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CHARACTERISTIC LIE RING OF ZHIBER-SHABAT-TZITZEICA EQUATION

A.U. SAKIEVA

Abstract. In this work we give a complete description of the characteristic Lie ring for Zhiber-Shabat-Tzitzeica equation. We construct the basis for the linear space of multiple commutators of arbitrary order. It is proven that the characteristic Lie ring is a ring of slow growth.

Keywords: Lie ring, nonlinear hyperbolic equation, integral.

1. Introduction

Characteristic Lie rings are an important tool for studying partial differential equations. At the first time the notion of a characteristic vector field lying in the base of the characteristic field was introduced by Goursat in [1]. The notion of characteristic algebra was introduced in the work of A.N. Leznov, V.G. Smirnov, A.B. Shabat [2]. The characteristic algebras and rings for differential equations were also studied in the works [3–6].

In this paper we deal with the problem of description of the characteristic Lie ring for the equation

$$u_{xy} = e^u + e^{-2u}. \tag{1}$$

Equation (1) was first found in the work of Tzitzeica [7] while studying the geometry of two-dimensional surfaces in $\mathbb{R}^3$. Later it was re-discovered by A.B. Shabst and A.V. Zhiber in [8] as a result of classification of integrable cases for Klein-Gordon equation. In the same work they constructed the hierarchy of higher symmetries and conservation laws. The Lax representations for (1) were found by A.V. Mikhailov (see [9]). Note that the higher symmetries of equation (1) have the order equalling $6n + 1$ and $6n - 1$, where $n \in \mathbb{N}$. A surprising fact is that exactly these numbers are distinguished in the description of the characteristic ring for equation (1). This fact seems to show a close connection between the algebra of higher symmetries of an equation and its characteristic ring, since exactly the same situation holds for Sine-Gordon equation (see [3, 4]).

In the work [4] for the equations

$$u_{xy} = f(u) \tag{2}$$

there were introduced the operators X_1 and X_2 generating the characteristic Lie ring for equation (2),

$$X_1 = \sum_{k=1}^{\infty} D^{k-1}(f)\frac{\partial}{\partial u_k}, \tag{3}$$

$$X_2 = \frac{\partial}{\partial u}, \tag{4}$$

where in our case $f = e^u + e^{-2u}$. Here D is the operator of total differentiation w.r.t. x. We observe that the operators X_1 and X_2 are linearly independent as $f(u) \neq 0$.

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Denote by L_i the linear space spanned on all commutators of length no more than $i-1$, where $i=2,3,\ldots$. And in this space we take linear combinations with the coefficients depending on smooth functions of a finite number of dynamical variables, and a set of the elements Z_1, Z_2, $\ldots$, Z_k is called linearly independent if there exists a set of the functions c_1, c_2, $\ldots$, c_k such that not all of them are zero and the identity $c_1Z_1+c_2Z_2+\ldots+c_kZ_k=0$ holds. Otherwise the set is linearly independent. For instance, $L_2=\{X_1,X_2\}$ is the linear space generated by the elements X_1,X_2, $\dim L_2=2$. We suppose that X_1 and X_2 the operators of length 1. Then L_3 consists of the elements of the space L_2 and the element $X_3=[X_2,X_1]$, i.e., $L_3=\{X_1,X_2,X_3\}$. Therefore, $L_4=L_3+\{[X_2,X_3],[X_1,X_3]\}$ and so forth.

Define $\delta(i)=\dim(L_i)-\dim(L_{i-1})$. It will be shown that the Lie ring for equation (1) is infinite-dimensional, and at that $\delta(i)=1$ if $i=6n-1, i=6n, i=6n+1, i=6n+3, n=1,2,\ldots$ and $\delta(i)=2$ as $i=6n+2, i=6n+4, n=1,2,\ldots$. Hence, the Lie ring for the equation (1) is the characteristic ring of slow growth. We observe that the structure of linear spaces L_i for $i\leqslant 10$ was studied in [4].

In what follows we shall make use of the next statement whose proof can found, for instance, in [4].

