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INTEGRABLE EVOLUTION EQUATIONS WITH CONSTANT SEPARANT

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Abstract. The survey provides classification results for integrable one-field evolution equations of orders 2, 3 and 5 with the constant separant. The classification is based on necessary integrability conditions following from the existence of the formal recursion operator for integrable equations. Recurrent formulas for the whole infinite sequence of necessary conditions are presented for the first time. The most of the classification statements can be found in papers by S.I. Svinilupov and V.V. Sokolov but the proofs have never been published before. The result concerning the fifth order equations is stronger than obtained before.

Keywords: evolution differential equation, integrability, generalized symmetry, conservation law, classification.

Introduction

This survey is devoted to the classification of integrable evolution equations

$$u_t = u_n + F(x, u, u_x, u_{xx}, \dots, u_{n-1}), \qquad u_i = \frac{\partial^i u}{\partial x^i}. \tag{0.1}$$

The equations with such dependence of the highest x-derivative are often referred to as equations with a constant separant.

Let us specify what we mean by the integrability in the present paper. Unfortunately, at present there exists no unified rigorous definition for the integrability of differential equations (for various approaches see, for instance, [1–3]). However, for some types of differential equations there are efficient criteria of the integrability, which can be not only checked for these equations, but also allow one to find all the equations from this class satisfying this criterion.

For evolution equations (0.1) with one temporary variable and one spatial variable the most effective integrability criterion is the existence of generalized local symmetries. In the works [4,7] a way of "excluding a symmetry" from this relation and obtaining necessary conditions for the existence of symmetries only in terms of the right hand side of the equation was suggested. These conditions which we call integrability conditions are written as so-called canonical conservation laws. Their main advantages are the independence of the conditions on the order of symmetry and their invariancy w.r.t. all point transformations not leading out of the class of equations (0.1).

It was shown in the papers [4–7] that necessary integrability conditions are implied by the existence of an infinite series of generalized symmetries or conservation laws for equation (0.1).

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In more details the technique of obtaining the conditions is discussed in the reviews [8, 9]. Here we do not deal with it. We note that there exists an alternative way [10, 11] for calculating canonical conservation laws by the logarithmic derivative of the formal eigenfunction of the linearization operator for equation (0.1) (see Appendix 3). The equivalency of these two approaches for the scalar equations follows from Theorem 2.9 in survey [12].

Let us describe the results of the work. In Chapter 1 by the simplest examples we show how canonical conservation laws look like and how one can classify integrable equations by employing them. In particular, in this chapter the problem of the classification for equations (0.1) with $n = 2$ is solved. General second order integrable evolution equations were classified in [13]. In [14] the results of the latter work were generalized for the case of weakly nonlocal symmetries.

In Chapter 2 we provide the solution of the classification problem for the integrable equations of the form

$$u_t = u_3 + F(x, u, u_1, u_2). \qquad (0.2)$$

The famous Korteweg-de Vries equation

$$u_t = u_3 + uu_1 \qquad (0.3)$$

belongs to this class. The case when the function F is independent on u_2 and x (see Section 1.2) was considered in [4, 15]. The results of Chapter 2 were announced in [5, 6], but the proof is published now for the first time. We also present for the first time a recurrent formula describing all infinite series of canonical densities. In the works [5, 6] only 4 first densities were written down explicitly which were indeed used then in the classification. Third order evolution integrable equations more general than (0.2) were studied in [9, 16, 17].

In Chapter 3 we consider a computationally complicated problem on the classification of integrable equations of the form

$$u_t = u_5 + F(u, u_1, u_2, u_3, u_4). \qquad (0.4)$$

In the note [18] a solution to this problem was announced under an additional assumptions that even canonical densities are trivial (see Remark 2). However, not only the proof but also any complete list of the found equations is absent in [18]. For the first time the list of equations (0.4) possessing generalized conservation laws was published in [9]. In the present work the condition of the triviality of even canonical densities is not employed and we solve thus a technically more complicated problem on the classification of equations (0.4) possessing generalized symmetries. The answer coincides in essence with the list in [9]. As in the case of the third order equations, a general formula for the whole infinite series of canonical densities is published for the first time in the present paper.

The results of the works [5,6,9,18] were obtained by hard calculations made “by hand”. This is why it was a non-zero probability of errors which could lead to losing integrable equations. Once computer systems like Maple, Mathematica, etc. appeared, an opportunity to automate partially the calculations rose. The results of the present paper were obtained by the program package Jet written by the first author. It was found no essential errors in the lists of the integrable equations but we found and corrected several misprints in [9].

At first glance, the problem of the classification of integrable equations (0.1) with arbitrary n seems to be far from the complete solution. This is not quite so. Each integrable equation together with all its symmetries form a so-called hierarchy of integrable equations. For the equations integrable by the inverse scattering problem method [19] all the equations of the hierarchy possess the same L-operator. This fact lies in the basis of the commutativity of the flows in hierarchies (each equation of the hierarchy is a symmetry for all others). A general statement on “almost” commutativity of the symmetries for equation (0.1) is contained in [20].

Assuming that the right hand side of equation (0.1) is polynomial and homogenous, it was proven in the works [21, 22] that the hierarchy of any such equation contains an equation of second, third, or fifth order. This statement looks very credible also without any additional restrictions for the right hand side of the equation. The proof in the general case is absent and this statement has a status of the conjecture well-known for experts. No counterexamples to this conjecture are known.

Up to this conjecture, in the survey we describe all the hierarchies of the integrable equations of the form (0.1). In other words, any integrable equation of order 4 or >5 is equivalent to a generalized symmetry of one of the equations given in this survey. We note that the calculation of symmetries for given equation is a linear problem, and there are several effective computer programs for solving it. Moreover, the generalized symmetries can be found by the use of quasilocal recursion operators (see [23] and the references therein).

Various results on the classification of integrable systems of evolution equations can be found in [8, 24–40]. Further references are contained, for instance, in the survey [41].

A separate difficult problem is the classification of integrable hyperbolic equations and systems [42–50].

1. Simplest classification problems

All necessary integrability conditions we shall use below are given in the form of local conservation laws. We remind [51] that a pair of functions ρ and θ depending on a finite number of the variables $x, u, u_1, \dots$ such that

$$\frac{d}{dt}(\rho) = \frac{d}{dx}(\theta) \tag{1.1}$$

is called a local conservation law for equation (0.1). Here

$$\frac{d}{dx} = \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u_0} + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + \cdots, \qquad u_0 = u, \tag{1.2}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + K_0\frac{\partial}{\partial u_0} + K_1\frac{\partial}{\partial u_1} + K_2\frac{\partial}{\partial u_2} + \cdots,$$

where

$$K_i = \frac{d^i}{dx^i}\Big(u_n + F(x, u_0, u_1, u_2, \dots, u_{n-1})\Big).$$

the operators $\frac{d}{dx}$ and $\frac{d}{dt}$ are often referred to as the total derivative w.r.t. x and the total derivative w.r.t. t in virtue of equation (0.1). The function ρ is called a density, and θ a flux of the conservation law.

Relation (1.1) is called a conservation law due to the following reason. Consider, for instance, the Korteweg-de Vries equation $u_t = u_3 + uu_1$. It is known that it possesses an infinite number of conservation laws. In particular, since the equation can be rewritten as

$$u_t = (u_2 + \frac{1}{2}u^2)_x,$$

the function u is the density of the conservation law. Suppose the solution $u(x,t)$ decays as $|x| \to \infty$. Then we have

$$\frac{d}{dt}\int_{-\infty}^{+\infty} u\,dx = 0,$$

i.e., the area under the graph of the solution is independent of t. In the same way, the integrals of others densities of the conservation laws are conserved.

It is clear that if ρ is a density of a conservation law, then $\rho_1 = \rho + \dfrac{d}{dx}(h)$ is also a density for any function h. We call two such densities equivalent and write $\rho \sim \rho_1$. A conservation law is called trivial if $\rho \sim 0$.

The order of the higher derivative, on which the function $f(x, u, u_1, \ldots, u_k)$ depends, is called **differential** order of this function. The differential order is usually indicated as $\operatorname{ord} f = k$. The minimum of the **differential** orders of equivalent densities is called the order of the conservation law.

The deduction of necessary integrability conditions as an infinite series of so-called canonical conservation laws was discussed in details in [8–11]; for an alternative version see Appendix 3. In this paper we often give appropriate formulas without proofs. But on the other hand we dwell on how to retrieve the complete list of integrable equations of the form (0.2) using these necessary conditions, and we describe point transformations necessary to reduce an arbitrary integrable equation to one of the canonical forms.

1.1. Integrable Burgers type equations.

Consider second order evolution equations

$$u_t = u_2 + f(x,\, u,\, u_1). \tag{1.3}$$

The canonical densities for this equation are defined by the recurrent formula

$$2\rho_{n+1} = \theta_n + \sum_{i=0}^{n} \rho_{n-i}\rho_i - \frac{\partial f}{\partial u_1}\rho_n + \frac{\partial f}{\partial u_1}\delta_{n,-1} + \frac{\partial f}{\partial u}\delta_{n0} - \frac{d}{dx}\rho_n, \quad n \geqslant -1. \tag{1.4}$$

Here $\rho_{-1} = 0$, δ_{ij} is the Kronecker delta. One of the ways of obtaining similar formulas is described in Appendix 3. The fluxes associated with these densities are calculated consequently in the process of classification. At that, the obstacles to their existence pose the restrictions for the right hand side of equation (0.2) that finally allow us to find all integrable equations (1.3).

Letting $n = -1, 0$ in (1.4), we find two first canonical conservation laws,

$$\frac{d}{dt}\frac{\partial f}{\partial u_1} = \frac{d}{dx}\sigma_1, \tag{1.5}$$

$$\frac{d}{dt}\left(\sigma_1 + 2\frac{\partial f}{\partial u} - \frac{1}{2}\left(\frac{\partial f}{\partial u_1}\right)^2\right) = \frac{d}{dx}\sigma_2, \tag{1.6}$$

where $\sigma_1 = 2\,\theta_0$ and $\sigma_2 = 4\,\theta_1 + \dfrac{d}{dx}\sigma_1$.

The former of these formulas means that for each integrable equation (1.3) the partial derivative w.r.t. u_1 of its right hand side is a density of a conservation law. For instance, for Burgers equation $u_t = u_2 + uu_1$ this formula yields a density $\rho = u$. In this case the function σ_1 is calculated easily,

$$\sigma_1 = u_2 + \frac{1}{2}u^2.$$

A general algorithm of calculating the flux for a given density is given below (see Remark 4).

Let us demonstrate the main modes for working with the conditions like (1.5), (1.6). In order to determine the character of the dependence of the right hand side on u_1, the most simplest way is to exclude the unknown function σ_1 in (1.5). For this we apply the Euler operator

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - \frac{d}{dx} \circ \frac{\partial}{\partial u_1} + \frac{d^2}{dx^2} \circ \frac{\partial}{\partial u_2} - \cdots$$

to both sides of (1.5). It is well known [51] that

$$\frac{\delta}{\delta u} \circ \frac{d}{dx} = 0,$$

and therefore,

$$0=\frac{\delta}{\delta u}\frac{d}{dt}\left(\frac{\partial f}{\partial u_1}\right)=-2u_4\frac{\partial^3 f}{\partial u_1^3}-4u_3\frac{d}{dx}\frac{\partial^3 f}{\partial u_1^3}+O(2), \qquad (1.7)$$

where the symbol $O(2)$ indicates terms whose order w.r.t. the derivatives are at most two. The latter identity must hold true for each solution (1.3). Since there exists no ordinary differential equation in x satisfied by all the solutions to equation (1.3), relation (1.7) must hold identically w.r.t. the variables $u, u_1, \ldots, u_4$. Equating the coefficient at u_4 to zero, we find that the equation reads as

$$u_t = u_2 + A(x,u)u_1^2 + B(x,u)u_1 + C(x,u). \qquad (1.8)$$

Thus, each integrable equation (1.3) is quadratic in u_1. It can be checked that for equation (1.8) condition (1.7) is equivalent to two equations

$$(C\varphi)_u = (B\varphi - \varphi_x)_x, \qquad \varphi_u = A\varphi,$$

where $\varphi = B_u - 2A_x$.

Taking into consideration that the integrability of any differential equation is preserved under point transformations, we simplify equation (1.8) by a point transformation $u = \psi(x,v)$ before we proceed to study the integrability conditions. Simple calculations lead us to the evolution equation for v

$$v_t = v_2 + v_1^2\left(\frac{\partial\psi}{\partial v}\right)^{-1}\left(\frac{\partial^2\psi}{\partial v^2}+A(x,\psi)\left(\frac{\partial\psi}{\partial v}\right)^2\right)+\bar B(x,v)v_1+\bar C(x,v).$$

It is obvious that the equation

$$\frac{\partial^2\psi}{\partial v^2}+A(x,\psi)\left(\frac{\partial\psi}{\partial v}\right)^2=0$$

has a solution depending on v forany function A. This is why by a point transformation one can vanish the function A in equation (1.8). This transformation is the first step in reducing any integrable equation to one of the canonical forms.

Condition (1.5) for the equation

$$u_t = u_2 + B(x,u)u_1 + C(x,u) \qquad (1.9)$$

becomes

$$B_u\big(u_2 + B(x,u)u_1 + C(x,u)\big) = \frac{d}{dx}\sigma_1. \qquad (1.10)$$

Since to use condition (1.6) we need to know completely or partially the function σ_1, instead of applying the variational derivative to both sides of (1.10) we employ an alternative approach which is a separation of a total x-derivative in the left hand side of (1.10). This approach is completely algorithmical and can be programmed in any language of symbolic computations (see Remark on page 113).

We have

$$B_u u_2 + B_u B u_1 + B_u C = \frac{d}{dx}\left(B_u u_1 + \frac{1}{2}B^2\right) - u_1(B_{uu}u_1 + B_{ux}) - BB_x + B_u C =$$
$$= \frac{d}{dx}\left(B_u u_1 + \frac{1}{2}B^2 - B_x\right) - B_{uu}u_1^2 + B_{xx} - BB_x + B_u C.$$

Substituting the last expression in (1.10), we obtain

$$-B_{uu}u_1^2 + B_{xx} - BB_x + B_u C = \frac{d}{dx}\left(\sigma_1 - B_u u_1 - \frac{1}{2}B^2 + B_x\right) \equiv \frac{d\psi}{dx}.$$

Since the left hand side depends only on x, u, u_1, then the function ψ may depend only on x and u. Substituting $\dfrac{d\psi}{dx} = \psi_x + \psi_u u_1$ and equating the coefficients at u_1^2 and u_1, we get

$$B_{uu} = 0, \qquad \psi_u = 0, \qquad B_{xx} - BB_x + B_u C = \psi_x.$$

Letting $B = \alpha(x)u + \beta(x)$, we find that each integrable equation (1.9) reads as

$$u_t = u_2 + \big(\alpha(x)u + \beta(x)\big)u_1 + C(x, u), \tag{1.11}$$

where

$$\alpha C(x, u) - \alpha\alpha' u^2 + (\alpha'' - \alpha\beta' - \alpha'\beta)u = \psi' + \beta\beta' - \beta''. \tag{1.12}$$

At that,

$$\sigma_1 = \psi + \alpha u_1 + \frac{1}{2}(\alpha u + \beta)^2 - \alpha' u - \beta'. \tag{1.13}$$

If $\alpha \neq 0$, then by (1.12) one can determine the function C. In this case equation (1.11) can be simplified by a point transformation $u \to u\, f_1(x) + f_2(x)$. By taking $f_1 = 1/\alpha$ $f_2 = 2\alpha'/\alpha^2 - \beta/\alpha$ we get $\alpha = 1$, $\beta = 0$. At that, equation (1.11) casts into the form

$$u_t = u_{xx} + uu_x + \psi'(x). \tag{1.14}$$

For this equation conditions (1.5), (1.6), as well as all other necessary integrability conditions, hold true. The Burgers equation (1.14) is reduced to the linear equation

$$v_t = v_{xx} + \varphi(x)v_x,$$

by the Cole-Hopf substitution $u = 2v_x/v + \varphi(x)$, where φ and ψ are related by the identity $\varphi'' + \varphi\varphi' = -\psi'$.

In the case $\alpha = 0$ the left hand side of equation (1.10) vanishes and this is why σ_1 is constant. Then condition (1.6) implies

$$\frac{\delta}{\delta u}\left(\sigma_1 + 2\frac{\partial f}{\partial u} - \frac{1}{2}\left(\frac{\partial f}{\partial u_1}\right)^2\right)_t = 0,$$

that is equivalent to the system of equations

$$C_{uuu} = 0, \quad CC_{uu} + C_{xuu} - (\beta C_u)_x + \psi'(x) = 0.$$

It yields $C = p(x)u + q(x)$, and we arrive at the linear equation

$$u_t = u_{xx} + \beta(x)u_x + p(x)u + q(x). \tag{1.15}$$

For this equation all necessary integrability conditions hold true.

Remark 1. Among obtained second order integrable equations (1.14) and (1.15) there is no the potential Burgers equation $u_t = u_{xx} + u_x^2$. The reason is that this equation is linearized by the point transformation $u = \ln v$. This transformation is a particular case of the point transformation that has to be applied to equation (1.8) for eliminating the function A.

1.2. Integrable KdV type equations. The list of integrable equations obtained in the previous section is quite poor. Let us consider a more substantial classification problem. Let us find all integrable evolution equations of the form

$$u_t = u_3 + f(u_1, u). \tag{1.16}$$

It turns out (see Section 2.1) that for each such integrable equation

$$\frac{d}{dt}\left(\frac{\partial f}{\partial u_1}\right) = \frac{d}{dx}(\sigma_1), \tag{1.17}$$

where σ_1 is a function depending on u, u_x, $\ldots$, u_3.

Example 1. For the mKdV equation $u_t = u_3 + u^2 u_1$ conservation law (1.17) reads as

$$(u^2)_t = (2uu_2 - u_1^2 + \frac{1}{2}u^4)_x. \qquad \square$$

Applying the Euler operator to both sides of (1.17), we obtain

$$0 = \frac{\delta}{\delta u}\left(\frac{\partial f}{\partial u_1}\right)_t = 3u_4\left(u_2\,\frac{\partial^4 f}{\partial u_1^4} + u_1\,\frac{\partial^4 f}{\partial u_1^3 \partial u}\right) + O(3). \tag{1.18}$$

The last identity must hold true for each solution of (1.16) and this is why it must be identity in the variables $u, u_1, \dots, u_4$. Equating the coefficient at u_4 to zero and employing that f is independent of u_2, we get

$$f(u_1, u) = \mu u_1^3 + A(u)u_1^2 + B(u)u_1 + C(u)$$

with some constant μ. It is easy to check that for each such function f condition (1.18) is equivalent to the system of ODEs

$$\mu A' = 0, \qquad B''' + 8\mu B' = 0, \qquad (B'C)' = 0, \qquad AB' + 6\mu C' = 0.$$

The next necessary integrability condition reads as

$$\frac{d}{dt}\left(\frac{\partial f}{\partial u}\right) = \frac{d}{dx}(\sigma_2)$$

that yields

$$\frac{\delta}{\delta u}\frac{d}{dt}\left(\frac{\partial f}{\partial u}\right) = 0. \tag{1.19}$$

The latter condition leads to additional equations

$$A' = 0, \qquad AC'' = 0, \qquad (C''' + 2\mu C')' = 0, \qquad (CC'')' = 0.$$

In the case $\mu \neq 0$ the obtained equations are sufficient to determine completely the functions A, B, and C. As a result, up to a scaling $u \to \mathit{const}\, u$, we arrive at the equations

$$u_t = u_{xxx} - \frac{1}{2}u_x^3 + \left(c_1 e^{2u} + c_2 e^{-2u} + c_3\right)u_x \tag{1.20}$$

and

$$u_t = u_{xxx} + c_1 u_x^3 + c_2 u_x^2 + c_3 u_x + c_4, \tag{1.21}$$

where c_i are arbitrary constants.

