-1} + g(u_1,\ldots,u_{n-2})), \end{gathered}$$

then $\operatorname{ord}(Y-X)F = n-1$, and therefore we obtain by the first relation in (2.63) that $\operatorname{ord} YXF = n-2$. Hence, if the original equation possesses a symmetry of an even order, then it should possess a second order symmetry. But no second order symmetry exists.

By the formulas (2.61) we get that

$$2F = (D - u_1D^{-1}u_1)(X-Y)F.$$

The latter due to (2.63) is written as

$$2F = (-D^2 + u_1D^{-1}u_1D)YXF = -LYXF.$$

Thus, the algebra of symmetries of Sine-Gordon equation is calculated by the recurrent formula

$$F^{(n+2)} = (D^2 - u_1^2 + u_1D^{-1}u_2)F^{(n)}, \quad F^{(1)} = u_1, \quad n = 1,3,5,\ldots. \tag{2.64}$$

2.7.3. Symmetries of Tzitzeica equation. We define the differentiations X and Y by the relation $e^uX + e^{-2u}Y = X_1$, where

$$X_1 = (e^u + e^{-2u})\frac{\partial}{\partial u_1} + D(e^u+e^{-2u})\frac{\partial}{\partial u_2} + \ldots.$$

Then for the functions $F(u_1,\ldots,u_n)$ the generating equation

$$D\overline{D}F = (e^u - 2e^{-2u})F$$

is equivalent to the sytstem

$$(D+u_1)XF = F, \quad (D-2u_1)YF = -2F. \tag{2.65}$$

A consequent applying of the operators X and Y to the equations (2.65) leads one to the formulas

$$\begin{gathered} (D-u_1)YXF = (Y-X)F, \quad DXYXF = 3YXF, \\ (D+u_1)X^2YXF = 3XYXF, \quad (D+2u_1)X^3YXF = 2X^2YXF, \\ (D-u_1)YX^2YXF = -X^2YXF, \\ DYX^3YXF = 2(YX^2YX - X^3YX)F, \\ (D+u_1)X(YX^3YXF) = YX^3YXF, \\ (D-2u_1)Y(YX^3YXF) = -2YX^3YXF. \end{gathered} \tag{2.66}$$

Let F be the symmetry of order n. Then it follows from the formulas (2.65), (2.66) that YX^3YXF is the symmetry of the original equations of order $n-6$. Then we rewrite the equations (2.65) as

$$D(2X+Y)F + 2u_1(X-Y)F = 0, \quad D(X-Y)F + u_1(X+2Y)F = 3F.$$

By the latter one can obtain the formula

$$3F = (D - u_1 - 2u_1D^{-1}u_1)(X-Y)F. \tag{2.67}$$

Employing now (2.66), we obtain a new representation for the symmetry (2.67)

$$27F = (D - u_1 - 2u_1D^{-1}u_1)(D-u_1)D(D+u_1)X^2YXF. \tag{2.68}$$

We write the forth and fifth identities in (2.66) as

$$[D,(X^3YX+2YX^2YX)+2u_1(X^3YX-YX^2YX)]F=0,$$
$$[D(X^3YX-YX^2YX)+u_1(2X^3YX+YX^2YX)-3X^2YX]F=0.$$

It follows from the last relations that

$$3X^2YXF=\left((D+u_1-2D^{-1}u_1)(X^3YX-YX^2YX)\right)F. \tag{2.69}$$

And finally, employing the sixth identity in (2.66) and (2.69), we can write the formula (2.68) as

$$162F=LYX^3YXF,$$

where the recurrence operator L is defined by the formula

$$L=(D-u_1-2u_1D^{-1}u_1)(D-u_1)D(D+u_1)(D+u_1-2u_1D^{-1}u_1)D. \tag{2.70}$$

The last relation gives the recurrent formula for the symmetries

$$F^{(n+6)}=LF^{(n)}. \tag{2.71}$$

Letting $F^{(1)}=u_1$ and $F^{(5)}=u_5+5(u_2-u_1^2)u_3-5u_1u_2^2+u_1^5$, we obtain from (2.71) two sequences of symmetries

$$\left\{F^{(1+6k)}\right\} \quad \text{and} \quad \left\{F^{(5+6k)}\right\}, \quad k=0,1,2,\dots$$

for the Tzitzeica equation.

2.7.4. Symmetries of modified Sine-Gordon equation. The modified Sine-Gordon equation (2.38), (2.39) (mSG) can be represented as

$$u_{xy}=s(u)b(u_1)\overline{b}(\overline{u}_1), \quad \text{where} \quad s''-2s^3-\mu s=0 \quad b'=-\frac{u_1}{b}, \quad \overline{b}'=-\frac{\overline{u}_1}{\overline{b}}, \quad \mu-const. \tag{2.72}$$

On the set of locally-analytic functions in $\Im$

$$\overline{D}F(u,u_1,u_2,\dots)=\overline{u}_1\tfrac{\partial}{\partial u}+sb\overline{b}\tfrac{\partial}{\partial u_1}+D(sb\overline{b})\tfrac{\partial}{\partial u_2}+\dots=$$
$$=\overline{u}_1\tfrac{\partial}{\partial u}+sb\overline{b}\tfrac{\partial}{\partial u_1}+(s'u_1b\overline{b}-s\tfrac{u_1u_2}{b}\overline{b}-s^2b^2\overline{u}_1)\tfrac{\partial}{\partial u_2}+\dots.$$

This is why the generators of the x-characteristic Lie algebra A of equation (2.72) read as

$$X=\frac{\partial}{\partial u}-s^2b^2\overline{u}_1\frac{\partial}{\partial u_2}+\dots, \quad Y=sb\frac{\partial}{\partial u_1}+(s'u_1b-s\frac{u_1u_2}{b})\frac{\partial}{\partial u_2}+\dots. \tag{2.73}$$

Then $\overline{D}=\overline{u}_1X+\overline{b}Y$.

Theorem 2.7. *The differential operator*

$$Y^2+s^2$$

maps the generalized symmetries of order n into the symmetries of order $n-2$. The recurrence operator

$$D^2+2\frac{u_1u_2}{b^2}D-u_1D^{-1}(\frac{u_3}{b^2}D+\frac{u_1u_2^2}{b^4}D+3s^2u_1D+$$
$$+3ss'u_1^2-ss'+\lambda u_2)+s^2+\lambda u_1^2$$

determines the algebra of the symmetries for the equation mSG (see [37]).

We observe that the recurrence operator was obtained in the work [28] by using Bäcklund transformation.

If $\mu=0$, i.e., $s'^2-ss''+s^4=0$, then the function s is determined as

$$s=\frac{\sqrt{\lambda}}{\cos(\sqrt{\lambda}u-c)}, \quad \lambda,c-const.$$

It happens that there exists an operator which maps the symmetries of the equation

$$u_{xy} = \frac{1}{\cos u}\sqrt{1-u_1^2}\sqrt{1-\overline{u}_1^2}. \tag{2.74}$$

into a y-integral.

Theorem 2.8. *The operator*

$$\frac{b}{s}Y + u_1$$

maps a symmetry F into an integral W of the equation (2.74). *And the operator*

$$\left(\frac{s'}{s} + \frac{u_2}{b^2}\right)^{-1}\left(D - \frac{s}{b}D\left(\frac{b}{s}\right)\right)$$

maps an integral into a symmetry.

3. System of hyperbolic equations

3.1. Symmetries. Characteristic ring.

3.1.1. Exponential systems of kind I and Cartan matrices. The integrability of the systems of equations $u_{z\overline{z}} = F(u)$ is determined by the properties of the characteristic Lie algebra defined by the vector field $F(u)$ (see [30]). In this connection the problem on classification of finite-dimensional (kind I) and possessing finite-dimensional representation (kind II) characteristic algebras appear. We consider exponential systems of equations. The exponential system with matrix of coefficients $A = (a_{ij})$ is written as

$$u^i_{z\overline{z}} = e^{v^i}, \quad v^i = a_{i1}u^1 + \ldots + a_{ir}u^r, \quad i = 1, \ldots, r. \tag{3.75}$$

If A is the Cartan matrix of a simple Lie algebra, then this system is integrated by qudrature (see [29, 57]).

For systems of equations (3.75) with arbitrary matrix A in the work [30] they made a conjecture on coinciding the characteristic algebra $\mathcal{X}(A)$ with a generated by positive roots subalgebra $G_+(A)$ of a countergraded Lie algebra canonically associated with the matrix A. It is known (see [25]) that a countergraded Lie algebra is finite-dimensional if and only if the matrix A is equivalent to one of Cartan matrices of a simple Lie algebra.

Or aim is the description of finite-dimensional characteristic algebras $\mathcal{X}(A)$ corresponding to non-degenerate matrices A. The elements of the algebra $\mathcal{X}(A)$ are the operators $\sum_{i,j} f_{ij}(u_1, u_2, \ldots)\frac{\partial}{\partial u^i_j}$ in the space of variables $u_j = (u^1_j, \ldots, u^r_j)$, $j \geq 1$. The generators $X_1, \ldots, X_r$ of Lie algebra $\mathcal{X}(A)$ are determined by the relations

$$X_j D = (D + a_j)\, X_j, \quad X_j u^k_1 = \delta^k_j, \tag{3.76}$$

where $D: \, u_j \to u_{j+1}, \; a_j = a_{j1}u^1_1 + \ldots + a_{jr}u^r_1$. Regarded as a vector space, the characteristic algebra is generated by the multiple commutator of special form

$$X_{\alpha_1,\ldots,\alpha_n} = ad_{\alpha_1} \ldots ad_{\alpha_{n-1}} X_{\alpha_n}, \quad ad_j : Y \to [X_j, Y]. \tag{3.77}$$

It is convenient to replace the non-degeneracy condition for the matrix A of system of equations (3.75) by the conditions

$$\begin{gathered} a_{ii} = 2, \quad a_{ij} = 0 \quad \Leftrightarrow \quad a_{ij} = 0, \\ a_{ij} = 0, -1, -2, \ldots \quad (i, j = 1, \ldots, r, i \neq j). \end{gathered} \tag{3.78}$$

We call the matrix satisfying these conditions (possible degenerate) a generalized Cartan matrix. Let us show that relations (3.78) are implications of the finite dimension of the algebra $\mathcal{X}(A)$ and the condition $\det A \neq 0$.

The finite dimension of the characteristic algebra means vanishing of commutators (3.77) of sufficiently high order n. It follows from the expansion

$$\mathcal{X}(A) \equiv \mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \ldots \oplus \mathcal{X}_n \oplus \ldots,$$

where $\mathcal{X}_j$ is the linear subspace spanned over commutators of order j. $\mathcal{X}_j \cap \mathcal{X}_k = \{0\}$, since the coefficients $X_\alpha u^i_m$ of the operator $X_\alpha \in \mathcal{X}_n$ are generalized homogenous polynomials of order $m - n$. For the operators $X_1, \ldots, X_r \in \mathcal{X}_1$ it holds true due to formula (3.76), and for commutators (3.77) it does due to the general formula

$$\begin{gathered} X_\alpha D = (D + a_{\alpha_1} + \ldots + a_{\alpha_n})\, X_\alpha + X_{[\alpha]}, \\ X_{[\alpha]} = -a_{\alpha_{n-1}\alpha_n} X_{\alpha/\alpha_n} + \sum_{j=1}^{n-1} c_j X_{\alpha/\alpha_j}, \quad c_j = \sum_{k=j+1}^{n} a_{\alpha_k\alpha_j}. \end{gathered} \tag{3.79}$$

where α/α_j is the multi-index obtained from α by crossing out the component with index j.

Formula (3.79) implies in particular the relation

$$X_{\alpha_1\ldots\alpha_n} u_n = X_{[\alpha]} u_{n-1}, \quad n \geq 2, \tag{3.80}$$

which yields that as $n \geq 1$

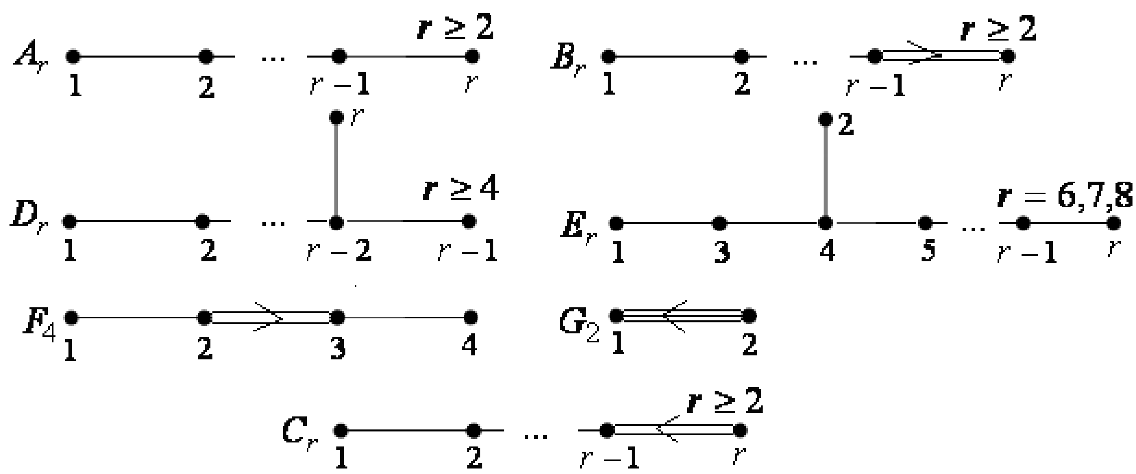
$$\begin{gathered} \left(ad^n_j X_k\right) u^i_{n+1} = n\left(a_{kj} + \tfrac{n-1}{2} a_{jj}\right) ad^{n-1}_j X_k u^i_n = \ldots = \\ = n! \prod_{p=2}^{n} \left(a_{kj} + \tfrac{p-1}{2} a_{jj}\right), \\ X_{jk} u^i_2 = n! \prod_{p=2}^{2} \left(a_{kj} + \tfrac{p-1}{2} a_{jj}\right) \left(a_{kj}\delta^i_k - a_{jk}\delta^k_j\right). \end{gathered} \tag{3.81}$$

Letting $a_{jj} = 0$, we obtain $ad^n_j X_k u^\alpha_{n+1} = n!\,(a_{kj})^{n-1}$. Thus, for a finite-dimensional algebra it follows from $a_{jj} = 0$ that $a_{1j} = a_{2j} = \ldots = a_{rj} = 0$, and it contradicts to the non-degeneracy of the matrix A. Hence, one can let $a_{jj} = 2$, $\forall j = 1, \ldots, r$. Formula (3.81) as $i = j$ implies $a_{jk}(a_{kj} + 1)(a_{kj} + 2) \ldots (a_{kj} + n) = 0$, $n \gg 1$. Relations (3.78) are proven.

The matrix A of order r is called expansible if for some partition of the index set $\{1, \ldots, r\} = I_1 \cup I_2$, $I_1 \cap I_2 = \varnothing$ the elements of the matrix A satisfy the conditions $a_{ij} = a_{ji} = 0$, $\forall i \in I_1, j \in I_2$. System of equations (3.75) with an expansible matrxi A splits into two independent subsystems. The matrices of systems (3.75) distinguishing only by the variables numeration are called equivalent.

Theorem 3.1. Description of finite-dimensional characteristic algebras *Non-expansible generalized Cartan matrices with a finite-dimensional characteristic algebra is equivalent to the Cartan matrix of a simple Lie algebra (table 1).*

Table 1.



In Table 1 we give the graphs (Dynkin schemes) of Cartan matrix. The vertices of the graph are numbered. The edge $\{i, j\}$ connects the vertices with the indexes i, j if $a_{ij}a_{ji} \neq 0$. The graphs given in the table determine uniquely the Cartan matrices (see [5]). The multiplicity of the edge $\{i, j\}$ indicates the value of the product $a_{ij}a_{ji} = 1, 2, 3$. The arrow determines the position of an element not equalling to -1. We note that the transition $i \leftrightarrow j$ of the graph vertices corresponds to the transition $u^i \leftrightarrow u^j$.

Remark 3.1. *The finite dimension of the characteristic algebra corresponding to one of the Cartan matrices follows from relations (see* (3.81)*)*

$$ad_j^{1-a_{kj}} X_k = 0, \quad j \neq k.$$

Indeed, similar relations determine completely the generated by positive roots subalgebra G_+ of the countergraded Lie algebra, which is finite-dimensional in the case of the Cartan matrices (see [25]).

The equation

$$\tfrac{\partial}{\partial \bar z}\omega\left(u_1, \ldots, u_n\right) = 0 \tag{3.82}$$

is called the characteristic equation of the system $u^i_{z\bar z} = F^i(u^1, \ldots, u^r), \quad i = 1, 2, \ldots, r$. The operator

$$\tfrac{\partial}{\partial \bar z} = F(u)\tfrac{\partial}{\partial u_1} + F_z(u)\tfrac{\partial}{\partial u_2} + F_{zz}(u)\tfrac{\partial}{\partial u_3} + \ldots \tag{3.83}$$

determines the characteristic Lie algebra $\mathcal{X}(F)$ of this system. The generators of the algebra $\mathcal{X}(F)$ are operators (3.83) associated with different values of the parameter $u = (u^1, \ldots, u^r)$. It is easy to see that in the case of exponential system (3.75) corresponding to the generalized Cartan matrix, the characteristic Lie algebra defined in this way coincides with the Lie algebra generated by operators (3.76).

Lemma 3.1. *Characteristic equation* (3.82) *of the system with a finite-dimensional algebra* $\mathcal{X}(F)$, $F = (F^1, \ldots, F^r)$ *has* r *solutions*

$$\omega^k = \omega^k\left(u_1, \ldots, u_{n_k}\right), \quad k = 1, \ldots, r,$$

satisfying the independence in general condition

$$\det\left[\frac{\partial\omega^1}{\partial u_{n_1}}, \ldots, \frac{\partial\omega^r}{\partial u_{n_r}}\right] \neq 0.$$

The main property of finite dimensional characteristic algebras $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \ldots$.

Lemma 3.2. *Let* A *be a generalized Cartan matrix,* $\dim \mathcal{X}(A) < \infty$*. Then any finite set* $\{X_\alpha = X_{\alpha_1 \ldots \alpha_m}\} \subset \mathcal{X}_m$ *satisfies the condition*

$$\sum_\alpha c_\alpha X_{[\alpha]} = 0 \quad \Rightarrow \quad \sum_\alpha c_\alpha \left(\sum_{k=1}^m a_{\alpha_k}\right) X_\alpha u^i_m = 0, \quad 1 \leqslant i \leqslant r.$$

Let us show that for any matrix A not containing in Table 1 (non-expansible and satisfying conditions (3.78)) either for some $n \leqslant 4 \quad \dim \mathcal{X}_{n+1}(A) > \dim \mathcal{X}_n(a)$ and Lemma 3.2 is applicable or the characteristic algebra $\mathcal{X}(A)$ has a finite dimensional subalgebra associated with a degenerate matrix.

Let A be a non-expansible generalized Cartan matrix of order $r = 2$. Due to formula (3.79),

$$X_{[112]} = 2\left(1 + a_{21}\right) X_{12}, \quad X_{[212]} = 2\left(1 + a_{12}\right) X_{12}.$$

Lemma 3.2 implies

$$\begin{gathered}\left(1 + a_{12}\right)\left(2a_1 + a_2\right) X_{112}u_3 - \left(1 + a_{21}\right)\left(a_1 + 2a_2\right) X_{212}u_3 = \\ = \left(1 + a_{12}\right) a_1 X_{112}u_3 - \left(1 + a_{21}\right) a_2 X_{212}u_3 = 0.\end{gathered}$$

By formula (3.80)

$$X_{112}u_3 = 2\left(1 + a_{21}\right) X_{12}u_2, \quad X_{212}u_3 = 2\left(1 + a_{12}\right) X_{12}u_2.$$

Hence,

$$(1 + a_{12})\,(1 + a_{21})\,(a_1 - a_2)\,X_{12}u_2 = 0.$$

Since the non-expansibility of the matrix A means $X_{12}u_2 \neq 0$, then

$$(1 + a_{12})\,(1 + a_{21}) = 0.$$

Assuming for definiteness $a_{12} = -1$, we obtain $X_{212} = X_{2112} = 0$ and

$$X_{[11112]} = 4\,(3 + a_{21}X_{1112})\,, \quad X_{[21112]} = -X_{[1112]}.$$

Lemma 3.2 yields

$$(1 + a_{21})\,(2 + a_{21})\,(3 + a_{21}) = 0.$$

The obtained result can be generalized. Considering the subalgebras with two generators, we make sure that the following holds true.

Remark 3.2. *The elements of the generalized Cartan matrix $A = (a_{ij})$ with a finite-dimensional characteristic algebra satisfy the condition*

$$\forall i \neq j \quad a_{ij}a_{ji} = 0, 1, 2, 3.$$

The proven statement exhausts the statement on the classification of second order matrix (see Table 1).

Remark 3.3. *The elements of a non-expansible generalized Cartan matrix $A = (a_{ij})$ $(r > 2,\ \dim \mathcal{X}(A) < \infty)$ satisfy the condition $a_{ij}a_{ji} \neq 3$.*

Remark 3.4. *Let $A = (a_{ij})$ be a non-expansible generalized Cartan of order $r \geq 3,\quad \dim \mathcal{X}(A) < \infty$. Then*

$$a_{ij}a_{ji} = 2 \quad \Rightarrow \quad a_{ik}a_{ki},\, a_{jk}a_{kj} \neq 2, \quad k \neq i, j.$$

The proof of the classification theorem is reduced to finding infinite subalgebras. In the process of proving it is found out that any infinite characteristic algebra satisfying the conditions given Remarks 3.2 – 3.4 contains the subalgebra corresponding to one of the matrices in Tables 2,3.

The matrices are formally divided into two tables. The infiniteness of the algebras in Table 2 is proven by Lemma 3.2. The matrices for which applying of Lemma 3.2 is complicated are moved to Table 3 of degenerating matrices (the infinite dimension of the corresponding algebras is checked independently).

Bearing in mind Table 2, let us write down the relations $\sum c_\alpha X_{[\alpha]} = 0$ indicating the applicability of Lemma 3.2. While using Lemma 3.2, some coefficients are inessential (see the proof of Remark 3.4); they are not written down explicitly.

Table 2.

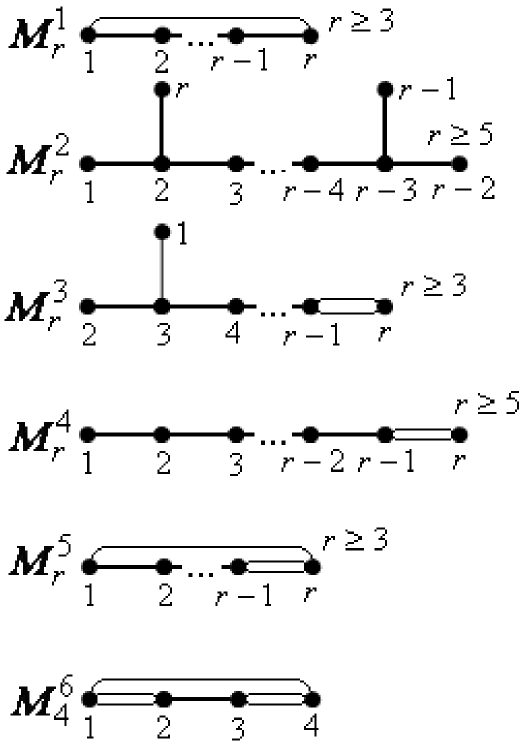
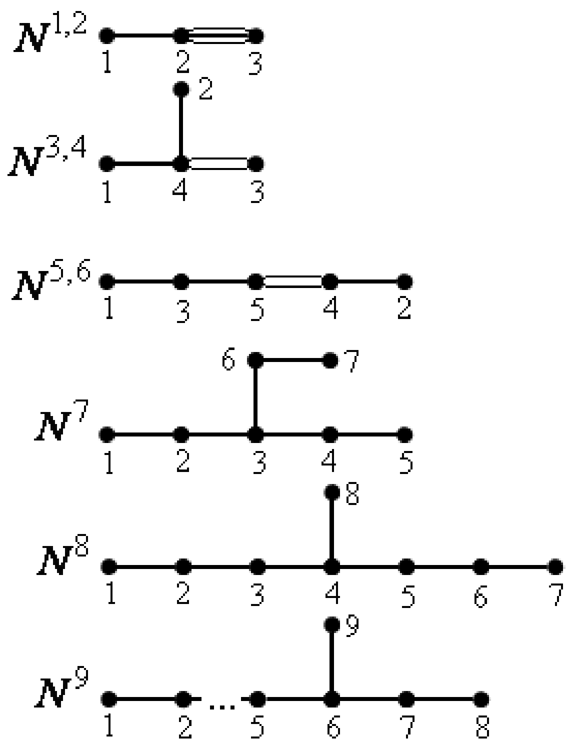


Table 3.



$$M_r^1: \quad X_{[r,1]} + \sum_{k=1}^{r-1} X_{[k,k+1]} = 0,$$

$$M_r^2: \quad \sum_{k=3}^{r-4} X_{[k-1,k,k+1]} + \frac{1}{2}\left(X_{[123]} + X_{[32r]} - X_{[12r]}\right) -$$

$$-\frac{1}{2}\left(X_{[r-1,r-3,r-2]} + X_{[r-1,r-3,r-4]} - X_{[r-4,r-3,r-2]}\right) = 0, \quad r \geq 6,$$

$$M_5^2: \quad X_{[312]} + X_{[512]} + X_{[423]} - X_{[524]} = 0,$$

$$M_r^3: \quad X_{[123]} + X_{[134]} + X_{[234]} + 2\sum_{k=4}^{r-2} X_{[k-1,k,k+1]} +$$

$$+ c_1 X_{[r-2,r-1,r]} + c_2\left(X_{[r-1,r-1,r]} + X_{[r,r,r-1]}\right) = 0,$$

$$M_r^4: \quad -(3+2a_{21})^{-1}\left(X_{[112]} + X_{[221]}\right) + 2X_{[123]} + 2a_{21}\sum_{k=3}^{r-2} X_{[k-1,k,k+1]} +$$

$$+ c_1 X_{[r-2,r-1,r]} + c_2\left(X_{[r-1,r-1,r]} + X_{[r,r,r-1]}\right) = 0,$$

$$M_r^5: \quad X_{[21r]} + \frac{1}{a_{r,r-1}} X_{[r-1,1,r]} + \sum_{k=2}^{r-2} X_{[k-1,k,k+1]} +$$

$$+ c_1 X_{[r-2,r-1,r]} + c_2\left(X_{[r-1,r-1,r]} + X_{[r,r,r-1]}\right) = 0, \quad r \geq 4,$$

$$M_4^6: \quad -\frac{a_{34}}{6+4a_{21}}\left(X_{[112]} + X_{[221]}\right) + a_{34}X_{[123]} + a_{21}X_{[234]} +$$

$$+ c\left(X_{[334]} + X_{[443]}\right) = 0.$$

3.1.2. Quadratic systems. The systems of equations

$$p^i_x = c^i_{jk}p^j q^k + c^i_k q^k, \quad q^k_y = d^k_{jl}p^j q^l + d^k_j p^j \tag{3.84}$$

will be called quadratic. Here $p^i = p^i(x,y), q^k = q^k(x,y),\ i = 1,2,\dots,n,\ k = 1,2,\dots,m$ are unknown functions; $c^i_{jk}, c_k, d^k_{jl}, d_j$ are constants.