Lemma 1. *Let a vector field Z be*

$$Z=\alpha_1\frac{\partial}{\partial u_1}+\alpha_2\frac{\partial}{\partial u_2}+\alpha_3\frac{\partial}{\partial u_3}+\ldots,\alpha_i=\alpha_i(u,u_1,u_2,\ldots),i=1,2,3,\ldots$$

Then $[D_x,Z]=0$ if and only if $Z=0$.

2. Characteristic ring for Zhiber-Shabat-Tzitzeica equation

We introduce the following notations for multiple commutators,

$$X_{i_1,\ldots i_n}=ad_{X_{i_1}}\ldots ad_{X_{i_{n-1}}}X_{i_n},\text{ where } ad_XY=[X,Y].$$

Theorem 1. *For Zhiber-Shabat-Tzitzeica equation (1) the identities*

$$\delta(i)=2,i=6n+2,i=6n+4,n=1,2,\ldots; \tag{5}$$

$$\delta(i)=1,i=6n-1,i=6n,i=6n+1,i=6n+3,n=1,2,\ldots. \tag{6}$$

hold. At that the following identities

$L_{6n+2}=L_{6n+1}\oplus\{X_{1\ldots121},X_{21\ldots121}\}$,
$L_{6n+4}=L_{6n+3}\oplus\{X_{1\ldots121},X_{21\ldots121}\}$,
$L_{6n-1}=L_{6n-2}\oplus\{X_{1\ldots121}\}$,
$L_{6n}=L_{6n-1}\oplus\{X_{1\ldots121}\}$,
$L_{6n+1}=L_{6n}\oplus\{X_{1\ldots121}\}$,
$L_{6n+3}=L_{6n+2}\oplus\{X_{1\ldots121}\}$

are valid. Id est, the operators X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, $\bar{X}_8$, X_9, X_{10}, $\bar{X}_{10}$, $\ldots X_{6n-1}$, X_{6n}, X_{6n+1}, X_{6n+2}, $\bar{X}_{6n+2}$, X_{6n+3}, X_{6n+4}, $\bar{X}_{6n+4}$, $\ldots$ form a basis of the characteristic Lie ring L of equation (1), where
$X_n=X_{i_1\ldots i_n}$ *at that* $i_1=\ldots=i_{n-2}=i_n=1, i_{n-1}=2$,
$\bar{X}_n=X_{i_1\ldots i_n}$ *at that* $i_2=\ldots=i_{n-2}=i_n=1, i_1=i_{n-1}=2$.

The operators X_1,X_2 are determined above. For X_1 and X_2 the relations

$$[D_x,X_1]=-(e^u+e^{-2u})X_2, \tag{7}$$

$$[D_x,X_2]=0 \tag{8}$$

hold true. We introduce an operator of length 2, $X_3=[X_2,X_1]$. Employing Jacobi identity and relations (7),(8), we obtain

$$[D_x,X_3]=-(e^u-2e^{-2u})X_2. \tag{9}$$

Assume that the operator X_3 is linearly expressed via X_1 and X_2, then we get

$$X_3=\lambda_1X_1+\lambda_2X_2. \tag{10}$$

We apply the operator D_x to both sides of the last identity; employing relations (7),(8),(9), we obtain

$$-(e^u - 2e^{-2u})X_2 = D_x(\lambda_1)X_1 - \lambda_1(e^u + e^{-2u})X_2 + D_x(\lambda_2)X_2. \tag{11}$$

We compare the coefficients at linearly independent operators X_2 and X_1, then we get

$$-(e^u - 2e^{-2u}) = -\lambda_1(e^u + e^{-2u}) + D_x(\lambda_2) \tag{12}$$

and

$$D_x(\lambda_1) = 0. \tag{13}$$

Identity (12) is inconsistent since $\lambda_N = \lambda_N(u, u_x, u_{xx}, ...)$, and $D_x(\lambda_2)$ contains $u_x, u_{xx}, ...$. Therefore, the operator $X_3 = X_{21}$ is not linearly expressed via X_1 and X_2. Hence, the linear space L_3 is three-dimensional, i.e., $L_3 = \{X_1, X_2, X_3\}$.