If $\mu = 0$, then solving the above system of ODEs for the functions A, B, C, we obtain that the equation reads as

$$u_t = u_{xxx} + c_0 u_x^2 + (c_1 u^2 + c_2 u + c_3)u_x + c_4 u + c_5,$$

where

$$c_0 c_1 = 0, \qquad c_0 c_2 = 0, \qquad c_4 c_1 = 0, \qquad c_4 c_2 = 0, \qquad c_1 c_5 = 0.$$

By the third integrability condition (see Section 2) we find additional relations,

$$c_0 c_4 = 0, \qquad c_2 c_5 = 0.$$

In the case $c_0 \neq 0$ we arrive at a particular case of equation (1.21). If $c_0 = 0$, two cases are possible; a) $c_1 \neq 0$ or $c_2 \neq 0$, $c_4 = c_5 = 0$ and b) $c_1 = c_2 = 0$. They lead us to two equations

$$u_t = u_{xxx} + (c_1 u^2 + c_2 u + c_3)u_x, \tag{1.22}$$

$$u_t = u_{xxx} + c_3 u_x + c_4 u + c_5. \tag{1.23}$$

The experts in nonlinear equation regard each linear equation as exactly integrable. Equations (1.20), (1.21), and (1.22) have been found by necessary integrability conditions. This is why it should be discussed independently in which exactly sense they are integrable. It is well-known that to all of these equations the method of the inverse scattering problem is applicable. Moreover, all of them are related with the Korteweg-de Vries equation $u_t = u_3 + uu_1$ by Miura type differential substitutions [52].

Remark 2. Conditions (1.18), (1.19) hold true for the equations (1.16) possessing generalized symmetries. If the equation possesses generalized conservation laws (at that, the existence of the symmetries is not assumed), condition (1.18) still holds, and condition (1.19) can be strengthen,

$$\frac{\delta}{\delta u}\left(\frac{\partial f}{\partial u}\right) = 0.$$

It is implied by the general statement [5] in accordance to which for the equations with generalized conservation laws the canonical densities with even indices are trivial.

1.3. On admissible point transformations. In the process of classification of integrable equations, as a rule, we use point transformations reducing integrable equation to one or another canonical form. For instance, in Section 1.1 we employed point transformations while reducing equation (1.8) to the form (1.9), and also for normalizing the functions $\alpha(x)$ and $\beta(x)$ in equation (1.11).

Let us describe point transformations we use in the classification of equations (0.1).

Each equation of the form (0.1) admits the transformations

$$\tilde{u} = \varphi(u, x). \tag{1.24}$$

Hereinafter, once transformation rules for some of the variables t, x, or u are not indicated in the formulas, this means that the corresponding variables are not changed. The scalings

$$\tilde{x} = ax, \qquad \tilde{t} = a^n t \tag{1.25}$$

are also admitted. Under such transformations

$$F(x, u, u_1, u_2, \dots) \to a^{-n} F(a^{-1}x, u, au_1, a^2u_2, \dots).$$

For some subclasses of equations (0.1) additional transformations depending on t are admitted. In particular, if $F(x, \lambda u, \lambda u_1, \dots, \lambda u_{n-1}) = \lambda F(x, u, u_1, \dots, u_{n-1})$, then for arbitrary constants a and b the transformation

$$\tilde{u} = u \exp(at + bx) \tag{1.26}$$

is applicable. Under this transformation $u_n \to (\partial_x - b)^n u, \quad F \to F + au$.

If, as in Section 1.2, it is assumed that the right hand side F of equation (0.1) is independent on the variable x, then the class of admissible transformations changes. Among (1.24), only the transformations

$$\tilde{u} = \varphi(u) \tag{1.27}$$

are admitted. At the same time additional point transformations appear. In particular, the Galilean transformation

$$\tilde{x} = x + ct \tag{1.28}$$

is always admissible; under this transformation $F \to F - cu_1$. If the function F is independent of u and x, then the transformation

$$\tilde{u} = u + c_1 x + c_2 t \tag{1.29}$$

is admissible. Under such transformation

$$F(u_1, u_2, u_3, \dots) \to F(u_1 - c_1, u_2, u_3, \dots) + c_2.$$

The equations related by aforementioned transformations are called *equivalent*. It is important to note that our classification is pure algebraic. We are not interesting in such properties of the solutions to the studied equations as being real. This is why the functions and constants being involved in formulas (1.24)–(1.26) can be both real and complex. For instance, the equations $u_t = u_3 - u_1^3$ and $u_t = u_3 + u_1^3$ are regarded as equivalent.

Integrable equations can involve arbitrary constants which can be eliminated by one or another transformation. Consider as an example equation (1.21), where $c_1 \neq 0$. By (possible complex) scaling $u \to \lambda u$ we fix a normalization $c_1 = 1$. Then the transformation $u \to u + \alpha x + \beta t$ leads us to the equation

$$u_t + \beta = u_{xxx} + (u_x + \alpha)^3 + c_2(u_x + \alpha)^2 + c_3(u_x + \alpha) + c_4.$$

It is easy to see that taking $\alpha = -c_2/3$ and $\beta = c_4 + \alpha^3 + c_2\alpha^2 + c_3\alpha$, we obtain $c_2 = 0$, $c_4 = 0$. The constant c_3 is eliminated by the Galilean transformation, and we obtain the potential modified Korteweg-de Vries equation,

$$u_t = u_{xxx} + u_x^3.$$

Similarly, the parameters in equation (1.22) are inessential.

2. Third order equations with constant separant

2.1. Integrability conditions. For the equations of the form (0.2) an infinite chain of the canonical conservation laws

$$\frac{d}{dt}(\rho_n) = \frac{d}{dx}(\theta_n), \quad n = 0, 1, \dots, \tag{2.1}$$

can be defined by the formulas (for the deduction see Appendix 3),

$$\rho_{n+2} = \frac{1}{3}\left[\theta_n - \delta_{n,0}F_u - F_{u_1}\rho_n - F_{u_2}\Big(\frac{d}{dx}\rho_n + 2\rho_{n+1} + \sum_{s=0}^{n}\rho_s\,\rho_{n-s}\Big)\right] - \sum_{s=0}^{n+1}\rho_s\,\rho_{n+1-s}$$
$$-\frac{1}{3}\sum_{0\leqslant s+k\leqslant n}\rho_s\,\rho_k\,\rho_{n-s-k} - \frac{d}{dx}\left[\rho_{n+1} + \frac{1}{2}\sum_{s=0}^{n}\rho_s\,\rho_{n-s} + \frac{1}{3}\frac{d}{dx}\rho_n\right], \quad n \geqslant 0, \tag{2.2}$$

where the first two elements of the sequence ρ_i read as

$$\rho_0 = -\frac{1}{3}F_{u_2}, \qquad \rho_1 = \frac{1}{9}F_{u_2}^2 - \frac{1}{3}F_{u_1} + \frac{1}{3}\frac{d}{dx}F_{u_2}.$$

Here $\delta_{i,j}$ is the Kronecker delta, $F_{u_i} = \partial F/\partial u_i$, where $i = 0, 1, 2$. The fluxes θ_n are calculated consequently in the process of classification. At that, the obstacles for its existence lead to differential equations, which must be satisfied by the right hand of integrable equation (0.2).

It is easy to check that first four conditions in this series are equivalent to the conditions

$$\frac{d}{dt}\frac{\partial F}{\partial u_2} = \frac{d}{dx}\sigma_0, \tag{2.3}$$

$$\frac{d}{dt}\left(3\frac{\partial F}{\partial u_1} - \left(\frac{\partial F}{\partial u_2}\right)^2\right) = \frac{d}{dx}\sigma_1, \tag{2.4}$$

$$\frac{d}{dt}\left(9\sigma_0 + 2\left(\frac{\partial F}{\partial u_2}\right)^3 - 9\left(\frac{\partial F}{\partial u_2}\right)\left(\frac{\partial F}{\partial u_1}\right) + 27\frac{\partial F}{\partial u}\right) = \frac{d}{dx}\sigma_2, \tag{2.5}$$

$$\frac{d}{dt}\sigma_1 = \frac{d}{dx}\sigma_3 \tag{2.6}$$

given in [6]. At that, $\sigma_0 = -3\theta_0,\ \sigma_1 = 3\dfrac{d}{dx}\sigma_0 - 9\theta_1, \ldots$ As it will be shown below, these four conditions are "almost" sufficient to obtain the complete list of integrable equations (0.2).

In order to employ efficiently the canonical series for the classification, it is useful to study first a possible structure of the densities of local conservation laws of small orders for the considered class of equations.

Lemma 1. *If a density ρ of a conservation law for equation (0.2) has the differential order* $\operatorname{ord}\rho = 2$*, then*

$$\rho = f_1 u_2^2 + f_2 u_2 + f_3, \tag{2.7}$$

where f_i are some functions in $x, u,\ u_1$, and

$$\frac{d}{dx} f_1 = \frac{2}{3} f_1 \frac{\partial F}{\partial u_2}. \tag{2.8}$$

Proof. Eliminating the terms u_5 and u_4 by the subtraction of total x-derivatives, we find that

$$\begin{aligned} \frac{d}{dt}\rho &= \frac{\partial \rho}{\partial u}(u_3 + F) + \frac{\partial \rho}{\partial u_1}\left(u_4 + \frac{d}{dx}F\right) + \frac{\partial \rho}{\partial u_2}\left(u_5 + \frac{d^2}{dx^2}F\right) \sim \\ &\sim \frac{u_3^3}{2}\frac{\partial^3 \rho}{\partial u_2^3} + \frac{3}{2}u_3^2\left(\frac{\partial^3 \rho}{\partial u_2^2 \partial u_1}u_2 + \frac{\partial^3 \rho}{\partial u_2^2 \partial u}u_1 + \frac{\partial^3 \rho}{\partial u_2^2 \partial x} - \frac{2}{3}\frac{\partial F}{\partial u_2}\frac{\partial^2 \rho}{\partial u_2^2}\right) + \cdots, \end{aligned} \tag{2.9}$$

where the dots indicate a linear in u_3 expression. By the definition of the conservation law, the last expression should read $\dfrac{d}{dx}\sigma$. It is clear that the function σ can not depend on the derivatives higher than u_2, and the degree in u_3 of the function $\dfrac{d}{dx}\sigma$ is at most one. Hence, equating the coefficients at u_3^3 and u_3^2 to zero, we obtain (2.7) and (2.8). □

Remark 3. We observe that in (2.7) the identity $f_1 = 0$ is possible. It concerns also other similar lemmas.

Remark 4. In the proof of Lemma 1 we used the following algorithm of checking whether a given function $S(x, u, u_1, \ldots, u_n)$ is a complete derivative w.r.t. x (i.e., whether it belongs to $\operatorname{Im}\dfrac{d}{dx}$). At first, S must be linear in the highest derivative u_n. If it holds true, then as one can see easily, we can subtract a total derivative from S so that the difference has the order less than n. Repeating this order lowering procedure, we either arrive at the situation when the function is nonlinear in its highest derivative, or we get zero.

Let us show how one can employ formulas (2.7) and (2.8) in the classification of equations (0.2).

Lemma 2. *Let for equation* (0.2) *the first integrability condition* (2.3) *holds. Then F is a polynomial in u_2 of at most second degree.*

Proof. According to condition (2.3), the function $\dfrac{\partial F}{\partial u_2}$ should be a density of a conservation law. Applying Lemma 1 to it, we write equations (2.7) and (2.8),

$$\frac{\partial F}{\partial u_2} = f_1 u_2^2 + f_2 u_2 + f_3,$$

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial u_0}u_1 + \frac{\partial f_1}{\partial u_1}u_2 = \frac{2}{3}f_1(f_1 u_2^2 + f_2 u_2 + f_3).$$

Since f_i are independent of u_2, then equating the coefficients at u_2^2, we obtain $f_1 = 0$. Integrating the equation $\dfrac{\partial F}{\partial u_2} = f_2 u_2 + f_3$ w.r.t. u_2, we arrive at the desired result. □

2.2. List of integrable equations. Our main aim is to prove the following statement [6].

Theorem 1. *Up to transformations of the form* (1.24)–(1.29) *each equation* (0.2) *satisfying integrability conditions* (2.1), (2.2) *for* $n = 0, 1, ..., 5$, *belongs to the list*

$$u_t = u_{xxx} + uu_x, \tag{2.10}$$

$$u_t = u_{xxx} + u^2 u_x, \tag{2.11}$$

$$u_t = u_{xxx} + u_x^2, \tag{2.12}$$

$$u_t = u_{xxx} - \frac{1}{2} u_x^3 + (c_1 e^{2u} + c_2 e^{-2u}) u_x, \tag{2.13}$$

$$u_t = u_{xxx} - \frac{3 u_x u_{xx}^2}{2(u_x^2 + 1)} + a_1 (u_x^2 + 1)^{3/2} + a_2 u_x^3, \tag{2.14}$$

$$u_t = u_{xxx} - \frac{3 u_{xx}^2}{2 u_x} + \frac{1}{u_x} - \frac{3}{2} \wp(u) u_x^3, \tag{2.15}$$

$$u_t = u_{xxx} - \frac{3 u_x u_{xx}^2}{2(u_x^2 + 1)} - \frac{3}{2} \wp(u) u_x (u_x^2 + 1), \tag{2.16}$$

$$u_t = u_{xxx} - \frac{3 u_{xx}^2}{2 u_x}, \tag{2.17}$$

$$u_t = u_{xxx} - \frac{3 u_{xx}^2}{4 u_x} + c_1 u_x^{3/2} + c_2 u_x^2, \quad c_1 \neq 0 \text{ or } c_2 \neq 0, \tag{2.18}$$

$$u_t = u_{xxx} - \frac{3\, u_{xx}^2}{4\, u_x} + \alpha(x) u_x, \tag{2.19}$$

$$\begin{aligned} u_t = u_{xxx} &- \frac{3\, u_{xx}^2}{4\, u_x} + \frac{3}{\xi} u_{xx} (\sqrt{\alpha'\, u_x} + u_x) + \frac{3\, u_x^3}{\xi^2} + \frac{6}{\xi^2}\, u_x^{5/2} \sqrt{\alpha'} \\ &+ \frac{3\, u_x^{3/2}}{\xi^2 \sqrt{\alpha'}} (\xi \alpha'' - 2\alpha'^2) + f\, u_x + c_0 + c_1\, u + c_2\, u^2, \end{aligned} \tag{2.20}$$

where $\xi = \alpha(x) - u, \quad f = -\dfrac{\alpha'''}{\alpha'} + \dfrac{3\alpha''^2}{4\alpha'^2} + 3\dfrac{\alpha''}{\xi} - 3\dfrac{\alpha'^2}{\xi^2} - \dfrac{c_0 + c_1\, \alpha + c_2\, \alpha^2}{\alpha'},$

$$u_t = u_{xxx} + 3\, u^2 u_{xx} + 9\, u u_x^2 + 3\, u^4 u_x + u_x \alpha(x) + \frac{1}{2}\, u \alpha'(x), \tag{2.21}$$

$$u_t = u_{xxx} + 3\, u u_{xx} + 3\, u_x^2 + 3\, u^2 u_x + (u \gamma(x))_x + \beta(x), \tag{2.22}$$

$$u_t = u_{xxx} + \alpha(x) u_x + \beta(x) u. \tag{2.23}$$

Here $(\wp')^2 = 4\wp^3 - g_2 \wp - g_3$, $a_1, a_2, c_0, c_1, c_2, g_2, g_3$ *are arbitrary constants,* α, β, *and* γ *are arbitrary functions.*

Remark 5. Quite often instead of equations (2.15) and (2.16) one considers point equivalent to them equations

$$u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{Q}{u_x}, \tag{2.24}$$

and

$$u_t = u_{xxx} - \frac{3}{8} \frac{\big((Q + u_x^2)_x\big)^2}{u_x\, (Q + u_x^2)} + \frac{1}{2} Q''\, u_x. \tag{2.25}$$

In both cases, $Q = c_0 + c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4$ is an arbitrary polynomial. If $Q' \neq 0$, then one can make the substitution $u = f(v)$ in equations (2.24) and (2.25), where $(f')^2 = Q(f)$. Then for v we get equations (2.15) and (2.16), respectively. At that,

$$g_2 = \frac{4}{3}c_2^2 - 4c_1c_3 + 16c_0c_4, \qquad g_3 = \frac{8}{27}c_2^3 - \frac{4}{3}c_1c_2c_3 - \frac{32}{3}c_0c_2c_4 + 4c_0c_3^2 + 4c_1^2c_4.$$

We observe that under the linear fractional transformations

$$u = \frac{z_1\tilde{u} + z_2}{z_3\tilde{u} + z_4} \tag{2.26}$$

the polynomial Q changes by the law

$$\tilde{Q}(\tilde{u}) = Q\left(\frac{z_1\tilde{u} + z_2}{z_3\tilde{u} + z_4}\right)(z_3\tilde{u} + z_4)^4(z_1z_4 - z_2z_3)^{-2}.$$

The expression g_2, g_3 are invariants of transformations group (2.26). Subject to the structure of the multiple roots, by a transformation (2.26) and scalings of x and t the polynomial Q can be reduced to one of following canonical forms, $Q(x) = x(x-1)(x-k)$, $Q(x) = x(x-1)$, $Q(x) = x^2$, $Q(x) = x$, $Q(x) = 1$ and $Q(x) = 0$.

Remark 6. In equation (2.15) the degenerate case $\wp = const$ is possible, and in the equation (2.16) the same degeneration leads to a special case of equation (2.14).

Let us prove Theorem 1. We note that the provided below proof contains the algorithm for reducing an arbitrary integrable equation (0.2) to one of canonical forms (2.10)–(2.23) by point transformations.

Proof. According to Lemma 2, each integrable equation can be written as follows,

$$u_t = u_{xxx} + A_2(u_x, u, x)u_{xx}^2 + A_1(u_x, u, x)u_{xx} + A_0(u_x, u, x). \tag{2.27}$$

It is easy to see that the density of conservation law (2.4) reads as (2.7), where $f_1 = 3\,A_{2,u_1} - 4\,A_2^2$. Relation (2.8) leads us to two equations

$$9\frac{\partial^2 A_2}{\partial u_1^2} - 36\,A_2\frac{\partial A_2}{\partial u_1} + 16\,A_2^3 = 0,$$

$$24\,A_2\left(\frac{\partial A_2}{\partial u}u_1 + \frac{\partial A_2}{\partial x}\right) + 2\,A_1\left(3\frac{\partial A_2}{\partial u_1} - 4\,A_2^2\right) - 9\frac{\partial^2 A_2}{\partial x\partial u_1} - 9\frac{\partial^2 A_2}{\partial u\partial u_1}u_1 = 0.$$

The first of the equations has a solution of the form

$$A_2 = -\frac{3}{4B}\frac{\partial B}{\partial u_1}, \qquad \text{where} \quad \frac{\partial^3 B}{\partial u_1^3} = 0, \tag{2.28}$$

at that, the second equation becomes

$$\left(2\,A_1B + 3\frac{\partial B}{\partial x} + 3\,u_1\frac{\partial B}{\partial u}\right)\frac{\partial^2 B}{\partial u_1^2} = 3\,B\frac{d}{dx}\frac{\partial^2 B}{\partial u_1^2}. \tag{2.29}$$

In view of the formula for the function A_2, it is clear that without loss of generality the leading coefficient of the polynomial $B(u_1)$ can be assumed to be one. This is why we have three cases,

$$\textbf{I}.\ B = u_1^2 + B_1(x,u)u_1 + B_0(x,u), \quad \textbf{II}.\ B = u_1 + B_0(x,u), \quad \textbf{III}.\ B = 1.$$

Equation (2.29) holds identically in the cases **II** and **III**, and in the first case it determines the function A_1,

$$A_1 = -\frac{3}{2B}\left(\frac{\partial B}{\partial x} + u_1\frac{\partial B}{\partial u}\right).$$

Case I. Under the point transformation $u = \varphi(x, v)$ the function B changes by the rule

$$\tilde{B}(x, v) = (\varphi_v v_1 + \varphi_x)^2 + B_1(x,\varphi)(\varphi_v v_1 + \varphi_x) + B_0(x,\varphi).$$

Therefore, taking for φ any solution to the equation $\varphi_x = -\frac{1}{2}B_1(x,\varphi)$, we reduce the issue to the case $B_1 = 0$.