For instance, the Liouville equation can be written as

$$p_x = pq, \quad q_y = p \quad (p = e^u, q = u_x). \tag{3.85}$$

the Sine-Gordon equation can be written as

$$p^1_x = p^1 q, \quad p^2_x = -p^2 q, \quad q_y = p^1 + p^2 \quad (p^1 = e^u, p^2 = e^{-u}, q = u_x). \tag{3.86}$$

Denote by a the algebra of smooth functions depending on a finite number of the variables $p^i, q^k, p^i_1, q^k_1, \dots, p^i_l, q^k_l, \dots,\ i = 1,2,\dots,n,\ k = 1,2,\dots,m$, where

$$p^i_{l+1} = \overline{D}p^i_l, \quad q^k_{l+1} = Dq^k_l, \quad p^i_0 = p^i, \quad q^k_0 = q^k.$$

By a_x we indicate the algebra of smooth functions depending on the variables q^k_l, $k = 1,2,\dots,m,\ l = 0,1,2,\dots$. In the same fashion we define the algebra a_y. If $f \in a_x$, then $\overline{D}f = f_0 + \sum_{j=1}^n p^j f_j$, where $f_j \in a_x,\ j = 1,2,\dots,n$. The mapping Y_i determined by the identities $Y_i f = f_i$ are the differentiations of the algebra a_x. Exactly in the same way we define the differentiations X_i of the algebra a_y.

Definition 3.1. *A generated by the elements Y_i subalgebra L_x of the algebra $Der\, a_x$ is called a characteristic algebra of system* (3.84) *along x.*

In the same way we define the characteristic Lie algebra L_y. In order to define the complete algebra of system (3.84) we consider the relations

$$\begin{gathered}[\overline{X}_0, \overline{Y}_i] = \sum_{l=1}^m d^l_i \overline{X}_l, \quad [\overline{X}_l, \overline{Y}_0] = -\sum_{i=1}^n c^i_l \overline{Y}_i, \\ [\overline{X}_0, \overline{Y}_0] = 0, \quad [\overline{X}_l, \overline{Y}_i] = -\sum_{j=1}^n c^j_{il}\overline{Y}_j + \sum_{k=1}^m d^k_{il}\overline{X}_k,\end{gathered} \tag{3.87}$$

where $i = 1,2,\dots,n,\ l = 1,2,\dots,m$.

Definition 3.2. *Let the Lie algebra $\overline{L}$ generated by the elements $\overline{X}_l, \overline{Y}_i$, $l = 0,1,\dots,m, i = 0,1,2,\dots,n$, regarded as a vector space be the direct sum $\overline{L} = \overline{L}_x \oplus \overline{L}_y$ of its subalgebras generated by the elements $\overline{Y}_i$ and $\overline{X}_l$, respectively. If the correspondences $X_l \to \overline{X}_l$ ($Y_i \to \overline{Y}_i$) generate isomorphisms of Lie algebras $L_y \to \overline{L}_y$ ($L_x \to \overline{L}_x$), then the algebra $\overline{L}$ is called a complete algebra of quadratic system* (3.84).

We note that relations (3.87) are equivalent to the identity

$$[D + \overline{X}_0 + q^k\overline{X}_k, \overline{D} + \overline{Y}_0 + p^i\overline{Y}_i] = 0 \tag{3.88}$$

if p^i, q^k are solutions to system (3.84). On the other hand, relations (3.87) and (3.88) generate system (3.84) under the condition of linear independence of the elements $\overline{X}_l$, $\overline{Y}_i$. In this case equation (3.88) is called the zero curvature representation ($L - A$–pair) for system of equations (3.84).

Definition 3.3. *The set of the functions $f^i, g^k \in a$ is called a symmetry of equation* (3.84) *if the equations*

$$p^i_\tau = f^i, \quad q^k_\tau = g^k, \quad i = 1,2,\dots,n, k = 1,2,\dots,m$$

are compatible with it.

Having differentiated system (3.84) w.r.t. the parameter τ, we obtain the system of equations for determining the symmetries,

$$\begin{aligned} Df^i &= c^i_{jk}(q^k f^j + p^j g^k) + c^i_k g^k, \\ \overline{D}g^k &= d^k_{jl}(q^l f^j + p^j g^l) + d^k_j f^j,\end{aligned} \tag{3.89}$$

where $i = 1, 2, \dots, n,\ l = 1, 2, \dots, m$.

Let S be a the linear space of symmetries and S_x (S_y) be the subset of the symmetries

$$f^i = f^i_0 + f^i_j p^j, \quad g^k \quad (f^i, \quad g^k = g^k_0 + g^k_j q^j),$$

for which $f^i_j, g^k \in a_x$ ($f^i, g^k_j \in a_y$).

The space of symmetries of the equations S seems to be the direct sum of its subspaces S_x and S_y.

For the symmetries of system of equations (3.84) in the space S_x determining system (3.89) casts into the form

$$\begin{gathered} Df^i_0 + c^i_k q^k f^i_l = c^i_{jk} q^k f^j_0 + c^i_k g^k, \quad Df^i_l + c^r_{lk} q^k f^i_r = c^i_{jk} q^k f^j_l + c^i_{lk} g^k, \\ Y_0 g^k = d^k_{jl} q^l f^j_0 + d^k_j f^j_0, \quad Y_1 g^k = d^k_{ji} q^i f^j_l + d^k_{li} g^i + d^k_j f^j_l, \end{gathered} \tag{3.90}$$

$i = 1, 2, \dots, n,\ l = 1, 2, \dots, n,\ k = 1, 2, \dots, m$.

With the notations $p^1 = e^{2u}, p^2 = e^u v, q^1 = u_x, q^2 = w$ the system of equations

$$u_{xy} = \alpha e^{2u} + e^u v w, \quad v_x = e^u w, \quad w_y = e^u v$$

casts into a quadratic representation

$$p^1_x = 2p^1 q^1, \quad p^2_x = p^2 p^1 + p^1 q^2, \quad q^1_y = \alpha p^1 + p^2 q^2, \quad q^2_y = p^2. \tag{3.91}$$

For system (3.91) as $\alpha = \frac{4}{9}$ in the work [31] it was obtained the zero curvature representation in Virasoro algebra. More precisely, the system (3.91) for $\alpha = \frac{4}{9}$ is the consequence of a incompletely defined system of equations followed by the zero curvature representation. In the paper another zero curvature representation is provided which is equivalent to this system.

The relations

$$[D, Y_i] = c^j_{ik} q^k Y_j, \quad [D, Y_0] = c^i_k q^k Y_i, \tag{3.92}$$

$$[\overline{D}, X_k] = d^l_{ki} p^i X_1, \quad [\overline{D}, X_0] = d^l_i p^i X_1, \tag{3.93}$$

implied by $[D, \overline{D}] = 0,\ \overline{D} = Y_0 + p^i Y_i,\ D = X_0 + q^i X_1$ are useful for the description of the characteristic algebra.

The following statement holds true.

Lemma 3.3. *If $Q \in Der a_x$, $[D, Q] = fQ$ and $Q(q^k) = 0$,$k = 1, 2, \dots, m$, then $Q = 0$.*

Proof. We have

$$Q(q^k_1) = QD(q^k) = (DQ - fQ)(q^k) = 0.$$

Then by the induction w.r.t. i we get $Q(q^k_i) = 0$. Thus, $Q = 0$. The lemma is proven.

System of equations (3.91)**.**

We restrict ourselves by treating the most interesting case $\alpha = \frac{4}{9}$.

Equations (3.92), (3.93) for system (3.91) are as follows,

$$\begin{gathered} [\overline{D}, X_0] = \tfrac{4}{9} p^1 X_1 + p^2 X_2, \quad [\overline{D}, X_1] = 0, \quad [\overline{D}, X_1] = p^2 X_1, \\ [D, Y_1] = -2q^1 Y_1 - q^2 Y_2, \quad [D, Y_2] = -q^1 Y_2, \quad Y_0 = 0. \end{gathered} \tag{3.94}$$

While describing the algebra L_y, we shall use the values of its generators X_k on the functions p^i,

$$\begin{gathered} X_0(p^1) = 0, \quad X_1(p^1) = 2p^1, \quad X_2(p^1) = 0, \\ X_0(p^2) = 0, \quad X_1(p^2) = p^2, \quad X_2(p^2) = p^1. \end{gathered} \tag{3.95}$$

From the identity $[\overline{D}, [X_1, X_2]] = p^2 X_1$ by formulas (3.94), (3.95) and Lemma 3.3 we get that $[X_1, X_2] = X_2$. Completely in the same way, employing the relation $[\overline{D}, [X_1, X_0]] = \frac{8}{9} p^1 X_1 + 2p^2 X_2$, we establish that $[X_1, X_0] = 2X_0$.

In what follows we let

$$U_0 = X_1, \quad U_1 = X_2, \quad U_2 = -X_0, \quad U_{i+2} = (ad X_2)^i U_2,\ i = 1, 2, \dots. \tag{3.96}$$

Lemma 3.4. *The formulas*

$$U_{i+2} = \frac{3i(i-1)}{2(i-2)}[X_0, U_i], \quad i = 3, 4, \dots \tag{3.97}$$

hold true.

It follows from the identities (3.96)–(3.97) that the elements $U_0, U_1, U_2, \dots$ form a basis of the characteristic algebra L_y. For description of the algebra L_x we introduce the elements

$$V_1 = Y_2, \quad V_2 = Y_1, \quad V_{i+2} = (adY_2)^i V_2, \quad i = 1, 2, \dots \tag{3.98}$$

Lemma 3.5. *The formulas*

$$V_{i+2} = \frac{3i(i-1)}{2(i-2)}[Y_1, V_i], \quad i = 3, 4, \dots \tag{3.99}$$

hold true.

Formulas (3.98) and (3.99) imply that the elements V_i, $i = 1, 2, \dots$ form a basis of the characteristic algebra L_x.

Relations (3.87) for system of equations (3.91) as $\alpha = \frac{4}{9}$ read

$$[\overline{X}_0, \overline{Y}_1] = \tfrac{4}{9}\overline{X}_1, \quad [\overline{X}_0, \overline{Y}_2] = \overline{X}_2, \quad [\overline{X}_1, \overline{Y}_1] = -2\overline{Y}_1;$$
$$[\overline{X}_1, \overline{Y}_2] = -\overline{Y}_2, \quad [\overline{X}_2, \overline{Y}_1] = -\overline{Y}_2, \quad [\overline{X}_2, \overline{Y}_2] = \overline{X}_1.$$

Then the structure of the algebras L_x and L_y point out that the representations for the generators $\overline{X}_0, \overline{X}_1, \overline{X}_2, \overline{Y}_1, \overline{Y}_2$ should be sought in the Virasoro algebra $\big([e_i, e_j] = (j-1)e_{i+j}$, $i = 0, \pm 1, \pm 2, \dots\big)$,

$$\overline{X}_0 = \frac{2}{3}\lambda^2 e_2, \quad \overline{X}_1 = e_0, \quad \overline{X}_2 = \lambda e_1, \quad \overline{Y}_2 = -\frac{1}{2\lambda}e_{-1}, \quad \overline{Y}_1 = -\frac{1}{6\lambda^2}e_{-2}.$$

The elements $\overline{U}_i, \overline{V}_i$ calculated by formulas (3.96) and (3.98) are as follows,

$$\overline{U}_i = \frac{2}{3}(i-2)!\lambda^i e_i, \quad \overline{V}_i = -\frac{1}{3}\left(\frac{1}{2}\right)^{i-1}(i-2)!\lambda^i e_{-1}, \quad i = 2, 3, \dots.$$

It is easy to check that they satisfy relations (3.97) and (3.99). Hence, the elements $\overline{U}_0, \overline{U}_i, \overline{V}_i$, $i = 1, 2, \dots$ form a basis of the complete algebra of system (3.91) as $\alpha = \frac{4}{9}$. This algebra is isomorphic to Virasoro algebra.

Zero curvature representation (3.88) for this systems is

$$[D + \frac{2}{3}\lambda^2 e_2 + q^1 e_0 + q^2 \lambda e_1, \overline{D} - \frac{1}{2\lambda}p^2 e_{-1} - \frac{1}{6\lambda^2}p^1 e_{-2}] = 0.$$

Symmetries of sysmtem (3.91)**.**

Let f^i, g^i, $i = 1, 2$ is a symmetry of system of equations (3.91) in the space S_x. Then employing formulas (3.94), it is easy to obtain from relations (3.90) that

$$f^1 = 2p^1 Y_2 g^2, \quad f^2 = (p^1 Y_1 + p^2 Y_2)g^2, \quad g^1 = DY_2 g^2, \tag{3.100}$$

where the function g^2 is a solution to the system of equations

$$\begin{gathered}(Y_1 - Y_2)g^2 = 0, \quad ((D + 2q^1)Y_1 Y_2 - 2\alpha Y_2)\, g^2 = 0, \\ ((D + q^1)Y_2^2 - q_2 Y_2 - 1)\, g^2 = 0.\end{gathered} \tag{3.101}$$

Applying differentiation Y_2 to latter equation (3.101), we obtain

$$\left((D + 2q^1)Y_2^3 - 2Y_2\right) g^2 = 0. \tag{3.102}$$

It follows from (3.101) and (3.102) that

$$(\alpha Y_2^3 - Y_1 Y_2)g^2 = 0. \tag{3.103}$$

Then we apply twice differentiation X_2 to the equation (3.102). We get

$$\left((D+4q^1)Y_2^5+5q^2Y_2^4\right)g^2=0. \qquad (3.104)$$

The following statement holds.

Lemma 3.6. *Let the function $\psi \in a_x$ be a solution to the equation*

$$\left((D+4q^1)Y_2^2+5q^2Y_2\right)\psi=0. \qquad (3.105)$$

Then $Y_2\psi=0$.

Employing formulas (3.100)–(3.102), we obtain that the symmetries of system (3.91) in the space S_x are calculated by the formulas

$$\begin{gathered} f^1=p^1\left(D+2q^1\right)\psi, \quad f^2=p^1q^2\psi+\tfrac{1}{2}p^2\left(D+2q^1\right)\psi, \\ g^1=\tfrac{1}{2}D\left(D+2q^1\right)\psi, \quad g^2=\left(D+2q^1\right)q^2\psi-\tfrac{1}{2}q^2\left(D+2q^1\right)\psi. \end{gathered} \qquad (3.106)$$

3.2. Characteristic Lie rings and Darboux integrability criterion for nonlinear hyperbolic systems of equations. We this section we consider the system of equations

$$u_{xy}=F(u,u_x,u_y)\ (u^i_{xy}=F^i,\ i=1,2,\ldots,n), \qquad (3.107)$$

possessing the complete set of x- and y-integrals.

It is known (see [7]) that the maximal number of independent $x-$integrals is equal to the order n of the original system.

Definition 3.4. *The system of equations* (3.107) *is called Darboux integrable if it possesses the maximal number of independent x- and y-integrals.*

Let us define x- and y-characteristic Lie rings for the system of equation (3.107). The operator $\overline{D}$ on the functions in the space of locally-analytic functions depending on a finite number of the variables $\overline{u}_1,u,u_1,u_2,\ldots,u_k\ldots$ acts as follows

$$\overline{D}=\overline{u}^i_2X_i+X_{n+1},$$

where

$$X_i=\frac{\partial}{\partial \overline{u}^i_1},\ i=1,2,\ldots,n,$$

$$X_{n+1}=\overline{u}^i_1\frac{\partial}{\partial u^i}+F^i\frac{\partial}{\partial u^i_1}+D(F^i)\frac{\partial}{\partial u^i_2}+\ldots+D^{k-1}(F^i)\frac{\partial}{\partial u^i_k}+\ldots.$$

The x-characteristic Lie ring of the equation (3.107) is the ring A generated by the vector fields $X_1,X_2,\ldots,X_{n+1}$. In the same way the $y-$characteristic Lie ring $\overline{A}$ is defined.

In the paper [27] the examples of the systems with the characteristic Lie ring A and $\overline{A}$ of dimension 5 are given. In the papers [30, 44] it was shown that the system $u^i_{xy}=F^i(u),\quad 1,2,\ldots,n$ possesses the complete set $x-$integrals if any only if the characteristic ring is finite-dimensional.

Theorem 3.2. *The system of equations* (3.107) *is Darboux integrable if and only if the characteristic Lie rings A and $\overline{A}$ are finite-dimensional. At that, if n_k is the number of k-th order x-integrals, $k=1,2,\ldots,m$, then*

$$\dim A=n+\sum_{i=1}^{m}in_i. \qquad (3.108)$$

Remark 3.5. *For the system of equations*

$$u^i_{xy} = F^i(x, y, u, u_x, u_y), \quad i = 1, 2, \dots, n \tag{3.109}$$

the x-characteristic Lie ring is generated by the operators

$$X_i = \tfrac{\partial}{\partial \overline{u}^i_1}, \quad i = 1, 2, \dots, n,$$
$$X_{n+1} = \tfrac{\partial}{\partial y} + \overline{u}^i_1 \tfrac{\partial}{\partial u^i} + F^i \tfrac{\partial}{\partial u^i_1} + D(F^i) \tfrac{\partial}{\partial u^i_2} + \dots + D^{k-1}(F^i) \tfrac{\partial}{\partial u^i_k} + \dots .$$

Then the system of equations (3.109) *is Darboux integrable if and only if the characteristic Lie rings A and $\overline{A}$ are finite-dimensional. At that, if s_i is the order of ith x-integral $i = 1, 2, \dots, n$, then*

$$\dim A = n + 1 + \sum_{i=1}^{n} s_i.$$

3.3. Nonlinear hyperbolic systems of equations with first order integrals. Consider the system of equations (3.107) with the complete set of x- and y-integrals $\omega^i(u, u_1)$, $\overline{\omega}^i(u, \overline{u}_1)$, $i = 1, 2, \dots, n$, i.e., with the x- and y-characteristic Lie rings A and $\bar{A}$ of dimension $2n$.

It follows from the equations

$$\overline{D}(\omega_i) = 0, \ D(\bar{\omega}_i) = 0, \ i = 1, 2, \dots, n$$

that the right hand side of the system (3.107) is

$$F^i(u, u_1, \bar{u}_1) = -\Gamma^i_{kj}(u) u^k_1 \overline{u}^j_1, \quad i = 1, 2, \dots, n, \tag{3.110}$$

where $\Gamma^i_{kj}(u)$ are Cristoffel symbols. The following statement holds.

Theorem 3.3. *The system of equations* (3.107), (3.110) *possesses the maximal number of first order x- and y-integrals if and only if the relations*

$$\begin{aligned} \tilde{R}^i_{pqj} &= \frac{\partial}{\partial u^q} \Gamma^i_{pj} - \frac{\partial}{\partial u^j} \Gamma^i_{pq} + \Gamma^s_{pj} \Gamma^i_{sq} - \Gamma^i_{vj} \Gamma^v_{pq} = 0, \\ R^i_{qpj} &= \frac{\partial}{\partial u^p} \Gamma^i_{jq} - \frac{\partial}{\partial u^j} \Gamma^i_{pq} + \Gamma^i_{ps} \Gamma^s_{jq} - \Gamma^i_{jv} \Gamma^v_{pq} = 0 \end{aligned} \tag{3.111}$$

hold true. Here R^i_{qpj} is the Riemann tensor, and $\tilde{R}^i_{pqj}$ is the adjoint Riemann tensor.

We observe that the x-integrals of the system (3.107), (3.110) are given by the formulas

$$\omega^i(u, u_1) = A^i_s(u) u^s_1, \ i = 1, 2, \dots, n,$$

where the functions $A^i_s(u)$ are a solution to the system of equations

$$\frac{\partial}{\partial u^k} A^i_s(u) - \Gamma^j_{sk} A^i_j(u) = 0.$$

The compatibility condition for the last system of equations is written as $\tilde{R}^i_{pqj} = 0$.

Theorem 3.4. *Each system of equations* (3.107) *($n = 2$) with the complete set of first order x- and y-integrals is reduced by a point transformation $u = \phi(v)$ to*

$$v^i_{xy} = v^2_x v^1_y - v^k_x v^k_y \frac{\partial}{\partial v^k} \ln\left(p(v^1) + q(v^2)\right), \ i = 1, 2. \tag{3.112}$$

The integrals of the system (3.112) are calculated by the formulas

$$\omega_1 = v^1_x - v^2_x, \ \omega_2 = \left[e^{-v^1} p(v^1) + s(v^1)\right] v^1_x + \left[e^{-v^1} q(v^2) - s(v^1)\right] v^2_x,$$

$$\bar{\omega}_1 = v^1_y - v^2_y, \ \bar{\omega}_2 = \left[e^{-v^2} p(v^1) - r(v^2)\right] v^1_y + \left[e^{-v^2} q(v^2) + r(v^2)\right] v^2_y,$$

where the functions $s(v^1),\ r(v^2),\ p(v^1)$ and $q(v^2)$ are related by the identities

$$s'(v^1) = e^{-v^1}p(v^1),\ r'(v^2) = e^{-v^2}q(v^2).$$

3.4. Two-component systems of equations with first and second order integrals. It was shown in the work [12] that any non-degenerate system of equations (3.107) as $n = 2$ with the integrals

$$\omega^1(u, u_1),\ \omega^2(u, u_1, u_2),\ \bar\omega^1(u, \bar u_1),\ \bar\omega^2(u, \bar u_1) \tag{3.113}$$

is reduced by a point transformation to one of the following types,

$$u^i_{xy} = -\Gamma^i_{kj}(u)u^k_1\bar u^j_1,\ i = 1, 2 \tag{3.114}$$

or

$$u^1_{xy} = u^1_1\bar u^2_1,\ u^2_{xy} = \bar r(u^1, \bar u^1_1, \bar u^2_1)u^1_1. \tag{3.115}$$

Here we consider the classification problem for system of equations (3.114) and (3.115) with integrals (3.113).

Lemma 3.7. *There exist no systems of equations* (3.114) *with integrals* (3.113). *The system of equations* (3.115) *possesses the integrals* (3.113) *if and only if the function* $\bar r$ *is a solution to the equation*

$$\frac{\partial\bar r}{\partial u^1} + \bar u^2_1\frac{\partial\bar r}{\partial\bar u^1_1} + \bar r\frac{\partial\bar r}{\partial\bar u^2_1} + \bar u^1_1\frac{P'(u^1)}{2} + P(u^1)\bar u^2_1 = 0. \tag{3.116}$$

At that,

$$\omega^1 = e^{-u^2}u^1_1,\ \omega^2 = u^2_2 - u^2_1\frac{D\omega^1}{\omega} - \frac{(u^2_1)^2}{2} + \frac{1}{2}P(u^1)e^{2u^2}(\omega^1)^2, \tag{3.117}$$

and the y-integrals $\bar\omega^1$ and $\bar\omega^2$ are determined by the first order partial differential equations

$$\left(\frac{\partial}{\partial u^1} + \bar u^2_1\frac{\partial}{\partial\bar u^1_1} + \bar r\frac{\partial}{\partial\bar u^2_1}\right)\bar\omega = 0,\ \frac{\partial}{\partial u^2}\bar\omega = 0. \tag{3.118}$$

In what follows we shall provide the conditions under those system of equations (3.107) as $n = 2$ possesses the integrals

$$\omega^1(u, u_1),\ \omega^2(u, u_1, u_2),\ \bar\omega^1(u, \bar u_1),\ \bar\omega^2(u, \bar u_1, \bar u_2). \tag{3.119}$$

Lemma 3.8. *System of equations* (3.107) *as $n = 2$ with the complete set of integrals* (3.119) *is reduced to one of the following systems,*

$$u^i_{xy} = A_i(u, u_1)\bar A_i(u, \bar u_1) + \Phi^i_{kj}(u)u^k_1\bar u^j_1,\ i = 1, 2, \tag{3.120}$$

$$\begin{cases} u^1_{xy} = B_1(u, u_1)\bar B_1(u, \bar u_1) + \Psi^1_{kj}(u)u^k_1\bar u^j_1 \\ u^2_{xy} = \bar u^k_1\alpha_k(u)B_2(u, u_1) + u^k_1\beta_k(u)\bar B_2(u, \bar u_1) + \Psi^2_{kj}(u)u^k_1\bar u^j_1, \end{cases} \tag{3.121}$$

$$u^i_{xy} = \bar u^k_1\gamma_k(u)C_i(u, u_1) + u^k_1\delta_k(u)\bar C_i(u, \bar u_1) + \Sigma^i_{kj}(u)u^k_1\bar u^j_1,\ i = 1, 2. \tag{3.122}$$

Next, on the first order integrals we impose the conditions

$$\begin{aligned} &\left(\frac{\partial}{\partial u^1_1}\left(\frac{\omega^1_{u^1_1}}{\omega^1_{u^2_1}}\right)\right)^2 + \left(\frac{\partial}{\partial u^2_1}\left(\frac{\omega^1_{u^1_1}}{\omega^1_{u^2_1}}\right)\right)^2 \neq 0, \\ &\left(\frac{\partial}{\partial\bar u^1_1}\left(\frac{\bar\omega^1_{\bar u^1_1}}{\bar\omega^1_{\bar u^2_1}}\right)\right)^2 + \left(\frac{\partial}{\partial\bar u^2_1}\left(\frac{\bar\omega^1_{\bar u^1_1}}{\bar\omega^1_{\bar u^2_1}}\right)\right)^2 \neq 0, \end{aligned} \tag{3.123}$$

which mean that the integrals ω^1 and $\bar\omega^1$ are not reduced to $\omega^1 = W(p, q, p_1),\ \bar\omega^1 = \bar W(p, q, \bar p_1)$ by the point transformation $u^1 = \varphi(p, q),\quad u^2 = \psi(p, q)$.