We introduce the operators of length 3, $X_4 = [X_1, X_3]$ and $\bar{X}_4 = [X_2, X_3]$, for which it holds

$$\left[D_x, \bar{X}_4\right] = 2\left[D_x, X_1\right] - \left[D_x, X_3\right] \tag{14}$$

and

$$[D_x, X_4] = (e^u - 2e^{-2u})X_3 - (e^u + e^{-2u})\left[X_2, X_3\right] = (2e^u - e^{-2u})X_3 - 2(e^u + e^{-2u})X_1. \tag{15}$$

Thus,

$$\bar{X}_4 = 2X_1 - X_3.$$

The operator $X_4 = X_{121}$ is not linearly expressed via operators of lower order, and we get $L_4 = \{X_1, X_2, X_3, X_4\}$.

Consider the operators of length 4, $X_5 = [X_1, X_4]$ and $\bar{X}_5 = [X_2, X_4]$. Employing Jacobi identity and relations (7), (8), and (15), we obtain $\bar{X}_5 = -X_4$ and

$$[D_x, X_5] = (2e^u - e^{-2u})X_4 - (e^u + e^{-2u})\left[X_2, X_4\right] = 3e^u X_4. \tag{16}$$

The operator $X_5 = X_{1121}$ is not linearly expressed via the operators of lower order, and therefore $L_5 = \{X_1, X_2, X_3, X_4, X_5\}$.

We introduce the operators of length 5, $X_6 = \left[X_1, X_5\right], \bar{X}_6$ and $[X_3, X_4]$. According to Jacobi identity, $[X_3, X_4] = X_5$. It is easy to show that for $\bar{X}_6$ the identity

$$\left[D_x, \bar{X}_6\right] = 0 \tag{17}$$

holds. Therefore, in accordance with Lemma 1, $\bar{X}_6 = 0$. For X_6 we obtain

$$[D_x, X_6] = [X_1, 3e^u X_4] - (e^u + e^{-2u})\left[X_2, X_5\right] = 3e^u X_5. \tag{18}$$

Hence, the operator $X_6 = X_{11121}$ is not linearly expressed via operators of lower order, and we have $L_6 = \{X_1, X_2, X_3, X_4, X_5, X_6\}$.

Consider the operators of length 6, $X_7 = \left[X_1, X_6\right], \bar{X}_7 = \left[X_2, X_6\right], \left[X_3, X_5\right]$. It is easy to show that $[X_3, X_5] = X_6, [X_2, X_6] = X_6$,

$$[D_x, X_7] = 3e^u X_6 - (e^u + e^{-2u})\left[X_2, X_6\right] = (2e^u - e^{-2u})X_6. \tag{19}$$

Therefore, $X_7 = X_{111121}$ is not linearly expressed via operators of lower order $L_7 = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$.

We introduce the operators of length 7, $X_8 = \left[X_1, X_7\right], \bar{X}_8 = \left[X_2, X_7\right], \left[X_3, X_6\right], \left[X_4, X_5\right]$. According to Jacobi identity, $[X_3, X_6] = \bar{X}_8 - X_7, [X_4, X_5] = 2X_7 - \bar{X}_8$. For X_8 and $\bar{X}_8$ the relations

$$\left[D_x, \bar{X}_8\right] = (4e^u + e^{-2u})X_6 \tag{20}$$

and

$$[D_x, X_8] = (2e^u - e^{-2u})X_7 - (e^u + e^{-2u})\bar{X}_8 \tag{21}$$

hold true. Id est, the space L_8 is obtained from L_7 by adding two linearly independent elements, $X_8 = X_{1111121}$ and $\bar{X}_8 = X_{2111121}$, i.e., $L_8 = L_7 \oplus \left\{X_8, \bar{X}_8\right\}$.

We consider the operators of length 8, $X_9 = [X_1, X_8]$, $\bar{X}_9 = [X_2, X_8]$, $\left[X_1, \bar{X}_8\right]$, $\left[X_2, \bar{X}_8\right]$, $[X_3, X_7]$, $[X_4, X_6]$.

According to Jacobi identity, $[X_3, X_7] = -X_8, [X_4, X_6] = X_8$.

It is also easy to show that $\left[X_2, \bar{X}_8\right] = 2X_7 + \bar{X}_8, \left[X_1, \bar{X}_8\right] = X_8$. $\left[D_x, \bar{X}_9\right] = 0$, therefore, due to Lemma 1, $\bar{X}_9 = [X_2, X_8] = 0$.