Returning back to the study of second integrability condition (2.4), we find that

$$\frac{d}{dt}\rho_1 \sim -\frac{u_2^4 B_{0,x}}{4\,(u_1^2+B_0)^3} - \frac{u_2^3}{6}\left[\frac{\partial^4 A_0}{\partial u_1^4} + \frac{3}{u_1^2+B_0}\left(u_1\frac{\partial^3 A_0}{\partial u_1^3} - \frac{\partial^2 A_0}{\partial u_1^2}\right) + \Phi(B_0,u,u_1)\right] + \qquad (2.30)$$
$$+Z_2 u_2^2 + Z_1 u_2 + Z_0,$$

where the expression Φ depends on the derivatives of the functions B_0 and vanishes as B_0 is constant; Z_i are some functions in x, u, u_1. Equating the coefficient at u_2^4 to zero, we find that $B_{0,x} = 0$ and therefore, $B = u_1^2 + B_0(u)$. By an appropriate point transformation $u \to \varphi(u)$ we convert B_0 to a constant c_0 being zero or one (case **I.1**), or zero (case **I.2**).

Equating the coefficient at u_2^3 in (2.30), where $\Phi = 0$, $B_0 = c_0$, to zero, we find the function A_0. As a result, equation (2.27) casts into the form,

$$u_t = u_{xxx} - \frac{3\,u_x u_{xx}^2}{2(u_x^2+c_0)} + A_0(u_1,u,x), \qquad (2.31)$$

where A_0 is defined by one of the following formulas,

$$\textbf{I.1}.\ c_0 = 1, \quad A_0 = a_0(u_1^2+1)^{3/2} + a_1 u_1(u_1^2+1) + a_2 u_1 + a_3,$$
$$\textbf{I.2}.\ c_0 = 0, \quad A_0 = \frac{a_0}{u_1} + a_1 u_1^3 + a_2 u_1 + a_3.$$

In both cases $a_i = a_i(x,u)$.

In the case **I.1** it follows from the further implications of the second integrability condition that a_0, a_2, and a_3 are constant and the function a_1 is independent of u. Moreover, $a_1''' = -8a_1a_1'$, $a_0a_1' = a_3a_1' = 0$. If $a_1' \neq 0$, then eliminating the constant a_2 by the Galilean transformation, we obtain equation (2.16). If $a_1' = 0$, then up to the Galilean transformation we get equation (2.14).

In the case **I.2**, equating the coefficient at u_2^2 in (2.30) to zero, we find the equation

$$5\frac{\partial a_1}{\partial x}\,u_1^4 - 4\frac{\partial a_2}{\partial u}\,u_1^3 - \frac{\partial a_2}{\partial x}\,u_1^2 + 2\frac{\partial a_3}{\partial x}\,u_1 + \frac{\partial a_0}{\partial x} = 0.$$

It yields that a_2 is constant and the functions a_0, a_1, and a_3 depend on u only. The constant a_2 is eliminated by the Galilean transformation, and one of the functions a_0,a_1, or a_3 can be made constant by an appropriate transformation $u \to \varphi(u)$.

I.2.1. If $a_0 \neq 0$, then without loss of generality we can assume that $a_0 = 1$. In this case the second integrability condition is equivalent to three equations, $a_3' = 0$, $a_3a_1' = 0$, $a_1''' + 8a_1a_1' = 0$. If $a_1' \neq 0$, then letting $a_1 = -3/2\wp$, we arrive at equation (2.15). If $a_1' = 0$, then the transformation $u \to u + a_3t$ is admissible and it eliminates the constant a_3. In this case we get the equation coinciding with (2.15) for a constant function $\wp$.

I.2.2. If $a_0 = 0$, then by a transformation $u \to \varphi(u)$ one can simplify a_3 or a_1. If $a_3 = 0$, then by such a transformation one can eliminate a_1, and we obtain equation (2.17). If $a_3 \neq 0$, then by a transformation $u \to \varphi(u)$ we make a_3 constant. Then the second integrability condition implies $a_1' = 0$ that allows us to employ the transformation $u \to u + a_3t$ eliminating a_3. That is, we arrive at the case $a_3 = 0$ considered above.

In the case **I** the complete classification has been obtained just by conditions (2.3) (Lemma 2) and (2.4). It has happened to be possible because ρ_1 is the density of high (second) order.

Case II. In this case $B = u_x + B_0(x,u)$. By a transformation $u \to \psi(x,u)$ one can eliminate the function B_0. Letting $B_0 = 0$, we find that

$$\rho_0 \sim A_1, \qquad \rho_2 \sim \frac{u_2^2}{u_1^2}\left(2\frac{\partial^2 A_1}{\partial u_1^2}\,u_1^2 - \frac{\partial A_1}{\partial u_1}\,u_1 + A_1\right) + O(1).$$

It is easy to check that

$$\frac{d}{dt}\rho_0 \sim \frac{u_2^3}{4\,u_1}\left(2\,u_1\frac{\partial^3 A_1}{\partial u_1^3}+3\frac{\partial^2 A_1}{\partial u_1^2}\right)+h_2u_2^2+h_0, \tag{2.32}$$

$$\frac{d}{dt}\rho_2 \sim \frac{u_3^2u_2}{4\,u_1^3}\left(2u_1^3\frac{\partial^3 A_1}{\partial u_1^3}+u_1^2\frac{\partial^2 A_1}{\partial u_1^2}+u_1\frac{\partial A_1}{\partial u_1}-A_1\right)+gu_3^2+O(2), \tag{2.33}$$

where h_i and g are some functions of u_1, u, x. Equating to zero the first terms in these expressions, we obtain a system being reduced to one second order equation

$$2u_1^2\frac{\partial^2 A_1}{\partial u_1^2}-u_1\frac{\partial A_1}{\partial u_1}+A_1=0.$$

It yields $A_1=a_1(x,u)u_1+a_2(x,u)\sqrt{u_1}$, and equation (2.27) reads as

$$u_t=u_{xxx}-\frac{3\,u_{xx}^2}{4\,u_x}+(a_1u_x+a_2\sqrt{u_x})u_{xx}+A_0(u_x,u,x). \tag{2.34}$$

Equating the term at u_2^2 in (2.32) to zero, we obtain two relations

$$3\frac{\partial a_2}{\partial u}=a_1a_2, \qquad 3\frac{\partial a_1}{\partial x}+a_2^2=0. \tag{2.35}$$

For the equations of the form (2.34) we have

$$\frac{d}{dt}\rho_1 \sim Z_3u_2^3+Z_2u_2^2+Z_1u_2+Z_0, \tag{2.36}$$

where $Z_i=Z_i(u_1,u,x)$. Equating the expression Z_3 to zero, we obtain a linear inhomogeneous fourth order equation for A_0, which determines the dependence of the function A_0 on u_1,

$$A_0=\frac{1}{9}\,{u_1}^3\left(6\,\frac{\partial a_1}{\partial u}+{a_1}^2\right)+\frac{2}{3}\,a_1\,a_2\,{u_1}^{5/2}+a_3\,{u_1}^2+a_4\,{u_1}^{3/2}+a_5\,u_1+a_6,$$

where $a_i=a_i(x,u)$. Now the dependence of all the coefficients A_i on u_1 is determined and this is why one can split the equations w.r.t. u_1 under the integrability conditions. For instance, the coefficient Z_2 at u_2^2 in (2.36) is linear in u_1, and this is why the identity $Z_2=0$ leads to two identities. These equations read as

$$3\frac{\partial a_4}{\partial u}=a_1a_4+2a_2\frac{\partial a_1}{\partial x}, \qquad 6\frac{\partial a_5}{\partial u}-3a_2a_4-12\frac{\partial a_3}{\partial x}+2\,a_2\frac{\partial a_2}{\partial x}=0. \tag{2.37}$$

The splitting w.r.t. u_1 in conditions (2.3)–(2.5) in view of equations (2.35) and (2.37) yields several additional equations. The simplest of them are

$$a_2a_3=0, \qquad a_2\left(3\frac{\partial a_5}{\partial u}-a_2a_4\right)=0, \tag{2.38}$$

$$27\frac{\partial^2 a_5}{\partial u^2}-18a_1\frac{\partial a_5}{\partial u}+2\,a_2^4=0, \tag{2.39}$$

$$\frac{\partial a_6}{\partial x}=0, \qquad 3\frac{\partial a_3}{\partial u}=2\,a_1a_3, \qquad a_2\left(2\frac{\partial a_2}{\partial x}-a_4\right)=0. \tag{2.40}$$

To analyze equations (2.35)–(2.40), it is natural to consider two cases, **II.1** $a_2=0$ or **II.2** $a_2\neq 0$.

II.1. If $a_2=0$, then it follows from (2.35) that $a_1=a_1(u)$. The point transformation $u\to\varphi(u)$, where φ satisfies the equation $3\,\varphi''+2\,(\varphi')^2a_1(\varphi)=0$, vanishes a_1. Taking into consideration relations (2.35)–(2.40), we can write equation (2.34) as

$$u_t=u_{xxx}-\frac{3\,u_{xx}^2}{4\,u_x}+a_3(x)u_x^2+a_4(x)u_x^{3/2}+a_5(x,u)u_x+a_6(u), \tag{2.41}$$

where $a_5 = \alpha(x) + 2\,u\,a_3'(x)$. For this equation conditions (2.3)–(2.6) are equivalent to simple relations

$$a_3 = c_3, \quad a_4 = c_4, \quad a_5 = \alpha(x), \quad a_6 = c_1 + c_2 u, \quad c_3 c_2 = c_4 c_2 = 0, \quad c_3 \alpha' = c_4 \alpha' = 0,$$

where c_i are arbitrary constants, α is an arbitrary function.

If $c_3 \neq 0$ or $c_4 \neq 0$, then a_5 and a_6 are constants which can be eliminated by the transformation $x \to x + a_5 t$, $u \to u + a_6 t$. As a result we get equation (2.18). If $c_3 = c_4 = 0$, then the equation reads as

$$u_t = u_{xxx} - \frac{3\,u_{xx}^2}{4\,u_x} + \alpha(x) u_x + c_1 + c_2 u.$$

If here $c_2 = 0$, then the transformation $u \to u + c_1 t$ eliminates the constant c_1. If $c_2 \neq 0$, then by the shift $u \to u - c_1/c_2$ we vanish c_1, and then by the transformation $u \to u \exp(c_2 t)$ we eliminate c_2. Thus, in all cases we arrive at equation (2.19).

II.2. In view of $a_2 \neq 0$ we set $a_2 = (3/\sqrt{2}) \exp(\psi/2)$, then equations (2.35) are reduced to $a_1 = 3/2\psi_u$ and to Liouville equation

$$\psi_{xu} + e^{\psi} = 0. \tag{2.42}$$

Then from (2.38) and (2.40) we get

$$a_3 = 0, \quad a_6 = a_6(u), \quad a_4 = \frac{3\,e^{\psi/2}}{\sqrt{2}} \frac{\partial \psi}{\partial x}, \quad \frac{\partial a_5}{\partial u} = \frac{3\,e^{\psi}}{2} \frac{\partial \psi}{\partial x}, \tag{2.43}$$

and equation (2.39) is reduced to (2.42). Besides the aforementioned relations, conditions (2.3) – (2.6) imply exactly one more equation,

$$\frac{\partial}{\partial u}(a_2^2 a_6) + \frac{\partial}{\partial x}\left[\left(\frac{\partial a_2}{\partial x}\right)^2 - 2\,a_2 \frac{\partial^2 a_2}{\partial x^2} - a_5 a_2^2\right] = 0. \tag{2.44}$$

The form of equation (2.42) slightly differs from the standard one, and this is why we provide its solution

$$\psi = \ln \frac{2\alpha' \nu'}{(\alpha - \nu)^2}, \qquad \alpha' \nu' \neq 0.$$

It yields the formula for a_2, which we write as

$$a_2 = 3 \frac{\sqrt{\alpha'(x) \nu'(u)}}{\alpha - \nu}.$$

Now it is easy to find a_1, a_4, and a_5,

$$a_5 = \frac{3\,\alpha''}{\alpha - \nu} - \frac{3\,\alpha'^2}{(\alpha - \nu)^2} + q(x),$$

where q is an arbitrary function. Substituting a_2 and a_5 into equation (2.44), we find the functions a_6 and q,

$$a_6 = \frac{c_0 + c_1 \nu + c_2 \nu^2}{\nu'}, \qquad q = \frac{3\,\alpha''^2}{\alpha'^2} - \frac{\alpha'''}{\alpha'} - \frac{c_0 + c_1 \alpha + c_2 \alpha^2}{\alpha'},$$

where c_i are the constants in the separation of variables.

Since $\nu' \neq 0$, by the point transformation $\tilde{u} = \nu(u)$ the result can be somewhat simplified and we arrive at equation (2.20).

Case III. As $B = 1$, it follows from (2.28) that $A_2 = 0$. Then the total derivative of ρ_0 w.r.t. t casts into the form

$$\frac{d}{dt}\rho_0 \sim u_2^3 \frac{\partial^3 A_1}{\partial u_1^3} + u_2^2 \left(3 \frac{\partial^3 A_1}{\partial u \partial u_1^2}\, u_1 + 3 \frac{\partial^3 A_1}{\partial x \partial u_1^2} - 2\,A_1 \frac{\partial^2 A_1}{\partial u_1^2}\right) + O(1).$$

Equating the expressions at u_2^3 and u_2^2 to zero, we obtain respectively

$$A_1 = a_0(x,u) + a_1(x,u)u_1 + a_2(x,u)u_1^2$$

and

$$3\frac{\partial a_2}{\partial u}\,u_1 + 3\frac{\partial a_2}{\partial x} - 2\,a_2(a_0 + a_1u_1 + a_2u_1^2) = 0.$$

Splitting the last relation w.r.t. u_1, we find that $a_2 = 0$.

Thus, equation (2.27) becomes

$$u_t = u_{xxx} + \big(a_0(x,u) + a_1(x,u)\,u_x\big)\,u_{xx} + A_0(u_x,u,x). \tag{2.45}$$

For this equation by a transformation $u \to \varphi(x,u)$ we can reduce the issue to $a_1 = 0$. After this simplification one can get easily that

$$\frac{d}{dt}\rho_1 \sim u_2^3\frac{\partial^4 A_0}{\partial u_1^4} + u_2^2\left(3\frac{\partial^4 A_0}{\partial u\partial u_1^3}\,u_1 + 3\frac{\partial^4 A_0}{\partial x\partial u_1^3} - 2\,a_0\frac{\partial^3 A_0}{\partial u_1^3}\right) + O(1).$$

By this, equating the coefficient at u_2^3 to zero, we find

$$A_0 = a_2u_1^3 + a_3u_1^2 + a_4u_1 + a_5,$$

where $a_i = a_i(x,u)$. Then the coefficient at u_2^2 gives an equation, which can be splitted w.r.t. u_1 into two following ones,

$$\frac{\partial a_2}{\partial u} = 0, \qquad 2\,a_0a_2 = 3\frac{\partial a_2}{\partial x}. \tag{2.46}$$

Returning back to the analysis of conditions (2.3)–(2.5), we can split now also w.r.t. u_1. In view of (2.46) it allows us, in particular, to obtain

$$\frac{\partial^3 a_0}{\partial u^3} = 0, \qquad \frac{\partial^2 a_3}{\partial u^2} = 0, \qquad a_2\left(3\frac{\partial a_2}{\partial x} - 2\frac{\partial a_3}{\partial u}\right) = 0. \tag{2.47}$$

Consider alternative cases **III.1** $a_2 \neq 0$ or **III.2** $a_2 = 0$.

III.1. By the transformation $u \to u\,\mu(x)$ we can normalize $a_2(x)$, $a_2 = -1/2$, which yields $a_0 = 0,\ a_3 = \alpha(x)$. Then by condition (2.4) we find

$$a_4 = f_0(x) + c_1e^{2u} + c_2e^{-2u}, \qquad a_5 = f_1(x) - \frac{u}{6}(4\,\alpha\alpha' + 3\,f_0'),$$

where c_1 and c_2 are constant and, in addition, $c_i\alpha = c_if_1 = c_if_0' = 0,\ i = 1,2$.

If $c_1 \neq 0$ or $c_2 \neq 0$, we obtain the equation differing from (2.13) by the Galilean transformation. If $c_1 = c_2 = 0$, then condition (2.6) determines f_0 and f_1,

$$f_0 = k_1 - \frac{2}{3}\,\alpha^2, \qquad f_1 = k_2 - \frac{2}{3}(k_1\alpha + \alpha'') + \frac{4}{27}\,\alpha^3,$$

where k_1 and k_2 are constant. Making the transformation $u \to u + \dfrac{2}{3}\displaystyle\int \alpha(x)dx$ in the obtained equation, we reduce it to the form $u_t = u_3 - u_1^3/2 + k_1u_1 + k_2$ being equivalent to a partial case of equation (2.13).

III.2. Equations (2.47) imply $a_0 = b_1(x)u^2 + b_2(x)u + b_3(x),\ a_3 = b_4(x)u + b_5(x)$. In this case the equation (2.45) is simplified by the transformation $u \to u\,f_1(x) + f_2(x)$. There are three non-equivalent cases **III.2.a** $a_0 = 3u^2 + b(x)$, **III.2.b** $a_0 = 3u$ and **III.2.c** $a_0 = 0$.

In the first two cases a simple check of conditions (2.3)–(2.6) leads us to equations (2.21) and (2.22), respectively.

In the case **III.2.c** the form of the equation is determined by three conditions (2.3)–(2.5),

$$u_t = u_{xxx} + a_3(x)u_x^2 + a_4(x,u)u_x + a_5(x,u),$$

where $a_4 = b_1u^2 + b_2u + b_3,\ a_5 = b_4u^3 + b_5u^2 + b_6u + b_7,\ b_i = b_i(x),\ 1 \leqslant i \leqslant 7$. Integrability conditions (2.3)–(2.6) yield a cumbersome system of equations for the functions b_i, studying of which yields several forks.

1. If $a_3 \neq 0$, then $a_3 = c_0,\ b_1 = b_2 = b_4 = b_5 = b_6 = 0,\ b_7 = c_1x + c_2 + \frac{1}{2}b_3'' + \frac{1}{4}b_3^2$. The transformation $u \to u - \dfrac{1}{2}\displaystyle\int b_3\,dx$ yields the equation $u_t = u_3 + c_0u_x^2 + c_1x + c_2$. The sixth integrability condition implies $c_1 = 0$; then by the transformation $u \to u + c_2t$ we eliminate c_2 and obtain (2.12).

2. If $a_3 = 0$, then $b_1 = \mathrm{const}$, $b_4 = 0$. Then again forks appear;

2.1. If $b_1 \neq 0$, then b_3, b_5, b_6, and b_7 are expressed in terms of $b_2(x)$ so that the transformation $u \to u - b_2/(2b_1)$ leads to the equation equivalent to (2.11).

2.2. If $b_1 = 0$, we obtain $b_2 = \mathrm{const},\ b_2(b_6 - b_3') = 0,\ b_2(b_3''' + b_3b_3' - b_2b_7) = 0$. If $b_2 \neq 0$, the transformation $u \to u - b_3/b_2$ leads us to the equation equivalent to (2.10). Otherwise we obtain linear equation (2.23). □

2.3. Comments to the list of integrable equations. It was shown in the previous section that each integrable equation (0.2) is reduced to one of equations (2.10)–(2.23) by a chain of point transformations. Although the answer as the list is not invariant w.r.t. point transformations, integrability conditions (2.1), (2.2) are so. This is why to check the integrability of a given equation it is not necessary to reduce the equation to one in the list. According to the proof of Theorem 1, it is sufficient to check four conditions (2.3)–(2.6) if the equation belongs to the classes **I** or **II**, and six conditions (2.2) if the equation belongs to the class **III**. It can be shown that if the right hand side of equation (0.2) is independent explicitly of x, then for the equations of the class **III** it is sufficient to check conditions (2.3)–(2.6).

The orders of canonical conservation laws (2.1), (2.2) are discrete invariants of the group of point transformations. The analysis of the structure of these conservation laws shows that these equations are divided into two groups. For the first group of the equations (let us call them S-integrable[1]) the canonical series contains conservation laws of arbitrarily high order. We observe that this property is more restrictive than just the condition of the existence of an infinite series of conservation laws for the equation. For instance, the linear equation $u_t = u_{xxx}$ possesses an infinite series of the conservation laws with the densities u_k^2, $k \in \mathbb{N}$. However, all its canonical laws are trivial.