Under conditions (3.123) with employing the equations $\bar{D}\omega^1 = 0$, $D\bar{\omega}^1 = 0$ it is possible to specify the right hand sides of systems (3.120)–(3.122). Namely, systems (3.120), (3.121) are reduced to

$$\begin{cases} u^1_{xy} = A(u,u_1)\bar{A}(u,\bar{u}_1) + \tilde{\Phi}^1_{kj}(u)u^k_1\bar{u}^j_1 \\ u^2_{xy} = \mu(u)A(u,u_1)\bar{A}(u,\bar{u}_1) + \bar{u}^k_1\varphi_k(u)A(u,u_1) + u^k_1\psi_k(u)\bar{A}(u,\bar{u}_1) + \\ \qquad + \tilde{\Phi}^2_{kj}(u)u^k_1\bar{u}^j_1, \end{cases} \tag{3.124}$$

and system (3.122) to

$$\begin{cases} u^1_{xy} = \bar{u}^k_1\chi^1_k(u)B(u,u_1) + u^k_1\epsilon^1_k(u)\bar{B}(u,\bar{u}_1) + \tilde{\Psi}^2_{kj}(u)u^k_1\bar{u}^j_1 \\ u^2_{xy} = \lambda(u)B(u,u_1)\bar{B}(u,\bar{u}_1) + \bar{u}^k_1\chi^2_k(u)B(u,u_1) + u^k_1\epsilon^2_k(u)\bar{B}(u,\bar{u}_1) + \\ \qquad + \tilde{\Psi}^2_{kj}(u)u^k_1\bar{u}^j_1. \end{cases} \tag{3.125}$$

Lemma 3.9. *Systems of equations* (3.124), (3.125) *with the complete set of integrals* (3.119) *satisfying condition* (3.123) *are reduced to the equations*

$$u^i_{xy} = -\Gamma^i_{kj}(u)u^k_1\bar{u}^j_1, \quad i = 1,2, \tag{3.126}$$

by point transformations.

For system (3.126) the x-characteristic Lie ring is generated by the operators

$$X_i = \frac{\partial}{\partial \bar{u}^i_1}, \quad X_3 = \bar{u}^p_1 Y_p,$$

where

$$Y_i = \frac{\partial}{\partial u^i} - \Gamma^p_{ki}u^k_1\frac{\partial}{\partial u^p_1} + \dots, \quad i = 1,2.$$

According to Theorem 3.2 if system of equations (3.126) possesses x-integrals (3.119), then $\dim A = 5$, which in its turn is equivalent to the fact that the vector fields Y_1, Y_2 and Y_3 ($Y_3 = [Y_1, Y_2]$) are linearly independent and

$$[Y_i, Y_3] = A_i(u,u_1,\bar{u}_1)Y_3. \tag{3.127}$$

Identity (3.127) can be rewritten as

$$[D,[Y_i,Y_3]] = A_i[D,Y_3] + D(A_i)Y_3. \tag{3.128}$$

Employing the equation $[D,\bar{D}] = 0$, we find

$$\begin{aligned} &[D, Y_i] = \Gamma^p_{kj}u^k_1 Y_p, \quad i = 1,2, \\ &[D, Y_3] = \tilde{R}^p_{k12}u^k_1 Y_p + (\Gamma^1_{k1} + \Gamma^2_{k2})u^k_1 Y_3. \end{aligned} \tag{3.129}$$

Now taking into consideration relations (3.127) and (3.129), we obtain that identity (3.128) is equivalent to the system

$$\begin{aligned} &\frac{\partial}{\partial u^i}\tilde{R}^p_{k12} + \tilde{R}^q_{k12}\Gamma^q_{qi} - \tilde{R}^p_{q12}\Gamma^q_{ki} = A_i(u)\tilde{R}^p_{k12}, \\ &\tilde{R}^2_{k12} + \frac{\partial}{\partial u^1}(\Gamma^1_{k1} + \Gamma^2_{k2}) - \Gamma^q_{k1}(\Gamma^1_{q1} + \Gamma^2_{q2}) = \frac{\partial}{\partial u^k}A_1(u) - \Gamma^q_{k1}A_q(u), \\ &-\tilde{R}^1_{k12} + \frac{\partial}{\partial u^2}(\Gamma^1_{k1} + \Gamma^2_{k2}) - \Gamma^q_{k2}(\Gamma^1_{q1} + \Gamma^2_{q2}) = \frac{\partial}{\partial u^k}A_2(u) - \Gamma^q_{k2}A_q(u). \end{aligned}$$

The last relations are necessary conditions for the existence of the x-integrals (3.119) for system of equations (3.126). In the same way one obtains the conditions for the existence of the y-integrals.

3.5. Quadratic systems of equations with first and second order integrals. In this subsection we consider the system of equations (3.126) (see [56]).

We note that under the transformation $u^i \to p^i(u^1, u^2),\ i = 1, 2$ system of equations (3.126) does not change the form, and the functions p^i can be chosen so that $\Gamma^1_{21} = \Gamma^1_{22} = 0$. In addition, we shall assume that $\Gamma^2_{11} = \Gamma^2_{21} = 0$. Hence, we consider the system of equations

$$u^1_{xy} = \Gamma^1_{1j} u^1_1 \bar{u}^j_1, \quad u^2_{xy} = \Gamma^2_{i2} u^i_1 \bar{u}^2_1 \tag{3.130}$$

with the complete set of integrals

$$\omega^1(u^1, u^2, u^1_1, u^2_1),\ \omega^2(u^1, u^2, u^1_1, u^2_1, u^1_2, u^2_2), \tag{3.131}$$

$$\bar{\omega}^1(u^1, u^2, \bar{u}^1_1, \bar{u}^2_1),\ \bar{\omega}^2(u^1, u^2, \bar{u}^1_1, \bar{u}^2_1, \bar{u}^1_2, \bar{u}^2_2). \tag{3.132}$$

The following statement holds.

Theorem 3.5. *System of equations* (3.130) *possesses the set of x-integrals* (3.131) *if and only if the relations*

$$\frac{\partial^2 \Gamma^2_{22}}{\partial u^1 \partial u^1} = \frac{\partial \Gamma^2_{22}}{\partial u^1} \cdot \frac{\partial \ln F}{\partial u^1}, \tag{3.133}$$

$$\frac{\partial^2 \Gamma^2_{22}}{\partial u^1 \partial u^2} = \frac{\partial \Gamma^2_{22}}{\partial u^1} \cdot \frac{\partial \ln F}{\partial u^2}, \tag{3.134}$$

$$-2\frac{\partial \Gamma^2_{22}}{\partial u^1} = \frac{\partial^2 \ln F}{\partial u^1 \partial u^2}, \tag{3.135}$$

$$\left(\frac{\partial}{\partial u^1} + \Gamma^1_{11} - \frac{\partial \ln F}{\partial u^1}\right)\left(\frac{\partial \Gamma^2_{12}}{\partial u^1} + \Gamma^1_{11}\Gamma^2_{12}\right) = 0, \tag{3.136}$$

$$-\Gamma^2_{22}\left(\frac{\partial \ln F}{\partial u^2} + \Gamma^2_{22}\right) = \frac{\partial}{\partial u^2}\left(\frac{\partial \ln F}{\partial u^2} + \Gamma^2_{22}\right), \tag{3.137}$$

$$\Gamma^2_{12}\left(F - \frac{\partial \Gamma^2_{22}}{\partial u^1}\right) - \left(\frac{\partial}{\partial u^2} - \Gamma^2_{22} + \Gamma^1_{12} - \frac{\partial \ln F}{\partial u^2}\right) \cdot \left(\frac{\partial \Gamma^2_{12}}{\partial u^1} + \Gamma^1_{11}\Gamma^2_{12}\right) = 0, \tag{3.138}$$

$$\left(\frac{\partial}{\partial u^2} + \Gamma^1_{12}\right)\left(\frac{\partial \ln F}{\partial u^1} + \Gamma^1_{11} + \Gamma^2_{12}\right) + \Gamma^2_{12}\left(\frac{\partial \ln F}{\partial u^2} + \Gamma^2_{22}\right) - F = 0, \tag{3.139}$$

$$\left(\frac{\partial}{\partial u^1} + \Gamma^1_{11}\right)\left(\frac{\partial \ln F}{\partial u^1} + \Gamma^1_{11} + \Gamma^2_{12}\right) + \frac{\partial \Gamma^2_{12}}{\partial u^1} + \Gamma^1_{11}\Gamma^2_{12} = 0 \tag{3.140}$$

hold, where

$$F(u^1, u^2) = \frac{\partial \Gamma^1_{12}}{\partial u^1} - \frac{\partial \Gamma^1_{11}}{\partial u^2}. \tag{3.141}$$

Considering the y-characteristic ring of system of equations (3.130), we obtain a “symmetric” version of Theorem 3.5.

Theorem 3.6. *System of equations* (3.130) *possesses the set of integrals* (3.132) *if and only if the relations*

$$\frac{\partial^2\Gamma^1_{11}}{\partial u^2\partial u^2} = \frac{\partial\Gamma^1_{11}}{\partial u^2}\cdot\frac{\partial\ln\bar F}{\partial u^2}, \tag{3.142}$$

$$\frac{\partial^2\Gamma^1_{11}}{\partial u^1\partial u^2} = \frac{\partial\Gamma^1_{11}}{\partial u^2}\cdot\frac{\partial\ln\bar F}{\partial u^1}, \tag{3.143}$$

$$-2\frac{\partial\Gamma^1_{11}}{\partial u^2} = \frac{\partial^2\ln\bar F}{\partial u^1\partial u^2}, \tag{3.144}$$

$$\left(\frac{\partial}{\partial u^2}+\Gamma^2_{22}-\frac{\partial\ln\bar F}{\partial u^2}\right)\left(\frac{\partial\Gamma^1_{12}}{\partial u^2}+\Gamma^2_{22}\Gamma^1_{12}\right)=0, \tag{3.145}$$

$$-\Gamma^1_{11}\left(\frac{\partial\ln\bar F}{\partial u^1}+\Gamma^1_{11}\right)=\frac{\partial}{\partial u^1}\left(\frac{\partial\ln\bar F}{\partial u^1}+\Gamma^1_{11}\right), \tag{3.146}$$

$$\Gamma^1_{12}\left(\bar F+\frac{\partial\Gamma^1_{11}}{\partial u^2}\right)+\left(\frac{\partial}{\partial u^1}-\Gamma^1_{11}+\Gamma^2_{12}-\frac{\partial\ln\bar F}{\partial u^1}\right)\cdot\left(\frac{\partial\Gamma^1_{12}}{\partial u^2}+\Gamma^2_{22}\Gamma^1_{12}\right)=0, \tag{3.147}$$

$$\left(\frac{\partial}{\partial u^1}+\Gamma^2_{12}\right)\left(\frac{\partial\ln\bar F}{\partial u^2}+\Gamma^1_{12}+\Gamma^2_{22}\right)+\Gamma^1_{12}\left(\frac{\partial\ln\bar F}{\partial u^1}+\Gamma^1_{11}\right)+\bar F=0, \tag{3.148}$$

$$\left(\frac{\partial}{\partial u^2}+\Gamma^2_{22}\right)\left(\frac{\partial\ln\bar F}{\partial u^2}+\Gamma^1_{12}+\Gamma^2_{22}\right)+\frac{\partial\Gamma^1_{12}}{\partial u^2}+\Gamma^2_{22}\Gamma^1_{12}=0, \tag{3.149}$$

hold, where

$$\bar F = \frac{\partial\Gamma^2_{22}}{\partial u^1}-\frac{\partial\Gamma^2_{12}}{\partial u^2}. \tag{3.150}$$

Thus, according to Theorems 3.5, 3.6 the classification of integrable system of equations (3.130) is reduced to the study of the compatibility for equations (3.133)–(3.140), (3.142)–(3.149) w.r.t. unknowns $\Gamma^1_{11},\Gamma^1_{12},\Gamma^2_{12},\Gamma^2_{22}$.

Theorem 3.7. *Assume the condition*

$$\frac{\partial\Gamma^1_{11}}{\partial u^2}\cdot\frac{\partial\Gamma^2_{22}}{\partial u^1}\neq 0 \tag{3.151}$$

hold. Then system (3.130) *with the complete set of integrals* (3.131), (3.132) *is reduced to one of the following types,*

$$\begin{gathered} u^1_{xy}=\frac{u^1_1\bar u^1_1}{X}+\left(\frac{1}{X}+\frac{1}{\alpha Y}\right)u^1_1\bar u^2_1,\quad u^2_{xy}=\frac{u^2_1\bar u^2_1}{Y}+\left(\frac{1}{\alpha X}+\frac{1}{\alpha^2 Y}\right)u^1_1\bar u^2_1,\\ X=u^1+u^2+c,\quad Y=\frac{u^1}{\alpha^2}+u^2-c,\end{gathered} \tag{3.152}$$

or

$$\begin{gathered} u^1_{xy}=\frac{u^2}{X}u^1_1\bar u^1_1+\left(\frac{1}{X}+\frac{1}{\alpha Y}\right)u^1u^1_1\bar u^2_1,\quad u^2_{xy}=\frac{u^1}{Y}u^2_1\bar u^2_1+\left(\frac{\alpha}{X}+\frac{1}{Y}\right)u^2u^1_1\bar u^2_1,\\ X=u^1u^2+d_2,\quad Y=u^1u^2+c_2,\quad \frac{\alpha+1}{\alpha}d_2=(\alpha+1)c_2,\end{gathered} \tag{3.153}$$

where c is an arbitrary constant, c_2, d_2, α are non-zero constants.

To solve the complete classification problem for systems of equations (3.130), it remains to treat the case when condition (3.151) is broken.

Lemma 3.10. *Let the condition*

$$\frac{\partial\Gamma^2_{22}}{\partial u^1}\cdot\frac{\partial\Gamma^1_{11}}{\partial u^2}=0$$

holds true, then there exist no systems of equations (3.130) *with the complete set of integrals* (3.131)*,* (3.132)*.*

Consider now the problem of constructing $x-$ and $y-$integrals for systems of equations (3.152), (3.153).

We note that the change $u^1 \to u^1 + \dfrac{2\alpha^2}{1-\alpha^2}c$, $u^2 \to u^2 - \dfrac{1+\alpha^2}{1-\alpha^2}c$ reduce system of equations (3.152) as $\alpha \neq 1$ to the system with zero constant c.

The following statements hold.

Theorem 3.8. *The integrals of system of equations* (3.152) *are given by the formulas, as* $\alpha = 1$,

$$\omega^1 = 2u^2 - \frac{u_1^2}{z} + 2c\ \ln z,\ \bar{\omega}^1 = 2u^1 - \frac{\bar{u}_1^1}{\bar{z}} - 2c\ \ln \bar{z}, \tag{3.154}$$

$$\omega^2 = \frac{z_1}{z} - z,\quad \bar{\omega}^2 = \frac{\bar{z}_1}{\bar{z}} - \bar{z}, \tag{3.155}$$

and as $\alpha \neq 1$ $(c = 0)$,

$$\omega^1 = \left(\frac{1}{\alpha} + 1\right) u^2 z^{1-\alpha} - u_1^2 z^{-\alpha},\quad \bar{\omega}^1 = \left(\frac{1}{\alpha} + 1\right) u^1 \bar{z}^{1-\alpha} - \bar{u}_1^1 \bar{z}^{-\alpha}, \tag{3.156}$$

$$\omega^2 = \frac{z_1}{z} - \frac{z}{\alpha},\quad \bar{\omega}^2 = \frac{\bar{z}_1}{\bar{z}} - \frac{\bar{z}}{\alpha}, \tag{3.157}$$

where

$$z = \frac{u_1^1}{X},\ \bar{z} = \frac{\bar{u}_1^2}{Y},\ z_1 = \frac{\partial z}{\partial x},\ \bar{z}_1 = \frac{\partial \bar{z}}{\partial y}.$$

Theorem 3.9. *The the integrals of systems of equations* (3.153) *are given by the formulas, as* $\alpha = -1$,

$$\omega^1 = \frac{(u^2)^2 z^2}{2}(d_2 - c_2) - c_2 u_1^2 z,\ \bar{\omega}^1 = \frac{(\bar{u}^1)^2 \bar{z}^2}{2}(c_2 - d_2) - d_2 \bar{u}_1^1 \bar{z}, \tag{3.158}$$

$$\omega^2 = \frac{z_1}{z} + \frac{d_2}{c_2} u^2 z,\ \bar{\omega}^2 = \frac{\bar{z}_1}{\bar{z}} + \frac{c_2}{d_2} u^1 \bar{z}, \tag{3.159}$$

and as $\alpha \neq -1$,

$$\omega^1 = \frac{u_1^2 - (u^2)^2 z\alpha}{z^\alpha},\ \bar{\omega}^1 = \frac{\bar{u}_1^1 - \dfrac{(u^1)^2 \bar{z}}{\alpha}}{\bar{z}^{\frac{1}{\alpha}}}, \tag{3.160}$$

$$\omega^2 = u^2 z - \frac{z_1}{z},\ \bar{\omega}^2 = u^1 \bar{z} - \frac{\bar{z}_1}{\bar{z}}, \tag{3.161}$$

where

$$z = \frac{u_1^1}{X},\ \bar{z} = \frac{\bar{u}_1^2}{Y},\ z_1 = \frac{\partial z}{\partial x},\ \bar{z}_1 = \frac{\partial \bar{z}}{\partial y}.$$

Theorem 3.10. *The general solution of system of equations* (3.152) *are given by the formulas,*

as $\alpha = 1$,

$$\begin{aligned} u^1(x,y) &= \tfrac{A(x)+B(y)}{(C(x)+D(y))^2} + c\ \ln \tfrac{1}{C(x)+D(y)} - \\ &\quad - \tfrac{B'(y)}{D'(y)(C(x)+D(y))} + \tfrac{c}{2}, \\ u^2(x,y) &= \tfrac{A(x)+B(y)}{(C(x)+D(y))^2} - c\ \ln \tfrac{1}{C(x)+D(y)} - \\ &\quad - \tfrac{A'(x)}{C'(x)(C(x)+D(y))} - \tfrac{c}{2}, \end{aligned} \tag{3.162}$$

and as $\alpha \neq 1$ $(c = 0)$,

$$\begin{aligned} u^1(x,y) &= \frac{\alpha A(x) + B(y)}{\alpha (C(x)+D(y))^{\alpha+1}} - \frac{B'(y)}{\alpha D'(y)(C(x)+D(y))^{\alpha}}, \\ u^2(x,y) &= \frac{A(x) + \alpha B(y)}{\alpha (C(x)+D(y))^{\alpha+1}} - \frac{A'(x)}{\alpha C'(x)(C(x)+D(y))^{\alpha}}. \end{aligned} \tag{3.163}$$

Theorem 3.11. *The general solution to system of equations* (3.153) *are given by the formulas,*

as $\alpha = -1$ *and* $c_2 + d_2 = 0$,

$$\begin{aligned} u^1(x,y) &= \left(\frac{B'(y)}{Y'(y)} + X(x)\right) e^{-A(x)-B(y)-X(x)Y(y)}, \\ u^2(x,y) &= -d_2 \left(\frac{A'(x)}{X'(x)} + Y(y)\right) e^{A(x)+B(y)+X(x)Y(y)}, \end{aligned} \tag{3.164}$$

as $\alpha = -1$ *and* $c_2 + d_2 \neq 0$,

$$\begin{aligned} u^1(x,y) &= \left(\tfrac{2d_2}{c_2+d_2} \cdot \tfrac{X(x)}{X(x)Y(y)+c} - \tfrac{\bar W'(y)}{\bar W(y) Y'(y)}\right) \times \\ &\quad \times (X(x)Y(y)+c)^{\frac{2c_2}{c_2+d_2}} \tfrac{\bar W(y)}{W(x)}, \\ u^2(x,y) &= \left(\tfrac{2c_2}{c_2+d_2} \cdot \tfrac{Y(y)}{X(x)Y(y)+c} - \tfrac{W'(x)}{W(x) X'(x)}\right) \times \\ &\quad \times (X(x)Y(y)+c)^{\frac{2d_2}{c_2+d_2}} \tfrac{W(x)}{\bar W(y)}, \end{aligned} \tag{3.165}$$

where

$$c = \frac{c_2 + d_2}{2},$$

and as $\alpha \neq -1$,

$$\begin{aligned} u^1(x,y) &= -\left(A(y) - (1+\alpha)B(y)D(x) - (1+\alpha)E(x)\right)^{-\frac{1}{1+\alpha}} \times \\ &\quad \times \tfrac{\alpha}{1+\alpha} \cdot \tfrac{c_2}{B'(y)} \left(A'(y) - (1+\alpha)B'(y)D(x)\right), \\ u^2(x,y) &= (A(y) - (1+\alpha)B(y)D(x) - \\ &\quad - (1+\alpha)E(x))^{-\frac{\alpha}{1+\alpha}} \left(B(y) + \tfrac{E'(x)}{D'(x)}\right). \end{aligned} \tag{3.166}$$

3.6. Linearization of exponential systems of rank 2. We consider the systems of equations (see [26])

$$u_{xy} = a_{i1} e^{u^1} + \ldots + a_{in} e^{u^n}, \; i = 1, 2, \ldots, n. \tag{3.167}$$

In the case $n = 2$

$$u_{xy} = a_{11} e^u + a_{12} e^v, \quad v_{xy} = a_{21} e^u + a_{22} e^v. \tag{3.168}$$

To solve the classification problem, we study the structure of the characteristic ring for the linearization of system of equations (3.168).

The linearization of the system of equations (3.168) reads as

$$p_{xy} = a_{11} e^u p + a_{12} e^v q, \quad q_{xy} = a_{21} e^u p + a_{22} e^v q. \tag{3.169}$$

In what follows we assume that u and v are given functions and $\Delta = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Let us define the $x-$ and $y-$characteristic Lie rings for the system of equations (3.169). The operator $\bar D$ on the space of locally analytic functions depending on a finite number of independent variables $x, y, p, q, p_1, q_1, p_2, q_2 \ldots$ acts as

$$\bar D = \bar p_1 Y_1^{(0)} + \bar q_1 Y_2^{(0)} + X_1,$$

where

$$Y_1^{(0)} = \frac{\partial}{\partial p}, \; Y_2^{(0)} = \frac{\partial}{\partial q},$$

$$X_1 = \frac{\partial}{\partial y} + (a_{11}e^u p + a_{12}e^v q)\frac{\partial}{\partial p_1} + (a_{21}e^u p + a_{22}e^v q)\frac{\partial}{\partial q_1} + \dots.$$

The x-characteristic Lie ring of system of equations (3.169) is the ring A generated by the vector fields $Y_1^{(0)},\ Y_2^{(0)},\ X_1$. In the same was the y-characteristic Lie ring $\bar{A}$ is defined.

Lemma 3.11. *Let*

$$Z = \sum_{i=1}^{\infty} \alpha_i \frac{\partial}{\partial p_i} + \sum_{i=1}^{\infty} \beta_i \frac{\partial}{\partial q_i},\ \ \alpha_i, \beta_i \in F,\ \ i = 1, 2, \dots.$$

Then the relation $[D, Z] = 0$ holds true if and only if $Z = 0$.

Consider the commutators

$$Y_1^{(1)} = [Y_1^{(0)}, X_1] = e^u[a_{11}\frac{\partial}{\partial p_1} + a_{21}\frac{\partial}{\partial q_1} + a_{11}u_1\frac{\partial}{\partial p_2} + a_{21}u_1\frac{\partial}{\partial q_2} + \dots],$$
$$Y_2^{(1)} = [Y_2^{(0)}, X_1] = e^v[a_{12}\frac{\partial}{\partial p_1} + a_{22}\frac{\partial}{\partial q_1} + a_{12}v_1\frac{\partial}{\partial p_2} + a_{22}v_1\frac{\partial}{\partial q_2} + \dots].$$

We introduce the notations

$$Y_1^{(0)} = Z_1^{(0)},\ Y_2^{(0)} = Z_2^{(0)},$$
$$Y_1^{(1)} = e^u Z_1^{(1)},\ Y_2^{(1)} = e^v Z_2^{(1)}.$$

Next, we define

$$Z_1^{(n+1)} = [Z_1^{(n)}, X_1],\ Z_2^{(n+1)} = [Z_2^{(n)}, X_1],\ n = 1, 2, \dots.$$

We note that the vector fields $X_1,\ Z_1^{(0)},\ Z_2^{(0)},\ Z_1^{(1)},\ Z_2^{(1)}$ are linearly independent. In view of the last notations the operator $\bar{D}$ becomes

$$\bar{D} = \bar{p}_1 Z_1^{(0)} + \bar{q}_1 Z_2^{(0)} + X_1.$$

It is easy to check that

$$[D, Z_1^{(0)}] = [D, Z_2^{(0)}] = 0,$$
$$[Z_1^{(i)}, [D, X_1]] = [Z_2^{(i)}, [D, X_1]] = 0,\ i = 1, 2, \dots.$$

The formulas

$$\begin{aligned}
&[D, X_1] = -(a_{11}e^u p + a_{12}e^v q)Z_1^{(0)} - (a_{21}e^u p + a_{22}e^v q)Z_2^{(0)},\\
&[D, Z_1^{(1)}] = -u_1 Z_1^{(1)} - a_{11}Z_1^{(0)} - a_{21}Z_2^{(0)},\\
&[D, Z_2^{(1)}] = -v_1 Z_2^{(1)} - a_{12}Z_1^{(0)} - a_{22}Z_2^{(0)},\\
&[D, Z_1^{(2)}] = -u_1 Z_1^{(2)} + a_{12}e^v Z_1^{(1)} - a_{21}e^v Z_2^{(1)},\\
&[D, Z_2^{(2)}] = -v_1 Z_2^{(2)} - a_{12}e^u Z_1^{(1)} + a_{21}e^u Z_2^{(1)}
\end{aligned} \tag{3.170}$$

hold true.

Lemma 3.12. *The operators $Z_1^{(2)}$ and $Z_2^{(2)}$ satisfy the relation*

$$e^u Z_1^{(2)} + e^v Z_2^{(2)} = 0. \tag{3.171}$$

For the sake of convenience in what follows we introduce the notations

$$Z_1^{(0)} = W_1^{(0)},\ Z_2^{(0)} = W_2^{(0)},\ Z_1^{(1)} = W_1^{(1)},\ Z_2^{(1)} = W_2^{(1)},$$
$$Z_1^{(2)} = e^v W_1^{(2)},\ Z_2^{(2)} = e^u W_2^{(2)}.$$

We define

$$W_1^{(n+1)} = [W_1^{(n)}, X_1],\ n = 2, 3, \dots.$$

At that, it is easy to show the validity of the identities

$$[W_1^{(n)}, [D, X_1]] = 0,$$
$$[D, W_1^{(n+1)}] = -[X_1, [D, W_1^{(n)}]].$$

We observe that the vector fields $X_1,\ W_1^{(0)},\ W_2^{(0)},\ W_1^{(1)},\ W_2^{(1)},\ W_1^{(2)}$ are linearly independent and the operators $W_2^{(2)}$ and $W_1^{(2)}$ are related by the formula

$$W_2^{(2)} = -W_1^{(2)}.$$

Lemma 3.13. *The relation*

$$\begin{aligned}
[D, W_1^{(n)}] &= -(u_1 + v_1)W_1^{(n)} + \\
&+ \sum_{i=2}^{n-1} (-1)^{n-i-1} C_{n-2}^{i-2} X_1^{n-i}(u_1 + v_1)W_1^{(i)} + \\
&+ \sum_{i=2}^{n-1} (-1)^{n-i-1} C_{n-3}^{i-2} X_1^{n-i-1}(a_{12}e^v + a_{21}e^u)W_1^{(i)},\ n = 3, 4, \ldots,
\end{aligned} \tag{3.172}$$

holds true.