For X_9 we obtain

$$[D_x, X_9] = (e^u - 2e^{-2u})X_8 - (e^u + e^{-2u})\,[X_2, X_8] = (e^u - 2e^{-2u})X_8. \tag{22}$$

Hence, $X_9 = X_{11111121}$ is not linearly expressed via operators of lower order, and $L_9 = L_8 \oplus \{X_9\}$.

We introduce the operators of length 9, $X_{10} = [X_1, X_9]$, $\bar{X}_{10} = [X_2, X_9]$, $\left[X_3, \bar{X}_8\right]$, $[X_3, X_8]$, $[X_4, X_7]$, $[X_5, X_6]$, for which the relations

$$[X_5, X_6] = 2X_9 + \bar{X}_{10}, [X_4, X_7] = -X_9 - \bar{X}_{10},$$
$$[X_3, X_8] = \bar{X}_{10}, \left[X_3, \bar{X}_8\right] = -3X_8$$

hold true. For the operators $X_{10}, \bar{X}_{10}$ we have

$$\left[D_x, \bar{X}_{10}\right] = (e^u + 4e^{-2u})X_8 + (e^u - 2e^{-2u})\,[X_2, X_8] = (e^u + 4e^{-2u})X_8 \tag{23}$$

and

$$[D_x, X_{10}] = (e^u - 2e^{-2u})X_9 - (e^u + e^{-2u})\bar{X}_{10}. \tag{24}$$

Thus, the operators $X_{10} = X_{111111121}$ and $\bar{X}_{10} = X_{211111121}$ are not linearly expressed via operators of lower order, and $L_{10} = L_9 \oplus \left\{X_{10}, \bar{X}_{10}\right\}$.

It can be shown that the basis of the characteristic ring generated by the elements X and Y can be always chosen among the elements of the form $ad_X^{k_1} ad_Y^{k_2} ... ad_X^{k_s} Y$.

We introduce the notations $X_n = [X_1, X_{n-1}]\,, \bar{X}_n = [X_2, X_{n-1}]$. We shall prove by the induction. Suppose that for $i = n - 1$ the identities

$$\left[D_x, X_{6(n-1)-1}\right] = (2e^u - e^{-2u})X_{6(n-1)-2} - (e^u + e^{-2u})\left[X_2, X_{6(n-1)-2}\right], \tag{25}$$

$$\left[D_x, X_{6(n-1)}\right] = 3e^u X_{6(n-1)-1} - (e^u + e^{-2u})\left[X_2, X_{6(n-1)-1}\right], \tag{26}$$

$$\left[D_x, X_{6(n-1)+1}\right] = 3e^u X_{6(n-1)} - (e^u + e^{-2u})\left[X_2, X_{6(n-1)}\right], \tag{27}$$

$$\left[D_x, X_{6(n-1)+2}\right] = (2e^u - e^{-2u})X_{6(n-1)+1} - (e^u + e^{-2u})\left[X_2, X_{6(n-1)+1}\right], \tag{28}$$

$$\left[D_x, X_{6(n-1)+3}\right] = (e^u - 2e^{-2u})X_{6(n-1)+2} - (e^u + e^{-2u})\left[X_2, X_{6(n-1)+2}\right], \tag{29}$$

$$\left[D_x, X_{6(n-1)+4}\right] = (e^u - 2e^{-2u})X_{6(n-1)+3} - (e^u + e^{-2u})\left[X_2, X_{6(n-1)+3}\right], \tag{30}$$

$$\bar{X}_{6(n-1)} = 0, \bar{X}_{6(n-1)-1} = -X_{6(n-1)-2}, \tag{31}$$

$$\bar{X}_{6(n-1)+1} = X_{6(n-1)}, \bar{X}_{6(n-1)+3} = 0, \tag{32}$$

$$\left[X_1, \bar{X}_{6(n-1)+2}\right] = X_{6(n-1)+2}, \left[X_2, \bar{X}_{6(n-1)+2}\right] = 2X_{6(n-1)+1} + \bar{X}_{6(n-1)+2}, \tag{33}$$

$$\left[X_1, \bar{X}_{6(n-1)+4}\right] = -X_{6(n-1)+4}, \left[X_2, \bar{X}_{6(n-1)+4}\right] = 2X_{6(n-1)+3} - \bar{X}_{6(n-1)+4} \tag{34}$$

are valid. Let us check identities (25) – (34) for $i = n$.