For the equations of the second group (C-integrable equations) among the canonical conservation laws there are only a few non-trivial ones. Equations (2.19)–(2.23) are S-integrable. The equations (2.19) and (2.23) have no non-trivial canonical conservation laws. Equation (2.20) has just one nontrivial canonical conservation law of the first order $\rho_0 \sim \dfrac{\sqrt{\alpha' u_1} + \alpha'}{u - \alpha(x)}$. Each of equations (2.21) and (2.22) has one canonical conservation law of zero order, $\rho_0 \sim u^2$ and $\rho_0 \sim u$, respectively.

All canonical conservation laws with even indexes of S-integrable equations are trivial [7], and the orders of odd laws increase by one, but the initial orders in these sequences are different. In Table 1 we provide the order of first four odd canonical conservation laws for all S-integrable equations.

For equation (2.18) we provide the orders of the densities in the case of generic constants. In the case $c_1 = 0$ we have $\rho_1 \sim 0$, and other orders remain to be unchanged. If $c_2 = 0$, the orders are equal to 1, 0 ($\rho_3 \sim 0$), 2, and 3.

Remark 7. Equations (2.10)–(2.18) are integrable by the method of the inverse scattering problem while (2.19)–(2.22) are linearizable by the differential substitutions (see Section 2.4).

[1]The terminology is due to F. Calogero.

Table 1. Orders of canonical conservation laws. For zero order conservation laws we indicate in the brackets to what the density is equivalent

ρ_i	(2.10)	(2.11)	(2.12)	(2.13)	(2.14)	(2.16)	(2.15)	(2.17)	(2.18)
ρ_1	0, ($\sim u$)	0, ($\sim u^2$)	0, (~ 0)	1	2	2	2	2	1
ρ_3	0, ($\sim u^2$)	1	1	2	3	3	3	3	2
ρ_5	1	2	2	3	4	4	4	4	3
ρ_7	2	3	3	4	5	5	5	5	4

If in the formulation of the original classification problem one assumes that the right hand side of equation (0.2) is independent explicitly of x, then the answer changes just for C-integrable equations. For equations (2.19), (2.21), (2.23) arbitrary functions are replaced by arbitrary constants and then these constants can be eliminated by point transformations.

Formula (2.20) contains two C-integrable equations independent of x explicitly,

$$\begin{aligned} u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x+1} - 3\,u_{xx}u^{-1}(\sqrt{u_x+1}+u_x+1) \\ +6\,u^{-2}u_x(u_x+1)^{3/2} + 3\,u^{-2}u_x\,(u_x+1)(u_x+2), \end{aligned} \tag{2.48}$$

$$\begin{aligned} u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x+1} - 3\,\frac{u_{xx}\,(u_x+1)\cosh u}{\sinh u} + 3\,\frac{u_{xx}\sqrt{u_x+1}}{\sinh u} \\ -6\,\frac{u_x(u_x+1)^{3/2}\cosh u}{\sinh^2 u} + 3\,\frac{u_x\,(u_x+1)(u_x+2)}{\sinh^2 u} + u_x^2(u_x+3). \end{aligned} \tag{2.49}$$

Equation (2.48) can be obtained from (2.20) with $\alpha = x,\ c_0 = c_1 = c_2 = 0$ by the transformation $u \to u + x$. Equation (2.49) is obtained in the case $\alpha = e^{2x}$, $c_0 = c_1 = c_2 = 0$ by the transformation $u \to e^{2(u+x)}$. It is impossible to obtain from (2.20) any other equation independent on x. It follows, for instance, from the results of independent classification of equations (0.2) without explicit dependence on x.

The situation with equation (2.22) is rather instructive. Let the functions γ and β be constant. Then the constant γ is eliminated by the Galilean transformation. It is easy to check that if $\beta \neq 0$, then the integrability conditions hold true but the canonical conservation laws depend explicitly on x. It is impossible if in the classification of equations possessing generalized symmetries we restrict the generalized symmetries not to depend explicitly on x. In this case $\beta = 0$ and equation (2.22) is just a third order symmetry for the Burgers equation. If the dependence on x for the symmetries is admitted, then the constant β in the answer should be preserved.

2.4. Differential substitutions relating the equations in the list. We say that a differential substitution

$$\tilde{u} = \Phi(x, u, u_1, \ldots, u_k) \tag{2.50}$$

acts from the equation

$$u_t = u_n + g(x, u, u_x, \ldots, u_{n-1}) \tag{2.51}$$

into the equation

$$\tilde{u}_t = \tilde{u}_n + f(x, \tilde{u}, \tilde{u}_x, \ldots, \tilde{u}_{n-1}) \tag{2.52}$$

if for each solution $u(x,t)$ of equation (2.51) formula (2.50) provides a solution for equation (2.52). The number k is called an order of the substitution. Since for $k > 0$ transformation (2.50) has no inverse transformation of the same form, equations (2.51) and (2.52) in this definition are not equal in rights. If a differential substitution reads as (2.50), where $\tilde{u}$ satisfies an equation (A) and u does equation (B), we express it as $(B) \to (A)$.

The most known differential substitution is the Miura transformation $\tilde u = u_x - u^2$ relating the Korteweg-de Vries equation

$$\tilde u_t = \tilde u_{xxx} + 6\,\tilde u\,\tilde u_x$$

and the modified Korteweg-de Vries equation

$$u_t = u_{xxx} - 6\,u^2\,u_x.$$

Other substitutions relating the main equations of the list were found in [52]. The issue on the invertibility of the differential substitutions was treated in [53].

The orders of possible differential substitutions relating S-integrable equations can be found by Table 1. Namely, if equations (2.51) and (2.52) are related by the substitution (2.50), then the orders of the canonical conservation laws with sufficiently large indices for (2.51) are greater by k than the orders of the canonical conservation laws with the same indices for (2.52). For instance, if equations (2.16) and (2.10) are related by a differential substitution, then it acts from (2.16) into (2.10) and is of the third order.

Below we provide the information on differential substitutions relating various integrable equations in the list. Since the superposition of differential substitution (2.50) and the Galilean transformation leads out of the class of the substitutions of the form (2.50), sometimes in order to find a substitution it is necessary to add a term $c\,u_x$ to the right hand side of equation (2.51). All such cases are mentioned below in the text.

I. S-integrable equations. It happens that for all S-integrable equations except the Krichever-Novikov equation (2.15) there exist substitutions acting in the Korteweg-de Vries equation. For equation (2.15) such substitution exists only when the Weierstrass function degenerates or, which is the same, the polynomial Q in formula (2.24) has multiple roots.

Let us provide all substitutions in the Korteweg-de Vries equation. If from a given equation there exists several substitutions into (2.10), we provide all of them. We replace equations (2.15) and (2.16) by (2.24) and (2.25), respectively, since after that the substitutions look more compact.

(2.11)→(2.10): $\tilde u = \pm i\sqrt{6}\,u_1 + u^2 + \lambda$; at that $u_t = u_{xxx} + u^2 u_x + \lambda u_x$.

(2.12)→(2.10): $\tilde u = 2\,u_1$.

(2.13)→(2.10): $\tilde u = 3\,u_2 - \dfrac{3}{2}\,u_1^2 + 2\sqrt{-6\,c_2}\,u_1\,e^{-u} + c_1 e^{2\,u} + c_2 e^{-2\,u}$.

(2.14)→(2.10): $$\tilde u = \frac{3\,u_3}{\sqrt{u_1^2+1}} - \frac{3\,u_1\,u_2^2}{(u_1^2+1)^{3/2}} - \frac{3\,u_2^2}{2(u_1^2+1)} - \frac{6\,c_0\,u_1\,u_2}{\sqrt{u_1^2+1}} + 6\,c_0\,u_2 + 3\,a_1\,u_1^2 + {}$$ $$+3\,a_1\,u_1\sqrt{u_1^2+1}, \text{ where } c_0 = \sqrt{(a_1-a_2)/2}\,.$$

(2.25)→(2.10): $$\tilde u = \frac{d}{dx}\left(6\,\frac{u_1+\sqrt{Q+u_1^2}}{u-a} - \frac{3}{2}\,\frac{Q'+2\,u_2}{\sqrt{Q+u_1^2}}\right) - \frac{3}{8}\,\frac{\left((Q+u_x^2)_x\right)^2}{u_1^2\,(Q+u_x^2)} + \frac{1}{2}Q'', \text{ where}$$ $Q(a) = 0$.

(2.18)→(2.10): $\tilde u = \sqrt{-3\,c_2}\,\dfrac{u_2}{\sqrt{u_1}} + 2c_2\,u_1 + \dfrac{3}{2}\,c_1\sqrt{u_1}$.

The above substitutions of higher orders are the superpositions of first order substitutions. These substitutions relate some of S-integrable equations. The first order substitutions are drawn on the graph (Fig. 1).

The arrows of the graph correspond to the following substitutions,

(2.14)→(2.13): $\tilde u = \ln\left(u_1 + \sqrt{1+u_1^2}\right)$. At that, in the equation (2.14) there should be an additional term $\dfrac{3}{2}\,a_2u_1$. The constants in the equations are related by the formulas $c_1 = \dfrac{3}{4}(a_1+a_2)$, $c_2 = \dfrac{3}{4}(a_2-a_1)$.
(2.25)→(2.13): $\tilde u = \ln\left(u_1 + \sqrt{Q+u_1^2}\right) - \ln(a_0 + 2\,a_1\,u + a_2\,u^2)$. At that, the polynomial Q is written in the factorized form, $Q(u) = (a_0 + 2\,a_1\,u + a_2\,u^2)(k_0 + 2\,k_1\,u + k_2\,u^2)$. Moreover, in equation (2.25) there should be an additional term $\dfrac{1}{2}(a_0k_2 + a_2k_0 - 2\,a_1k_1)u_1$, and the constants c_1 and c_2 in equation (2.13) are determined by the formulas $c_1 = \dfrac{3}{2}(a_0a_2 - a_1^2)$, $c_2 = \dfrac{3}{2}(k_0k_2 - k_1^2)$.
(2.13)→(2.11): $\tilde u = \pm\dfrac{i}{2}\sqrt{6}\,u_1 + \sqrt{c_1}\,e^u + \sqrt{c_2}\,e^{-u}$, at that, in equation (2.13) there should be an additional term $2\sqrt{c_1c_2}\,u_1$.
(2.18)→(2.11): $\tilde u = a + b\sqrt{u_1}$, $b \neq 0$, where $c_1 = \dfrac{4}{3}ab$, $c_2 = \dfrac{1}{2}b^2$. At that, in equation (2.18) there should be an additional term a^2u_1.

If in equation (2.24) the polynomial Q has multiple roots, then there exists the following substitutions from this equation not indicated in the graph,
(2.24)→(2.10):
1) $\tilde u = \dfrac{d}{dx}\left(-3\dfrac{u_2}{u_1} + \dfrac{12\,u_1}{u-a}\right) - \dfrac{3\,u_2^2}{2\,u_1^2} - \dfrac{Q}{u_1^2}$, $Q = (u-a)^2\big(c_0 + c_1u + c_2u^2\big)$.

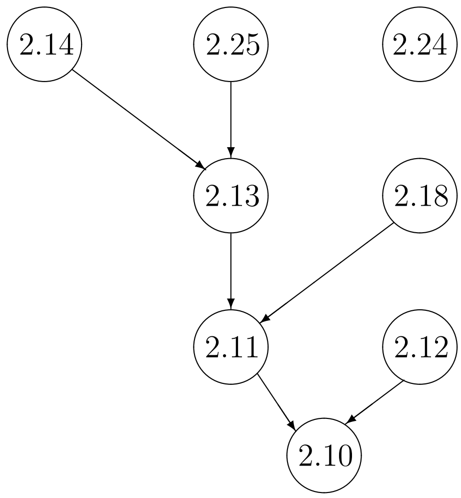

Figure 1. Graph of the substitutions for S-integrable third order equations

2) $\tilde u = \dfrac{d}{dx}\left(3\dfrac{u_2}{u_1} - \dfrac{12\,h}{u_1}\right) - \dfrac{3\,u_2^2}{2\,u_1^2} - \dfrac{Q}{u_1^2}$, $\qquad h = c_0 + c_1u + c_2u^2$, $Q = 6\,h^2$.
(2.24)→(2.13): $\tilde u = \ln u_1 - \ln h$, $\qquad h = a_0 + a_1u + a_2u^2$, $Q = -c_2h^2$, $c_1 = \dfrac{3}{2}(4\,a_0a_2 - a_1^2)$.

In the case $Q = 0$ equation (2.24) coincides with Schwarz-Korteweg-de Vries equation (2.17). Equation (2.17) is related with the Korteweg-de Vries equations by three different substitutions,
(2.17)→(2.10): 1) $\tilde u = 3\,\dfrac{u_3}{u_1} - \dfrac{9\,u_2^2}{2\,u_1^2}$; 2) $\tilde u = -3\,\dfrac{u_3}{u_1} + \dfrac{3\,u_2^2}{2\,u_1^2}$; 3) $\tilde u = -3\,\dfrac{u_3}{u_1} + \dfrac{3\,u_2^2}{2\,u_1^2} + 12\left(\dfrac{u_2}{u} - \dfrac{u_1^2}{u^2}\right)$.
All of them are superpositions of first order substitutions. Besides aforementioned, in these superpositions the following substitutions from equation (2.17) are involved,
(2.17)→(2.13): 1) $\tilde u = \ln(u_1)$, $c_1 = c_2 = 0$, 2) $\tilde u = \ln(u_1) - \ln(u^2 + c_1/6)$, $c_2 = 0$. One more substitution is obtained from 2) by the replacement $\tilde u \to -\tilde u$, $c_2 \leftrightarrow c_1$.

II. C-integrable equations.

(2.19)→(2.23): $\tilde u = \sqrt{u_1}$, at that, in equation (2.23) $\beta = \dfrac{1}{2}\alpha'$.
(2.19)→(2.21): $\tilde u = \sqrt{u_1/(2\,u)}$.
(2.23)→(2.22): $\tilde u = u_1/u$, at that, in equation (2.23) $\alpha = \gamma$.

(2.20)→(2.22): $\tilde u = \dfrac{1}{\xi}(\alpha' + \sqrt{u_x\alpha'}\,) - \dfrac{1}{2}\,\alpha''(\alpha')^{-1}$, where $\xi = \alpha(x) - u$. At that, in equation (2.22)

$$\gamma = \frac{1}{2}\,\alpha'''(\alpha')^{-1} - \frac{3}{4}\,(\alpha''/\alpha')^2 - (c_0 + c_1\alpha + c_2\alpha^2)/\alpha',$$
$$\beta = \frac{1}{2}\,\alpha^{(4)}(\alpha')^{-1} - 2\,\alpha''\alpha'''(\alpha')^{-2} + \frac{3}{2}\,(\alpha''/\alpha')^3 - \frac{1}{2}\,\alpha''(\alpha')^{-2}(c_0 + c_1\alpha + c_2\alpha^2).$$

3. Fifth order equations

In this section we find all equations (0.4) possessing infinite sequences of local generalized symmetries. In the process of classification we employed necessary integrability conditions which follow from the existence of formal symmetry [4, 7] and are written as canonical conservation laws. It is easy to check that each third order integrable equation (2.10)–(2.23) has a fifth order symmetry (0.4). It turns out that if one exclude these symmetries from the consideration, then the list of other integrable equations coincides (up to the equivalency) with the list obtained in works [9, 18], where the equations possessing generalized conservation laws were studied.

In Subsection 3.1 we provide the complete list of integrable equations (0.4) not being symmetries of lower order equations. The equations in the list differ slightly by the form from equivalent equations in [9, 18]. Subsection 3.2 contains a new recurrent formula for the integrability conditions. We note that in the works [9, 18] only several simplest conditions were provided. For a given equation (0.4) the integrability conditions can be easily checked one by one by computer.

In Subsection 3.3 we adduce a schematic proof of the classification theorem. It contains an algorithm for reducing an integrable equation (0.4) to one of the canonical forms from Subsection 3.1 by point transformations (1.24)–(1.26). In other words, once we employ point transformations, we indicate what we normalize by them. We hope that following this algorithm and guidelines given in the text, the interested reader can easily recover all the details of rather hard calculations. Post factum it follows from the proof that if equation (0.4) satisfies first ten integrability conditions, then it is integrable. We observe that the equation $u_t = u_5 + uu_1$ satisfies fist nine integrability condition, but does not tenth.

3.1. List of integrable equations.

Theorem 2. *Suppose nonlinear equation* (0.4) *satisfies two conditions*,
1) *there exists an infinite sequence of generalized symmetries*

$$u_{\tau_i} = G_i(u, ..., u_{n_i}), \qquad i = 1, 2, ..., \qquad n_{i+1} > n_i > \cdots > 5; \tag{3.1}$$

2) *there exist no symmetries* (3.1) *of orders* $1 < n_i < 5$.
Then the equation is equivalent to one in the list

$$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1, \tag{3.2}$$
$$u_t = u_5 + 5uu_3 + \frac{25}{2}u_1u_2 + 5u^2u_1, \tag{3.3}$$
$$u_t = u_5 + 5u_1u_3 + \frac{5}{3}u_1^3, \tag{3.4}$$
$$u_t = u_5 + 5u_1u_3 + \frac{15}{4}u_2^2 + \frac{5}{3}u_1^3, \tag{3.5}$$
$$u_t = u_5 + 5(u_1 - u^2)u_3 + 5u_2^2 - 20uu_1u_2 - 5u_1^3 + 5u^4u_1, \tag{3.6}$$
$$u_t = u_5 + 5(u_2 - u_1^2)u_3 - 5u_1u_2^2 + u_1^5, \tag{3.7}$$