Suppose now that the characteristic Lie ring of system of equations (3.169) is finite-dimensional. It means that there exists $n \geq 2$, for which the operators $X_1, W_1^{(0)},\ W_2^{(0)},\ W_1^{(1)},\ W_2^{(1)},\ W_1^{(2)},\ W_1^{(3)}, \ldots\ W_1^{(n)}$ form a basis of this ring. Then the operator $W_1^{(n+1)}$ is a linear combination of the elements of this basis.

Since

$$W_1^{(0)} = \frac{\partial}{\partial p},\quad W_2^{(0)} = \frac{\partial}{\partial q},$$

and the higher order operators have the structure

$$\alpha_i \frac{\partial}{\partial p_i} + \beta_i \frac{\partial}{\partial q_i} + \ldots,\ i = 1, 2, \ldots,$$

then

$$W_1^{(n+1)} = \sum_{k=1}^{n} A_k W_1^{(k)} + B_1 W_2^{(1)},$$

where $A_k,\ B_1$ are functions of the variables $u, v, u_1, v_1, \bar{u}_1, \bar{v}_1, \ldots$.
The last relation is equivalent to the identity

$$[D, W_1^{(n+1)}] = \sum_{k=1}^{n} D(A_k)W_1^{(k)} + D(B_1)W_2^{(1)} + \sum_{k=1}^{n} A_k[D, W_1^{(k)}] + B_1[D, W_2^{(1)}].$$

By Lemma 3.13 we obtain

$$\begin{gathered}
D(A_1)W_1^{(1)} + D(B_1)W_2^{(1)} + A_1(-u_1W_1^{(1)} - a_{11}W_1^{(0)} - a_{21}W_2^{(0)}) + \\
+ A_2(a_{12}W_1^{(1)} - a_{21}W_2^{(1)}) + B_1(-v_1W_2^{(1)} - a_{12}W_1^{(0)} - a_{22}W_2^{(0)}) = 0.
\end{gathered}$$

Comparing the coefficients at the vector field $W_1^{(0)}, W_2^{(0)}, W_1^{(1)}, W_2^{(1)}$ in the left and right hand sided of the last identity, we get the system

$$\begin{gathered}
-a_{11}A_1 - a_{12}B_1 = 0, \\
-a_{21}A_1 - a_{22}B_1 = 0, \\
D(A_1) + a_{12}A_2 - u_1A_1 = 0, \\
D(B_1) - a_{21}A_2 - v_1B_1 = 0.
\end{gathered}$$

It implies $A_1 = B_1 = 0$ and $A_2 = 0$. Thus, we have proven the following statement.

Lemma 3.14. *The x-characteristic algebra A of system of equations* (3.169) *is finite-dimensional if and only if either $W_1^{(3)} = 0$ or*

$$W_1^{(n+1)} = \sum_{k=3}^{n} A_k W_1^{(k)}, \;\; A_k = A_k(u, v, u_1, v_1, \bar{u}_1, \bar{v}_1, \ldots), \;\; n = 3, 4, \ldots.$$

At that, either $\dim A = 6$ *or* $\dim A = n + 4$, $n = 3, 4, \ldots$, *respectively.*

Employing now Lemmas 3.13 and 3.14, let us right down necessary and sufficient conditions for the characteristic ring of the system (3.169) to be finite-dimensional.

As $\dim A = 6$, we obtain

$$X_1(u_1 + v_1) + a_{12}e^v + a_{21}e^u = 0, \tag{3.173}$$

and in the case $\dim A = n + 4$ ($n \geq 3$) we have

$$\begin{gathered}
(-1)^{n-2} X_1^{n-2}((a_{11} + 2a_{21})e^u + (a_{22} + 2a_{12})e^v) = \\
= \sum_{p=3}^{n} A_p (-1)^{p-3} X_1^{p-3}((a_{11} + 2a_{21})e^u + (a_{22} + 2a_{12})e^v), \\
(-1)^{n-i} \left(C_{n-1}^{i-2} X_1^{n-i+1}(u_1 + v_1) + C_{n-2}^{i-2} X_1^{n-i}(a_{12}e^v + a_{21}e^u) \right) = \\
= \sum_{p=i+1}^{n} A_p (-1)^{p-i-1} \left(C_{p-2}^{i-2} X_1^{p-i}(u_1 + v_1) + C_{p-3}^{i-2} X_1^{p-i-1}(a_{12}e^v + a_{21}e^u) \right) + \\
+ D(A_i), \;\; i = 3, 4, \ldots, n-1, \\
(n-1) X_1(u_1 + v_1) + a_{12}e^v + a_{21}e^u = D(A_n).
\end{gathered} \tag{3.174}$$

It can be shown that for system (3.174) the unknowns A_i are the functions of the variables $\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_{n-i+1}, \bar{v}_{n-i+1}$, $i = 3, 4, \ldots, n-1$.

Theorem 3.12. *If the characteristic Lie algebra of system of equations* (3.169) *is finite-dimensional, then system* (3.168) *is reduced to*

$$u_{xy} = 2e^u + a_{12}e^v, \quad v_{xy} = -e^u + 2e^v. \tag{3.175}$$

We proceed to systems (3.175).

We remind that the Lie algebra A for linearized system of equations (3.169) is generated by the vector fields X_1, $W_1^{(0)}$, $W_2^{(0)}$, $W_1^{(1)}$, $W_2^{(1)}$, $W_1^{(2)}$ and thus $\dim A \geq 6$.

In what follows we study the systems of equations for which $\dim A \leqslant 9$.

Theorem 3.13. *The dimension of the x-characteristic algebra A for linearized system of equations* (3.169) *does not exceed* 9 *if and only if the coefficient* a_{12} *takes one of the values* -1, -2, *or* -3. *At that,* $\dim A = 6$, 7, 9, *respectively.*

We have obtained all the equations for which the dimension of the characteristic ring of the linearization does not exceed 9. It has been shown that the right hand sides of these equations are determined by the Cartan matrices of a simple Lie algebra.

4. Differential-difference hyperbolic equations

In this section we consider the chains of differential-difference equations

$$t_x(n+1) = f(t(n), t(n+1), t_x(n)), \tag{4.176}$$

where an unknown function $t = t(n, x)$ depends on a discrete variable n and a continuous variable x. Chain (4.176) can be regarded as an infinite system of ordinary differential equations with the sequence of unknown functions $\{t(n)\}_{n=-\infty}^{n=+\infty}$. The function $f(t, t_1, t_x)$ is assumed to be locally analytic w.r.t. all three arguments, and in a some domain the condition

$$\frac{\partial f}{\partial t_x} \neq 0 \tag{4.177}$$

holds true. We use the subscript to indicate the shift of the discrete argument $t_k = t(n+k, x)$ ($t_0 = t$), and also to denote the derivatives w.r.t. x,

$$t_x = \frac{\partial}{\partial x} t(n,x), \quad t_{xx} = \frac{\partial^2}{\partial x^2} t(n,x).$$

Denote by D and D_x the shift operator and the operator of total derivative w.r.t. x, respectively. For instance, $Dh(n,x) = h(n+1,x)$ and $D_x h(n,x) = \frac{\partial}{\partial x} h(n,x)$. As the dynamical variables we choose the variables $\{t_k\}_{k=-\infty}^{\infty}$ and $\{D_x^m t\}_{m=1}^{\infty}$. Below we regard the dynamical variables as independent.

4.1. Liouville type differential-difference equations. The functions I and F depending on x and finite number of dynamical variables are called respectively n- and x-integrals of the equation (4.176) if the identities $DI = I$ and $D_x F = 0$ hold. The integrals $I = I(x)$, $F = const$ are called trivial integrals.

Definition 4.1. *Chain* (4.176) *is called Darboux integrable if it possesses non-trivial x- and n-integrals.*

It should be noted that a Darboux integrable chain is reduced to a pair of equations, an ordinary difference and an ordinary differential equations. Indeed, it follows from the definition that an n-integral can depend only on x and an x-integral only on n. This is why each solution of chain (4.176) satisfies two equations

$$I(x, t, t_x, t_{xx}, \ldots) = p(x), \quad F(x, t, t_{\pm 1}, t_{\pm 2}, \ldots) = q(n)$$

with appropriately chosen functions $p(x)$ and $q(n)$.

At present discrete nonlinear models have important applications in physics and are actively studied. The detailed discussion of the applications and the overview of the literature can be found in the works [1, 23, 58, 62].

In this chapter we suggest an algorithm for classification of Darboux integrable chains (4.176) based on the notion of the characteristic Lie ring (see [43, 50–54]).

We introduce the notion of the characteristic ring L_n of chain (4.176) in the direction of n. We observe that

$$D^{-j} \frac{\partial}{\partial t_1} D^j I = 0 \tag{4.178}$$

for any n-integral and $j \geq 1$. Indeend, the identity $DI = I$ can be rewritten in the expanded form,

$$I(x, t_1, f, f_x, f_{xx}, \ldots) = I(x, t, t_x, t_{xx}, \ldots). \tag{4.179}$$

The left hand side of the last identity depends on the variable t_1, while the right hand side does not. Therefore,

$$\frac{\partial}{\partial t_1} DI = 0$$

that yields

$$D^{-1} \frac{\partial}{\partial t_1} DI = 0.$$

Arguing in this way, it is easy to obtain formula (4.178). We introduce the vector fields

$$Y_j = D^{-j} \frac{\partial}{\partial t_1} D^j, \quad j \geq 1 \tag{4.180}$$

and

$$X_j = \frac{\partial}{\partial t_{-j}}, \quad j \geq 1. \tag{4.181}$$

Thus, we see that each n-integral I lies in the kernel of the operators X_j and Y_j for each $j \geq 1$. The next theorem contains the definition of the characteristic ring L_n for (4.176) (see [43]).

Theorem 4.1. *If equation* (4.176) *possesses a non-trivial n-integral, then the following two conditions hold,*

- the linear span of the operators $\{Y_j\}_{j=1}^{\infty}$ has a finite dimension. Denote this dimension by N.

- the Lie ring L_n over field of locally differentiable functions generated by the operators $Y_1, Y_2, \ldots, Y_N, X_1, X_2, \ldots, X_N$, has a finite dimension, We call L_n a characteristic Lie ring in the direction of n.

Let us introduce the notion of a characteristic ring L_x for chain (4.176) in the direction of x. In order to it, we observe that by condition (4.177) chain (4.176) can be rewritten as

$$t_x(n-1) = g(t(n), t(n-1), t_x(n)).$$

By definition, the x-integral $F(x, t, t_{\pm 1}, t_{\pm 2}, \ldots)$ satisfies the equation $D_x F = 0$, i.e., $K_0 F = 0$, where

$$K_0 = \frac{\partial}{\partial x} + t_x \frac{\partial}{\partial t} + f \frac{\partial}{\partial t_1} + g \frac{\partial}{\partial t_{-1}} + f_1 \frac{\partial}{\partial t_2} + g_{-1} \frac{\partial}{\partial t_{-2}} + \ldots. \tag{4.182}$$

But since F can not depend on t_x, we get $XF = 0$, where

$$X = \frac{\partial}{\partial t_x}. \tag{4.183}$$

Then it is obvious that F lies in the kernel of each operator in the Lie ring generated by the pair of the operators X and K_0 over field of locally analytic functions.

It is possible to prove the following important statement (see [9]).

Theorem 4.2. *Chain* (4.176) *possesses a non-trivial x-integral if and only if its characteristic Lie ring L_x has a finite dimension.*

4.2. Classification of Darboux integrable chains of special form. Consider the problem on description of all chains

$$t_{1x} = t_x + d(t, t_1), \tag{4.184}$$

possessing nontrivial x- and n-integrals. The complete list of chains (4.184) possessing x-integrals is provided in the next theorem.

Theorem 4.3. *Chain* (4.184) *possesses a nontrivial x-integral if and only if $d(t, t_1)$ belongs to one of the classes*

(1) $d(t, t_1) = A(t - t_1)$,

(2) $d(t, t_1) = c_0(t - t_1)t + c_2(t - t_1)^2 + c_3 t - c_3 t_1$,

(3) $d(t, t_1) = A(t - t_1)e^{\alpha t}$,

(4) $d(t, t_1) = c_4(e^{\alpha t_1} - e^{\alpha t}) + c_5(e^{-\alpha t_1} - e^{-\alpha t})$,

where $A = A(t - t_1)$, $c_i = const, i = 0, \ldots, 5$, $c_0 \neq 0, c_4 \neq 0, c_5 \neq 0$ and $\alpha - const$, $\alpha \neq 0$.

At that, the x-integrals read as

(i) $F = x + \int^{\tau} \frac{du}{A(u)}$ *if* $A(u) \neq 0$ *and* $F = t_1 - t$ *if* $A(u) \equiv 0$,

(ii) $F = \frac{1}{-c_2 - c_0} \ln\left|\frac{-(c_2+c_0)\tau_1}{\tau_2} + c_2\right| + \frac{1}{c_2} \ln\left|\frac{c_2\tau_1}{\tau} - c_2 - c_0\right|$ *as* $c_2(c_2 + c_0) \neq 0$, $F = \ln \tau_1 - \ln \tau_2 + \frac{\tau_1}{\tau}$ *as* $c_2 = 0$ *and* $F = \frac{\tau_1}{\tau_2} - \ln \tau + \ln \tau_1$ *as* $c_2 = -c_0$,

(iii) $F = \int^{\tau} e^{-\alpha u} \frac{du}{A(u)} - \int^{\tau_1} \frac{du}{A(u)}$,

(iv) $F = \frac{(e^{\alpha t} - e^{\alpha t_2})(e^{\alpha t_1} - e^{\alpha t_3})}{(e^{\alpha t} - e^{\alpha t_3})(e^{\alpha t_1} - e^{\alpha t_2})}$,

where $\tau = t - t_1$, $\tau_1 = t_1 - t_2$, $\tau_2 = t_2 - t_3$.

Let us discuss some necessary conditions for the existence of x-integral. Denote by F the class of locally analytic functions each of those depends on a finite number of dynamical variables. In particular, we obtain $f(t, t_1, t_x) \in F$. In what follows we shall deal with the vector fields defined as formal series

$$Y = \sum_{-\infty}^{\infty} y_k \frac{\partial}{\partial t_k} \tag{4.185}$$

with the coefficients $y_k \in F$. Let us specify how the linear dependence and linear independence is understood for vector fields (4.185). Let P_N be the projector defined in the class of formal series (4.185),

$$P_N(Y) = \sum_{-N}^{N} y_k \frac{\partial}{\partial t_k}. \tag{4.186}$$

Consider first the vector fields defined by a finite sum,

$$Z = \sum_{-N}^{N} z_k \frac{\partial}{\partial t_k}. \tag{4.187}$$

Vector fields $Z_1, Z_2, \ldots, Z_m$ of the form (4.187) are linearly dependent in a some open domain Ω if there exists a set of function $\lambda_1, \lambda_2, \ldots, \lambda_m$ defined in Ω such that the function $|\lambda_1|^2 + |\lambda_2|^2 + \ldots + |\lambda_m|^2$ is not identically zero and for all points of the domain Ω the identity

$$\lambda_1 Z_1 + \lambda_2 Z_2 + \ldots + \lambda_m Z_m = 0 \tag{4.188}$$

holds true.

We call the set of vector fields $Y_1, Y_2, \ldots, Y_m$ of the form (4.185) linearly dependent in the domain Ω if for each natural N the set of the vector fields $P_N(Y_1), P_N(Y_2), \ldots, P_N(Y_m)$ defined by finite sums is linearly dependent in this domain. Otherwise the set $Y_1, Y_2, \ldots, Y_m$ is called linearly independent.

In an obvious was the definition of linear dependence of vector fields implies the following statement.

Remark 4.1. *If a vector field Y is a linear combination*

$$Y = \lambda_1 Y_1 + \lambda_2 Y_2 + \ldots + \lambda_m Y_m, \tag{4.189}$$

where the vector fields $Y_1, Y_2, \ldots, Y_m$ are linearly independent in Ω, and the coefficients of all vector fields $Y, Y_1, Y_2, \ldots, Y_m$ belong to F and are defined in Ω, then the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ also belong to F.

Let us return back to chains (4.184). In this case the ring L_x splits into the direct sum of two subrings. Indeed, since $f = t_x + d$ and $g = t_x - d_{-1}$, then $f_k = t_x + d + \sum_{j=1}^{k} d_j$ and $g_{-k} = t_x - \sum_{j=1}^{k+1} d_{-k}$ as $k \geq 1$, where $d = d(t, t_1), d_j = d(t_j, t_{j+1})$. This is why it is easy to see that $K_0 = t_x \widetilde{X} + Y$, where

$$\widetilde{X} = \frac{\partial}{\partial t} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_{-1}} + \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_{-2}} + \ldots \tag{4.190}$$

and

$$Y = \frac{\partial}{\partial x} + d\frac{\partial}{\partial t_1} - d_{-1}\frac{\partial}{\partial t_{-1}} + (d + d_1)\frac{\partial}{\partial t_2} - (d_{-1} + d_{-2})\frac{\partial}{\partial t_{-2}} + \ldots. \tag{4.191}$$

It follows from the relations $[X, \widetilde{X}] = 0$ and $[X, Y] = 0$ that $\widetilde{X} = [X, K_0] \in L_x$ so that $Y \in L_x$. Hence, $L_x = \{X\} \oplus L_{x1}$, where L_{x1} is the Lie ring generated by the operators $\widetilde{X}$ and Y.

Lemma 4.1. *If equation* (4.184) *has a nontrivial x-integral, then it has x-integral independent explicitly on x.*

Proof. Suppose there exists a nontrivial x-integral of chain (4.184). Then the Lie ring L_x is finite-dimensional. We choose its basis as follows,

$$T_1 = \frac{\partial}{\partial x} + \sum_{k=-\infty}^{\infty} a_{1,k} \frac{\partial}{\partial t_k}, \quad T_j = \sum_{k=-\infty}^{\infty} a_{j,k} \frac{\partial}{\partial t_k}, \quad 2 \leqslant j \leqslant N.$$

Moreover, there exists a x-integral $F(x, t, t_1, \ldots, t_{N-1})$ satisfying the system of equations

$$\frac{\partial F}{\partial x} + \sum_{k=0}^{N-1} a_{1,k} \frac{\partial F}{\partial t_k} = 0, \quad \sum_{k=0}^{N-1} a_{j,k} \frac{\partial F}{\partial t_k} = 0, \quad 2 \leqslant j \leqslant N.$$

Due to the known Jacobi theore (see [30]), there exists a change of variables $\theta_j = \theta_j(t, t_1, \ldots, t_{N-1})$ reducing the system to

$$\frac{\partial F}{\partial x} + \sum_{k=0}^{N-1} \widetilde{a}_{1,k} \frac{\partial F}{\partial \theta_k} = 0, \quad \frac{\partial F}{\partial \theta_k} = 0, \quad 2 \leqslant j \leqslant N-2,$$

which is equivalent to the equation

$$\frac{\partial F}{\partial x} + \widetilde{a}_{1,N-1} \frac{\partial F}{\partial t_{N-1}} = 0$$

for $F = F(x, \theta_{N-1})$.

Here two cases are possible, (1) $\widetilde{a}_{1,N-1} = 0$ and (2) $\widetilde{a}_{1,N-1} \neq 0$. In the case (1) we find $\frac{\partial F}{\partial x} = 0$, and in the second

$$F = x + H(\theta_{N-1}) = x + H(t, t_1, \ldots, t_{N-1})$$

for some function H. It is obvious that $F_1 = DF = x + H(t_1, t_2, \ldots, t_N)$ is also an x-integral. This is why $F_1 - F$ is a non-trivial x-integral independent explicitly on x. The lemma is proven.

By Lemma 4.1 one can seek x-integral depending only on the variables $t, t_{\pm 1}, t_{\pm 2}, \ldots$. In other words, one restrict himself by the study of the Lie ring generated by the vector fields $\widetilde{X}$ and $\widetilde{Y}$,

$$\widetilde{Y} = d\frac{\partial}{\partial t_1} - d_{-1}\frac{\partial}{\partial t_{-1}} + (d + d_1)\frac{\partial}{\partial t_2} - (d_{-1} + d_{-2})\frac{\partial}{\partial t_{-2}} + \ldots. \tag{4.192}$$

It can be shown that the linear operator acting as $Z \to DZD^{-1}$ defines an automorphism of the characteristic ring L_x. This automorphism plays a key role in studying the chains. A straightforward calculation shows that

$$D\widetilde{X}D^{-1} = \widetilde{X}, \quad D\widetilde{Y}D^{-1} = -d\widetilde{X} + \widetilde{Y}. \tag{4.193}$$

Lemma 4.2. *Let the vector field $Z = \sum a(j)\frac{\partial}{\partial t_j}$ with the coefficients $a(j) = a(j, t, t_{\pm 1}, t_{\pm 2}, \ldots)$ depending on a finite number of the dynamical variables satisfies the condition $DZD^{-1} = \lambda Z$ and let $a(j) = 0$ for some $j = j_0$, then $Z = 0$.*

Proof. Applying the shift automorphism to the operator Z, we obtain $DZD^{-1} = \sum D(a(j))\frac{\partial}{\partial t_{j+1}}$. Now to complete the proof we compare the coefficients at $\frac{\partial}{\partial t_j}$ in the identity $DZD^{-1} = \lambda Z$. The lemma is proven.

Let us construct an infinite sequence of multiple commutators of the vector fields $\widetilde{X}$ and $\widetilde{Y}$,

$$\widetilde{Y}_1 = [\widetilde{X}, \widetilde{Y}], \quad \widetilde{Y}_k = [\widetilde{X}, \widetilde{Y}_{k-1}] \quad \text{for} \quad k \geq 2. \tag{4.194}$$

Lemma 4.3. *The identity*

$$D\widetilde{Y}_k D^{-1} = -\widetilde{X}^k(d)\widetilde{X} + \widetilde{Y}_k, \quad k \geq 1 \tag{4.195}$$

holds true.

We prove the lemma by induction. As $k = 1$, it follows from (4.193) and (4.194) that

$$D\widetilde{Y}_1 D^{-1} = D[\widetilde{X}, \widetilde{Y}]D^{-1} = [D\widetilde{X}D^{-1}, D\widetilde{Y}D^{-1}] = [\widetilde{X}, -d\widetilde{X} + \widetilde{Y}] = -\widetilde{X}(d)\widetilde{X} + \widetilde{Y}_1.$$

Suppose now that the statement holds true for $k-1$, then we get

$$D\widetilde{Y}_k D^{-1} = [D\widetilde{X}D^{-1}, D\widetilde{Y}_{k-1}D^{-1}] = [\widetilde{X}, -\widetilde{X}^{k-1}(d)\widetilde{X} + \widetilde{Y}_{k-1}] = -\widetilde{X}^k(d)\widetilde{X} + \widetilde{Y}_k.$$

The lemma is proven.

Since the vector fields $X, \widetilde{X}$ and $\widetilde{Y}$ are linearly independent, then the dimension of the Lie ring L_x is at least three. By (4.195) the case $\widetilde{Y}_1 = 0$ means $\widetilde{X}(d) = 0$ or $d_t + d_{t_1} = 0$ that implies $d = A(t - t_1)$. Here $A(\tau)$ is an arbitrary function of one variable.

Suppose chain (4.184) possesses a nontrivial x-integral and $\widetilde{Y}_1 \neq 0$. Consider the sequence of vector fields $\{\widetilde{Y}_1, \widetilde{Y}_2, \widetilde{Y}_3, \ldots, \}$. Since L_x is of finite dimension, there exists a natural N such that

$$\widetilde{Y}_{N+1} = \gamma_1\widetilde{Y}_1 + \gamma_2\widetilde{Y}_2 + \ldots + \gamma_N\widetilde{Y}_N, \quad N \geq 1, \tag{4.196}$$

and $\widetilde{Y}_1, \widetilde{Y}_2, \ldots, \widetilde{Y}_N$ are linearly independent. Therefore,

$$D\widetilde{Y}_{N+1}D^{-1} = D(\gamma_1)D\widetilde{Y}_1D^{-1} + D(\gamma_2)D\widetilde{Y}_2D^{-1} + \ldots + D(\gamma_N)D\widetilde{Y}_ND^{-1}, \quad N \geq 1.$$

By Lemma 4.3 and (4.196) the last equation can be rewritten as

$$\begin{gathered}-\widetilde{X}^{N+1}(d)\widetilde{X} + \gamma_1\widetilde{Y}_1 + \gamma_2\widetilde{Y}_2 + \ldots + \gamma_N\widetilde{Y}_N = D(\gamma_1)(-\widetilde{X}(d)\widetilde{X} + \widetilde{Y}_1) + \\ +D(\gamma_2)(-\widetilde{X}^2(d)\widetilde{X} + \widetilde{Y}_2) + \ldots + D(\gamma_N)(-\widetilde{X}^N(d)\widetilde{X} + \widetilde{Y}_N).\end{gathered}$$

Comparing the coefficients at the linearly independent operators $\widetilde{X}, \widetilde{Y}_1, \widetilde{Y}_2, \ldots, \widetilde{Y}_N$, we obtain the following system of equations,

$$\begin{gathered}\widetilde{X}^{N+1}(d) = D(\gamma_1)\widetilde{X}(d) + D(\gamma_2)\widetilde{X}^2(d) + \ldots + D(\gamma_N)\widetilde{X}^N(d), \\ \gamma_1 = D(\gamma_1), \quad \gamma_2 = D(\gamma_2), \ldots, \gamma_N = D(\gamma_N).\end{gathered}$$

Since the coefficients at the vector-fields $\widetilde{Y}_j$ depend only on variables $t, t_{\pm1}, t_{\pm2}, \ldots$, then the coefficients γ_j can depend only on these variables (see Remark 4.1). Moreover, it follows from the last system that the coefficients γ_k are constant for all $1 \leqslant k \leqslant N$, and the functions $d = d(t, t_1)$ satisfy the differential equation

$$\widetilde{X}_{N+1}(d) = \gamma_1\widetilde{X}(d) + \gamma_2\widetilde{X}^2(d) + \ldots + \gamma_N\widetilde{X}^N(d), \quad \widetilde{X}(d) = d_t + d_{t_1}. \tag{4.197}$$

Employing the change of variables $s = t$ and $\tau = t - t_1$, we rewrite equation (4.197) as

$$\frac{\partial^{N+1}d}{\partial s^{N+1}} = \gamma_1\frac{\partial d}{\partial s} + \gamma_2\frac{\partial^2 d}{\partial s^2} + \ldots + \gamma_N\frac{\partial^N d}{\partial s^N}. \tag{4.198}$$

Therefore, the following statement holds.