We introduce the operators of length $6n - 2$, $X_{6n-1} = X_{6(n-1)+5} = \left[X_1, X_{6(n-1)+4}\right]$ and $\bar{X}_{6n-1} = \bar{X}_{6(n-1)+5} = \left[X_2, X_{6(n-1)+4}\right]$. We have

$$\left[D_x, \bar{X}_{6n-1}\right] = \left[D_x, \left[X_2, X_{6(n-1)+4}\right]\right] = -\left[D_x, X_{6(n-1)+4}\right], \tag{35}$$

hence, $\bar{X}_{6n-1} = -X_{6(n-1)+4}$. For X_{6n-1} it holds

$$[D_x, X_{6n-1}] = (2e^u - e^{-2u})X_{6n-2} - (e^u + e^{-2u})\,[X_2, X_{6n-2}] = 3e^u X_{6n-2}. \tag{36}$$

It means that the operator $X_{6n-1} = X_{1...121}$ is not linearly expressed via operators of lower order, and $L_{6n-1} = L_{6n-2} \oplus \{X_{6n-1}\}$, so, $\delta(6n - 1) = 1$.

We consider the operators of length $6n - 1$, $X_{6n} = [X_1, X_{6n-1}]\,, \bar{X}_{6n} = [X_2, X_{6n-1}]$. We have

$$\left[D_x, \bar{X}_{6n}\right] = 0, \tag{37}$$

and therefore in accordance with Lemma 1 $\bar{X}_{6n} = 0$. We also have

$$[D_x, X_{6n}] = [X_1, 3e^u X_{6n-2}] - (e^u + e^{-2u})\,[X_2, X_{6n-1}] = 3e^u X_{6n-1}. \tag{38}$$

Therefore, the operator $X_{6n} = X_{1\dots121}$ is not linearly expressed via operators of lower order, and $L_{6n} = L_{6n-1} \oplus \{X_{6n}\}$. Thus, $\delta(6n) = 1$.

We introduce the operators of length $6n$, $X_{6n+1} = [X_1, X_{6n}]\,, \bar{X}_{6n+1} = [X_2, X_{6n}]$ for which

$$\left[D_x, \bar{X}_{6n+1}\right] = 3e^u X_{6n-1}; \tag{39}$$

therefore, $\bar{X}_{6n+1} = X_{6n}$. It is easy to show that

$$[D_x, X_{6n+1}] = (2e^u - e^{-2u})X_{6n}. \tag{40}$$

It means that the operator $X_{6n+1} = X_{1\dots121}$ is not linearly expressed via operators of lower order, and $L_{6n+1} = L_{6n} \oplus \{X_{6n+1}\}$. We get $\delta(6n+1) = 1$.

We consider the operators of length $6n+1$, $X_{6n+2} = [X_1, X_{6n+1}]\,, \bar{X}_{6n+2} = [X_2, X_{6n+1}]$. We have

$$\left[D_x, \bar{X}_{6n+2}\right] = (4e^u + e^{-2u})X_{6n} \tag{41}$$

and

$$[D_x, X_{6n+2}] = (2e^u - e^{-2u})X_{6n+1} - (e^u + e^{-2u})\bar{X}_{6n+2}. \tag{42}$$

Therefore, the operators $X_{6n+2} = [X_1, X_{6n+1}] = X_{1\dots121}$ and $\bar{X}_{6n+2} = [X_2, X_{6n+1}] = X_{21\dots121}$ are not linearly expressed via operators of lower order, $L_{6n+2} = L_{6n+1} \oplus \left\{X_{6n+2}, \bar{X}_{6n+2}\right\}$. Thus, $\delta(6n+2) = 2$.

We introduce the operators of length $6n+2$, $X_{6n+3} = [X_1, X_{6n+2}]\,, \bar{X}_{6n+3} = [X_2, X_{6n+2}]\,, \left[X_1, \bar{X}_{6n+2}\right]\,, \left[X_2, \bar{X}_{6n+2}\right]$.