$$
\begin{aligned}
u_t &= u_5 + 5(u_2 - u_1^2 + \lambda_1 e^{2u} - \lambda_2^2 e^{-4u})\, u_3 - 5u_1u_2^2 + 15(\lambda_1 e^{2u} + 4\lambda_2^2 e^{-4u})\, u_1u_2 \\
&\quad + u_1^5 - 90\lambda_2^2 e^{-4u}\, u_1^3 + 5(\lambda_1 e^{2u} - \lambda_2^2 e^{-4u})^2\, u_1,
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
u_t &= u_5 + 5(u_2 - u_1^2 - \lambda_1^2 e^{2u} + \lambda_2 e^{-u})\, u_3 - 5u_1u_2^2 - 15\lambda_1^2 e^{2u}\, u_1u_2 \\
&\quad + u_1^5 + 5(\lambda_1^2 e^{2u} - \lambda_2 e^{-u})^2\, u_1, \quad \lambda_2 \neq 0,
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
u_t &= u_5 - 5\frac{u_2u_4}{u_1} + 5\frac{u_2^2u_3}{u_1^2} + 5\left(\frac{\mu_1}{u_1} + \mu_2 u_1^2\right) u_3 - 5\left(\frac{\mu_1}{u_1^2} + \mu_2 u_1\right) u_2^2 \\
&\quad - 5\frac{\mu_1^2}{u_1} + 5\mu_1\mu_2 u_1^2 + \mu_2^2 u_1^5,
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
u_t &= u_5 - 5\frac{u_2u_4}{u_1} - \frac{15}{4}\,\frac{u_3^2}{u_1} + \frac{65}{4}\,\frac{u_2^2u_3}{u_1^2} + 5\left(\frac{\mu_1}{u_1} + \mu_2 u_1^2\right)\, u_3 - \frac{135}{16}\frac{u_2^4}{u_1^3} \\
&\quad - 5\left(\frac{7\mu_1}{4u_1^2} - \frac{\mu_2 u_1}{2}\right) u_2^2 - 5\frac{\mu_1^2}{u_1} + 5\mu_1\mu_2 u_1^2 + \mu_2^2 u_1^5,
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
u_t &= u_5 - \frac{5}{2}\,\frac{u_2u_4}{u_1} - \frac{5}{4}\,\frac{u_3^2}{u_1} + 5\frac{u_2^2u_3}{u_1^2} + \frac{5\,u_2u_3}{2\sqrt{u_1}} - 5(u_1 - 2\mu u_1^{1/2} + \mu^2)\, u_3 - \frac{35}{16}\,\frac{u_2^4}{u_1^3} \\
&\quad - \frac{5}{3}\,\frac{u_2^3}{u_1^{3/2}} + 5\Big(\frac{3\mu^2}{4u_1} - \frac{\mu}{\sqrt{u_1}} + \frac{1}{4}\Big)\, u_2^2 + \frac{5}{3}\, u_1^3 - 8\mu u_1^{5/2} + 15\mu^2 u_1^2 - \frac{40}{3}\,\mu^3 u_1^{3/2},
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
u_t &= u_5 + \frac{5}{2}\,\frac{f - u_1}{f^2}\, u_2u_4 + \frac{5}{4}\,\frac{2\,f - u_1}{f^2}\, u_3^2 + 5\,\mu\,(u_1 + f)^2 u_3 \\
&\quad + \frac{5}{4}\,\frac{4\,{u_1}^2 - 8\,u_1 f + f^2}{f^4}\, u_2^2u_3 + \frac{5}{16}\,\frac{2 - 9\,u_1^3 + 18\,u_1^2 f}{f^6}\, u_2^4 \\
&\quad + \frac{5\mu}{4}\,\frac{(4\,f - 3\,u_1)(u_1 + f)^2}{f^2}\, u_2^2 + \mu^2(u_1 + f)^2\big(2\,f(u_1 + f)^2 - 1\big),
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
u_t &= u_5 + \frac{5}{2}\,\frac{f - u_1}{f^2}\, u_2u_4 + \frac{5}{4}\,\frac{2\,f - u_1}{f^2}\, u_3^2 - 5\,\omega\,(f^2 + u_1^2) u_3 \\
&\quad + \frac{5}{4}\,\frac{4\,u_1^2 - 8\,u_1 f + f^2}{f^4}\, u_2^2u_3 + \frac{5}{16}\,\frac{2 - 9\,u_1^3 + 18\,u_1^2 f}{f^6}\, u_2^4 \\
&\quad + \frac{5}{4}\,\omega\,\frac{5\,u_1^3 - 2\,u_1^2 f - 11\,u_1 f^2 - 2}{f^2}\, u_2^2 - \frac{5}{2}\,\omega'\,(u_1^2 - 2\,u_1 f + 5\,f^2) u_1u_2 \\
&\quad + 5\,\omega^2 u_1 f^2 (3\,u_1 + f)(f - u_1),
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
u_t &= u_5 + \frac{5}{2}\,\frac{f - u_1}{f^2}\, u_2u_4 + \frac{5}{4}\,\frac{2\,f - u_1}{f^2}\, u_3^2 + \frac{5}{4}\,\frac{4\,u_1^2 - 8\,u_1 f + f^2}{f^4}\, u_2^2u_3 \\
&\quad + \frac{5}{16}\,\frac{2 - 9\,{u_1}^3 + 18\,{u_1}^2 f}{f^6}\, u_2^4 + 5\,\omega\,\frac{2\,{u_1}^3 + {u_1}^2 f - 2\,u_1 f^2 + 1}{f^2}\, u_2^2 \\
&\quad - 10\,\omega\, u_3(3\,u_1 f + 2\,u_1^2 + 2\,f^2) - 10\,\omega'(2\,f^2 + u_1 f + {u_1}^2)\, u_1u_2 \\
&\quad + 20\,\omega^2 u_1({u_1}^3 - 1)(u_1 + 2\,f),
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
u_t = {} & u_5 + \frac{5}{2}\,\frac{f-u_1}{f^2}\,u_2u_4 + \frac{5}{4}\,\frac{2\,f-u_1}{f^2}\,u_3^2 - 5\,c\frac{f^2+u_1^2}{\omega^2}\,u_3 \\
& + \frac{5}{4}\,\frac{4\,u_1^2-8\,u_1f+f^2}{f^4}\,u_2^2u_3 + \frac{5}{16}\,\frac{2-9\,{u_1}^3+18\,u_1^2f}{f^6}\,u_2^4 \\
& - 10\,\omega\,(3\,u_1f+2\,u_1^2+2\,f^2)\,u_3 - \frac{5}{4}\,c\,\frac{11\,u_1f^2+2\,u_1^2f+2-5\,u_1^3}{\omega^2f^2}\,u_2^2 \\
& + 5\,\omega\,\frac{2\,u_1^3+u_1^2f-2\,u_1f^2+1}{f^2}\,u_2^2 + 5\,c\,\omega'\,\frac{u_1^2+5\,f^2-2\,u_1f}{\omega^3}\,u_1u_2 \\
& - 10\,\omega'\,(2\,f^2+u_1f+{u_1}^2)\,u_1u_2 + 20\,\omega^2u_1(u_1^3-1)(u_1+2\,f) \\
& + 40\,\frac{c\,u_1f^3(2\,u_1+f)}{\omega} + 5\,\frac{c^2u_1f^2(3\,u_1+f)(f-u_1)}{\omega^4}, \quad c\neq 0.
\end{aligned}
\tag{3.16}
$$

Here λ_1, λ_2, μ, μ_1, μ_2, and c are parameters, the function $f(u_1)$ solves the algebraic equation

$$(f+u_1)^2(2f-u_1)+1=0, \tag{3.17}$$

and $\omega(u)$ is any nonconstant solution to the differential equation

$$\omega'^2 = 4\,\omega^3 + c. \tag{3.18}$$

Remark 8. All the equations in the list of Theorem 2 are S-integrable. In the proof of the theorem it has been established that each C-integrable equation (0.4) is a symmetry of some third order C-integrable equation in list (2.10) – (2.23).

Remark 9. If $\lambda_1=\lambda_2=0$ in equation (3.8), it coincides with (3.7). If $\lambda_1=\lambda_2=0$ in equation (3.9), it also coincides with (3.7), and if $\lambda_2=0$, then (3.9) coincides with (3.8) as $\lambda_2=0$ up to the change $\lambda_1\to-\lambda_1^2$ in the latter.

Remark 10. If $\mu_2\neq 0$ in equation (3.10), the substitution $u=c^{-1}\ln v$ reduces it to

$$v_t = v_5 - 5\frac{v_2v_4}{v_1} + 5\frac{v_2^2v_3}{v_1^2} + 5\mu\left(\frac{vv_3-v_1v_2}{v_1} - \frac{vv_2^2}{v_1^2}\right) - 5\mu^2\frac{v^2}{v_1}, \tag{3.19}$$

where $\mu=\mu_1c,\ c=\sqrt{-\mu_2}$.

Remark 11. If $\mu_2\neq 0$ in equation (3.11), the transformation $u=c^{-1}\ln v$ reduces it to

$$v_t = v_5 - 5\frac{v_2v_4}{v_1} - \frac{15v_3^2}{4v_1} + 65\frac{v_2^2v_3}{4v_1^2} - \frac{135v_2^4}{16v_1^3} + 5\mu\left(\frac{vv_3}{v_1} + \frac{1}{2}v_2 - \frac{7}{4}\frac{vv_2^2}{v_1^2}\right) - 5\mu^2\frac{v^2}{v_1}, \tag{3.20}$$

where $\mu=\mu_1c,\ c=2\sqrt{-\mu_2}$.

3.2. Integrability conditions. The following conditions can be borrowed from the works [9,18].

Lemma 3. *For nonlinear integrable equations (0.4) the first four integrability conditions can be written as*

$$\frac{d}{dt}\frac{\partial F}{\partial u_4} = \frac{d}{dx}\sigma_0, \tag{3.21}$$

$$\frac{d}{dt}\left(2\left(\frac{\partial F}{\partial u_4}\right)^2 - 5\frac{\partial F}{\partial u_3}\right) = \frac{d}{dx}\sigma_1, \tag{3.22}$$

$$\frac{d}{dt}\left(15\frac{\partial F}{\partial u_3}\frac{\partial F}{\partial u_4} - 25\frac{\partial F}{\partial u_2} - 4\left(\frac{\partial F}{\partial u_4}\right)^3\right) = \frac{d}{dx}\sigma_2, \tag{3.23}$$

$$\frac{d}{dt}\left[25\left(\frac{d}{dx}\frac{\partial F}{\partial u_4}\right)^2 + 5\frac{\partial F}{\partial u_4}\left(5\frac{d}{dx}\frac{\partial F}{\partial u_3} + 10\frac{\partial F}{\partial u_2} - 7\frac{\partial F}{\partial u_3}\frac{\partial F}{\partial u_4}\right) + \right. \tag{3.24}$$

$$+ 7\left(\frac{\partial F}{\partial u_4}\right)^4 + 25\left(\frac{\partial F}{\partial u_3}\right)^2 - 125\frac{\partial F}{\partial u_1}\Bigg] = \frac{d}{dx}\sigma_3,$$

where $\frac{d}{dx}$ *is the operator of the total derivative w.r.t.* x*, and* $\frac{d}{dt}$ *is the evolution derivation w.r.t.* t *by virtue of equation (0.4).*

The given conditions follow from the existence of a formal symmetry. But technically it is more convenient to employ the described in Appendix 3 method of calculating the densities of canonical conservation laws

$$\frac{d}{dt}\rho_n = \frac{d}{dx}\theta_n, \quad n = 0, 1, \dots \tag{3.25}$$

by the logarithmic derivative of a formal eigenfunction of the linearization operator for equation (0.4). This approach allows us to obtain the recurrent formula

$$\begin{aligned}
\rho_{n+4} =& \frac{1}{5}\theta_n - \frac{1}{5}\left[F_{u_0}\delta_{n,0} + F_{u_1}\delta_{n,-1} + F_{u_2}\delta_{n,-2} + F_{u_3}\delta_{n,-3} + F_{u_4}\delta_{n,-4} + F_{u_1}\rho_n\right] \\
& - 2\sum_0^{n+3}\rho_i\rho_j - 2\sum_0^{n+2}\rho_i\rho_j\rho_k - \frac{1}{5}\sum_0^{n}\rho_i\rho_j\rho_k\rho_l\rho_m + \sum_0^{n+1}\left(\frac{d}{dx}\rho_i\right)\frac{d}{dx}\rho_j \\
& + \sum_0^{n}\rho_i\left(\frac{d}{dx}\rho_j\right)\frac{d}{dx}\rho_k - \sum_0^{n+1}\rho_i\rho_j\rho_k\rho_l - \frac{1}{5}F_{u_2}\left[\frac{d}{dx}\rho_n + 2\rho_{n+1} + \sum_0^{n}\rho_i\rho_j\right] \\
& - \frac{1}{5}F_{u_3}\left[\frac{d^2}{dx^2}\rho_n + 3\frac{d}{dx}\rho_{n+1} + 3\rho_{n+2} + \frac{3}{2}\frac{d}{dx}\sum_0^{n}\rho_i\rho_j + 3\sum_0^{n+1}\rho_i\rho_j + \sum_0^{n}\rho_i\rho_i\rho_k\right] \\
& - \frac{1}{5}F_{u_4}\Bigg[\frac{d^3}{dx^3}\rho_n + 4\frac{d^2}{dx^2}\rho_{n+1} + 6\frac{d}{dx}\rho_{n+2} + 4\rho_{n+3} + 2\frac{d^2}{dx^2}\sum_0^{n}\rho_i\rho_j + 6\frac{d}{dx}\sum_0^{n+1}\rho_i\rho_j + \\
& +6\sum_0^{n+2}\rho_i\rho_j + 4\sum_0^{n+1}\rho_i\rho_j\rho_k - \sum_0^{n}\left(\frac{d}{dx}\rho_i\right)\frac{d}{dx}\rho_j + 2\frac{d}{dx}\sum_0^{n}\rho_i\rho_j\rho_k + \sum_0^{n}\rho_i\rho_j\rho_k\rho_l\Bigg] - \\
& - \frac{d}{dx}\Bigg[\frac{1}{5}\frac{d^3}{dx^3}\rho_n + \frac{d^2}{dx^2}\rho_{n+1} + 2\frac{d}{dx}\rho_{n+2} + \sum_0^{n}\rho_i\frac{d^2}{dx^2}\rho_j + 2\frac{d}{dx}\sum_0^{n+1}\rho_i\rho_j + 3\sum_0^{n+2}\rho_i\rho_j \\
& +2\rho_{n+3} + \frac{2}{3}\frac{d}{dx}\sum_0^{n}\rho_i\rho_j\rho_k + \frac{1}{2}\sum_0^{n}\left(\frac{d}{dx}\rho_i\right)\frac{d}{dx}\rho_j + 2\sum_0^{n+1}\rho_i\rho_j\rho_k + \frac{1}{2}\sum_0^{n}\rho_i\rho_j\rho_k\rho_l\Bigg],
\end{aligned} \tag{3.26}$$

where $n \geqslant -4$, $\rho_k = 0$, $\forall k < 0$, δ_{ij} is the Kronecker delta, $F_{u_i} = \partial F/\partial u_i$. In formula (3.26) the notation

$$\sum_m^n a_i b_j \dots p_z = \sum_{\substack{i+j+\dots+z=n \\ i\geqslant m,\, j\geqslant m,\dots,\, z\geqslant m}} a_i b_j \dots p_z$$

was employed for multiple sums. The summation indices in formula (3.26) are non-negative.

Recurrent formula (3.26) is published at the first time. It is easy to check that the first four integrability conditions in sequence (3.25) are equivalent to conditions (3.21) – (3.24).

Conditions (3.26), (3.25) can be used more efficiently for the classification if one first studies the structure of the densities of local conservation laws for equations (0.4). We recall that the symbol $O(n)$ indicates a function of the differential order at most n. Moreover, we employ the symbol $P_n(u_k)$ to denote a polynomial of the degree n of the variable u_k, whose coefficients have differential order less than k. In what follows we employ regularly the equivalence $f\frac{d}{dx}g \sim -g\frac{d}{dx}f$ implied by $\frac{d}{dx}(fg) \sim 0$. For instance, we have

$$u_{n+1}f(u,\dots,u_n) \sim -\sum_{i=0}^{n-1} u_{i+1}\frac{\partial}{\partial u_i}\int f\,d\,u_n,$$

which yields, in particular, $u_{n+1}O(n) \sim O(n)$.

Lemma 4. *If a density ρ of a local conservation law for equation (0.4) has differential order $n \geqslant 3$, then the equation*

$$\frac{d}{dx}\frac{\partial^2\rho}{\partial u_n^2} = \frac{2}{5}\frac{\partial^2\rho}{\partial u_n^2}\frac{\partial F}{\partial u_4} \tag{3.27}$$

holds true.

Proof. By definition we have

$$\frac{d}{dt}\rho = \sum_{k=0}^{n}\frac{\partial\rho}{\partial u_k}\left(u_{k+5} + \frac{d^k}{dx^k}F\right).$$

Using the equivalence, we can lower the order of this expression up to $n+2$. First we show that

$$\sum_{k=0}^{n-2}\frac{\partial\rho}{\partial u_k}\left(u_{k+5} + \frac{d^k}{dx^k}F\right) \sim O(n+1).$$

For this it is sufficient to convert the highest order term. Assuming $n \geqslant 3$, we get

$$\begin{aligned}\frac{\partial\rho}{\partial u_{n-2}}u_{n+3} &\sim -u_{n+2}\frac{d}{dx}\frac{\partial\rho}{\partial u_{n-2}} = -u_{n+2}u_{n+1}\frac{\partial^2\rho}{\partial u_n\partial u_{n-2}} + u_{n+2}O(n)\\ &\sim \frac{1}{2}u_{n+1}^2\frac{d}{dx}\frac{\partial^2\rho}{\partial u_n\partial u_{n-2}} - u_{n+1}\frac{d}{dx}O(n) = O(n+1).\end{aligned}$$

Thus,

$$\frac{d}{dt}\rho \sim \frac{\partial\rho}{\partial u_n}u_{n+5} + \frac{\partial\rho}{\partial u_{n-1}}u_{n+4} + \frac{\partial\rho}{\partial u_n}\frac{d^n}{dx^n}F + \frac{\partial\rho}{\partial u_{n-1}}\frac{d^{n-1}}{dx^{n-1}}F + O(n+1). \tag{3.28}$$

Let us convert the first term,

$$\begin{aligned}a_1 &\stackrel{\text{def}}{=} \frac{\partial\rho}{\partial u_n}u_{n+5} \sim u_{n+3}\frac{d^2}{dx^2}\frac{\partial\rho}{\partial u_n} = u_{n+3}\frac{d}{dx}\sum_{i=0}^{n}\frac{\partial^2\rho}{\partial u_n\partial u_i}u_{i+1}\\ &= u_{n+3}\left(\sum_{i=0}^{n}\frac{\partial^2\rho}{\partial u_n\partial u_i}u_{i+2} + \sum_{i,j=0}^{n}\frac{\partial^3\rho}{\partial u_n\partial u_i\partial u_j}u_{i+1}u_{j+1}\right)\\ &\sim -\frac{1}{2}u_{n+2}^2\frac{d}{dx}\frac{\partial^2\rho}{\partial u_n^2} - u_{n+2}\frac{d}{dx}\left(\sum_{i=0}^{n-1}\frac{\partial^2\rho}{\partial u_n\partial u_i}u_{i+2} + \sum_{i,j=0}^{n}\frac{\partial^3\rho}{\partial u_n\partial u_i\partial u_j}u_{i+1}u_{j+1}\right)\\ &\sim -u_{n+2}^2\left(\frac{1}{2}\frac{d}{dx}\frac{\partial^2\rho}{\partial u_n^2} + \frac{\partial^2\rho}{\partial u_n\partial u_{n-1}} + 2\sum_{i=0}^{n}\frac{\partial^3\rho}{\partial u_n^2\partial u_i}u_{i+1}\right) + u_{n+2}O(n+1)\\ &\sim -\frac{5}{2}u_{n+2}^2\frac{d}{dx}\frac{\partial^2\rho}{\partial u_n^2} - \frac{\partial^2\rho}{\partial u_n\partial u_{n-1}}u_{n+2}^2 + O(n+1).\end{aligned}$$

The second term in (3.28) is converted in the same way,

$$a_2 \stackrel{\text{def}}{=} \frac{\partial\rho}{\partial u_{n-1}} u_{n+4} \sim u_{n+2}\left(\sum_{i=0}^{n} \frac{\partial^2\rho}{\partial u_{n-1}\partial u_i} u_{i+2} + \sum_{i,j=0}^{n} \frac{\partial^3\rho}{\partial u_{n-1}\partial u_i\partial u_j} u_{i+1}u_{j+1}\right)$$
$$= \frac{\partial^2\rho}{\partial u_n\partial u_{n-1}} u_{n+2}^2 + u_{n+2}O(n+1) \sim \frac{\partial^2\rho}{\partial u_n\partial u_{n-1}} u_{n+2}^2 + O(n+1).$$

The above calculations are correct if $n \geqslant 2$. To convert the remaining two terms in (3.28), it is important that $n \geqslant 3$,

$$a_3 \stackrel{\text{def}}{=} \frac{\partial\rho}{\partial u_n}\frac{d^n}{dx^n} F \sim \left(\frac{d^2}{dx^2}\frac{\partial\rho}{\partial u_n}\right)\frac{d^{n-2}}{dx^{n-2}} F$$
$$= \left(\frac{\partial^2\rho}{\partial u_n^2} u_{n+2} + O(n+1)\right)\left(\frac{\partial F}{\partial u_4} u_{n+2} + O(n+1)\right)$$
$$= \frac{\partial^2\rho}{\partial u_n^2}\frac{\partial F}{\partial u_4} u_{n+2}^2 + u_{n+2}O(n+1) + O(n+1) \sim \frac{\partial^2\rho}{\partial u_n^2}\frac{\partial F}{\partial u_4} u_{n+2}^2 + O(n+1),$$
$$a_4 \stackrel{\text{def}}{=} \frac{\partial\rho}{\partial u_{n-1}}\frac{d^{n-1}}{dx^{n-1}} F \sim -\left(\frac{d}{dx}\frac{\partial\rho}{\partial u_{n-1}}\right)\frac{d^{n-2}}{dx^{n-2}} F = -\left(\frac{d}{dx}\frac{\partial\rho}{\partial u_{n-1}}\right)\left(\frac{\partial F}{\partial u_4} u_{n+2} + O(n+1)\right)$$
$$= u_{n+2}O(n+1) + O(n+1) \sim O(n+1).$$

Summing up the obtained expressions for $a_1, \ldots, a_4$, we find

$$\frac{d}{dt}\rho \sim u_{n+2}^2\left(\frac{\partial^2\rho}{\partial u_n^2}\frac{\partial F}{\partial u_4} - \frac{5}{2}\frac{d}{dx}\frac{\partial^2\rho}{\partial u_n^2}\right) + O(n+1).$$

Since a quadratic in higher derivative expression can not be a total derivative of any function, we obtain (3.27). □

Corollary. *If we have $n > 3$ in Lemma 4, then the density ρ is quadratic in u_n. Indeed, the left hand side of equation (3.27) contains the term $\rho_{u_n u_n u_n} u_{n+1}$ whose differential order is greater than 4 if $n > 3$. The order of other terms is less and hence $\rho_{u_n u_n u_n} = 0$.*

We apply the obtained result to the classification of equations (0.4).