Theorem 4.4. *The needed function $d = d(t, t_1)$ reads as*

$$d(t, t_1) = \sum_k \left(\sum_{j=0}^{m_k-1} \lambda_{k,j}(t - t_1)t^j\right) e^{\alpha_k t}, \tag{4.199}$$

where $\lambda_{k,j}(t - t_1)$ are some functions, α_k are the characteristic roots of the multiplicity m_k for equation (4.198).

Let $\alpha_0 = 0, \alpha_1, \ldots, \alpha_s$ are the distinct roots of the characteristic equation. Then equation (4.197) can be represented as

$$\begin{gathered}\Lambda(\widetilde{X})d = \widetilde{X}^{m_0}(\widetilde{X} - \alpha_1)^{m_1}(\widetilde{X} - \alpha_2)^{m_2}\ldots(\widetilde{X} - \alpha_s)^{m_s}d = 0,\\ m_0 + m_1 + \ldots + m_s = N + 1, \quad m_0 \geq 1.\end{gathered} \tag{4.200}$$

Starting with formula (4.192), we introduce the mapping $h \to Y_h$, which maps a function $h = h(t, t_{\pm 1}, t_{\pm 2}, \ldots)$ into the vector field

$$Y_h = h\frac{\partial}{\partial t_1} - h_{-1}\frac{\partial}{\partial t_{-1}} + (h + h_1)\frac{\partial}{\partial t_2} - (h_{-1} + h_{-2})\frac{\partial}{\partial t_{-2}} + \ldots.$$

For each polynomial with constant coefficients $P(\lambda) = c_0 + c_1\lambda + \ldots + c_m\lambda^m$ we have the formula

$$P(ad_{\widetilde{X}})\widetilde{Y} = Y_{P(\widetilde{X})d}, \quad ad_X Y = [X, Y], \tag{4.201}$$

which makes an isomorphism between the linear space V of all solutions to equation (4.198) and the linear span $\widetilde{V}$ of the vector fields $\widetilde{Y}, \widetilde{Y}_1, \ldots, \widetilde{Y}_N$.

We represent function (4.199) as the sum $d(t, t_1) = P(t, t_1) + Q(t, t_1)$ of a polynomial term $P(t, t_1) = \sum_{j=0}^{m_0-1} \lambda_{0,j}(t - t_1)t^j$ and an "exponential" one $Q(t, t_1) = \sum_{k=1}^{s}\left(\sum_{j=0}^{m_k-1}\lambda_{k,j}(t - t_1)t^j\right)\mathrm{e}^{\alpha_k t}$.

Lemma 4.4. *Let equation* (4.184) *possess a nontrivial x-integral. Then at least one of the functions $P(t, t_1)$ and $Q(t, t_1)$ is identically zero.*

Proof. Suppose the opposite, i.e., none of the functions is identically zero. We first show that in this case the ring L_x contains the vector fields $T_0 = Y_{A(\tau)e^{\alpha_k t}}$ and $T_1 = Y_{B(\tau)}$ with some functions $A(\tau)$ and $B(\tau)$. As T_0 we choose the vector field $\Lambda_0(ad_{\widetilde{X}})\widetilde{Y} = Y_{\Lambda_0(\widetilde{X})d} \in L_x$, where $\Lambda_0(\lambda) = \frac{\Lambda(\lambda)}{\lambda - \alpha_k}$. It is obvious that the function $\widetilde{A}(t, t_1) = \Lambda_0(\widetilde{X})d$ satisfies the equation $(\widetilde{X} - \alpha_k)\widetilde{A}(t, t_1) = \Lambda(\widetilde{X})d = 0$, which implies immediately that $\widetilde{A}(t, t_1) = A(\tau)e^{\alpha_k t}$.

In the same way one can construct the field $T_1 = Y_{B(\tau)} \in L_x$. We observe that in accordance with our assumption the functions $A(\tau)$ and $B(\tau)$ are not identically zero.

We consider an infinite sequence of the vector fields defined by the rule

$$T_2 = [T_0, T_1], \quad T_3 = [T_0, T_2], \ldots, T_n = [T_0, T_{n-1}], \quad n \geq 3.$$

It can be shown that

$$\begin{gathered}[\widetilde{X}, T_0] = \alpha_k T_0, \quad [\widetilde{X}, T_1] = 0, \quad [\widetilde{X}, T_n] = \alpha_k(n-1)T_n, \quad n \geq 2,\\ DT_0D^{-1} = -Ae^{\alpha_k t}\widetilde{X} + T_0, \quad DT_1D^{-1} = -B\widetilde{X} + T_1, \ldots,\\ DT_nD^{-1} = T_n - \tfrac{(n-1)(n-2)}{2}\alpha_k Ae^{\alpha_k t}T_{n-1} + b_n\widetilde{X} + \textstyle\sum_{k=0}^{n-2} a_k^{(n)}T_k, \quad n \geq 2.\end{gathered}$$

Since the algebra is finite-dimensional and $\widetilde{X}, T_0, T_1, \ldots, T_N$ are linearly independent, there exists a number N such that

$$T_{N+1} = \lambda\widetilde{X} + \mu_0T_0 + \mu_1T_1 + \ldots + \mu_NT_N. \tag{4.202}$$

We have

$$\begin{gathered}DT_{N+1}D^{-1} = D(\lambda)\widetilde{X} + D(\mu_0)\left(-Ae^{\alpha_k t}\widetilde{X} + T_0\right) + \ldots +,\\ +D(\mu_N)\left(T_N - \tfrac{(N-1)(N-2)}{2}\alpha_k Ae^{\alpha_k t}T_{N-1} + \ldots\right).\end{gathered}$$

Comparing the coefficients at the operator T_N in the last equation, we find

$$\mu_N - \frac{N(N-1)}{2}\alpha_k A(\tau)e^{\alpha_k t} = D(\mu_N).$$

It follows that μ_N is a function depending on t only. Applying the operator $ad_{\widetilde{X}}$ to both sides of equation (4.202), we get

$$N\alpha_k T_{N+1} = [\widetilde{X}, T_{N+1}] = \widetilde{X}(\lambda)\widetilde{X} + \Big(\widetilde{X}(\mu_0) + \mu_0\alpha_k\Big) T_0 + \ldots + \\ + \Big(\widetilde{X}(\mu_N) + \mu_N(N-1)\alpha_k\Big) T_N.$$

Again comparing the coefficients at T_N, we find

$$N\alpha_k\mu_N = \widetilde{X}(\mu_N) + (N-1)\alpha_k\mu_N \quad \text{or} \quad \widetilde{X}(\mu_N) = \alpha_k\mu_N.$$

Therefore, $\mu_N = A_1 e^{\alpha_k t}$, where A_1 is a non-zero constant and this is why $A(\tau)e^{\alpha_k t} = A_2 e^{\alpha_k t} - A_2 e^{\alpha_k t_1}$, $A_2 = const.$

We have $T_0 = A_2 e^{\alpha_k t}\widetilde{X} - A_2 S_0$, where $S_0 = \sum_{j=-\infty}^{\infty} e^{\alpha_k t_j}\frac{\partial}{\partial t_j}$. And also

$$[\widetilde{X}, S_0] = \alpha_k S_0, \quad DS_0D^{-1} = S_0.$$

We consider a new sequence of vector fields

$$P_1 = S_0, \quad P_2 = [T_1, S_0], \quad P_3 = [T_1, P_2], \quad P_n = [T_1, P_{n-1}], \quad n \geq 3.$$

It can be shown that

$$[\widetilde{X}, P_n] = \alpha_k P_n, \quad DP_nD^{-1} = P_n - \alpha_k(n-1)BP_{n-1} + \\ + b_n\widetilde{X} + a_nS_0 + \textstyle\sum_{j=2}^{n-2} a_j^{(n)}P_j, \quad n \geq 2.$$

Since the algebra L_x is finite-dimensional, there exists a number M such that

$$P_{M+1} = \lambda^*\widetilde{X} + \mu_2^*P_2 + \ldots + \mu_M^*P_M, \tag{4.203}$$

where the fields $\widetilde{X}, P_2, \ldots, P_M$ are linearly independent. Then

$$DP_{M+1}D^{-1} = D(\lambda^*)\widetilde{X} + D(\mu_2^*)(P_2 + \ldots) + \ldots + \\ + D(\mu_M^*)(P_M - \alpha_k(M-1)BP_{M-1} + \ldots).$$

Comparing the coefficients at P_M in the last relations, we obtain

$$\mu_M^* - M\alpha_k B(\tau) = D(\mu_M^*). \tag{4.204}$$

Hence, μ_M^* is a function depending on t only.

We apply the operator $ad_{\widetilde{X}}$ to both sides of equation (4.203), then we get

$$\alpha_k P_{M+1} = [\widetilde{X}, P_{M+1}] = \widetilde{X}(\lambda^*)\widetilde{X} + (\widetilde{X}(\mu_2^*) + \alpha_k\mu_2^*)P_2 + \\ + \ldots + (\widetilde{X}(\mu_M^*) + \alpha_k\mu_M^*)P_M.$$

Afresh, comparing the coefficients at P_M and knowing that $\alpha_k\mu_M^*(t) = \widetilde{X}(\mu_M^*(t)) + +\alpha_k\mu_M^*(t)$, we obtain that μ_M^* is constant. It follows from equation (4.204) that $B(\tau) = 0$. This contradiction implies that at least one of the functions $P(t, t_1)$ and $Q(t, t_1)$ is identically zero. The lemma is proven.

Further specification of the function $d(t, t_1)$ and the complete proof of Theorem 4.3 can be found in the work [53].

The result of the complete classification of equation (4.184) is contained in the next statement (see [52]).

Theorem 4.5. *The chain* (4.184) *possessing simultaneously nontrivial x- and n-integrals belongs to one of the types,*

(1) $d(t, t_1) = A(t_1 - t)$, *where* $A(t_1 - t) = \frac{d}{d\theta}P(\theta)$, $t_1 - t = P(\theta)$, $P(\theta)$ *is a quasipolynomial w.r.t.* θ,
(2) $d(t, t_1) = C_1(t_1^2 - t^2) + C_2(t_1 - t)$,
(3) $d(t, t_1) = \sqrt{C_3e^{2\alpha t_1} + C_4e^{\alpha(t_1+t)} + C_3e^{2\alpha t}}$,
(4) $d(t, t_1) = C_5(e^{\alpha t_1} - e^{\alpha t}) + C_6(e^{-\alpha t_1} - e^{-\alpha t})$,

where $\alpha \neq 0$, C_i, $1 \leqslant i \leqslant 6$ are arbitrary constant. At that the corresponding integrals of minimal order can be reduced to

i) $F = x - \int^{t_1 - t} \frac{ds}{A(s)}$, $I = L(D_x)t_x$, *where $L(D_x)$ is a differential operator vanishing on $\frac{d}{d\theta}P(\theta)$. At that $D_x\theta = 1$.*

ii) $F = \frac{(t_3-t_1)(t_2-t)}{(t_3-t_2)(t_1-t)}$, $I = t_x - C_1t^2 - C_2t$,

iii) $F = \int^{t_1-t} \frac{\mathrm{e}^{-\alpha s}ds}{\sqrt{C_3\mathrm{e}^{2\alpha s}+C_4\mathrm{e}^{\alpha s}+C_3}} - \int^{t_2-t_1} \frac{ds}{\sqrt{C_3\mathrm{e}^{2\alpha s}+C_4\mathrm{e}^{\alpha s}+C_3}}$, $I = 2t_{xx} - \alpha t_x^2 - \alpha C_3\mathrm{e}^{2\alpha t}$,

iv) $F = \frac{(\mathrm{e}^{\alpha t}-\mathrm{e}^{\alpha t_2})(\mathrm{e}^{\alpha t_1}-\mathrm{e}^{\alpha t_3})}{(\mathrm{e}^{\alpha t}-\mathrm{e}^{\alpha t_3})(\mathrm{e}^{\alpha t_1}-\mathrm{e}^{\alpha t_2})}$, $I = t_x - C_5e^{\alpha t} - C_6e^{-\alpha t}$.

4.3. S-integrable differential-difference equations. Employing coordinate representations (4.182), (4.183) of the characteristic vector fields, it is possible to construct the characteristic Lie ring $L_x = \{X, K_0\}$ associated with an arbitrary differential-difference equation (4.176).

In what follows we study in detail the characteristic Lie ring of the chain

$$t_{1x} = t_x + A_1(\mathrm{e}^{\alpha t_1} + \mathrm{e}^{\alpha t}) + A_2(\mathrm{e}^{-\alpha t} + \mathrm{e}^{-\alpha t_1}), \tag{4.205}$$

which is a differential-difference analogue of the Sine-Gordon equation. $u_{xy} = \sin u$. Since equation (4.205) read as (4.184), then as the generators of the rings one choose the operators $\widetilde{X}, \widetilde{Y}$ (see (4.190), (4.192)). Then we employ identity (4.201), in which we let $d = A_1(\mathrm{e}^{\alpha t_1} + \mathrm{e}^{\alpha t}) + A_2(\mathrm{e}^{-\alpha t} + \mathrm{e}^{-\alpha t_1})$. We let $P_0(\lambda) = \frac{1}{2\alpha A_1}(\lambda + \alpha)$, $P_1(\lambda) = -\frac{1}{2\alpha A_2}(\lambda - \alpha)$.

We introduce two operators $S_0^* = P_0(ad_{\widetilde{X}})\widetilde{Y}$ and $S_1^* = P_1(ad_{\widetilde{X}})\widetilde{Y}$,

$$\begin{gathered} S_0^* = (\mathrm{e}^{\alpha t_1} + \mathrm{e}^{\alpha t})\frac{\partial}{\partial t_1} - (\mathrm{e}^{\alpha t_{-1}} + \mathrm{e}^{\alpha t})\frac{\partial}{\partial t_{-1}} + \\ +(\mathrm{e}^{\alpha t} + 2\mathrm{e}^{\alpha t_1} + \mathrm{e}^{\alpha t_2})\frac{\partial}{\partial t_2} - (\mathrm{e}^{\alpha t} + 2\mathrm{e}^{\alpha t_{-1}} + \mathrm{e}^{\alpha t_{-2}})\frac{\partial}{\partial t_{-2}} + \ldots, \end{gathered} \tag{4.206}$$

$$\begin{gathered} S_1^* = (\mathrm{e}^{-\alpha t_1} + \mathrm{e}^{-\alpha t})\frac{\partial}{\partial t_1} - (\mathrm{e}^{-\alpha t_{-1}} + \mathrm{e}^{-\alpha t})\frac{\partial}{\partial t_{-1}} + \\ +(\mathrm{e}^{-\alpha t} + 2\mathrm{e}^{-\alpha t_1} + \mathrm{e}^{-\alpha t_2})\frac{\partial}{\partial t_2} - (\mathrm{e}^{-\alpha t} + 2\mathrm{e}^{-\alpha t_{-1}} + \mathrm{e}^{-\alpha t_{-2}})\frac{\partial}{\partial t_{-2}} + \ldots. \end{gathered} \tag{4.207}$$

It follows from the obvious identities $[\widetilde{X}, S_0^*] = \alpha S_0^*$, $[\widetilde{X}, S_1^*] = -\alpha S_1^*$, $\widetilde{Y} = A_1S_0^* + A_2S_1^*$ that $L_{x1} = \{\widetilde{X}\} \oplus L_{x2}$, where L_{x2} is the Lie ring generated by the operators S_0^*, S_1^*.

Let us construct the basis of the space consisting of the elements of the ring L_{x2}. We replace the dependent variables as $\tau_j = t_j - t_{j+1}$, then τ_j and $t = t_0$ are new variable and the identities $\frac{\partial}{\partial t_j} = -\frac{\partial}{\partial \tau_{j-1}} + \frac{\partial}{\partial \tau_j}$ hold true that allows us to rewrite the operators S_0^*, S_1^* as $S_0^* = -\mathrm{e}^{\alpha t}S_0$, $S_1^* = -\mathrm{e}^{-\alpha t}S_1$, where

$$S_0 = \sum_j A(\tau_j)\mathrm{e}^{\alpha\rho(j)}\frac{\partial}{\partial \tau_j}, \quad S_1 = \sum_j B(\tau_j)\mathrm{e}^{-\alpha\rho(j)}\frac{\partial}{\partial \tau_j}, \tag{4.208}$$

and also

$$A(\tau) = 1 + \mathrm{e}^{-\alpha\tau}, \quad B(\tau) = 1 + \mathrm{e}^{\alpha\tau}, \tag{4.209}$$

$$\rho(j) = \begin{cases} -\tau - \tau_1 - \ldots - \tau_{j-1}, & \text{if} \quad j \geq 1; \\ 0, \quad \text{if} \quad j = 0; \\ \tau_{-1} + \tau_{-2} + \ldots + \tau_j, & \text{if} \quad j \leqslant -1. \end{cases} \tag{4.210}$$

Employing the identity $D\rho(j) = \rho(j+1) + \tau$, it is easy to check that

$$DS_0D^{-1} = \mathrm{e}^{\alpha\tau}S_0, \quad DS_1D^{-1} = \mathrm{e}^{-\alpha\tau}S_1. \tag{4.211}$$

As expected, the characteristic ring L_{x2} has an infinite dimension. The ring L_{x2} (as well as L_{x1}, L_x) is the ring of minimal growth. In other words, the dimension of the linear space of multiple commutators increases by one as the multiplicity increases, and by two subject to the parity. For instance, if V_j is the linear space of all the commutators of the multiplicity at most j, then a basis of V_{2k} consists of the operators $\{S_0, S_1, P_1, P_2, P_3, \ldots, P_{2k}, Q_2, Q_4, \ldots, Q_{2k}\}$, and a basis of V_{2k+1} does of the operators $\{S_0, S_1, P_1, P_2, P_3, \dots, P_{2k+1}, Q_2, Q_4, \dots, Q_{2k}\}$. Here the operators P_j, Q_j are defined consequently,

$$\begin{array}{ll}
P_1 = [S_0, S_1] + \alpha S_0 + \alpha S_1, & Q_1 = P_1, \\
P_2 = [S_1, P_1], & Q_2 = [S_0, Q_1], \\
P_3 = [S_0, P_2] + \alpha P_2, & Q_3 = [S_1, Q_2] - \alpha Q_2, \\
P_{2j} = [S_1, P_{2j-1}], & Q_{2j} = [S_0, Q_{2j-1}], \\
P_{2j+1} = [S_0, P_{2j}] + \alpha P_{2j}, & Q_{2j+1} = [S_1, Q_{2j}] - \alpha Q_{2j},
\end{array}$$

for $j \geq 1$. The calculations show that

$$\begin{gathered}
DP_1D^{-1} = P_1 - 2\alpha(S_0 + S_1), \\
DP_2D^{-1} = \mathrm{e}^{-\alpha\tau}(P_2 + 2\alpha P_1 - 2\alpha^2(S_0 + S_1)), \\
DP_3D^{-1} = P_3 + 2\alpha Q_2 - 2\alpha P_2 - 4\alpha^2 P_1 + 4\alpha^3(S_0 + S_1), \\
DP_4D^{-1} = \mathrm{e}^{-\alpha\tau}(P_4 + 2\alpha Q_3 - 4\alpha^2 P_2 + 4\alpha^2 Q_2 - \\
-4\alpha^3 P_1 + 4\alpha^4(S_0 + S_1)), \\
DQ_2D^{-1} = \mathrm{e}^{\alpha\tau}(Q_2 - 2\alpha P_1 + 2\alpha^2(S_0 + S_1)), \\
DQ_3D^{-1} = Q_3 + 2\alpha Q_2 - 2\alpha P_2 - 4\alpha^2 P_1 + 4\alpha^3(S_0 + S_1), \\
DQ_4D^{-1} = \mathrm{e}^{\alpha\tau}(Q_4 - 2\alpha P_3 + 2\alpha^2(P_2 - Q_2) + \\
+4\alpha^3 P_1 - 4\alpha^4(S_0 + S_1)), \\
P_3 = Q_3, \quad [S_1, P_2] = -\alpha P_2, \quad [S_0, Q_2] = \alpha Q_2, \\
[S_1, P_4] = -\alpha P_4, \quad [S_0, Q_4] = \alpha Q_4.
\end{gathered} \tag{4.212}$$

The coefficient at $\dfrac{\partial}{\partial\tau}$ in all vector fields DP_iD^{-1}, DQ_iD^{-1}, $1 \leqslant i \leqslant 4$ is zero.

Lemma 4.5. *For each $j \geq 1$ the identities*

(1) $DP_{2j+1}D^{-1} + 2\alpha \mathrm{e}^{\alpha\tau} DP_{2j}D^{-1} = P_{2j+1} + 2\alpha Q_{2j}$,
(2) $\mathrm{e}^{\alpha\tau} DP_{2j+2}D^{-1} - \alpha DP_{2j+1}D^{-1} = P_{2j+2} + \alpha Q_{2j+1}$,
(3) $DQ_{2j+1}D^{-1} - 2\alpha \mathrm{e}^{-\alpha\tau} DQ_{2j}D^{-1} = Q_{2j+1} - 2\alpha P_{2j}$,
(4) $\mathrm{e}^{-\alpha\tau} DQ_{2j+2}D^{-1} + \alpha DQ_{2j+1}D^{-1} = Q_{2j+2} - \alpha P_{2j+1}$,
(5) $P_{2j+1} = Q_{2j+1}$,
(6) $[S_1, P_{2j+2}] = -\alpha P_{2j+2}$,
(7) $[S_0, Q_{2j+2}] = \alpha Q_{2j+2}$

hold true. Moreover, the coefficient at $\dfrac{\partial}{\partial\tau}$ in all vector fields DP_kD^{-1}, DQ_kD^{-1} is zero.

Proof. By the induction in j. By (4.212) it is clear that the statement of the lemma holds as $j = 1$. Suppose (1) – (7) are valid for all j, $1 \leqslant j \leqslant k$. Let us show that (1) is valid for

$j=k+1$.

$$\begin{aligned}
DP_{2j+3}D^{-1} &= D([S_0,P_{2j+2}]+\alpha P_{2j+2})D^{-1}=[\mathrm{e}^{\alpha\tau}S_0,DP_{2j+2}D^{-1}]+\alpha DP_{2j+2}D^{-1}=\\
&=[\mathrm{e}^{\alpha\tau}S_0,\alpha\mathrm{e}^{-\alpha\tau}DP_{2j+1}D^{-1}+\mathrm{e}^{-\alpha\tau}P_{2j+2}+\alpha\mathrm{e}^{-\alpha\tau}Q_{2j+1}]+\alpha DP_{2j+2}D^{-1}=\\
&=-\alpha^2(1+\mathrm{e}^{-\alpha\tau})DP_{2j+1}D^{-1}+\alpha\mathrm{e}^{-\alpha\tau}[\mathrm{e}^{\alpha\tau}S_0,DP_{2j+1}D^{-1}]-\alpha(1+\mathrm{e}^{-\alpha\tau})P_{2j+2}-\\
&-\alpha^2(1+\mathrm{e}^{-\alpha\tau})Q_{2j+1}+P_{2j+3}-\alpha P_{2j+2}+\alpha Q_{2j+2}+\alpha DP_{2j+2}D^{-1}=\\
&=-\alpha^2(1+\mathrm{e}^{-\alpha\tau})DP_{2j+1}D^{-1}+\alpha\mathrm{e}^{-\alpha\tau}D[S_0,Q_{2j+1}]D^{-1}-\alpha(2+\mathrm{e}^{-\alpha\tau})P_{2j+2}-\\
&-\alpha^2(1+\mathrm{e}^{-\alpha\tau})Q_{2j+1}+P_{2j+3}+\alpha Q_{2j+2}+\alpha DP_{2j+2}D^{-1}=\\
&=-\alpha^2(1+\mathrm{e}^{-\alpha\tau})DP_{2j+1}D^{-1}+\alpha Q_{2j+2}-\alpha^2P_{2j+1}-\alpha^2DQ_{2j+1}D^{-1}-\\
&-\alpha(2+\mathrm{e}^{-\alpha\tau})P_{2j+2}-\alpha^2(1+\mathrm{e}^{-\alpha\tau})Q_{2j+1}-2\alpha^2Q_{2j+1}-2\alpha P_{2j+2}+P_{2j+3}=\\
&=-2\alpha^2DP_{2j+1}D^{-1}+2\alpha Q_{2j+2}-2\alpha^2Q_{2j+1}-2\alpha P_{2j+2}+P_{2j+3}=\\
&=2\alpha P_{2j+2}+2\alpha^2Q_{2j+1}-2\alpha\mathrm{e}^{\alpha\tau}DP_{2j+2}D^{-1}+2\alpha Q_{2j+2}-2\alpha^2Q_{2j+1}-\\
&-2\alpha P_{2j+2}+P_{2j+3}=-2\alpha\mathrm{e}^{\alpha\tau}DP_{2j+2}D^{-1}+2\alpha Q_{2j+2}+P_{2j+3}.
\end{aligned}$$

The condition (3) is proven exactly in the same way as (1). Let us show that (5) is valid for $j=k+1$. It is obvious that we have

$$\begin{aligned}
DP_{2j+3}D^{-1} &= -2\alpha\mathrm{e}^{\alpha\tau}DP_{2j+2}D^{-1}+2\alpha Q_{2j+2}+P_{2j+3}=\\
&= -2\alpha(\alpha DP_{2j+1}D^{-1}+P_{2j+2}+\alpha Q_{2j+1})+2\alpha Q_{2j+2}+P_{2j+3},
\end{aligned}$$

and

$$\begin{aligned}
DQ_{2j+3}D^{-1} &= 2\alpha\mathrm{e}^{-\alpha\tau}DQ_{2j+2}D^{-1}-2\alpha P_{2j+2}+Q_{2j+3}=\\
&= 2\alpha(-\alpha DQ_{2j+1}D^{-1}+Q_{2j+2}-\alpha P_{2j+1})-2\alpha P_{2j+2}+Q_{2j+3}.
\end{aligned}$$

By (5) $P_{2j+1}=Q_{2j+1}$, and hence

$$D(P_{2j+3}-Q_{2j+3})D^{-1}=-2\alpha P_{2j+2}-2\alpha Q_{2j+2}+2\alpha Q_{2j+2}+2\alpha P_{2j+2}=0.$$

therefore, $P_{2j+3}=Q_{2j+3}$.