It is easy to show the validity of the identity

$$\left[D_x, \left[X_2, \bar{X}_{6n+2}\right]\right] = (8e^u - e^{-2u})X_{6n}, \tag{43}$$

hence, $\left[X_2, \bar{X}_{6n+2}\right] = 2X_{6n+1} + \bar{X}_{6n+2}$.

$$\left[D_x, \left[X_1, \bar{X}_{6n+2}\right]\right] = (2e^u - e^{-2u})X_{6n+1} - (e^u + e^{-2u})\bar{X}_{6n+2}, \tag{44}$$

and therefore, $\left[X_1, \bar{X}_{6n+2}\right] = X_{6n+2}$.

For the operators X_{6n+3} and $\bar{X}_{6n+3}$ we have

$$\left[D_x, \bar{X}_{6n+3}\right] = 0 \tag{45}$$

and

$$[D_x, X_{6n+3}] = (e^u - 2e^{-2u})X_{6n+2}, \tag{46}$$

Then due to Lemma 1, $\bar{X}_{6n+3} = 0$, and it means that on this step into the basis of the characteristic one operator $X_{6n+3} = X_{1\dots121}$ is added, and therefore $L_{6n+3} = L_{6n+2} \oplus \{X_{6n+3}\}$. Thus, $\delta(6n+3) = 1$.

We consider the operators of length $6n+3$, $X_{6n+4} = [X_1, X_{6n+3}]\,, \bar{X}_{6n+4} = [X_2, X_{6n+3}]\,,$ for which it holds

$$\left[D_x, \bar{X}_{6n+4}\right] = (e^u + 4e^{-2u})X_{6n+2}, \tag{47}$$

$$[D_x, X_{6n+4}] = (e^u - 2e^{-2u})X_{6n+3} - (e^u + e^{-2u})\bar{X}_{6n+4}. \tag{48}$$

Hence, the space L_{6n+4} is obtained from L_{6n+3} by adding two elements $X_{6n+4} = X_{1\dots121}$ and $\bar{X}_{6n+4} = X_{21\dots121}$, i.e., $L_{6n+4} = L_{6n+3} \oplus \left\{X_{6n+4}, \bar{X}_{6n+4}\right\}$. Thus, $\delta(6n+4) = 2$.

We introduce the operators of length $6n+4$,

$X_{6(n+1)-1} = [X_1, X_{6n+4}]\,, \bar{X}_{6(n+1)-1} = [X_2, X_{6n+4}]\,, \left[X_1, \bar{X}_{6n+4}\right]\,, \left[X_2, \bar{X}_{6n+4}\right]$.

The relation

$$\left[D_x, \left[X_2, \bar{X}_{6n+4}\right]\right] = (e^u - 8e^{-2u})X_{6n+2} \tag{49}$$

holds true, and hence $\left[X_2, \bar{X}_{6n+4}\right] = 2X_{6n+3} - \bar{X}_{6n+4}$.

We also have

$$\left[D_x, \left[X_1, \bar{X}_{6n+4}\right]\right] = (-e^u + 2e^{-2u})X_{6n+3} + (e^u + e^{-2u})\bar{X}_{6n+4} \tag{50}$$

that yields $\left[X_1, \bar{X}_{6n+4}\right] = -X_{6n+4}$.

It follows from the identity

$$\left[D_x, \bar{X}_{6(n+1)-1}\right] = (-e^u + 2e^{-2u})X_{6n+3} + (e^u + e^{-2u})\bar{X}_{6n+4} \tag{51}$$

that $\bar{X}_{6(n+1)-1} = -X_{6n+4}$.

For $X_{6(n+1)-1}$ we have

$$\left[D_x, X_{6(n+1)-1}\right] = 3e^u X_{6n+4}. \tag{52}$$

Hence, the operator $X_{6(n+1)-1} = X_{1\ldots121}$ is not linearly expressed via operators of lower order, and $L_{6(n+1)-1} = L_{6n+4} \oplus \left\{X_{6(n+1)-1}\right\}$. Thus, $\delta(6(n+1)-1) = 1$.

Theorem is proven.

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Al'fiya Uralovna Sakieva,
Institute of Mathematics with Computer Center of Ufa Scientific Center, RAS,
Chernyshevkogo str., 112,
450008, Ufa, Russia
E-mail: `alfiya85.85@mail.ru`