Lemma 5. *Assume equation (0.4) satisfies condition (3.21). Then the function F is quadratic in u_4.*

Proof. Applying Corollary of Lemma 4 to the canonical density $\rho = F_{u_4}$, we obtain

$$\frac{\partial F}{\partial u_4} = f_1 + f_2 u_4 + f_3 u_4^2,$$

where the functions f_1, f_2 and f_3 are independent of u_4. Substituting this expression into (3.27), we find

$$\frac{d}{dx} f_3 = \frac{2}{5} f_3(f_1 + f_2 u_4 + f_3 u_4^2).$$

The left hand side of this equation is linear in u_4, and the right hand side is quadratic; thus, $f_3 = 0$. It yields $F = f_0 + f_1 u_4 + \frac{1}{2} f_2 u_4^2$, where the functions f_i are independent of u_4. □

3.3. Scheme of proof of main theorem.

Lemma 6. *Suppose equation (0.4) satisfies integrability conditions (3.21), (3.22), and (3.23). Then the function F is linear in u_4.*

Proof. According to Lemma 5, the function F is quadratic in u_4, $F = f_0 + f_1 u_4 + \frac{1}{2} f_2 u_4^2$. It can be easily verified that

$$\rho_2 \sim u_4^3 f_2\left(16 f_2^2 - 15\frac{\partial f_2}{\partial u_3}\right) + Z_1 u_4^2 + O(3).$$

In accordance with Corollary of Lemma 4, a cubic in u_4 term should vanish and therefore

$$\frac{\partial f_2}{\partial u_3} = \frac{16}{15}\, f_2^2.$$

In view of this equation we find

$$\rho_1 \sim u_4^2\, f_2^2 + O(3).$$

For this density relation (3.27) read as follows,

$$\frac{d}{dx}\, f_2 = \frac{1}{5}\, f_2(f_1 + f_2\, u_4).$$

It implies the equation

$$\frac{\partial f_2}{\partial u_3} = \frac{1}{5}\, f_2^2,$$

which together with the previous one yields $f_2 = 0$. □

Thus, if integrability conditions (3.21) – (3.23) hold true, then equation (0.4) reads as

$$u_t = u_5 + u_4 f_1(u, u_1, u_2, u_3) + f_0(u, u_1, u_2, u_3). \tag{3.29}$$

Lemma 7. *If a function of third differential order $\rho(u, u_1, u_2, u_3)$ is a density of a conservation law for equation (3.29), then it is at most quadratic in u_3.*

Proof. Letting $n = 3$ and $F = f_0 + f_1 u_4$ in (3.27) $n = 3$, we obtain

$$\frac{d}{dx}\frac{\partial^2 \rho}{\partial u_3^2} = \frac{2}{5}\,\frac{\partial^2 \rho}{\partial u_3^2} f_1. \tag{3.30}$$

Taking into account that f_1 is independent of u_4, we find that $\rho_{u_3 u_3 u_3} = 0$. □

Corollary 1. *The function f_1 in (3.29) is linear in u_3.*

Proof. Indeed, it follows from $F = f_0 + f_1 u_4$ and (3.21) that f_1 is the density of a conservation law for equation (3.29). Thus, as it was proven above, this function reads as $f_1 = g_1 + g_2 u_3 + g_3 u_3^2$, where $g_i = g_i(u, u_1, u_2)$. Substituting this expression into (3.30) instead of ρ, we obtain $g_3 = 0$. □

Corollary 2. *If $f_1 = g_1 + g_2 u_3$, $g_i = g_i(u, u_1, u_2)$ in equation (3.29) and this equation possesses a conservation law with a density ρ of second differential order, then the equation*

$$\frac{d}{dx}\frac{\partial^2 \rho}{\partial u_2^2} = \frac{2}{5}\,\frac{\partial^2 \rho}{\partial u_2^2}(g_1 + g_2 u_3) \tag{3.31}$$

holds true.

Proof. The statement can be easily checked by straightforward calculations. □

Proposition 1. *If integrability conditions (3.21)–(3.23) hold true, then equation (0.4) reads as*

$$u_t = u_5 + A_1 u_2 u_4 + A_2 u_4 + A_3 u_3^2 + (A_4 u_2^2 + A_5 u_2 + A_6) u_3 + A_7 u_2^4 + A_8 u_2^3 + A_9 u_2^2 + A_{10} u_2 + A_{11}, \tag{3.32}$$

where $A_i = A_i(u, u_1)$.

Proof. By Corollary 1 of Lemma 7, equation (0.4) reads as (3.29), where $f_1 = g_1(u, u_1, u_2) + g_2(u, u_1, u_2) u_3$. Consider then a density ρ_1 of conservation law (3.22). It is easy to check that

$$\rho_1 \sim \frac{2}{5}\, f_1^2 + \frac{\partial f_1}{\partial u_0} u_1 + \frac{\partial f_1}{\partial u_1} u_2 + \frac{\partial f_1}{\partial u_2} u_3 - \frac{\partial f_0}{\partial u_3}.$$

In this expression all terms except the last one are at most quadratic in u_3. By Lemma 7 the considered density must be quadratic in u_3 and therefore, the function f_0 is cubic in u_3,

$$f_0 = g_4 + g_5 u_3 + g_6 u_3^2 + g_7 u_3^3, \qquad g_i = g_i(u, u_1, u_2).$$

Due to the obtained results, densities of conservation laws (3.22) and (3.23) are equivalent to the expressions

$$\rho_1 \sim u_3^2\left(5\frac{\partial g_2}{\partial u_2} + 2\, g_2^2 - 15\, g_7\right) + O(2),$$

$$\rho_2 \sim u_3^3\left(50\frac{\partial g_7}{\partial u_2} - 25\frac{\partial^2 g_2}{\partial u_2^2} + 30\, g_2\frac{\partial g_2}{\partial u_2} + 8\, g_2^3 - 90\, g_2 g_7\right) + P_2(u_3).$$

According to Lemma 7, the coefficient at u_3^3 in the second formula should vanish. Moreover, the condition $\dfrac{d}{dt}\rho_1 \sim 0$ leads to extra four equations relating the functions g_2, g_7 and their derivatives w.r.t. u_2. By these equations it is easy to obtain $g_2 = g_7 = 0$.

Thus, $F = g_1 u_4 + g_4 + g_5 u_3 + g_6 u_3^2$. Now the density of conservation law (3.21) equals g_1 and we can substitute $\rho = g_1$ and $g_2 = 0$ into (3.31),

$$\frac{d}{dx}\frac{\partial^2 g_1}{\partial u_2^2} = \frac{2}{5}\frac{\partial^2 g_1}{\partial u_2^2} g_1.$$

As above, by this we obtain the linear in higher derivative function $g_1 = A_1(u, u_1)u_2 + A_2(u, u_1)$.

In view of the obtain results, condition (3.23) yields $g_6 = A_3(u, u_1)$. Then by condition (3.22) we get $\dfrac{\partial^3 g_5}{\partial u_2^3} = 0$. And finally, bearing in mind all the obtained results, we find by condition (3.23) that $\dfrac{\partial^5 g_4}{\partial u_2^5} = 0$. □

In studying equation (3.32) the following lemma will be useful.

Lemma 8. *Equation (3.32) preserves its form under point transformations $u = \varphi(v)$.*

Some of the formulas for the conversion of the coefficients A_i look simple,

$$\tilde{A}_1(v) = \varphi' A_1(u), \qquad \tilde{A}_2(v) = A_2(u) + \varphi'' v_1^2 A_1(u) + 5\varphi''(\varphi')^{-1} v_1,$$
$$\tilde{A}_3(v) = \varphi' A_3(u), \qquad \tilde{A}_4(v) = \varphi'^2 A_4(u), \qquad \tilde{A}_7(v) = \varphi'^3 A_7(u). \tag{3.33}$$

Other formulas are much more bulky and we omit them.

It can be checked that the first six densities of the canonical conservation laws for equation (3.32) are equivalent to

$$\rho_0 = -\frac{1}{5}(A_1 u_2 + A_2), \qquad \rho_1 \sim R_1 = \psi_1 u_2^2 + \psi_2 u_2 + \psi_3, \tag{3.34}$$
$$\rho_2 \sim R_2 = \psi_4 u_2^3 + \psi_5 u_2^2 + \psi_6 u_2 + \psi_7, \qquad \rho_3 \sim R_3 = \psi_8 u_3^2 + \psi_9 u_2^4 + \psi_{10} u_2^3 + \dots, \tag{3.35}$$
$$\rho_4 \sim R_4 = \psi_{11} u_2 u_3^2 + \psi_{12} u_3^2 + \psi_{13} u_2^5 + \dots, \tag{3.36}$$
$$\rho_5 \sim R_5 = \psi_1 u_4^2 + \psi_{15} u_3^3 + (\psi_{16} u_2^2 + \psi_{17} u_2 + \psi_{18}) u_3^2 + \psi_{19} u_2^6 + \dots, \tag{3.37}$$

where the coefficients ψ_k are expressed in terms of the functions A_i and their derivatives. For instance, ψ_1 in (3.34) and (3.37) reads as

$$\psi_1 = \frac{1}{25}\left(2A_1^2 - 5A_4 + 10\frac{\partial A_3}{\partial u_1}\right).$$

Lemma 9. *If equation (3.32) has a conservation law with a density $\rho(u, u_1, \dots, u_n)$ of differential order $n \geqslant 2$, then*

$$\rho \sim \alpha_1(u, u_1) u_n^2 + \alpha_2(u, u_1, \dots, u_{n-1}), \tag{3.38}$$

at that,

$$5\frac{\partial \alpha_1}{\partial u_1} = 2\alpha_1 A_1, \qquad 5\frac{\partial \alpha_1}{\partial u_0}\, u_1 = 2\alpha_1 A_2. \tag{3.39}$$

For $n = 3$ and $n = 4$ the structure of the densities can be easily specified. If $n = 3$, then

$$\rho \sim \alpha_1 u_3^2 + \alpha_2 u_2^4 + \alpha_3 u_2^3 + \alpha_4 u_2^2 + \alpha_5, \quad \alpha_i = \alpha_i(u, u_1),$$

and $\alpha_1(A_1 - 2A_3) = 0$. If $n = 4$, then

$$\rho \sim \alpha_1 u_4^2 + \alpha_2 u_3^3 + (\alpha_3 u_2^2 + \alpha_4 u_2 + \alpha_5) u_3^2 + \beta(u, u_1, u_2), \quad \alpha_i = \alpha_i(u, u_1),$$

where β is a polynomial of sixth degree in u_2.

Corollary. *The coefficients ψ_4 in (3.35), ψ_{11} and ψ_{13} in (3.36) are zero.*

The form of equation (3.32) depends essentially on the orders of its canonical conservation laws. Among integrable equations (3.32), there can be equations of the two following types,

I. Equations possessing no generalized canonical conservation laws. In other words, all canonical densities for the equations of the first type are equivalent to densities of zero or first differential order.

II. Equations possessing generalized canonical conservation laws of orders $\geqslant 2$.

In the case **I** one should equate all nontrivial terms of higher order in the densities of the canonical conservation laws to zero. This is why in expressions (3.34)–(3.37) there should be $\psi_1 = \psi_4 = \psi_5 = \psi_8 = \psi_9 = \psi_{10} = \cdots = 0$. In particular,

$$A_4 = \frac{2}{5} A_1^2 + 2 \frac{\partial A_3}{\partial u_1}.$$

By the equation $\psi_4 = 0$ one can express A_7 in terms of A_1 and A_3, and by $\psi_5 = 0$ one can express A_8 in terms of A_1, A_2, A_3 and A_5. From six conditions $\rho_i \sim h_i(u, u_1),\ i = 1, \ldots, 6$ one can extract a cumbersome system of differential equations for the remaining functions A_i. In this system there is the following closed subsystem of the equations for A_1 and A_3,

$$A_3 = \frac{1}{2} A_1, \qquad \frac{\partial A_1}{\partial u_1} = \frac{2}{5} A_1^2. \tag{3.40}$$

The latter of these equations has two solutions $A_1 = 0$ and $A_1 = -\frac{5}{2}(u_1 + a(u))^{-1}$. If $a(u) \neq 0$, then by point transformation $u \to \varphi(u)$ one can normalize $a = 1$. Thus, there are three possible cases,

$$\textbf{I.a}.\ \ A_1 = 0; \qquad \textbf{I.b}.\ \ A_1 = -\frac{5}{2} u_1^{-1}; \qquad \textbf{I.c}.\ \ A_1 = -\frac{5}{2}(u_1 + 1)^{-1}.$$

Case I.a. It follows from the equations $\psi_i = 0$ that $A_2 = g_1(u) + g_2(u) u_1$. Employing the point transformation $u \to \varphi(u)$, we can assume that $g_2 = 0$ (see (3.33)). After that all remaining functions $A_i(u, u_1)$ are happened to be polynomials with constant coefficients. To determine these coefficients we check 10 integrability conditions (3.25). We find out that there exist only three integrable equations of the considered type,

$$u_t = u_5 + u_4 c_1 + c_2 u_3 + c_3 u_2 + c_4 u_1 + c_5 u + c_6,$$

$$u_t = u_5 + 5u^2 u_4 + 10u\, u_3 (u^3 + 4u_1) + 25u\, u_2^2 + 10 u_2 (5u_1^2 + 12 u^3 u_1 + u^6) + 140 u^2 u_1^3 + 70 u^5 u_1^2 + 5u^8 u_1,$$

$$u_t = u_5 + 5u\, u_4 + 10u^2 u_3 + 15 u_1 u_3 + 10 u_2^2 + 10 u^3 u_2 + 50 u\, u_1 u_2 + 5u^4 u_1 + 30 u^2 u_1^2 + 15 u_1^3.$$

The second of these equations is a symmetry for equation (2.21), where $\alpha = 0$. The third equation is the symmetry of the Burgers equation $u_t = u_2 + 2uu_1$ (as well as a symmetry of equation (2.22), where $\beta = \gamma = 0$).

Case I.b. There exists only one integrable equation in this class,

$$u_t = u_5 - \frac{5u_2 u_4}{2u_1} + 5\frac{u_2^2 u_3}{u_1^2} - \frac{5u_3^2}{4u_1} - \frac{35 u_2^4}{16 u_1^3} + k\, u.$$

It is a symmetry of equation (2.19) as $\alpha(x) = c$.

Case I.c. By the integrability conditions one can find $A_2 = f(u)(u_1 + 1) + g(u)\sqrt{u_1 + 1}$, where f and g are arbitrary functions. If $g = 0$, all the functions A_i are independent of u, and the transformation $u \to u - x$ is admissible. It reduces this case to the case **I.b.**

If $g \neq 0$, there exist two very cumbersome C-integrable equations being the symmetries of equations (2.48) and (2.49), respectively.

In the case **II** equation (3.32) possesses at least one generalized conservation law, and thus, in accordance with (3.39), we can write A_1 and A_2 as

$$A_1 = \frac{5}{2f_0}\frac{\partial f_0}{\partial u_1}, \qquad A_2 = \frac{5}{2f_0}\frac{\partial f_0}{\partial u}u_1, \quad f_0 = f_0(u, u_1). \tag{3.41}$$

As a result, equation (3.32) casts into the form

$$u_t = u_5 + \frac{5}{2}(\ln f_0)_x u_4 + A_3 u_3^2 + (A_4 u_2^2 + A_5 u_2 + A_6)u_3 + A_7 u_2^4 + A_8 u_2^3 + A_9 u_2^2 + A_{10} u_2 + A_{11}, \tag{3.42}$$

where $A_i = A_i(u, u_1)$. The first canonical conservation law for this equation is trivial,

$$\rho_0 = -\frac{1}{2}\frac{d}{dx}\ln f_0, \qquad \theta_0 = -\frac{1}{2}\frac{d}{dt}\ln f_0.$$

Second integrability condition (3.22) is reduced to

$$\frac{d}{dt}\rho_1 \sim u_4^2 f_0 \frac{d}{dx}\left(\frac{A_4}{f_0} - \frac{2}{f_0}\frac{\partial A_3}{\partial u_1} - \frac{5}{2\,f_0^3}\left(\frac{\partial f_0}{\partial u_1}\right)^2\right) + u_3^3\,Z_1 + u_3^2\,Z_2 + O(2) \sim 0.$$

Equating the coefficient at u_4^2 to zero, we obtain

$$A_4 = 2\frac{\partial A_3}{\partial u_1} + \frac{5}{2f_0^2}\left(\frac{\partial f_0}{\partial u_1}\right)^2 + c_1 f_0, \tag{3.43}$$

where c_1 is an integration constant. The function Z_1 is linear in u_2, this is why the identity $Z_1 = 0$ implies two equations,

$$c_1\left[25\,f_0\frac{\partial^2 f_0}{\partial u_1^2} - 45\left(\frac{\partial f_0}{\partial u_1}\right)^2 + 10\,A_3\,f_0\frac{\partial f_0}{\partial u_1} - 14\,f_0^2\frac{\partial A_3}{\partial u_1} - 6\,c_1\,f_0^3\right] = 0, \tag{3.44}$$

$$c_1\left[25\,f_0\,u_1\frac{\partial^2 f_0}{\partial u_1 \partial u} - 30\,u_1\frac{\partial f_0}{\partial u_1}\frac{\partial f_0}{\partial u} + 5\,f_0\frac{\partial f_0}{\partial u}(3 + 2\,u_1 A_3) - 2\,f_0^2\,u_1\frac{\partial A_3}{\partial u} - 3\,f_0^2 A_5\right] = 0. \tag{3.45}$$

The function Z_2 is cubic in u_2 and therefore the identity $Z_2 = 0$ implies four equations involving also the factor c_1. This is why it is natural to consider two cases, $c_1 = 0$ and $c_1 \neq 0$. Moreover, in view of Lemma 9, there appears one more fork, $A_1 - 2A_3 = 0$ or $A_1 - 2A_3 \neq 0$. Thus, we have the four cases

II.a. $c_1 = 0$, $A_3 = \frac{1}{2}A_1$; **II.c**. $c_1 = 0$, $A_3 = \frac{1}{2}A_1 + f_1$;

II.b. $c_1 \neq 0$, $A_3 = \frac{1}{2}A_1$, **II.d**. $c_1 \neq 0$, $A_3 = \frac{1}{2}A_1 + f_1$,

where $f_1 = f_1(u, u_1)$, $f_1 \neq 0$.

Case II.a. In this case the density in condition (3.23) can be written as

$$\rho_2 \sim u_2^3\,f_0^{-3}\left[5\,f_0^2\frac{\partial^3 f_0}{\partial u_1^3} + 5\,f_0\frac{\partial f_0}{\partial u_1}\frac{\partial^2 f_0}{\partial u_1^2} - 5\left(\frac{\partial f_0}{\partial u_1}\right)^3 - 16\,A_7\,f_0^3\right] + P_2(u_2).$$

By Lemma 9 the coefficient at u_2^3 should vanish, and it determines the function A_7,

$$A_7 = \frac{5}{16\,f_0^3}\left[f_0^2\frac{\partial^3 f_0}{\partial u_1^3} + f_0\frac{\partial f_0}{\partial u_1}\frac{\partial^2 f_0}{\partial u_1^2} - \left(\frac{\partial f_0}{\partial u_1}\right)^3\right]. \tag{3.46}$$

In view of (3.46), forth integrability condition (3.24) is reduced to

$$\frac{d}{dt}\rho_3 \sim u_5^2 f_0 \frac{d}{dx}\left[f_0^{-2}\frac{\partial^2 f_0}{\partial u_1^2} - 2\, f_0^{-3}\left(\frac{\partial f_0}{\partial u_1}\right)^2\right] + P_1 u_4^3 + u_4^2 u_3 (P_2 u_2 + P_3) + u_4^2 O(2) + O(3) \sim 0,$$

where P_i are some functions of first differential order. Equating the term at u_5^2 to zero, we obtain $f_0 = \big(c\, u_1^2 + \alpha(u) u_1 + \beta(u)\big)^{-1}$, where c is a constant, and α and β are arbitrary functions. As a result, the equation $P_1 = 0$ holds automatically, and $P_2 = 0$ yields $c = 0$. Thus,

$$f_0 = \big(\alpha(u) u_1 + \beta(u)\big)^{-1}, \quad A_1 = -\frac{5}{2}\frac{\alpha}{\alpha u_1 + \beta}, \quad A_2 = -\frac{5}{2}\frac{\alpha' u_1^2 + \beta' u_1}{\alpha u_1 + \beta}.$$

In view of transformation formulas (3.33) for A_1 and A_2 one can see that the change $u \to \varphi(u)$ allows one to simplify f_0. If $\alpha = 0$, without loss of generality we put $f_0 = 1$; if $\beta = 0$, without loss of generality we can let $\alpha = 1$; if $\alpha\beta \neq 0$, we may assume that $\beta = \alpha$.