Let us show that (2) is valid as $j=k+1$. We have

$$\begin{gathered}
\mathrm{e}^{\alpha\tau}DP_{2j+1}D^{-1}=\mathrm{e}^{\alpha\tau}D[S_1,P_{2j+3}]D^{-1}=\mathrm{e}^{\alpha\tau}[\mathrm{e}^{-\alpha\tau}S_1,DP_{2j+3}D^{-1}]=\\
=\mathrm{e}^{\alpha\tau}[\mathrm{e}^{-\alpha\tau}S_1,-2\alpha\mathrm{e}^{\alpha\tau}DP_{2j+2}D^{-1}+2\alpha Q_{2j+2}+P_{2j+3}]=\\
=\mathrm{e}^{\alpha\tau}(-2\alpha^2(1+\mathrm{e}^{\alpha\tau})DP_{2j+2}D^{-1})-2\alpha\mathrm{e}^{2\alpha\tau}[\mathrm{e}^{-\alpha\tau}S_1,DP_{2j+2}D^{-1}]+\\
+P_{2j+4}+2\alpha Q_{2j+3}+2\alpha^2Q_{2j+2}=-2\alpha^2(\mathrm{e}^{\alpha\tau}+\mathrm{e}^{2\alpha\tau})DP_{2j+2}D^{-1}+\\
+2\alpha^2\mathrm{e}^{2\alpha\tau}DP_{2j+2}D^{-1}+P_{2j+4}+2\alpha Q_{2j+3}+2\alpha^2Q_{2j+2}=\\
=-2\alpha^2\mathrm{e}^{\alpha\tau}DP_{2j+2}D^{-1}+P_{2j+4}+2\alpha Q_{2j+3}+2\alpha^2Q_{2j+2}=\\
=\alpha DP_{2j+3}D^{-1}-\alpha P_{2j+3}-2\alpha^2Q_{2j+2}+P_{2j+4}+2\alpha Q_{2j+3}+\\
+2\alpha^2Q_{2j+2}=\alpha DP_{2j+3}D^{-1}+\alpha Q_{2j+3}+P_{2j+4}.
\end{gathered}$$

The proof of (4) is similar to that of (2).

Let us prove (6) for $j = k + 1$.

$$\begin{gathered}D[S_1, P_{2j+4}]D^{-1} = [\mathrm{e}^{-\alpha\tau} S_1, \alpha \mathrm{e}^{-\alpha\tau} DP_{2j+3}D^{-1} + \mathrm{e}^{-\alpha\tau} P_{2j+4} + \alpha \mathrm{e}^{-\alpha\tau} Q_{2j+3}] = \\ = [\mathrm{e}^{-\alpha\tau} S_1, \alpha \mathrm{e}^{-\alpha\tau}(-2\alpha \mathrm{e}^{\alpha\tau} DP_{2j+2}D^{-1} + P_{2j+3} + 2\alpha Q_{2j+2}) + \\ + \mathrm{e}^{-\alpha\tau} P_{2j+4} + \alpha \mathrm{e}^{-\alpha\tau} Q_{2j+3}] = [\mathrm{e}^{-\alpha\tau} S_1, -2\alpha^2 DP_{2j+2}D^{-1} + \\ +2\alpha \mathrm{e}^{-\alpha\tau} P_{2j+3} + 2\alpha^2 \mathrm{e}^{-\alpha\tau} Q_{2j+2} + \mathrm{e}^{-\alpha\tau} P_{2j+4}] = -2\alpha^2 D[S_1, P_{2j+2}]D^{-1} - \\ -2\alpha^2 \mathrm{e}^{-2\alpha\tau}(1 + \mathrm{e}^{\alpha\tau})P_{2j+3} - 2\alpha^3 \mathrm{e}^{-2\alpha\tau}(1 + \mathrm{e}^{\alpha\tau})Q_{2j+2} + 2\alpha \mathrm{e}^{-2\alpha\tau} P_{2j+4} + \\ +2\alpha^2 \mathrm{e}^{-2\alpha\tau} Q_{2j+3} + 2\alpha^3 \mathrm{e}^{-2\alpha\tau} Q_{2j+2} - \alpha \mathrm{e}^{-2\alpha\tau}(1 + \mathrm{e}^{\alpha\tau})P_{2j+4} + \\ + \mathrm{e}^{-2\alpha\tau}[S_1, P_{2j+4}] = 2\alpha^3 DP_{2j+2}D^{-1} - 2\alpha^2 \mathrm{e}^{-\alpha\tau} P_{2j+3} + \alpha(\mathrm{e}^{-2\alpha\tau} - \\ - \mathrm{e}^{-\alpha\tau})P_{2j+4} - 2\alpha^3 \mathrm{e}^{-\alpha\tau} Q_{2j+2} + \mathrm{e}^{-2\alpha\tau}[S_1, P_{2j+4}] = \\ = \alpha^2 \mathrm{e}^{-\alpha\tau} P_{2j+3} + 2\alpha^3 \mathrm{e}^{-\alpha\tau} Q_{2j+2} - \alpha^2 \mathrm{e}^{-\alpha\tau} DP_{2j+3}D^{-1} - 2\alpha^2 \mathrm{e}^{-\alpha\tau} P_{2j+3} + \\ + \alpha(\mathrm{e}^{-2\alpha\tau} - \mathrm{e}^{-\alpha\tau})P_{2j+4} - 2\alpha^3 \mathrm{e}^{-\alpha\tau} Q_{2j+2} + \mathrm{e}^{-2\alpha\tau}[S_1, P_{2j+4}] = \\ = -\alpha^2 \mathrm{e}^{-\alpha\tau} P_{2j+3} + \alpha(\mathrm{e}^{-2\alpha\tau} - \mathrm{e}^{-\alpha\tau})P_{2j+4} - \alpha DP_{2j+4}D^{-1} + \alpha \mathrm{e}^{-\alpha\tau} P_{2j+4} + \\ + \alpha^2 \mathrm{e}^{-\alpha\tau} Q_{2j+3} + \mathrm{e}^{-2\alpha\tau}[S_1, P_{2j+4}].\end{gathered}$$

Hence,

$$\begin{gathered}D[S_1, P_{2j+4}]D^{-1} = \mathrm{e}^{-2\alpha\tau}[S_1, P_{2j+4}] + \alpha \mathrm{e}^{-2\alpha\tau} P_{2j+4} - \alpha DP_{2j+4}D^{-1} \\ D([S_1, P_{2j+4}] + \alpha P_{2j+4})D^{-1} = \mathrm{e}^{-2\alpha\tau}([S_1, P_{2j+4}] + \alpha P_{2j+4}).\end{gathered}$$

Therefore, $[S_1, P_{2j+4}] = -\alpha P_{2j+4}$.

The proof of (7) is similar to that of (6). The lemma is proven.

Remark 4.2. *The identities*

$$\begin{gathered}\mathrm{e}^{-\alpha\tau} DQ_{2j}D^{-1} + \mathrm{e}^{\alpha\tau} DP_{2j}D^{-1} = Q_{2j} + P_{2j}, \\ DP_{2j+1}D^{-1} = P_{2j+1} + \sum_{k=1}^{j}(\mu_{2k}^{(2j+1)} P_{2k} + \nu_{2k}^{(2j+1)} Q_{2k}) + \\ + \textstyle\sum_{k=0}^{j-1} \mu_{2k+1}^{(2j+1)} P_{2k+1} + \mu_0^{(2j+1)} S_0 + \nu_0^{(2j+1)} S_1, \\ DP_{2j}D^{-1} = \mathrm{e}^{-\alpha\tau}(P_{2j} + \sum_{k=1}^{j-1}(\mu_{2k}^{(2j)} P_{2k} + \nu_{2k}^{(2j)} Q_{2k}) + \\ + \textstyle\sum_{k=0}^{j-1} \mu_{2k+1}^{(2j)} P_{2k+1} + \mu_0^{(2j)} S_0 + \nu_0^{(2j)} S_1), \\ DQ_{2j}D^{-1} = \mathrm{e}^{\alpha\tau}(Q_{2j} - \sum_{k=1}^{j-1}(\mu_{2k}^{(2j)} P_{2k} + \nu_{2k}^{(2j)} Q_{2k}) - \\ - \textstyle\sum_{k=0}^{j-1} \mu_{2k+1}^{(2j)} P_{2k+1} - \mu_0^{(2j)} S_0 - \nu_0^{(2j)} S_1)\end{gathered}$$

hold true. Moreover, $\mu_{2j}^{(2j+1)} = -2\alpha$, $\nu_{2j}^{(2j+1)} = 2\alpha$, $\mu_{2j-1}^{(2j)} = 2\alpha$.

Suppose L_x is finite-dimensional. Then there are three possibilities,

1) $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, \ldots, P_{2j-1}$ are linearly independent and $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, \ldots, P_{2j-1}, P_{2j}$ are linearly dependent,
2) $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, \ldots, P_{2j-1}, P_{2j}$ are linearly independent and $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, \ldots, P_{2n-1}, P_{2j}, Q_{2j}$ are linearly dependent,
3) $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, \ldots, P_{2j}, Q_{2j}$ are linearly independent and $S_0, S_1, P_1, P_2, Q_2, P_3, P_4, Q_4, \ldots, P_{2j}, Q_{2j}, P_{2j+1}$ are linearly dependent.

In the case 1),

$$P_{2j} = \gamma_{2j-1} P_{2j-1} + \gamma_{2j-2} P_{2j-2} + \eta_{2j-2} Q_{2j-2} + \ldots$$

and

$$\begin{gathered}DP_{2j}D^{-1} = D(\gamma_{2j-1}) DP_{2j-1}D^{-1} + \\ + D(\gamma_{2j-2}) DP_{2j-2}D^{-1} + D(\eta_{2j-2}) DQ_{2j-2}D^{-1} + \ldots .\end{gathered} \tag{4.213}$$

We employ Remark 4.2 for comparing the coefficients at P_{2j-1} in (4.213) and obtain the inconsistent equation

$$\mathrm{e}^{-\alpha\tau}(\gamma_{2j-1}+2\alpha)=D(\gamma_{2j-1}).$$

It shows that the case 1) does not realize.

In the case 2),

$$Q_{2j}=\gamma_{2j}P_{2j}+\gamma_{2j-1}P_{2j-1}+\eta_{2j-2}Q_{2j-2}+\dots$$

and

$$\begin{gathered}DQ_{2j}D^{-1}=D(\gamma_{2j})DP_{2j}D^{-1}+\\+D(\gamma_{2j-1})DP_{2j-1}D^{-1}+D(\eta_{2j-2})DQ_{2j-2}D^{-1}+\dots.\end{gathered}\tag{4.214}$$

We again employ Remark 4.2 for comparing the coefficients at P_{2j-1} in (4.214) and arrive to the inconsistent condition

$$\mathrm{e}^{\alpha\tau}(\gamma_{2j-1}-2\alpha)=D(\gamma_{2j-1}),$$

which shows that the case 2) is impossible.

In the case 3)

$$P_{2j+1}=\eta_{2j}Q_{2j}+\gamma_{2j}P_{2j}+\dots$$

and

$$DP_{2j+1}D^{-1}=D(\eta_{2j})DQ_{2j}D^{-1}+D(\gamma_{2j})DP_{2j}D^{-1}+\dots.\tag{4.215}$$

We employ Remark 4.2 for comparing the coefficients at P_{2j-1} in (4.215) and and arrive at the contradiction

$$(\gamma_{2j}-2\alpha)=D(\gamma_{2j})\mathrm{e}^{-\alpha\tau}.$$

This is why the case 3) is impossible. Therefore, the characteristic Lie ring L_x has an infinite dimension.

5. Fully discrete equations

At present the discrete models

$$u_{1,1}=f(m,n,u,u_1,\overline{u}_1)\tag{5.216}$$

called also the equations on a square graph are studied intensively due to their important applications in physics, discrete geometry, architecture, biology, etc. In equation (5.216) the sought function $u=u(m,n)$ depends on two independent discrete variables. The subscripts and the bar accent over a letter indicate the shift of the arguments,

$$u_k=u(m+k,n),\quad \overline{u}_k=u(m,n+k),\quad u_{i,j}=u(m+i,n+j).$$

The function f is supposed to be smooth and defined in some domain $\mathbb{R}^3$. It is also assumed that equation (5.216) can be solvable at least locally w.r.t. each of three variables $u,u_1,\overline{u}_1$, i.e., there exist functions $f^{i,j}$ such that

$$\begin{aligned}u&=f^{-1,-1}(m,n,u_{1,1},\overline{u}_1,u_1),\\u_1&=f^{1,-1}(m,n,\overline{u}_1,u_{1,1},u),\\\overline{u}_1&=f^{-1,1}(m,n,u_1,u,u_{1,1}).\end{aligned}$$

5.1. Liouville type discrete equations.

In this subsection we consider the equations of the form (5.216) possessing integrals.

Definition 5.1. *As n-integral of equation* (5.216) *we call a sequence of the functions* $\{I_{(i)}(m,n,u_{-j},u_{-j+1},\dots,u_k)\}_{i=-\infty}^{+\infty}$ *depending on m, n, and a finite number of dynamical variables* $\{u_i\}$ *such that the relation*

$$\overline{D}I_{(i)}(m,n,u_{-j},u_{-j+1},\dots,u_k)=I_{(i+1)}(m,n,u_{-j},u_{-j+1},\dots,u_k)$$

holds, where $\overline{D}$ *is the operator of argument shift such that* $\overline{D}h(m,n)=h(m,n+1)$.

Remark 5.1. *In process of proving Theorem 5.1 (see below) it is found out that a n-integral can be represented as $I = I(m, n, G)$, where $G = G(u, u_1, \dots, u_N)$ is a some function.*

Example 5.1. *Consider the equation of the form* (5.216),

$$u_{1,1} = \frac{1}{u_1},$$

whose n-integral is the sequence of the functions $I_{(i)} = I_{(i)}(u_1)$ such that

$$I_{(i)} = \begin{cases} u_1, & i is even; \\ \frac{1}{u_1}, & i is odd. \end{cases}$$

Indeed,

$$\begin{array}{l} \overline{D}I_{(2m)} = \overline{D}u_1 = u_{1,1} = \frac{1}{u_1} = I_{(2m+1)}, \\ \overline{D}I_{(2m+1)} = \overline{D}\frac{1}{u_1} = \frac{1}{u_{1,1}} = u_1 = I_{(2m+2)}. \end{array}$$

The coordinate representation the equation $\overline{D}I_{(i)} = I_{(i+1)}$ is

$$I_{(i)}(m, n, r_{-j+1}, r_{-j+2}, \dots, r, \overline{u}_1, f, f_1, \dots, f_{k-1}) = I_{(i+1)}(m, n, u_{-j}, u_{-j+1}, \dots, u_k), \qquad (5.217)$$

where $r = f^{-1,1}(m, n, u, u_{-1}, \overline{u}_1)$. As dynamical (independent) variables we choose $\{u_j\}_{j=-\infty}^{+\infty}$ and $\{\overline{u}_j\}_{j=-\infty}^{+\infty}$. Then the function $r_{-1} = D^{-1}(r)$ can be rewritten as

$$r_{-1} = f^{-1,1}(m-1, n, u_{-1}, u_{-2}, u_{-1,1}) = f^{-1,1}(m-1, n, u_1, u, f^{-1,1}(m, n, u, u_{-1}, \overline{u}_1)).$$

Here D is the operator of the shift of the variable m, $Dy(m, n) = y(m+1, n)$. In the same way all the shifts in (5.217) can be represented as a superposition of the functions depending only on the dynamical variables. We note the right hand side of identity (5.217) is independent of the variable $\overline{u}_1$, and the condition $\frac{\partial}{\partial \overline{u}_1}\overline{D}I_{(i)} = 0$ is thus satisfied, or, which is the same, $Y_1 I_{(i)} = 0$, where $Y_1 = \overline{D}^{-1}\frac{\partial}{\partial \overline{u}_1}\overline{D}$. In the expanded form the operator Y_1 reads as

$$\begin{array}{c} Y_1 = \frac{\partial}{\partial u} + \overline{D}^{-1}\left(\frac{\partial f}{\partial \overline{u}_1}\right)\frac{\partial}{\partial u_1} + \overline{D}^{-1}\left(\frac{\partial r}{\partial \overline{u}_1}\right)\frac{\partial}{\partial u_{-1}} + \\ + \overline{D}^{-1}\left(\frac{\partial f_1}{\partial \overline{u}_1}\right)\frac{\partial}{\partial u_2} + \overline{D}^{-1}\left(\frac{\partial r_{-1}}{\partial \overline{u}_1}\right)\frac{\partial}{\partial u_{-2}} + \dots \end{array} \qquad (5.218)$$

We introduce the notations $x = \overline{D}^{-1}\frac{\partial f(u, u_1, \overline{u}_1)}{\partial \overline{u}_1} = -\frac{\partial f^{1,-1}(u, u_1, \bar{u}_{-1})/\partial u}{\partial f^{1,-1}(u, u_1, \bar{u}_{-1})/\partial u_1}$.

Lemma 5.1. *The identities*

$$\begin{array}{c} \frac{\partial r}{\partial \overline{u}_1} = \frac{1}{D^{-1}\left(\frac{\partial f}{\partial \overline{u}_1}\right)}, \\ \frac{\partial f_j}{\partial \overline{u}_1} = \frac{\partial f}{\partial \overline{u}_1} \cdot D\left(\frac{\partial f}{\partial \overline{u}_1}\right) \cdot \dots \cdot D^j\left(\frac{\partial f}{\partial \overline{u}_1}\right) \end{array}$$

hold true.

Proof. The second of the relations in the lemma is an obvious implication of the formula for derivative of a composite function. For instance, for $j = 1$ we have

$$\frac{\partial f_1}{\partial \overline{u}_1} = \frac{\partial}{\partial \overline{u}_1} f(u_1, u_2 f(u, u_1, \overline{u}_1)) = D\left(\frac{\partial f}{\partial \overline{u}_1}\right) \cdot \frac{\partial f}{\partial \overline{u}_1}.$$

To prove the first relation it is sufficient to differentiated the identity

$$\overline{u}_1 = f^{-1,1}(u_1, u, f(u, u_1, \overline{u}_1))$$

w.r.t. the variable $\overline{u}_1$ and to get

$$1 = D\left(\frac{\partial f^{-1,1}}{\partial \overline{u}_1}\right) \cdot \frac{\partial f}{\partial \overline{u}_1}.$$

The lemma is proven.

Employing the lemma, the operator Y_1 can be rewritten as

$$Y_1 = \frac{\partial}{\partial u} + x\frac{\partial}{\partial u_1} + \frac{1}{x_{-1}}\frac{\partial}{\partial u_{-1}} + xx_1\frac{\partial}{\partial u_2} + \frac{1}{x_{-1}x_{-2}}\frac{\partial}{\partial u_{-2}} + \dots. \tag{5.219}$$

We call the operator Y_1 a characteristic vector field.

It is clear now that n-integral in a solution to the first order partial differential equation $Y_1 I_{(i)} = 0$, whose coefficients are expressed in terms of the variable x and its shifts, and this is why they depend, generally speaking, on the variable $\overline{u}_{-1}$, while the function $I_{(i)}$ itself can not depend on $\overline{u}_{-1}$, i.e., $X_1 I_{(i)} = 0$, where $X_1 = \frac{\partial}{\partial \overline{u}_{-1}}$. It is notable that in a general case except these two equations and their differential consequences the n-integral I satisfies also other equations which is a distinguishing property of a discrete equation. Indeed, the it follows from the identity $\overline{D} I_{(i)} = I_{(i+1)}$ that each integer k $\overline{D}^k I_{(i)} = I_{(i+k)}$. In the last identity under the condition $k > 0$ the right hand side is independent of the variable $\overline{u}_1$, while in the left hand side $\overline{u}_1$ appears formally; hence, we have $\overline{D}^{-k}\frac{\partial}{\partial \overline{u}_1}\overline{D}^k I_{(i)} = 0$, $k \geq 0$. Straightforward calculations show that

$$\overline{D}^{-k}\frac{\partial}{\partial \overline{u}_1}\overline{D}^k = X_{k-1} + Y_k, \quad k \geq 2,$$

where

$$\begin{gathered} Y_{j+1} = \overline{D}^{-1}(Y_j f)\tfrac{\partial}{\partial u_1} + \overline{D}^{-1}(Y_j r)\tfrac{\partial}{\partial u_{-1}} + \\ + \overline{D}^{-1}(Y_j f_1)\tfrac{\partial}{\partial u_2} + \overline{D}^{-1}(Y_j r_{-1})\tfrac{\partial}{\partial u_{-2}} + \dots, \\ X_j = \tfrac{\partial}{\partial \overline{u}_{-j}}, \quad j \geq 1. \end{gathered} \tag{5.220}$$

Denote by N^* the dimension of linear space generated by the operators $\{Y_j\}_1^\infty$. We shall the Lie ring over the field of locally analytic functions generated by the operators $\{Y_j\}_1^{N^*} \cup \{X_j\}_1^{N^*}$ a characteristic Lie ring L_n of equation (5.216) in the direction of n.

Theorem 5.1. *The equation* (5.216) *possesses a nontrivial n-integral if and only if* $\dim L_n < \infty$.

Proof. Suppose equation (5.216) possesses a nontrivial n-integral $I = I_{(i)}(m, n, u_{-j}, u_{-j+1}, \dots, u_k)$, where $\frac{\partial I}{\partial u_{-j}} \neq 0$, $\frac{\partial I}{\partial u_k} \neq 0$. We introduce the Lie ring M generated by the vector fields $\{Y_j\}_1^\infty \cup \{X_j\}_1^{N_2}$, where the number N_2 will be determined below. We let

$$M^{(j,k)} = \{T^{j,k} = P_{j,k}(T) : T \in M\},$$

where $P_{j,k}$ is the projector defined as

$$P_{i,m} : \sum_{s=-N_2}^{-1} a_s\frac{\partial}{\partial \overline{u}_s} + \sum_{-\infty}^{+\infty} b_s\frac{\partial}{\partial u_s} \to \sum_{s=-N_2}^{-1} a_s\frac{\partial}{\partial \overline{u}_s} + \sum_{s=-i}^{m} b_s\frac{\partial}{\partial u_s},$$

$i, m = 1, 2, 3, \dots$.

Denote by N_1 the dimension of the space$M^{(j,k)}$. It is obvious that $N_1 \leqslant N_2 + k + j + 1$. Let the set of the operator $\{T_{01}, T_{02}, \dots, T_{0N_1}\}$ form a basis in $M^{(j,k)}$. We indicate by $T_i = \sum_{s=-N_2}^{-1} a_s(T_j)\frac{\partial}{\partial \overline{u}_s} + \sum_{-\infty}^{+\infty} b_s(T_j)\frac{\partial}{\partial u_s}$ the vector field in M such that $P_{j,k}(T_j) = T_{0j}$, $j = 1, 2, \dots, N_1$. Let us show that the set of the operators $\{T_1, T_2, \dots, T_{N_1}\}$ forms a basis in M.

We take an arbitrary vector field $T = \sum_{s=-N_2}^{-1} a_s(T)\frac{\partial}{\partial \overline{u}_s} + \sum_{-\infty}^{+\infty} b_s(T)\frac{\partial}{\partial u_s}$ in M. Since $P_{j,k}(T) \in M^{(j,k)}$, then $P_{j,k}(T) = \sum_{m=1}^{N_1} \beta_m T_{0m}$. Let us check that $T = \sum_{m=1}^{N_1} \beta_m T_{jm}$ which is equivalent to the identity $Z = 0$, where $Z = T - \sum_{m=1}^{N_1} \beta_m T_{jm}$. By definition we have

$P_{j,k}(Z) = 0$. Since I is an n-integral depending on $m, n, u_{-j}, u_{-j+1}, \dots, u_k$, then DI is an n-integral depending on $m, n, u_{-j+1}, u_{-j+2}, \dots, u_{k+1}$. Indeed, $\overline{D}(DI) = D(\overline{D}I) = DI$. Therefore,

$$0 = Z(DI) = P_{j,k}(Z)DI + \left(a_{k+1}(T) - \sum_{s=1}^{N_1} \beta_s a_{k+1}(T_s)\right) \frac{\partial}{\partial u_{k+1}} DI = \\ = \left(a_{k+1}(T) - \sum_{s=1}^{N_1} \beta_s a_{k+1}(T_s)\right) \frac{\partial}{\partial u_{k+1}} DI.$$

Since $\frac{\partial}{\partial u_{k+1}} DI = D\left(\frac{\partial}{\partial u_k} I\right) \neq 0$, then $a_{k+1}(T) = \sum_{s=1}^{N_1} \beta_s a_{k+1}(T_s)$, and it means that $P_{j,k+1}(Z) = 0$. Applying the operator Z consequently to the integrals $D^2 I$, $D^3 I$, ..., and also to the integrals $D^{-1} I$, $D^{-2} I$, ..., we find that $P_{i,m}(Z) = 0$ for any natural numbers i, m. Therefore, $Z = 0$. It proves that the ring M is of finite dimension for any choice of the number N_2. Then the linear span of the vector fields $\{Y_j\}_1^\infty$ has a finite dimension; denote it by N. Let us specify now the value of the number N_2 by choosing $N_2 \geq N$. Then we have that the ring L_n generated by the operators $\{Y_j\}_1^N \cup \{X_j\}_1^N$ is a subring of a finite-dimensional ring M. This is why the ring L_n is finite-dimensional.

Suppose the dimension of the characteristic Lie ring L_n is finite; denote it by N_1. Let N be the dimension of the span of the vector fields $\{Y_j\}_1^\infty$. Then the set $\{Y_1, Y_2, \dots, Y_N\}$ forms a basis in it. We let $N_2 = N_1 - N$. Introduce

$$L_n^{(N_2)} = \{T^{(m)} = P_{N_2}^{(N)}(T) : T \in L_n\},$$

where the projector $P_{N_2}^{(N)}$ acts according the rule

$$P_{N_2}^{(N)} \left(\sum_{s=-N}^{-1} a_s \frac{\partial}{\partial \overline{u}_s} + \sum_{s=0}^{\infty} b_s \frac{\partial}{\partial u_s}\right) = \\ = \sum_{s=-N}^{-1} a_s \frac{\partial}{\partial \overline{u}_s} + \sum_{s=0}^{N_2} b_s \frac{\partial}{\partial u_s}. \tag{5.221}$$

Let $\{T_{0i}\}_{i=1}^{N_1}$ form a basis in the linear space $L_n^{(N_2)}$. Then we have N_1 equations of the form $T_{0i} G = 0$ for a function G of $N_1 + 3$ variables $m, n, u, u_1, \dots, u_{N_2}, \overline{u}_{-1}, \overline{u}_{-2}, \dots, \overline{u}_{-N}$. At that m and n are involved as parameters in the coefficients of the equation. According to Jacobi theorem, the considered system of equations has a non-constant solution G. By the equations $X_j G = 0$ this function is independent on the variables $\overline{u}_1, \overline{u}_2, \dots, \overline{u}_N$ and satisfies the condition $TG = 0$ for any $T \in L_n$. The function G is not defined uniquely, any other solution of the system depending on the same variables $m, n, u, u_1, \dots, u_{N_2}$ can be represented as $h(m, n, G)$ for some function h.

Since $\overline{D}^{-1} Y_1 \overline{D} = X_1 + Y_2$, $\overline{D}^{-1} X_j \overline{D} = X_{j+1}$, $j \geq 1$, $\overline{D}^{-1} Y_k \overline{D} = Y_{k+1}$, $k \geq 2$, for any vector filed Z in L_n we have $\overline{D}^{-1} Z \overline{D} = Z^* + \lambda X_{N+1}$ for some $Z^* \in L_n$ and some function λ. Therefore,

$$Z\overline{D}G = \overline{D}(\overline{D}^{-1} Z \overline{D} G) = \overline{D}(Z^* + \lambda X_{N+1})G = 0$$

for each $Z \in L_n$. This is why $\overline{D}G$ is also a solution to the aforementioned system of partial differential equations that implies $\overline{D}G = h(m, n, G)$.

In the same one can show that $\overline{D}^{-1} G = g(m, n, G)$ for some function g. To construct the desired n-integral I, it is sufficient now to let

$$G(m, n, u, u_1, \dots, u_N) = I_{(0)}(m, n, u, u_1, \dots, u_N), \\ \overline{D}^i G(m, n, u, u_1, \dots, u_N) = I_{(i)}(m, n, u, u_1, \dots, u_N), \\ \overline{D}^{-i} G(m, n, u, u_1, \dots, u_N) = I_{(-i)}(m, n, u, u_1, \dots, u_N), \quad i \geq 1.$$

The constructed in this way sequence of the functions $I_{(i)}(m, n, u, u_1, \dots, u_N)$ is an n-integral, since it satisfies the relation $\overline{D} I_{(i)} = I_{(i+1)}$.

5.2. General discrete equations. Employing explicit expressions (5.218) – (5.220), one can determine the characteristic vector fields $Z_k = X_{k-1} + Y_k$, $k \geq 2$ for an arbitrary equation of the form (5.216). According to Theorem 5.1, the ring L_n is infinite-dimensional provided there exists no n-integral. It is obvious that the operators $\{Z_k\}_1^\infty$ are linearly independent.

Lemma 5.2. *The commutation relations*

$$\begin{array}{lll} 1) & [Z_k, Z_j] = 0 \quad \textit{for all} \quad k, j \geq 1; \\ 2) & [X_k, Z_j] = 0 \quad \textit{for all} \quad k > j \end{array}$$

hold true, where $Z_1 := Y_1$.

Proof. Let $j > k$, the identity $[\frac{\partial}{\partial \overline{u}_1}, Z_{j-k}] = 0$ is valid since the coefficients of the operators Z_i, $i \geq 1$ are independent of the variable $\overline{u}_1$. Applying the conjugation operator (it is not an automorphism of the ring)

$$Z \to \overline{D}^{-1} Z \overline{D} \tag{5.222}$$

k times, we obtain

$$[\overline{D}^{-k} \frac{\partial}{\partial \overline{u}_1} \overline{D}^k, \overline{D}^{-k} Z_{j-k} \overline{D}^k] = [Z_k, Z_j] = 0.$$

The second part of the lemma is implied by the fact that $X_k = \frac{\partial}{\partial \overline{u}_{-k}}$, and the coefficients of the operator Z_j are independent of u_{-k} as $k > j$. The lemma is proven.

The key role in the description of the ring L_n is played by the automorphism defined by the rule

$$Z \to D Z D^{-1}, \tag{5.223}$$

where D is the operator of shift of the argument n. Let us show that X_1 and Y_1 regarded as the operators on the set of the functions depending on a finite number of the variables in a restricted dynamical set $S_N = \{\overline{u}_{-N}, \overline{u}_{-N+1}, \ldots, \overline{u}_{-1}, u, u_{\pm 1}, u_{\pm 2}, \ldots\}$ satisfy the relations

$$D X_1 D^{-1} = p X_1 + p(1) X_2 + \ldots + p(N-1) X_N, \tag{5.224}$$

$$D Y_1 D^{-1} = \frac{1}{x} Y_1, \tag{5.225}$$

where $p = D X_1 f^{-1,-1}$, $p(k) = D X_1 \overline{D}^{-k} f^{-1,-1}$, and $f^{-1,-1} = f^{-1,-1}(u, u_{-1}, \overline{u}_{-1})$. We observe that the coefficients of the operators $Y_1, Y_2, \ldots, Y_N$ depend only on the variables in the set S_N. Identity (5.224) can be easily checked by applying both sides of the identity to the dynamical variables. Exactly in the same way one can prove (5.225).

We introduce similar identities for generalized characteristic operators Y_j, X_j, $j \geq 1$. It is convenient to begin with the operator $Y_0 = \frac{\partial}{\partial \overline{u}_1}$. We first specify the action of the operator on the functions depending on all dynamical variables. It is obvious that

$$D Y_0 D^{-1} = \xi(1) \frac{\partial}{\partial \overline{u}_1} + \xi(2) \frac{\partial}{\partial \overline{u}_2} + \ldots + \xi(j) \frac{\partial}{\partial \overline{u}_j}, \tag{5.226}$$

where $\xi(k) = D Y_0 \overline{D}^{k-1} f^{-1,1}$, $f^{-1,1} = f^{-1,1}(u, u_{-1}, \overline{u}_1)$. The last identity can be checked easily by applying to the variables $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_j, \ldots$. It is also clear that all other dynamical variables lie in the kernel of operator (5.226). We apply now conjugation operator (5.222) to identity (5.226) and obtain as a result, taking into consideration the identities $D\overline{D} = \overline{D} D$, $\overline{D}^{-1} Y_0 \overline{D} = Y_1$, $\overline{D}^{-1} \frac{\partial}{\partial \overline{u}_k} \overline{D} = \frac{\partial}{\partial \overline{u}_{k-1}}$ as $k \geq 2$, the relation

$$D Y_1 D^{-1} = \overline{\xi}_{-1}(1) Y_1 + \overline{\xi}_{-1}(2) Y_0 + \overline{\xi}_{-1}(3) \frac{\partial}{\partial \overline{u}_2} + \ldots, \tag{5.227}$$

where $\overline{\xi}_{-1}(j) = \overline{D}^{-1} \xi(j)$, that proves in particular formula (5.225). It remains to check that $\overline{\xi}_{-1}(1) = \frac{1}{x}$. Indeed, differentiating the identity $\overline{u}_1 = f(u_{-1}, u, f^{-1,1}(u, u_{-1}, \overline{u}_1))$ w.r.t. the variable $\overline{u}_1$, we find $D^{-1}\left(\frac{\partial f}{\partial \overline{u}_1}\right)\cdot\frac{\partial f^{-1,1}}{\partial \overline{u}_1}=1$. It yields that $\xi(1)\cdot\overline{x}_1=1$. Hence, $\overline{D}^{-1}\xi(1)=\frac{1}{x}$. Applying now repeatedly operator (5.222) to identity (5.227), we find

$$\begin{aligned}DZ_kD^{-1}&=\overline{\xi}_{-k}(1)Z_k+\overline{\xi}_{-k}(2)Z_{k-1}+\ldots+\\&+\overline{\xi}_{-k}(k)Y_1+\overline{\xi}_{-k}(k+1)Y_0+\overline{\xi}_{-k}(k+2)\tfrac{\partial}{\partial\overline{u}_2}+\ldots,\end{aligned}\tag{5.228}$$

where $Z_k=\overline{D}^{1-k}Y_1\overline{D}^{k-1}=Y_k+X_{k-1}$ as $k\geq 2$. On the restricted set of the dynamical variables S_N identity (5.228) casts into the form

$$DZ_kD^{-1}=\overline{\xi}_{-k}(1)Z_k+\overline{\xi}_{-k}(2)Z_{k-1}+\ldots+\overline{\xi}_{-k}(k)Y_1.\tag{5.229}$$

For instance, as $k=2$ we have

$$DZ_2D^{-1}=\frac{1}{\overline{x}_{-1}}Z_2+\overline{\xi}_{-2}(2)Y_1.\tag{5.230}$$

On whole the set of the dynamical variables formula (5.224) is extended as

$$DX_1D^{-1}=pX_1+\sum_{i=1}^{\infty}p(i)X_{i+1}.$$

We apply conjugation operator (5.222) on this identity; bearing in mind the conditions $\overline{D}^{-1}X_j\overline{D}=X_{j+1}$, $j\geq 1$, we get

$$DX_jD^{-1}=\overline{p}_{1-j}X_j+\overline{p}_{1-j}(1)X_{j+1}+\ldots+\overline{p}_{1-j}(k)X_{k+1}+\ldots,$$

whose restriction on S_N yields

$$DX_jD^{-1}=\overline{p}_{1-j}X_j+\overline{p}_{1-j}(1)X_{j+1}+\ldots+\overline{p}_{1-j}(N-1)X_N\tag{5.231}$$

as $j\leqslant N$.

Lemma 5.3. *Suppose that $Z=\sum_{-\infty}^{+\infty}b(j)\frac{\partial}{\partial u_j}\in L_n$ satisfies two conditions, $DZD^{-1}=cZ$ for some function c and $b(j_0)\equiv 0$ for some fixed value $j=j_0$. Then $Z=0$.*

The proof is carried out by simple calculations (see [42]).

Example 5.2. *As an example we consider one of the discrete versions of the Liouville equation*

$$e^{u_{1,1}+u}=e^{u_1+\overline{u}_1}+1.\tag{5.232}$$

Let us calculate the functions x and p for equation (5.232). *We have*

$$u_{1,-1}=\ln\left(e^{\overline{u}_{-1}+u_{+1}}-1\right)-u.\tag{5.233}$$

Therefore, $r=f^{-1,1}(u,u_{-1},\overline{u}_1)=\ln\left(e^{\overline{u}_1+u_{-1}}-1\right)-u$. Employing the identity

$$x=\overline{D}^{-1}\left(\frac{\partial f(u,u_1,\overline{u}_1)}{\partial\overline{u}_1}\right)=-\frac{\frac{\partial f^{1,-1}(u,u_1,\overline{u}_{-1})}{\partial u}}{\frac{\partial f^{1,-1}(u,u_1,\overline{u}_{-1})}{\partial u_1}},\tag{5.234}$$

we find $x=1-e^{-u_1-\overline{u}_{-1}}$. Then we find $D^{-1}\left(\frac{1}{x}\right)=\frac{1}{x_{-1}}=1+e^{-u_{-1}-\overline{u}_{-1}}$.

To describe p, we employ the identity

$$p=D\left(\frac{\partial f^{-1,-1}(u,u_{-1},\overline{u}_{-1})}{\partial\overline{u}_{-1}}\right)=-\frac{1}{\frac{\partial f^{1,-1}(u,u_1,\overline{u}_{-1})}{\partial\overline{u}_{-1}}}.\tag{5.235}$$

As a result we get $p=x$. This is why

$$DX_1D^{-1}=(1-e^{-u_1-\overline{u}_{-1}})X_1,\quad DY_1D^{-1}=\frac{1}{1-e^{-u_1-\overline{u}_{-1}}}Y_1.$$

On the operators $R_1 = [X_1, Y_1]$, $P_1 = [X_1, R_1]$, $Q_1 = [Y_1, R_1]$ mapping (5.223) *acts by the rule*

$$\begin{aligned} DR_1D^{-1} &= R_1 + \tfrac{x-1}{x}Y + (x-1)X_1, \\ DP_1D^{-1} &= xP_1 + (x-1)R_1 - \tfrac{x-1}{x}Y - (x-1)X_1, \\ DQ_1D^{-1} &= \tfrac{1}{x}Q_1 - \tfrac{x-1}{x}R_1 - \tfrac{x-1}{x}Y - (x-1)X_1. \end{aligned} \tag{5.236}$$

It follows from formulas (5.236) *that $D(P_1+R_1)D^{-1} = x(P_1+R_1)$ and $D(Q_1+R_1) = \frac{1}{x}(Q_1+R_1)$. The last relations by Lemma 5.3 imply the identities $P_1 = -R_1$, $Q_1 = -R_1$.*

In the same way one can check that

$$Z_2 = X_1 - \left(1 + e^{u-\overline{u}_{-2}}\right) R_1. \tag{5.237}$$

For this it is sufficient to compare the identity

$$DZ_2D^{-1} = \frac{1}{\overline{x}_{-1}}Z_2 + \left(\frac{1}{x\overline{x}_{-1}} - 1\right) Y_1$$

with the first of formulas (5.236) *taking into consideration the identity $DX_1D^{-1} = xX_1$.*

By Lemma 5.3 it follows from (5.237) *that $Y_2 = -\left(1 + e^{u-\overline{u}_{-2}}\right) R_1$. Therefore, the characteristic algebra L_n for equation* (5.232) *is three-dimensional as the linear space spanned on the vectors X_1, Y_1, R_1; the n-integral of minimal order depends on three variables, for instance, $I = I(u, u_1, u_{-1})$.*

In order to find I, we solve the linear system $Y_1I = 0$, $R_1I = 0$, or in the expanded form

$$\begin{aligned} \tfrac{\partial I}{\partial u} + \left(1 - e^{-u_1-\overline{u}_{-1}}\right) \tfrac{\partial I}{\partial u_1} + \left(1 + e^{-u_{-1}-\overline{u}_{-1}}\right) \tfrac{\partial I}{\partial u_{-1}} &= 0, \\ e^{-u_1-\overline{u}_{-1}} \tfrac{\partial I}{\partial u_1} - e^{-u_{-1}-\overline{u}_{-1}} \tfrac{\partial I}{\partial u_{-1}} &= 0. \end{aligned}$$

It implies easily that $I = e^{u_{-1}-u} + e^{u_1-u}$.

Let us find out how the characteristic algebra changes under the change of variables in a discrete equation. The most general point transformation in the equation (5.216) is defined by the function

$$u(m,n) = \phi(m, n, v(m,n)). \tag{5.238}$$

Change (5.238) reduces (5.216) to the equation

$$v_{1,1} = \widetilde{f}(m, n, v, v_1, \overline{v}_1), \tag{5.239}$$

where $\widetilde{f} = \phi^{-1}(m, n, f(m, n, \phi(m, n, v), \phi(m+1, n, v_1), \phi(m, n+1, \overline{v}_1)))$.

Let u find out how the characteristic vector fields X_j, Z_j and $\widetilde{X}_j$, $\widetilde{Z}_j$ of equations (5.238) and (5.239) are related.

Lemma 5.4. *The identities*

$$x = -\frac{\frac{\partial f^{1,-1}}{\partial u}}{\frac{\partial f^{1,-1}}{\partial u_1}}, \qquad \frac{1}{x_{-1}} = -\frac{\frac{\partial f^{-1,-1}}{\partial u}}{\frac{\partial f^{-1,-1}}{\partial u_{-1}}}$$

hold true.

Proof. Let us prove the second identity in the statement. Differentiating an obvious identity

$$u_{-1} = f^{-1,-1}\left(\overline{u}_1, f^{-1,1}(u, u_{-1}, \overline{u}_1), u\right)$$

w.r.t. the variable $\overline{u}_1$, we find

$$0 = \overline{D}\left(\frac{\partial f^{-1,-1}(u, u_{-1}, \overline{u}_{-1})}{\partial u}\right) + \overline{D}\left(\frac{\partial f^{-1,-1}(u, u_{-1}, \overline{u}_{-1})}{\partial u_{-1}}\right) \cdot \frac{\partial f^{-1,1}(u, u_{-1}, \overline{u}_{-1})}{\partial \overline{u}_1}.$$

It yields

$$\frac{1}{x_{-1}} = \overline{D}^{-1}\frac{\partial f^{-1,1}}{\partial \overline{u}_1} = -\frac{\frac{\partial f^{-1,-1}}{\partial u}}{\frac{\partial f^{-1,-1}}{\partial u_{-1}}}.$$

The first identity in the statement is proven in the same way by differentiating the identity

$$u_1 = f^{1,-1}(\overline{u}_1, f(u, u_1, \overline{u}_1), u)$$

w.r.t. $\overline{u}_1$. The lemma is proven.

By Lemma 5.4 we have

$$\widetilde{x} = -\frac{\frac{\partial \tilde{f}^{1,-1}}{\partial v}}{\frac{\partial \tilde{f}^{1,-1}}{\partial v_1}} = -\frac{\frac{\partial f^{1,-1}}{\partial u}\cdot\phi'(v)}{\frac{\partial f^{1,-1}}{\partial u_1}\cdot\phi'(v_1)} = \frac{\phi'(v)}{\phi'(v_1)}x,$$
$$\frac{1}{\widetilde{x}_{-1}} = -\frac{\frac{\partial \tilde{f}^{-1,-1}}{\partial v}}{\frac{\partial \tilde{f}^{-1,-1}}{\partial v_{-1}}} = -\frac{\frac{\partial f^{-1,-1}}{\partial u}\cdot\phi'(v)}{\frac{\partial f^{-1,-1}}{\partial u_{-1}}\cdot\phi'(v_{-1})} = \frac{\phi'(v)}{\phi'(v_{-1})}\cdot\frac{1}{x_{-1}}$$

This is why $\frac{\partial}{\partial v} = \phi'(v)\frac{\partial}{\partial u}$, and also

$$\widetilde{x}\cdot\widetilde{x}_1\cdot\ldots\cdot\widetilde{x}_j\frac{\partial}{\partial v_{j+1}} = \phi'(v)\cdot x\cdot x_1\cdot\ldots\cdot x_j\frac{\partial}{\partial u_{j+1}} \tag{5.240}$$

and

$$\frac{1}{\widetilde{x}_{-1}}\cdot\frac{1}{\widetilde{x}_{-2}}\cdot\ldots\cdot\frac{1}{\widetilde{x}_{-j}}\frac{\partial}{\partial v_{-j}} = \phi'(v)\cdot\frac{1}{x_{-1}}\cdot\frac{1}{x_{-2}}\cdot\ldots\cdot\frac{1}{x_{-j}}\frac{\partial}{\partial u_{-j}}. \tag{5.241}$$

By (5.240), (5.241) explicit expression (5.219) and the formulas

$$\widetilde{Y}_1 = \frac{\partial}{\partial v} + \widetilde{x}\frac{\partial}{\partial v_1} + \frac{1}{\widetilde{x}_{-1}}\frac{\partial}{\partial v_{-1}} + \widetilde{x}\widetilde{x}_1\frac{\partial}{\partial v_2} + \frac{1}{\widetilde{x}_{-1}\widetilde{x}_{-2}}\frac{\partial}{\partial v_{-2}} + \ldots$$

we find the desired relation

$$\widetilde{Y}_1 = \phi'(m,n,v)Y_1. \tag{5.242}$$

It is obvious that $X_1 = \frac{\partial}{\partial \overline{u}_{-1}}$ and $\widetilde{X}_1 = \frac{\partial}{\partial \overline{v}_{-1}}$ are related by the identity

$$\widetilde{X}_1 = \phi'(m,n-1,\overline{v}_{-1})X_1. \tag{5.243}$$

Applying conjugation operator (5.222) to (5.242), (5.243) and employing the identities

$$Z_{j+1} = \overline{D}^{-j}Y_1\overline{D}^j, \quad X_{j+1} = \overline{D}^{-j}X_j\overline{D},$$

we get

$$\widetilde{Z}_{j+1} = \phi(m,n-j,\overline{v}_{-j})Z_{j+1}, \quad \widetilde{X}_{j+1} = \phi(m,n-j-1,\overline{v}_{-j-1})X_{j+1}. \tag{5.244}$$

5.3. *S*-integrable discrete equations. In this section we study the characteristic operators of S-integrable discrete equations of the form (5.216), i.e., of soliton type equations. Let a Lie ring T be generated by the vector fields X and Y. Denote by V_j, $j \geq 0$ the linear space over the field of locally analytic functions spanned on X, Y, and all multiple commutators of the operators X, Y of order less or equal j so that

$$V_0 = \{X, Y\}, \quad V_1 = \{X, Y, [X, Y]\},$$
$$V_2 = \{X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]]\}, \ldots.$$

We introduce the function $\triangle(k) = \dim V_{k+1} - \dim V_k$.

Definition 5.2. *We call T a ring of minimal growth if there exists a sequence of natural numbers $\{t_k\}_{k=1}^{\infty}$, for which $\triangle(t_k) \leqslant 1$.*

Denote by T_{kj} the Lie rings generated by the operators X_k, Y_j. The following conjecture looks credible.

Proposition 5.1. *Suppose equation* (5.216) *is S-integrable, then for all $k, j \geq 1$ the associated ring T_{kj} is a ring of minimal growth.*

As an example we consider the discrete potential KdV equation

$$u_{1,1} = u + \frac{1}{u_1 - \overline{u}_1}. \tag{5.245}$$

We represent (5.245) in two different ways $(u - u_{-1,-1})(\overline{u}_{-1} - u_{-1}) = 1$ and $(u_1 - \overline{u}_{-1})(u_{1,-1} - u) = 1$. It yields

$$u_{-1,-1} = u + \frac{1}{u_{-1} - \overline{u}_{-1}} := f^{-1,-1}, \quad u_{1,-1} = u + \frac{1}{u_1 - \overline{u}_{-1}} := f^{1,-1}.$$

Hence,

$$x = -\frac{\frac{\partial f^{1,-1}}{\partial u}}{\frac{\partial f^{1,-1}}{\partial u_1}} = (u_1 - \overline{u}_{-1})^2,$$
$$\frac{1}{x_{-1}} = -\frac{\frac{\partial f^{-1,-1}}{\partial u}}{\frac{\partial f^{-1,-1}}{\partial u_{-1}}} = (u_{-1} - \overline{u}_{-1})^2,$$
$$p = \frac{1}{\frac{\partial f^{1,-1}}{\partial \overline{u}_{-1}}} = (u_1 - \overline{u}_{-1})^2.$$

This is why

$$Y_1 = \frac{\partial}{\partial u} + (u_1 - \overline{u}_{-1})^2 \frac{\partial}{\partial u_1} + (u_{-1} - \overline{u}_{-1})^2 \frac{\partial}{\partial u_{-1}} + \dots. \tag{5.246}$$

It is easy to see that

$$\begin{aligned}
&Y_1 x = Y_1(u_1 - \overline{u}_{-1})^2 = 2(u_1 - \overline{u}_{-1})^3 = 2x\sqrt{x},\\
&Y_1 x_{-1} = Y_1(u_{-1} - \overline{u}_{-1})^{-2} = 2\sqrt{x_{-1}},\\
&X_1 x = \tfrac{\partial}{\partial \overline{u}_{-1}}(u_1 - \overline{u}_{-1})^2 = -2(u_1 - \overline{u}_{-1}) = -2\sqrt{x},\\
&X_1 x_{-1} = \tfrac{\partial}{\partial \overline{u}_{-1}}(u_{-1} - \overline{u}_{-1})^{-2} = -2(u_{-1} - \overline{u}_{-1})^{-3} = -2x_{-1}\sqrt{x_{-1}}.
\end{aligned}$$

Consider the ring $T_{1,1}$ generated by the operators (see (5.246)) Y_1 and $X_1 = \frac{\partial}{\partial \overline{u}_{-1}}$. We construct the sequence of multiple commutators,

$$R_1 = [X_1, Y_1], \quad P_1 = [X_1, R_1], \quad Q_1 = [Y_1, R_1],$$
$$R_{k+1} = [X_1, Q_k], \quad P_k = [X_1, R_k], \quad Q_k = [Y_1, R_k], \quad k \geq 1.$$

Theorem 5.2. *The sequence $X_1, Y_1, R_1, P_1, Q_1, R_2, P_2, Q_2, \dots$ forms a basis of the ring $T_{1,1}$ (see [42]).*

Proof. We employ the identities $DX_1D^{-1} = xX_1$ and $D(yY_1)D^{-1} = Y_1$, where $y = x_{-1}$, and write $[DX_1D^{-1}, D(yY_1)D^{-1}] = [xX_1, Y_1]$. We reduce the last identity to

$$D(R_1 - 2\sqrt{y}Y_1)D^{-1} = R_1 - 2\sqrt{x}X_1. \tag{5.247}$$

The symmetric expression is the most simple and convenient one. We commute (5.247), preserving the symmetricity with $DX_1D^{-1} = xX_1$,

$$[D(R_1 - 2\sqrt{y}Y_1)D^{-1}, DX_1D^{-1}] = [R_1 - 2\sqrt{x}X_1, xX_1].$$

The last identity is reduced to

$$D(P_1 - 2\sqrt{y}R_1 + 2yY_1)D^{-1} = x(P_1 + 2X_1). \tag{5.248}$$

Commuting (5.247) with $D(yY_1)D^{-1} = Y_1$, we obtain

$$D(y(Q_1 - 2Y_1))D^{-1} = Q_1 + 2\sqrt{x}R_1 - 2xX_1. \tag{5.249}$$

We commute $DX_1D^{-1} = xX_1$ with identity (5.248), then

$$D\left([X_1, P_1] - 2\sqrt{y}P_1 + 4yR_1 - 4y\sqrt{y}Y_1\right)D^{-1} = x^2[X_1, P_1] - 2x\sqrt{x}P_1 - 4x\sqrt{x}X_1.$$

From the last identity we deduct term by term identity (5.248) multiplied by $2\sqrt{x}$, and as a result we get $D[X_1, P_1] = x^2[X_1, P_1]$. By Lemma 5.3 it implies $[X_1, P_1] = 0$. In the same one can check that $[Y_1, Q_1] = 0$.

It can be checked that the action of automorphism (5.223) on the operator R_2, P_2, Q_2 is written as

$$D(R_2-2\sqrt{y}Q_1)D^{-1}=R_2+2\sqrt{x}P_1,$$
$$D(P_2+2\sqrt{y}R_2-2yQ_1)D^{-1}=x(P_2-2P_1),$$
$$D(y(Q_2-2Q_1))D^{-1}=Q_2+2\sqrt{x}R_2+2xP_1.$$

By induction one can prove that for all $j>1$ the relations

$$D(R_j-2\sqrt{y}Q_{j-1})D^{-1}=R_j+2\sqrt{x}P_{j-1},$$
$$D(P_j+2(-1)^j\sqrt{y}R_j+2(-1)^{j-1}yQ_{j-1})D^{-1}=x(P_j-2P_{j-1}),$$
$$D(y(Q_j-2Q_{j-1}))D^{-1}=Q_j+2\sqrt{x}R_j-2xP_{j-1}X$$

hold, and $[X_1,P_j]=0$, $[Y_1,Q_j]=0$, $[Y_1,P_j]=[X,Q_j]$, $[R_j,P_k]=P_{k+j}$, $[R_j,Q_k]=-Q_{k+j}$, $[R_j,R_k]=0$, $[P_j,Q_k]=-R_{k+j+1}$, $[P_j,P_k]=0$, $[Q_j,Q_k]=0$. The theorem is proven.

Corollary 5.1. *The ring $T_{1,1}$ is that of minimal growth.*

Proof. By construction we have $V_0=\{X_1,Y_1\}$, $V_1=V_0\oplus\{R_1\}$, $V_2=V_1\oplus\{P_1,Q_1\}$, $\ldots$, $V_{2k-1}=V_{2k-2}\oplus\{R_k\}$, $V_{2k}=V_{2k-1}\oplus\{P_k,Q_k\}$, $\ldots$. Hence, $\triangle(2k+1)=\dim V_{2k+2}-\dim V_{2k+1}=2$, $\triangle(2k)=\dim V_{2k+1}-\dim V_{2k}=1$ for each $k\geq 0$.