Thus, there appear the following three non-equivalent cases,

$$\textbf{II.a.1}.\ \ f_0 = 1; \qquad \textbf{II.a.2}.\ \ f_0 = \frac{1}{u_1}; \qquad \textbf{II.a.3}.\ \ f_0 = \frac{a(u)}{u_1 + 1}.$$

Case II.a.1. The identities $c_1 = 0$, $f_0 = 1$ lead to the relations $A_1 = A_2 = A_3 = = A_4 = A_7 = 0$. Third integrability condition (3.23) reads as

$$\frac{d}{dt}\rho_2 \sim u_4^2 \frac{d}{dx}\left(3\, A_8 - \frac{\partial A_5}{\partial u_1}\right) + \frac{1}{5}\, u_3^3\, A_5 \left(3\, A_8 - \frac{\partial A_5}{\partial u_1}\right) + P_2(u_3) \sim 0.$$

In this expression the coefficients at u_4^2 and u_3^3 should be equated to zero. At the same time, the density in condition (3.24) reads as

$$\rho_3 \sim u_2^3\left(2\frac{\partial A_8}{\partial u_1} - \frac{\partial^2 A_5}{\partial u_1^2}\right) + P_2(u_2).$$

The coefficient at u_2^3 should vanish by Lemma 9. The mentioned three identities imply $A_8 = A_8(u)$, $A_5 = 3\,(A_8 + c_2) u_1 + q_1(u)$; $c_2\,(A_8 + c_2) = 0$, $c_2\, q_1 = 0$, where c_2 is a constant.

In view of the above results we find

$$\rho_4 \sim u_2^3\, A_8' + P_2(u_2).$$

By Lemma 9 it yields $A_8' = 0$, and we hence get

$$A_8 = c_3, \quad A_5 = 3\,(c_2 + c_3) u_1 + q_1(u); \quad c_2\,(c_2 + c_3) = 0, \quad c_2\, q_1 = 0.$$

Now conditions (3.22) and (3.24) are written as

$$\frac{d}{dt}\rho_1 \sim u_3^2 \frac{d}{dx}\left(\frac{\partial^2 A_6}{\partial u_1^2} - 2\, q_1'\right) + P_5(u_2) \sim 0,$$

$$\frac{d}{dt}\rho_3 \sim u_4^2 \frac{d}{dx}\left(\frac{\partial A_9}{\partial u_1} - \frac{\partial^2 A_6}{\partial u_1^2} + \frac{9}{5}(c_2^2 - c_3^2) u_1^2 - \frac{6}{5}\, c_3\, q_1\, u_1 - \frac{1}{5}\, q_1^2\right) + P_3(u_3) \sim 0.$$

Equating the expressions at u_3^2 and u_4^2 to zero, we find A_6 and A_9,

$$A_6 = (q_1' + c_4) u_1^2 + q_2\, u_1 + q_3, \quad A_9 = \frac{3}{5}(c_3^2 - c_2^2) u_1^3 + \frac{3}{5} c_3\, q_1 u_1^2 + \frac{1}{5}(c_5 + 10\, q_1' + q_1^2) u_1 + q_4,$$

where $q_i = q_i(u)$ are arbitrary functions.

Then it follows from the third and fifth integrability conditions that $c_3 = c_2 = 0$. The third integrability condition determines the function A_{10} as a polynomial of third degree in u_1, and the forth integrability condition determines the function A_{11} as a polynomial of fifth degree in u_1. In order to determine the coefficients of the polynomials A_6, A_9, A_{10}, and A_{11} we check 10 integrability conditions. This work, being technically not difficult, requires the examination of a great number of options while solving equations. The result is S-integrable equations (3.2) – (3.9), as well as integrable equations being symmetries of the equations (2.10) – (2.13).

Case II.a.2. The above obtained formulas for A_1, A_2, A_3, and also (3.43) and (3.46) remain to be true. Substituting there $c_1 = 0$ and $f_0 = u_1^{-1}$, we obtain

$$A_1 = -\frac{5}{2\,u_1}, \quad A_2 = 0, \quad A_3 = -\frac{5}{4\,u_1}, \quad A_4 = \frac{5}{u_1^2}, \quad A_7 = -\frac{35}{16\,u_1^3}.$$

It is easy to check that integrability condition (3.23) can be written as

$$\frac{d}{dt}\rho_2 \sim u_4^2\left[u_2\,u_1^{-1}\left(6\,A_8 + 6\,u_1\frac{\partial A_8}{\partial u_1} + \frac{\partial A_5}{\partial u_1} - 2\,u_1\frac{\partial^2 A_5}{\partial u_1^2}\right) + 6\,u_1\frac{\partial A_8}{\partial u} + 3\frac{\partial A_5}{\partial u} - 2\,u_1\frac{\partial^2 A_5}{\partial u_1 \partial u}\right] +$$
$$+ \frac{1}{6}u_3^3 u_2 u_1^{-2}\left(78\,A_8 - 42\,u_1\frac{\partial A_8}{\partial u_1} - 60\,u_1^2\frac{\partial^2 A_8}{\partial u_1^2} + 20\,u_1^2\frac{\partial^3 A_5}{\partial u_1^3} - 16\,u_1\frac{\partial^2 A_5}{\partial u_1^2} + 13\frac{\partial A_5}{\partial u_1}\right) +$$
$$+ u_3^3\,Q(u, u_1) + P_3(u_3) \sim 0.$$

Moreover,

$$\rho_3 \sim u_2^3 u_1^{-2}\left(4\,u_1^2\frac{\partial A_8}{\partial u_1} + 12\,A_8\,u_1 - 2\,u_1^2\frac{\partial^2 A_5}{\partial u_1^2} - 3\,u_1\frac{\partial A_5}{\partial u_1} + 4\,A_5\right) + P_2(u_2).$$

The coefficients at u_4^2 and $u_3^3\,u_2$ in (3.23) as well as the coefficient at u_2^3 in ρ_3 should vanish. It gives us four equations, whose solution reads as

$$A_5 = q_1 + \frac{q_2}{\sqrt{u_1}}, \quad A_8 = -\frac{q_1}{2\,u_1} - \frac{2\,q_2}{3\,u_1^{3/2}},$$

where $q_i = q_i(u)$. A slightly more cumbersome integrability condition (3.22) yields

$$A_6 = c_2 + q_3 u_1 + 2\,c_3\sqrt{u_1} + 2\,q_2' u_1^{3/2} + q_1' u_1^2.$$

Due to these results, conservation law (3.24) becomes

$$\frac{d}{dt}\rho_3 \sim u_4^2 u_2\left(\frac{\partial^2 A_9}{\partial u_1^2} + \frac{3}{u_1}\frac{\partial A_9}{\partial u_1} - \frac{q_1^2 + 15\,q_1'}{5\,u_1} - \frac{5\,q_2' + q_1 q_2}{5\,u_1^{3/2}} - \frac{3}{4}(c_2 u_1^{-3} + c_3 u_1^{-5/2})\right) +$$
$$+ u_4^2\left(\frac{\partial^2 A_9}{\partial u_1 \partial u}\,u_1 + 2\frac{\partial A_9}{\partial u} - \frac{2}{5}\sqrt{u_1}\,(5\,q_2' + q_1 q_2)' - \frac{1}{5}\,u_1(2\,q_1 q_1' + 15\,q_1'') - \frac{2}{5}\,q_2 q_2' + \frac{1}{4}\,q_3'\right)$$
$$+ P_3(u_3) \sim 0.$$

The terms with u_4^2 should vanish that implies

$$A_9 = c_4 - \frac{1}{8}\,q_3 + \frac{q_2^2}{10} + \frac{q_4}{u_1^2} + \frac{q_1^2}{15}\,u_1 + q_1' u_1 + \frac{4}{25}\sqrt{u_1}\,(5\,q_2' + q_1 q_2) - \frac{c_3}{\sqrt{u_1}} - \frac{3\,c_2}{4\,u_1}.$$

Then from conditions (3.22) – (3.24) we find A_{10} and A_{11}, but we do not write these expressions because they are bulky.

To specify constant coefficients and the structure of the functions $q_i(u)$ we check ten integrability conditions. These conditions are satisfied by the equation (3.12) and an equation being a symmetry of equation (2.18).

Case II.a.3. The way of arguing in this case is exactly the same as in **II.a.2**, but there are small differences in the formulas. General in the case **II.a** formulas become here

$$A_1 = -\frac{5}{2\,\xi}, \quad A_2 = \frac{5\,a'\,u_1}{2\,a}, \quad A_3 = -\frac{5}{4\,\xi}, \quad A_4 = \frac{5}{\xi^2}, \quad A_7 = -\frac{35}{16\,\xi^3},$$

where $a = a(u)$ is an arbitrary function, $\xi = u_1 + 1$.

Integrability condition (3.23) reads as

$$\frac{d}{dt}\rho_2 \sim u_4^2 u_2\,\xi^{-1}\left(6\,A_8 + 6\,\xi\frac{\partial A_8}{\partial u_1} + \frac{\partial A_5}{\partial u_1} - 2\,\xi\frac{\partial^2 A_5}{\partial u_1^2} + \frac{15\,a'}{2\,a\,\xi^2}\right) + u_4^2 u_1\left(6\frac{\partial A_8}{\partial u} + 3\,\xi^{-1}\frac{\partial A_5}{\partial u}\right.$$
$$\left. -2\frac{\partial^2 A_5}{\partial u_1\partial u} + \frac{2\,a'}{a}\frac{\partial A_5}{\partial u_1} - 6\frac{a'}{a}A_8 - \frac{3\,a'}{a\,\xi}A_5 + \frac{15}{2\,a^2\xi^2}\left(2\,a'^2 - aa''\right)\right) +$$
$$+ Q_1(u,u_1)u_3^3 u_2 + Q_2(u,u_1)u_3^3 + P_2(u_3) \sim 0.$$

This condition together with the formula for the density of conservation law (3.24)

$$\rho_3 \sim u_2^3\xi^{-2}\left(4\,\xi^2\frac{\partial A_8}{\partial u_1} + 12\,\xi\,A_8 - 2\,\xi^2\frac{\partial^2 A_5}{\partial u_1^2} - 3\,\xi\frac{\partial A_5}{\partial u_1} + 4\,A_5 - \frac{5\,a'}{a\,\xi}\right) + P_2(u_2)$$

and by the relation $Q_1 = 0$ lead us to four equations with the solution

$$A_5 = q_1 + \frac{q_2}{\sqrt{\xi}}, \qquad A_8 = -\frac{q_1}{2\,\xi} - \frac{2\,q_2}{3\,\xi^{3/2}} + \frac{4\,a'}{5\,a\,\xi^2},$$

where $q_i = q_i(u)$. Then, by integrability condition (3.22) we determine the A_6, and by integrability condition (3.24) we find the function A_9. After that we determine A_{10} and A_{11} by conditions (3.22) – (3.24). All these expressions involving arbitrary functions of u are rather bulky and we omit them.

It follows from the fifth integrability condition $\frac{d}{dt}\rho_4 \sim 0$ that $a' = 0$, $q_1 = 0$, $q_2' = 0$ and so forth. Only in the A_{11} their remain two arbitrary functions of u. The integrability conditions 5 – 7 yield a vast system of algebraic equations for the constants and for the two remaining functions. It follows from this system that all the functions A_i are independent of u. This is why we can apply a transformation $u \to u - x$ leading to the case **II.a.2.** Thus, in the considered case there are no new integrable equations.

Case II.b differs from the previous ones by that canonical conservation law (3.22) has the second order. It follows from condition (3.22) that

$$f_0 = -\frac{5}{2c_1}(u_1^2 + a(u)u_1 + b(u))^{-1}, \qquad ab' = 2a'b,$$

and the functions A_5, A_7, A_8, and A_9 are expressed in terms of f_0,

$$A_5 = \frac{15}{2\,f_0}\frac{\partial^2 f_0}{\partial u\partial u_1}u_1 + \frac{5}{f_0^2}\frac{\partial f_0}{\partial u}\left(f_0 - \frac{\partial f_0}{\partial u_1}u_1\right),$$
$$A_7 = \frac{c_1}{4}\frac{\partial f_0}{\partial u_1} + \frac{5}{8\,f_0}\frac{\partial^3 f_0}{\partial u_1^3} - \frac{35}{32\,f_0^2}\frac{\partial^2 f_0}{\partial u_1^2}\frac{\partial f_0}{\partial u_1} + \frac{5}{8\,f_0^3}\left(\frac{\partial f_0}{\partial u_1}\right)^3,$$
$$A_8 = \frac{5}{24\,f_0^2}\left(14\,f_0 - 3\frac{\partial f_0}{\partial u_1}u_1\right)\frac{\partial^2 f_0}{\partial u\partial u_1} + \frac{5\,u_1}{12\,f_0^2}\left(5\,f_0\frac{\partial^3 f_0}{\partial u\partial u_1^2} - \frac{\partial^2 f_0}{\partial u_1^2}\frac{\partial f_0}{\partial u}\right) +$$
$$+ \frac{c_1}{3}\frac{\partial f_0}{\partial u}u_1 - \frac{5}{24\,f_0^3}\frac{\partial f_0}{\partial u}\frac{\partial f_0}{\partial u_1}\left(3\,f_0 + 8\frac{\partial f_0}{\partial u_1}u_1\right).$$

The formula for A_9 is omitted since it is bulky.

Taking into account formulas (3.33) and an explicit form of the functions A_1 and A_2, one can observe that if functions a and b are nonzero, then we can make them constant by a point transformation $u \to \varphi(u)$. If $a = 0$, then up to a point transformation we have either $b = 0$ or $b = 1$. If $a \neq 0$, by letting $a = 2$ we get $b' = 0$. Thus, there are the following three possible cases,

$$\textbf{II.b.1}.\ \ f_0 = -\frac{5}{2c_1u_1^2}; \qquad \textbf{II.b.2}.\ \ f_0 = -\frac{5}{2c_1(u_1^2+1)};$$
$$\textbf{II.b.3}.\ \ f_0 = -\frac{5}{2c_1}((u_1+1)^2+c)^{-1},$$

where c is a constant.

Case II.b.1. It follows from conditions (3.22) and (3.24) that

$$A_5 = A_8 = 0, \quad A_6 = c_2 + q_1u_1^2 + q_2u_1^{-2}, \quad A_7 = -\frac{45}{8}\,u_1^{-3},$$
$$A_{10} = q_1'u_1^3 + q_2'u_1^{-1}, \quad A_9 = -\frac{1}{2}\,q_1u_1 - \frac{3}{2}\,c_2u_1^{-1} - \frac{5}{2}\,q_2u_1^{-3},$$
$$A_{11} = \frac{1}{5}\left(q_1'' + \frac{3}{10}\,q_1^2\right) + \frac{c_2}{5}\,q_1u_1^3 - \frac{3}{5}\,c_2\,q_2u_1^{-1} - \frac{1}{10}\,q_2^2u_1^{-3} + \left(\frac{1}{15}\,q_1q_2 - \frac{1}{3}\,q_2''\right)u_1 + q_3,$$

where $q_i = q_i(u)$, c_2 is a constant.

The check of the conditions 6 – 10 shows that there exist only two integrable equations, which are the symmetries of equations (2.15) and (2.17).

Case II.b.2. It follows from conditions (3.22) and (3.24) that

$$A_5 = A_8 = 0, \quad A_6 = q + c_2u_1\sqrt{u_1^2+1} + (3\,q + c_3)u_1^2, \quad A_7 = \frac{5}{8}\,u_1\frac{19 - 9\,u_1^2}{(u_1^2+1)^3}, \quad q = q(u),$$
$$A_9 = \frac{3}{2}\,u_1\frac{2\,q + c_3}{u_1^2+1} + \frac{c_2}{\sqrt{u_1^2+1}} - \frac{1}{2}\,c_2\sqrt{u_1^2+1} - \frac{1}{2}\,(3\,q + c_3)\,u_1, \quad A_{10} = q'u_1(3\,u_1^2 + 2),$$
$$A_{11} = \frac{3}{25}\,c_2(3\,q + c_3)(u_1^2+1)^{5/2} - \frac{1}{5}\,c_2(2\,q + c_3)(u_1^2+1)^{3/2} + \frac{1}{10}(3\,q^2 + 2\,c_4\,q)\,u_1 +$$
$$+\frac{3}{50}\big(10\,q'' + (3\,q + c_3)^2 + c_2^2\big)u_1^5 + \frac{1}{10}\big(5\,q'' + 6\,q^2 + 5\,c_3\,q + c_2^2\big)\,u_1^3 + c_5, \quad c_2\,q' = 0.$$

The check of the conditions 6 – 10 shows that there exist only two integrable equations, which are the symmetries of equations (2.14) and (2.16).

Case II.b.3. Second integrability condition (3.22) allows us to show that all the functions A_i are independent of u. This is why by the transformation $u \to u - x$ equation (3.42) is reduced to the equations from the cases **II.b.1** if $c = 0$ and **II.b.2** if $c \neq 0$. Thus, in the considered case there is no new integrable equations.

Case II.c. The density in (3.23) is equivalent to the cubic in u_2 expression (3.35). The condition $\psi_4 = 0$ allows us to express A_7 in terms f_0 and f_1. Then we find

$$\frac{d}{dt}\rho_2 \sim (Z_1u_2 + Z_2)\,u_4^2 + O(3).$$

From $Z_1 = 0$, $Z_2 = 0$ we deduce two equations of the form

$$\frac{\partial A_8}{\partial u_1} = F_1(f_0, f_1), \qquad \frac{\partial A_8}{\partial u} = F_2(f_0, f_1),$$

which can be integrated explicitly. Substituting A_7 and A_8 in all the expressions, we get $\rho_3 \sim \alpha\,u_3^2 + O(2)$. Since $2\,A_3 - A_1 = 2\,f_1 \neq 0$, by Lemma 9 we have $\alpha = 0$ that yields the Riccati equation

$$\frac{\partial f_1}{\partial u_1} = \varphi_1(f_0)f_1^2 + \varphi_2(f_0)f_1 + \varphi_3(f_0),$$

where φ_2 and φ_3 depend both on f_0 and on the first and second order derivatives of f_0 w.r.t. u_1.

As above, we obtain $\rho_4 \sim u_3^2(Q_1 u_2 + Q_2) + O(2)$ and consider the equations $Q_1 = 0$, $Q_2 = 0$. The second of these equations determines A_5, and the first implies an ordinary differential equation with the derivatives of f_0 w.r.t. u_1 containing f_1. By the forth integrability condition

$$\frac{d}{dt}\rho_3 \sim u_5^2(P_1\, u_2 + P_2) + u_4^3\, P_3 + u_4^2 u_3 (P_4\, u_2 + P_5) + u_4^2(P_6\, u_2^2 + P_7\, u_2 + P_8) + u_3^4 P_9 + \\ + u_3^3(P_{10}\, u_2^3 + P_{11}\, u_2^2 + P_{12}\, u_2 + P_{13}) + u_3^2 O(2) + O(2) \sim 0$$

we obtain equations $P_i = 0$, $i = 1, \dots, 13$, among those there are many equations involving only f_1, f_0, and the derivatives of f_0 w.r.t. u_1. Expressing all the derivatives of f_0 from some of the equations and substituting them in other equations, we obtain the contradiction $f_0 f_1 = 0$.

It means that under the conditions **II.c** there exist no integrable equations.

Case II.d. We recall that in this case the functions A_1 and A_2 read as (3.41) that ensures the triviality of the first canonical conservation law. The function A_4 is given by formula (3.43), and since $c_1 \neq 0$, we have extra two equations (3.44) and (3.45). Moreover, $A_3 = A_1/2 + f_1$, $f_1 \neq 0$.