In works [42,51] the connection between the integrability of equation (5.216) and the property of minimal growth for the rings $T_{j,k}$ was studied.

In work [42] the following statement was proven.

Theorem 5.3. *Assume the Lie ring $T_{1,1}$ of the discrete equation*

$$u_{1,1}=u+\phi(u_1-\overline{u}_1) \tag{5.250}$$

satisfies the condition that there exists at least one natural number j such that $\Delta(k)\leqslant 1$. Then by a point change the equation is reduced to one of the following equations,

$$\begin{array}{ll}(1) & u_{1,1}=u+c(u_1-\overline{u}_1-\beta),\\ (2) & (u_{1,1}-u-\alpha)(u_1-\overline{u}_1-\beta)=\gamma,\\ (3) & (\alpha u_1+\beta\overline{u}_1)u_{1,1}+u(\gamma u_1-\delta\overline{u}_1)=0.\end{array}$$

We note that in this theorem on the Lie ring there imposed a very weak condition, namely, the existence of a sequence of natural numbers for which $\Delta(k)\leqslant 1$ is replaced the condition that at least one such number exists. At that a certain list of the equations is obtained and all of them integrable. The equation (1) is linear, the equation (2) is the discrete potential Korteweg-de Vries equation, and the equation (3) belongs to the known list of Adler, Bobenko, Suris (ABS) (see [45]).

In the work [51] the equation

$$u_{1,1}+u=\phi(u_1+\overline{u}_1) \tag{5.251}$$

is studied under a similar restriction.

Theorem 5.4. *Let the ring $T_{1,1}$ of discrete equation* (5.251) *satisfies the condition that there exists at least one natural k such that $\Delta(k)\leqslant 1$. Then equation* (5.251) *is reduced by a point change to one of the following equations,*

$$\begin{array}{ll}(1) & u_{1,1}+u=c(u_1-\overline{u}_1-\beta),\\ (2) & (u_{1,1}+u-\alpha)(u_1+\overline{u}_1-\beta)=\gamma,\\ (3) & \alpha_1 u\overline{u}_1u_{1,1}+\alpha_2uu_{1,1}+\alpha_3u_1\overline{u}_1+\alpha_4=0.\end{array}$$

We observe that the equation (2) is integrable as $\alpha=\beta$, since it is reduced to the potential KdV equation, and the equation (3) as $\alpha_3=\pm\alpha_2$ is reduced to a known equation from the list of ABS. In other cases these equations are non-integrable.

6. Perspectives of algebraic method

6.1. Characteristic ring of "n-waves" equations. We consider the system of hyperbolic partial differential equations

$$(\frac{\partial}{\partial t} + a_i \frac{\partial}{\partial x})u^i = \phi_i(u^1, u^2, \dots, u^n), \quad i = 1, 2, \dots, n. \tag{6.252}$$

Here a_i are arbitrary constants and ϕ_i are arbitrary functions. As the functions ϕ_i are quadratic, we deal with the system of n-waves equations [63]. In order to determine two characteristic directions, we introduce independent variables ξ and η as follows,

$$\frac{\partial}{\partial t} + a_{i_0}\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial t} + a_{i_1}\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta}.$$

In terms of new variables the system casts into the form

$$\begin{aligned} p_\xi &= f(p,q,r), \\ q_\eta &= \phi(p,q,r), \\ r_\xi &= r_\eta A + \psi(p,q,r), \end{aligned} \tag{6.253}$$

where $f = (f^1, f^2, \dots, f^s)$, $\phi = (\phi^1, \phi^2, \dots, \phi^l)$, $\psi = (\psi^1, \psi^2, \dots, \psi^m)$, $p = (u^{i_1}, u^{i_2}, \dots, u^{i_s})$, $q = (u^{j_1}, u^{j_2}, \dots, u^{j_l})$, $r = (u^{k_1}, u^{k_2}, \dots, u^{k_m})$, $A = diag(\lambda_1, \lambda_2, \dots, \lambda_m)$, $\forall i\ \lambda_i \neq 0$, where $p = (p^1, p^2, \dots, p^s)$, $q = (q^1, q^2, \dots, q^l)$, $r = (r^1, r^2, \dots, r^m)$. Denote by F ($\bar F$) the set of locally analytic functions depending on a finite number of the variables $p, q, r, q_1, r_1, q_2, r_2, \dots, q_i, r_i, \dots$ $(p, q, r, \bar p_1, \bar r_1, \bar p_2, \bar r_2, \dots, \bar p_i, \bar r_i, \dots)$. Here $q_i = D^i q$, $r_i = D^i r$, $\bar p_i = \bar D^i p$, $\bar r_i = \bar D^i r$, $i = 1, 2, \dots$, $D = \frac{d}{d\xi}$, $\bar D = \frac{d}{d\eta}$. The operator of total differentiation $\bar D$ w.r.t. the variable η on the set F is defined as

$$\begin{aligned} \bar D &= \sum_{i=1}^s \bar p_1^i \frac{\partial}{\partial p^i} + \sum_{i=1}^l \phi^i(p,q,r)\frac{\partial}{\partial q^i} + \sum_{i=1}^m [\frac{1}{\lambda_i} r_1^i - \frac{1}{\lambda_i}\psi^i(p,q,r)]\frac{\partial}{\partial r^i} + \\ &+ \sum_{i=1}^l D\phi^i(p,q,r)\frac{\partial}{\partial q_1^i} + \sum_{i=1}^m [\frac{1}{\lambda_i} r_2^i - \frac{1}{\lambda_i} D\psi^i(p,q,r)]\frac{\partial}{\partial r_1^i} + \dots \\ &+ \sum_{i=1}^l D^n\phi^i(p,q,r)\frac{\partial}{\partial q_n^i} + \sum_{i=1}^m [\frac{1}{\lambda_i} r_{n+1}^i - \frac{1}{\lambda_i} D^n\psi^i(p,q,r)]\frac{\partial}{\partial r_n^i} + \dots . \end{aligned} \tag{6.254}$$

Considering the vector fields $X_i = \frac{\partial}{\partial p^i}$, $i = 1, 2, \dots, s$ and

$$\begin{aligned} X_{s+1} &= \sum_{i=1}^l \phi^i(p,q,r)\frac{\partial}{\partial q^i} + \sum_{i=1}^m [\frac{1}{\lambda_i} r_1^i - \frac{1}{\lambda_i}\psi^i(p,q,r)]\frac{\partial}{\partial r^i} + \\ &+ \sum_{i=1}^l D\phi^i(p,q,r)\frac{\partial}{\partial q_1^i} + \sum_{i=1}^m [\frac{1}{\lambda_i} r_2^i - \frac{1}{\lambda_i} D\psi^i(p,q,r)]\frac{\partial}{\partial r_1^i} + \dots \\ &+ \sum_{i=1}^l D^n\phi^i(p,q,r)\frac{\partial}{\partial q_n^i} + \sum_{i=1}^m [\frac{1}{\lambda_i} r_{n+1}^i - \frac{1}{\lambda_i} D^n\psi^i(p,q,r)]\frac{\partial}{\partial r_n^i} + \dots , \end{aligned} \tag{6.255}$$

we obtain that $\bar D = \sum_{i=1}^s \bar p_1^i X_i + X_{s+1}$.

Definition 6.1. *The Lie ring R_ξ over the field F generated by the vector fields $X_1, X_2, \dots, X_{s+1}$ is called the characteristic Lie ring in the directions of ξ of the system of equations* (6.252).

In the similar way we define the characteristic Lie ring R_η in the direction of η. It is generated by the following vector fields,

$$\begin{aligned} Y_i &= \frac{\partial}{\partial q^i}, \quad i = 1, 2, \ldots, l, \\ Y_{l+1} &= \sum_{i=1}^{s} f^i(p,q,r)\frac{\partial}{\partial p^i} + \sum_{i=1}^{m}[\lambda_i \bar{r}_1^i + \psi^i(p,q,r)]\frac{\partial}{\partial r^i} + \ldots \\ &+ \sum_{i=1}^{s} \bar{D}^n f^i(p,q,r)\frac{\partial}{\partial \bar{p}_n^i} + \sum_{i=1}^{m}[\lambda_i \bar{r}_{n+1}^i + \bar{D}^n\psi^i(p,q,r)]\frac{\partial}{\partial \bar{r}_n^i} + \ldots . \end{aligned} \tag{6.256}$$

In this case the operator of total differentiation D w.r.t. the variable ξ on the set $\bar{F}$ reads as $D = \sum_{i=1}^{l} q_1^i Y_i + Y_{l+1}$.

6.2. Evolution equations.

6.2.1. Lie rings of evolution equations. We consider the evolution equations

$$\frac{\partial u}{\partial t} = f(u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^n u}{\partial x^n}). \tag{6.257}$$

In order to determine the vector fields generating the Lie ring of equation (6.257), we shall study the auxiliary equation

$$\frac{\partial^2 u}{\partial t \partial x} = F(u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{n+1} u}{\partial x^{n+1}}), \tag{6.258}$$

where $F = Df$, D is the operator of total differentiation w.r.t. the variable x. We define the operator $\bar{D}$ in the space of locally analytic functions depending on a finite number of the variables $u, u_1, u_2, \ldots, u_i, \ldots$ ($u_n = \frac{\partial^n u}{\partial x^n}$) by the rule

$$\bar{D} = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + F\frac{\partial}{\partial u_1} + DF\frac{\partial}{\partial u_2} + \ldots + D^{n-1}F\frac{\partial}{\partial u_n} + \ldots .$$

We introduce the vector fields

$$X_1 = \frac{\partial}{\partial u}, \quad X_2 = F\frac{\partial}{\partial u_1} + DF\frac{\partial}{\partial u_2} + \ldots + D^{n-1}\frac{\partial}{\partial u_n} + \ldots .$$

Definition 6.2. *The Lie ring R generated by the vector fields X_1 and X_2 is called a characteristic Lie ring of the equation* (6.257).

The following statement holds.

Lemma 6.1. *If* $\dim R < \infty$*, then the right hand side* $F(u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{n+1} u}{\partial x^{n+1}})$ *of the equation* (6.258) *is a quasipolynom w.r.t. the variable u.*

Proof. Since $[D, \bar{D}] = 0$, employing $[D, \bar{D}] = [D, \frac{\partial u}{\partial t}X_1 + X_2]$, we have

$$[D, X_1] = 0, \qquad [D, X_2] = FX_1. \tag{6.259}$$

We let $X_3 = [X_1, X_2]$ and, employing Jacobi identity and relations (6.259), we obtain

$$[D, X_3] = \frac{\partial F}{\partial u}X_1. \tag{6.260}$$

We define the sequence of vector fields X_i, $i = 4, 5, \ldots$ as $X_i = [X_1, X_{i-1}]$. As above, we get that

$$[D, X_i] = \frac{\partial^{i-2} F}{\partial u^{i-2}}X_1, \quad i = 4, 5, \ldots . \tag{6.261}$$

Let the ring R be finite-dimensional. Then there exists m such that the vector fields $X_2, X_3, \ldots, X_m$ are linearly independent and

$$X_{m+1} = \sum_{i=2}^{m} \alpha_i X_i, \tag{6.262}$$

where the coefficients α_i, $i = 2, 3, \ldots, m$ are the functions of the variables $u, u_1, u_2, \ldots$.

By (6.262) we have $[D, X_{m+1}] = \sum_{i=2}^{m} D(\alpha_i) X_i + \sum_{i=2}^{m} \alpha_i [D, X_i]$. In accordance with (6.261), we rewrite the latter as

$$\frac{\partial^{m-1} F}{\partial u^{m-1}} X_1 = \sum_{i=2}^{m} D(\alpha_i) X_i + \sum_{i=2}^{m} \alpha_i \frac{\partial^{i-2} F}{\partial u^{i-2}} X_1.$$

Since the vector fields $X_1, X_2, \ldots, X_m$ are linearly independent, we obtain

$$D(\alpha_i) = 0, \quad i = 2, 3, \ldots, m, \tag{6.263}$$

$$\frac{\partial^{m-1} F}{\partial u^{m-1}} = \sum_{i=2}^{m} \alpha_i \frac{\partial^{i-2} F}{\partial u^{i-2}}. \tag{6.264}$$

These equations imply that α_i is constant and F is a quasipolynom w.r.t. the variable u. The lemma is proven.

Remark 6.1. *If the Lie ring R of the evolution equation is finite-dimensional, then according to* (3.141) *the right hand side $f(u, u_1, \ldots, u_n)$ is the solution to the partial differential equation*

$$\frac{\partial^{m-1}}{\partial u^{m-1}} \left(\sum_{i=0}^{n} u_{i+1} \frac{\partial f}{\partial u_i} \right) = \sum_{i=2}^{m} \alpha_i \left(\frac{\partial^{i-2}}{\partial u^{i-2}} \sum_{k=0}^{n} u_{k+1} \frac{\partial f}{\partial u_k} \right).$$

Let us give examples of Lie rings of evolution equations.

Example 6.1. *We consider the equation*

$$u_t = u_x + u^2.$$

Applying the operator D, we get $u_{xt} = u_{xx} + 2uu_x$.

It follows from the relation

$$D_t F(u, u_1, u_2, \ldots) = (u_t \frac{\partial}{\partial u} + f \frac{\partial}{\partial u_1} + Df \frac{\partial}{\partial u_2} + \ldots) F = (u_t X_1 + X_2) F$$

that

$$D_t = u_t X_1 + X_2, \tag{6.265}$$

where $f = u_{xx} + 2uu_x$.

Lemma 6.2. *The vector field $Y = a_1(u, u_1, \ldots, u_{n_1}) \frac{\partial}{\partial u_1} + a_2(u, u_1, \ldots, u_{n_2}) \frac{\partial}{\partial u_2} + \ldots$ commutes with the operator D if and only if $Y = 0$.*

The proof is implied by the formula $[D, Y] = (Da_1 \frac{\partial}{\partial u_1} + Da_2 \frac{\partial}{\partial u_2} + Da_3 \frac{\partial}{\partial u_3} + \ldots) - -a_1 \frac{\partial}{\partial u} - a_2 \frac{\partial}{\partial u_1} - a_3 \frac{\partial}{\partial u_2} - \ldots$.

Due to (6.265) and $[D, D_t] = 0$ we have

$$fX_1 + u_t [D, X_1] + [D, X_2] = 0.$$

The last relation splits into to equations $[D, X_1] = 0$ and $[D, X_2] = -fX_1$.

We introduce the operators $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3]$, $X_5 = [X_2, X_3]$. It is easy to show that $[D, X_3] = -2u_1 X_1$ and $[D, X_4] = 0$. It follows from the lemma that $X_4 = 0$.

Since the operator $X_3 = 2u_1 \frac{\partial}{\partial u_1} + 2u_2 \frac{\partial}{\partial u_2} + \ldots$, then

$$[D, X_5] = (X_3 f) X_1 + [X_2, -2u_1 X_1] = (4u_1 u + 2u_2) X_1 + 2u_1 X_3 - 2f X_1,$$

or $[D, X_5] = 2u_1X_3$.

Let us prove that a basis of the ring consists of the operators X_1, X_2, X_3, X_5. One can see that $[X_1, X_5] = 0$. Consider $X_7 = [X_2, X_5]$. By straightforward calculations we obtain that $[D, X_7] = -4u_1^2X_1 + 2u_1X_5 + 2fX_3$, hence $X_7 = 2u_1X_3 + 2uX_5$. Consider now the operator $X_8 = [X_3, X_5]$. Calculate $[D, X_8]$,

$$[D, X_8] = -[X_5, [D, X_3]] + [X_3, [D, X_5]] = 2X_5(u_1)X_1 + 2X_3(u_1)X_3 = 4u_1X_3.$$

Comparing the relations $[D, X_8] = 4u_1X_3$ and $[D, X_5] = 2u_1X_3$, we get $X_8 = 2X_5$. It yields that the Lie ring of this equation is four-dimensional and the elements X_1, X_2, X_3, X_5 are linearly dependent.

Example 6.2. *Burgers equation*

$$u_t = u_{xx} + 2uu_x.$$

The corresponding equation (6.258) *reads as*

$$u_{xt} = u_3 + 2uu_2 + 2u_1^2. \tag{6.266}$$

The characteristic vector fields

$$X_1 = \tfrac{\partial}{\partial u}, \quad X_2 = (u_3 + 2uu_2 + 2u_1^2)\tfrac{\partial}{\partial u_1} + (u_4 + 2uu_3 + 6u_1u_2)\tfrac{\partial}{\partial u_2} + \ldots + \\ +(u_{n+1} + 2uu_n + \ldots)\tfrac{\partial}{\partial u_n} + \ldots .$$

Here

$$X_3 = [X_1, X_2] = 2D - 2u_1X_1, \tag{6.267}$$

where $D = u_1\frac{\partial}{\partial u} + u_2\frac{\partial}{\partial u_1} + \ldots + u_n\frac{\partial}{\partial u_{n-1}} + \ldots$.

It follows from the relation $[D, \bar{D}] = 0$ *that*

$$[D, u_tX_1 + X_2] = (u_3 + 2uu_2 + 2u_1^2)X_1 + u_t[D, X_1] + [D, X_2] = 0.$$

Then

$$[D, X_1] = 0 \quad \text{and} \quad [D, X_2] = -(u_3 + 2uu_2 + 2u_1^2)X_1. \tag{6.268}$$

Employing (6.267) *and* (6.268), *we obtain*

$$X_4 = [X_1, X_3] = [X_1, 2D - 2u_1X_1] = 0, \\ X_5 = [X_2, X_3] = [X_2, 2D - 2u_1X_1] = \\ = 2(u_3 + 2uu_2 + 2u_1^2)X_1 - 2(u_3 + 2uu_2 + 2u_1^2)X_1 + 2u_1X_3.$$

It yields $X_4 = 0$, $X_5 = 2u_1X_3$. *Thus, a basis of the characteristic ring for Burgers equations consists of the operators* X_1, X_2, X_3.

Example 6.3. *Consider the Korteweg-de Vries equation* $u_t = u_{xxx} + uu_x$. *The equation* (6.258) *becomes*

$$u_{xt} = u_4 + uu_2 + u_1^2. \tag{6.269}$$

For the equation (6.269) *it is easy to show that* $X_4 = [X_1, X_3] = 0$, $X_5 = [X_2, X_3] = = u_1X_3$. *Therefore, a basis of the characteristic Lie ring of the Korteweg-de Vries equations consists of the operators* X_1, X_2, X_3.

Example 6.4. *For the modified Korteweg-de Vries equation* $u_t = u_{xxx} + u^2u_x$ *the equation* (6.258) *casts into the form*

$$u_{xt} = u_4 + u^2u_2 + 2uu_1^2.$$

The operators X_1, X_2, $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3]$ *form a basis of the characteristic Lie ring of the the modified Korteweg-de Vries equation*

6.2.2. Associated Lie algebras. As it follows from the examples given in Subsection 6.2.1, the characteristic Lie ring determines the dependence of the right hand side $f = f(u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^n u}{\partial x^n})$ of equation (6.257) on the variable u. Here we are going to introduce the definition of a Lie ring which would take into account also the dependence of f on the derivatives $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots, \frac{\partial^n u}{\partial x^n}$. In order to do it, we rewrite equation (6.257) as

$$u_t^1 = f^1(u^1, u^2, u^3, \dots, u^n), \tag{6.270}$$

letting $u^1 = u$, $u^2 = u_x$, $u^3 = u_{xx}, \dots, u^n = \frac{\partial^n u}{\partial x^n}$.

Then by consequent differentiating w.r.t. x we obtain from (6.270) the system of equations

$$\begin{aligned} u_t^1 &= f^1(u^1, u^2, \dots, u^n), \\ u_t^2 &= f^2(u^1, u^2, \dots, u^n, u_x^n), \\ u_t^3 &= f^3(u^1, u^2, \dots, u^n, u_x^n, u_{xx}^n), \\ &\dots, \\ u_t^n &= f^n(u^1, u^2, \dots, u^n, u_x^n, u_{xx}^n, \dots, \frac{\partial^{n-1} u^n}{\partial x^{n-1}}). \end{aligned} \tag{6.271}$$

Thus, from equation (6.257) we pass to evolution system of equations (6.271) w.r.t. unknown functions u^1, u^2, $\dots$, u^n. Now, as in Subsection 6.2.1, to define the characteristic Lie ring of system (6.271), we consider the system

$$u_{xt}^i = F^i, \quad F^i = Df^i, \quad i = 1, 2, \dots, n. \tag{6.272}$$

The characteristic Lie ring of system (6.271) is defined by the operator $\overline{D}$,

$$\overline{D} = \frac{\partial u^k}{\partial t} \cdot \frac{\partial}{\partial u^k} + F^k \frac{\partial}{\partial u_1^k} + DF^k \frac{\partial}{\partial u_2^k} + \dots,$$

namely, by the vector fields

$$X_1 = \frac{\partial}{\partial u^1}, \quad X_2 = \frac{\partial}{\partial u^2}, \quad \dots, \quad X_n = \frac{\partial}{\partial u^n},$$
$$X_{n+1} = F^k \frac{\partial}{\partial u_1^k} + DF^k \frac{\partial}{\partial u_2^k} + \dots.$$

And, finally, we shall call the characteristic Lie ring of system (6.271) an associated Lie ring of evolution equation (6.257).

For example, for the Burgers equation

$$u_t = u_{xx} + 2uu_x$$

we have $u_x = v$, $u_{xx} = w$. Then system (6.271) and (6.272) become

$$\begin{aligned} u_t &= w + 2uv, \\ v_t &= w_x + 2u_x v + 2uv_x, \\ w_t &= w_{xx} + 4u_x v_x + 2u_{xx} v + 2uv_{xx}, \end{aligned}$$

and

$$\begin{aligned} u_{xt} &= w_x + 2uv_x + 2u_x v, \\ v_{xt} &= w_{xx} + 2u_{xx} v + 2uv_{xx} + 4u_x v_x, \\ w_{xt} &= w_{xxx} + 6u_{xx} v_x + 6u_x v_{xx} + 2u_{xxx} v, \end{aligned}$$

respectively.

6.3. Systems of ordinary differential equations. Here we consider the system of ordinary differential equations

$$\frac{du^i}{dy} = f_i(x, y, u^1, u^2, \dots, u^n), \quad i = 1, 2, \dots, n. \tag{6.273}$$

In order to define the notion of the characteristic Lie ring for equations (6.273), we shall assume that the solution $u^1, u^2, \dots, u^n$ depend on the parameter x. Then by differentiation equations (6.273) w.r.t. the variable x, we obtain the system of equations

$$\frac{\partial^2 u^i}{\partial y \partial x} = \frac{\partial f_i}{\partial x} + \sum_{k=1}^{n} \frac{\partial f_i}{\partial u^k} \cdot \frac{\partial u^k}{\partial x}. \tag{6.274}$$

It is known (see, for instance, [56]) that hyperbolic system (6.274) possesses a pair of characteristic Lie ring, namely, the x-characteristic Lie ring X is generated by the vector field

$$X_1 = \frac{\partial}{\partial u^1}, \quad X_2 = \frac{\partial}{\partial u^2}, \dots, X_n = \frac{\partial}{\partial u^n},$$
$$X_{n+1} = \frac{\partial}{\partial y} + F_i \frac{\partial}{\partial u_1^i} + DF_i \frac{\partial}{\partial u_2^i} + D^2 F_i \frac{\partial}{\partial u_3^i} + \dots,$$

and the y-characteristic Lie ring Y by the fields

$$Y_1 = \frac{\partial}{\partial u_1^1}, \quad Y_2 = \frac{\partial}{\partial u_1^2}, \dots, Y_n = \frac{\partial}{\partial u_1^n},$$
$$Y_{n+1} = \frac{\partial}{\partial x} + u_1^i \frac{\partial}{\partial u^i} + F_i \frac{\partial}{\partial \overline{u}_1^i} + \overline{D} F_i \frac{\partial}{\partial \overline{u}_2^i} + \dots,$$

where $D(\overline{D})$ is the operator of total differentation w.r.t. the variable $x(y)$, $u_k^i = D^k u^i$, $\overline{u}_k^i = \overline{D}^k u^i$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots$.

We call now the x- and y-characteristic Lie rings of system (6.274) Lie rings for system of differential equations (6.273).

The study of system (6.273) is based on considering of the ring X.

We note that if $\dim X < \infty$, then the right hand sides f_i of system (6.273) are quasipoynomials w.r.t. the variables $u^1, u^2, \dots, u^n$.

As an example we consider the ordinary differential equation

$$u_y = f(y, u). \tag{6.275}$$

It is easy to show that if the characteristic Lie ring of equation (6.275) is finite-dimensional, then the right hand side $f(y, u)$ is a quasipolynom w.r.t. the variable u.

For instance, the dimension of the Lie ring of the equation

$$u_y = \alpha_0(y) + \alpha_1(y) u + \alpha_2 u^2 \tag{6.276}$$

equals 4, and if u is a solution to equation (6.276) depending on the parameter x, then the expression $\frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2}$ is independent of y, i.e.,

$$\frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} = f(x).$$

Let us adduce an example of the Riccati equation (6.276) with the Lie ring of dimension 3. Such an example is the equation

$$u_y = \alpha_1(y) u + u^2. \tag{6.277}$$

A solution of Riccati equation (6.277) depending on the parameter x satisfies the relation

$$\frac{u_{xx}}{u_x} - 2\frac{u_x}{u} = f(x).$$

Remark 6.2. *Another approach of determining the characteristic Lie ring of system* (6.273) *is based on the change*

$$u^i = \frac{\partial v^i}{\partial x}, \quad i = 1, 2, \ldots, n.$$

Then system (6.273) *becomes*

$$\frac{\partial^2 v^i}{\partial x \partial y} = f_i \left(x, y, \frac{\partial v^1}{\partial x}, \ldots, \frac{\partial v^n}{\partial x} \right). \tag{6.278}$$

We call the x- and y-characteristic Lie rings for system of hyperbolic equations (6.278) Lie rings of original system of ordinary differential equations (6.273).

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Anatolii Vasilevich Zhiber,
Institute of Mathematics with Computer Center of Ufa Scientific Center, RAS,
Chernyshevskogo str., 112,
450008, Ufa, Russia
E-mail: `zhiber@mail.com`

Regina Dimovna Murtazina,
Ufa State
Aviation Technical University,
K. Marx st., 12
450000, Ufa, Russia
E-mail: `ReginaUFA@yandex.ru`

Ismagil Talgatovich Habibullin,
Institute of Mathematics with Computer Center of Ufa Scientific Center, RAS,
Chernyshevskogo str., 112,
450008, Ufa, Russia
E-mail: `habibullinismagil@gmail.com`

Alexei Borisovich Shabat,
Landau Institute for Theoretical Physics, RAS,
Kosygina str., 2,
119334, Moscow, Russia
E-mail: `shabatab@mail.ru`