After the exclusion of A_3, equation (3.44) casts into the form

$$15\, f_0\, \frac{\partial^2 f_0}{\partial u_1^2} - 30 \left(\frac{\partial f_0}{\partial u_1}\right)^2 + 20 f_0\, f_1\, \frac{\partial f_0}{\partial u_1} - 28\, f_0^2 \frac{\partial f_1}{\partial u_1} - 12\, c_1 f_0^3 = 0, \tag{3.47}$$

and equation (3.45) allows us to express A_5 in terms of f_0 and f_1,

$$A_5 = \frac{15}{2\, f_0}\, u_1\, \frac{\partial^2 f_0}{\partial u \partial u_1} - \frac{5}{3\, f_0^2}\, \frac{\partial f_0}{\partial u} \left(3\, u_1 \frac{\partial f_0}{\partial u_1} - 3\, f_0 - 2\, u_1\, f_0\, f_1\right) - \frac{2}{3}\, u_1\, \frac{\partial f_1}{\partial u}. \tag{3.48}$$

In view of said above, second integrability condition (3.22) becomes

$$\frac{d}{dt}\rho_1 \sim u_3^2(Z_1\, u_2^3 + Z_2\, u_2^2 + Z_3\, u_2 + Z_4) + Z_5\, u_2^7 + P_6(u_2) \sim 0.$$

We express A_7 from the equation $Z_1 = 0$,

$$A_7 = \frac{55\, f_0^{-1}}{112}\, \frac{\partial^3 f_0}{\partial u_1^3} - \frac{f_0^{-2}}{1568} \left(185\, \frac{\partial f_0}{\partial u_1} + 84\, f_0\, f_1\right) \frac{\partial^2 f_0}{\partial u_1^2} - \\ - \frac{f_0^{-3}}{392} \left(205 \left(\frac{\partial f_0}{\partial u_1}\right)^3 - 230\, f_0\, f_1 \left(\frac{\partial f_0}{\partial u_1}\right)^2 - 44\, c_1\, f_0^3 \frac{\partial f_0}{\partial u_1}\right), \tag{3.49}$$

and A_8 from the equation $Z_2 = 0$,

$$A_8 = \frac{55}{28}\, f_0^{-1}\, u_1 \frac{\partial^3 f_0}{\partial u \partial u_1^2} - \frac{5\, f_0^{-2}}{84}\, u_1 \frac{\partial f_0}{\partial u}\, \frac{\partial^2 f_0}{\partial u_1^2} + \frac{f_0^{-1}}{126}(30\, c_1\, u_1\, f_0 - 7\, f_1) \frac{\partial f_0}{\partial u} \\ - \frac{f_0^{-2}}{168} \left(25\, u_1 \frac{\partial f_0}{\partial u_1} - 490\, f_0 + 36\, u_1\, f_0\, f_1\right) \frac{\partial^2 f_0}{\partial u \partial u_1} - \frac{55}{21}\, f_0^{-3} u_1 \left(\frac{\partial f_0}{\partial u_1}\right)^2 \frac{\partial f_0}{\partial u} \\ + \frac{5\, f_0^{-2}}{504}(272\, u_1\, f_1 - 63) \frac{\partial f_0}{\partial u_1} \frac{\partial f_0}{\partial u} - \frac{f_0^{-1}}{63}\, \frac{\partial f_1}{\partial u} \left(31\, u_1\, \frac{\partial f_0}{\partial u_1} - 7\, f_0\right). \tag{3.50}$$

By the equations $Z_3 = 0$ and $Z_4 = 0$ we can express the functions A_9 and A_{10}, respectively, in terms of the functions f_0, f_1, A_6 and their derivatives. We omit these expressions since they are cumbersome.

The equation $Z_5 = 0$ is an ordinary fifth order differential equation for f_0 w.r.t. the variable u_1. Other implications of the second integrability condition involve the derivatives of the functions f_0, f_1, A_6, and A_{11} w.r.t. two variables u_0 and u_1 and are too complicated for the analysis.

Then, by Lemma 9 it follows from expressions (3.35) for ρ_2 and ρ_3 that $\psi_4 = 0$ and $\psi_8 = 0$. These two equations read as

$$70\, f_0^2 \frac{\partial^3 f_0}{\partial u_1^3} - f_0 \left(405 \frac{\partial f_0}{\partial u_1} - 28\, f_0\, f_1\right) \frac{\partial^2 f_0}{\partial u_1^2} + 6 \frac{\partial f_0}{\partial u_1} \left(65 \left(\frac{\partial f_0}{\partial u_1}\right)^2 - 6\, f_0\, f_1 \frac{\partial f_0}{\partial u_1} - 2\, c_1\, f_0^3\right) = 0, \tag{3.51}$$

$$25\, f_0 \frac{\partial^2 f_0}{\partial u_1^2} - 10 \frac{\partial f_0}{\partial u_1} \left(5 \frac{\partial f_0}{\partial u_1} - f_0\, f_1\right) + f_0^2 (15\, c_1\, f_0 + 28\, f_1^2) = 0. \tag{3.52}$$

Expressing the second derivative of f_0 from (3.52) and substituting it into (3.51), in view of (3.47) we obtain $f_0 = -4/(5c_1) f_1^2$. After the exclusion of f_0, equations (3.47) and (3.52) are reduced to the equation

$$25\, f_1 \frac{\partial^2 f_1}{\partial u_1^2} - 75 \left(\frac{\partial f_1}{\partial u_1}\right)^2 + 10\, f_1^2 \frac{\partial f_1}{\partial u_1} + 8\, f_1^4 = 0, \tag{3.53}$$

and equation (3.51) together with the mentioned equation $Z_5 = 0$ are the implications of equation (3.53). Substituting $f_1 = 5/(4\, f)$ into (3.53), we obtain the equation

$$2 \frac{\partial}{\partial u_1} \left(f \frac{\partial f}{\partial u_1}\right) + \frac{\partial f}{\partial u_1} = 1, \tag{3.54}$$

whose general integral is written as

$$(f + u_1 + a)^2 (2f - u_1 - a) + b = 0, \tag{3.55}$$

where a and b are arbitrary functions of the variable u.

In view of all obtained results including differential consequences of equation (3.55), it is easy to check that

$$\frac{d}{dt} \rho_1 \sim u_2^6\, f^{-10} (3a + 3u_1 - 5\, f)(3a'b - ab') + P_5(u_2),$$

where the prime denotes the derivative w.r.t. u. Thus, $3\, a'\, b = ab'$ that implies $a = c\, b^{1/3}$, $c = \text{const}$ if $b \neq 0$.

This result allows one to convert both these functions into constants by a point transformation $u = \varphi(v)$. Indeed, since $2\, f_1 = 2\, A_3 - A_1$, in accordance with formulas (3.33), $\tilde{f}_1(v) = \varphi' f_1(u)$. Therefore, the function $f \sim f_1^{-1}$ is transformed by the law $\tilde{f}(v) = \varphi'^{-1} f(u)$. Making the transformation in equation (3.55), we obtain

$$[\tilde{f} + v_1 + a(u) \varphi'^{-1}]^2 [2 \tilde{f} - v_1 - a(u) \varphi'^{-1}] + b(u) \varphi'^{-3} = 0. \tag{3.56}$$

If $a = b = 0$, then no transformation is needed and we have

$$(f + u_1)^2 (2f - u_1) = 0.$$

If $b = 0$ and $a \neq 0$, then letting $\varphi' = a$, we reduce equation (3.55) to

$$(f + u_1 + 1)^2 (2f - u_1 - 1) = 0.$$

If $b(u) \neq 0$, then $a(u) = k\, b^{1/3}(u)$, where k is a constant. Choosing $\varphi' = b^{1/3}$, we obtain equation (3.55) in the form

$$(f + u_1 + a)^2 (2f - u_1 - a) + 1 = 0, \tag{3.57}$$

where a is a constant.

Thus, up to a point transformation, the quantities a and b in (3.55) are constants, and the following three cases are possible,

II.d.1. $f = -u_1 - a$; **II.d.2**. $f = \frac{1}{2}(u_1 + a)$; **II.d.3**. $f(u_1)$ satisfies (3.57).

In each of these cases the parameter a takes one of the values, $a = 0$ or $a = 1$.

Employing equation (3.54), we can exclude higher derivatives of f from the expressions for the functions A_i found above. It leads us to rather compact expressions,

$$A_1 = -\frac{5}{f}f', \quad A_2 = 0, \quad A_3 = \frac{5}{4f}(1-2f'), \quad A_4 = \frac{5}{4f^2}(16f'^2-3),$$
$$A_5 = 0, \quad A_7 = -\frac{5}{16}f^{-3}(2f'-1)(28f'^2+20f'+1), \quad A_8 = 0,$$
$$A_9 = \frac{1}{2}f^2\frac{\partial^3 A_6}{\partial u_1^3} + \frac{1}{4}f(6f'+1)\frac{\partial^2 A_6}{\partial u_1^2} - \frac{1}{4}\frac{\partial A_6}{\partial u_1} - \frac{3f'}{2f}A_6,$$
$$A_{10} = u_1 f^2\frac{\partial^3 A_6}{\partial u\partial u_1^2} + \frac{1}{2}u_1 f(2f'+1)\frac{\partial A_6}{\partial u\partial u_1} - \frac{1}{2}(f+u_1+2ff')\frac{\partial A_6}{\partial u},$$

and these formulas hold true for each of three cases II.d.1, II.d.2 and II.d.3.

Case II.d.1. If $a = 1$, then it follows from the integrability conditions that $A_i = A_i(u_1), \forall i$. Thus, the transformation $u \to u - x$ is admissible, and we arrive at the case $a = 0$. In the case $a = 0$ numerous forks lead to the only integrable equation (3.11).

Case II.d.2. If $a = 1$, then exactly as in the previous case we arrive at the case $a = 0$, and in the case $a = 0$ we get equation (3.10).

Case II.d.3. Let us consider this case in more details. In view of the above results, the second and forth integrability conditions read as

$$\frac{d}{dt}\rho_1 \sim u_2^5 Q_1 + u_2^4 Q_2 + u2^3 Q_3 + u_2^2 Q_4 + O(1) \sim 0, \tag{3.58}$$
$$\frac{d}{dt}\rho_3 \sim u_4^2(P_1+u_2P_2) + u_3^3(P_3+u_2P_4) + \tag{3.59}$$
$$+ u_3^2(P_5 + u_2P_6 + u_2^2P_7 + u2^3P_8) + u_2^7P_9 + P_6(u_2) \sim 0.$$

Here the functions Q_i and P_j depend on u_0 and u_1 only. For the equivalence of these expressions to zero one needs the identities $Q_i = 0$, $P_j = 0$ for all i, j. The conditions $Q_1 = 0$, $P_2 = 0$, $P_4 = 0$, $P_8 = 0$, and $P_9 = 0$ are homogeneous ordinary differential equations for the functions $A_6(u_1)$, and u in involved as a parameter. The first two equations are of the fifth order, the orders of the others are 6, 7, and 9, respectively. By excluding higher derivatives from the first two equations, we arrive at the equation

$$2f^2f'\frac{\partial^2 A_6}{\partial u_1^2} + f(f'+1)(2f'-1)\frac{\partial A_6}{\partial u_1} + (1-3f')A_6 = 0.$$

All remaining equations are its differential consequences. The general solution of the above equation is given by

$$A_6 = \gamma(u)(f+u_1+a)^2 + 10\,\omega(u)(u_1+a)f, \tag{3.60}$$

where γ and ω are arbitrary functions.

Substituting solution (3.60) into the equation $Q_2 = 0$, we obtain

$$70\,a\omega'(u_1+a)f^3 + a\,\gamma'\Big[7f^4 + 14(u_1+a)f^3 + 7(u_1+a)^2f^2 + f - u_1 - a\Big] = 0. \tag{3.61}$$

Calculating the resultant of the polynomials (3.61) and (3.57) w.r.t. the variable u_1 yields

$$a\Big[34300\,\omega'^2(20\,\omega'+9\gamma')f^{12} + 980\,\omega'(165\,\omega'\gamma' + 350\,\omega'^2 + 3\gamma'^2)f^9$$
$$- 7\gamma'(930\,\omega'\gamma' + 2100\,\omega'^2 - \gamma'^2)f^6 + \gamma'^2(210\,\omega' + 59\gamma')f^3 - \gamma'^3\Big] = 0.$$

Since ω and γ are functions of u, and f is a non-constant function of u_1, all the coefficients of this polynomial should vanish. It implies

$$a\gamma' = 0, \qquad a\omega' = 0.$$

Next we consider the equations involving A_{11}. These are $Q_3 = 0$, $Q_4 = 0$, $P_5 = 0$, and $P_6 = 0$. First two of them have the second order, from the first we can express $\partial^2 A_{11}/\partial u_1^2$, and $\partial^2 A_{11}/\partial u \partial u_1$ from the second. By $Q_3 = 0$ we can exclude A_{11} from $P_6 = 0$, and it implies the equation for the function ω,

$$\omega'' = 6\omega^2.$$

Together with $a\omega' = 0$ it implies that $a\omega = 0$.

Excluding higher derivatives of A_{11} from $P_5 = 0$, we obtain the equation

$$\frac{\partial A_{11}}{\partial u} = P_1(f, u_1, \gamma', \omega') P_2^{-1}(f, u_1, \gamma', \omega'),$$

where P_1 and P_2 are polynomials in the variables f and u_1, such that $P_1(f, u_1, 0, 0) = 0$ and $P_2(f, u_1, 0, 0) \neq 0$. It means that if $a \neq 0$ and $\gamma' = \omega' = 0$, then A_6 and A_{11} are independent of u. It leads to the fact that all A_i depend only on u_1. Thus, the transformation $u \to u - ax$ is admitted and it eliminates a in equation (3.57). Therefore, it is sufficient to consider only the case $a = 0$.

As $a = 0$, the equations for A_{11} become not very bulky,

$$\begin{aligned}
\frac{\partial^2 A_{11}}{\partial u_1^2} =& \frac{1}{2\,f}\,(\gamma'' + 4\,\gamma\omega)(8\,f^4 + 16\,u_1\,f^3 + 8\,{u_1}^2 f^2 + 3\,f + u_1) \\
&+ \frac{2\,\gamma^2}{5\,f}(9\,f^4 + 18\,u_1\,f^3 + 9\,{u_1}^2 f^2 + 4\,f + u_1) \\
&- \frac{10}{f}\omega^2(14\,f^4 + 23\,u_1\,f^3 - 31\,{u_1}^2 f^2 + 9\,f - 3\,u_1),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial A_{11}}{\partial u} =& 40\,f^2 \frac{\omega'\,\omega\,(2\,f^6 + f^5 u_1 - f^4 {u_1}^2 - 3\,f^3 - 7\,u_1\,f^2 - 2)}{2\,f^3 + u_1\,f^2 - {u_1}^2 f + 1} \\
&- 2\,f^2 \frac{(2\,f^3 + 3\,u_1\,f^2 + {u_1}^2 f + 1)(2\,\gamma\omega' + 3\,\gamma'\,\omega)}{2\,f^3 + u_1\,f^2 - {u_1}^2 f + 1}.
\end{aligned}$$

Integrating the first equation,[1] we obtain

$$\begin{aligned}
A_{11} =& \frac{1}{10}\,(8\,f^5 + 2\,f^2 - {u_1}^2 + 16\,u_1\,f^4 + 8\,{u_1}^2 f^3)(4\,\gamma\omega + \gamma'') \\
&+ \frac{1}{25}\,\gamma^2(18\,{u_1}^2 f^3 + 36\,u_1\,f^4 + 7\,f^2 + 18\,f^5 - {u_1}^2) \\
&- 4\,\omega^2(2\,f^5 + 3\,f^2 - u_1\,f^4 - 8\,{u_1}^2 f^3 + {u_1}^2) + \alpha\,u_1 + \beta,
\end{aligned}$$

where α and β are arbitrary functions of u. It follows from the second equation that α and β are constants. Using the Galilean transformation, we can assume that $\alpha = 0$.

Substituting the expression for $\partial A_{11}/\partial u$ into the equation $Q_4 = 0$, we arrive at extra two equations for γ and ω. Finally the system of equations for these functions reads as

$$\omega'' = 6\omega^2, \tag{3.62}$$

$$\gamma''' = 8\gamma'\omega + 4\gamma\omega', \tag{3.63}$$

$$(\gamma + 15\omega)\gamma' + 10(\gamma + 10\omega)\omega' = 0, \tag{3.64}$$

[1]The method of integration is described in Appendix 2.

Finding all these functions A_i explicitly and employing equations (3.62) – (3.64), it is easy to check completely the integrability conditions 1–4. These conditions lead us to the only restriction $\beta\omega' = 0$, where β is the constant involved in A_{11}.

If $\omega' \neq 0$, then $\beta = 0$. If $\omega' = 0$, it follows from (3.62) that $\omega = 0$, and from (3.64) that $\gamma = const$. In this case the coefficients of equation (3.42) are independent of u, and the transformation $u \to u + \beta t$ eliminating the constant β in A_{11} is admissible. Thus, $\beta = 0$ for all ω and γ.

If $\omega = 0$, letting $\gamma = 5\mu$, we obtain equation (3.13).

If $\omega \neq 0$, it follows from (3.62) that $\omega' \neq 0$. In this case the order of equation (3.62) lowers and we obtain the equation $\omega'^2 = 4\,\omega^3 + c$ coinciding with (3.18). Since $\omega' \neq 0$, then (3.64) implies $\gamma + 15\,\omega \neq 0$, and hence one can express γ' from (3.64). It allows us to exclude the derivatives of the functions γ and ω from (3.63). As a result we obtain the equation

$$(\gamma + 30\,\omega)(\gamma + 5\,\omega)(\gamma + 20\,\omega)\big[(\gamma + 20\,\omega)(\gamma + 5\,\omega)^2 + 125\,c\big] = 0, \tag{3.65}$$

where c is a constant in (3.18).

If $\gamma = -30\,\omega$, then it follows from (3.64) that $\omega = 0$, which contradicts the assumption. If $\gamma = -5\,\omega$, then we get the equation (3.14), and if $\gamma = -20\,\omega$, then we obtain equation (3.15).

Consider the case

$$(\gamma + 20\,\omega)(\gamma + 5\,\omega)^2 + 125\,c = 0. \tag{3.66}$$

Cubic curve (3.66) is rational and is parameterized by the substitution

$$\omega = \tilde\omega + \tilde c\,\tilde\omega^{-2}, \quad \gamma = -5\,\tilde c\,\tilde\omega^{-2} - 20\,\tilde\omega,$$

where $c = -27\,\tilde c$. Substituting these expressions into (3.62) – (3.64), we find that $\tilde\omega$ satisfies equation (3.18) with the constant $\tilde c$ instead of c. Substituting the found functions A_i into equation (3.32), employing the expressions for ω and γ, and redenoting $\tilde\omega \to \omega$, $\tilde c \to c$, we obtain equation (3.16).

3.4. Differential substitutions relating equations in the list.

As it was noted in Section 2.4, while calculating differential substitutions, it is useful to know the orders of the canonical conservation laws. In Table 2 we provide the orders of several canonical conservation laws for equations in list (3.2) – (3.16).

Even densities are not indicated in Table 2, since they happened to be trivial $\rho_{2n} \sim 0$. For equation (3.12) the orders of ρ_1 and ρ_9 are provided for the case of generic constants; if $\mu = 0$, then $\rho_1 \sim 0$, $\rho_9 \sim 0$.

The differential substitutions admitted by fifth order S-integrable equations are shown on Figure 2.

Below we provide the substitutions for the equations with generic constants.

(3.13)→(3.6): $\tilde u = \dfrac{u_2}{2f} + \sqrt{-\mu}\,(f + u_1)$.

(3.15)→(3.9): $\tilde u = \ln(f + u_1) - \ln\varphi$. At that, $\omega = \dfrac{\lambda_1^2}{4\,\varphi^2} + \dfrac{1}{2}\,\lambda_2\,\varphi$, and the constant c in the equation (3.18) satisfied by ω equals $c = -\dfrac{27}{16}\,\lambda_1^2\lambda_2^2$.

Table 2. The orders of canonical conservation laws. For zero order conservation laws we indicate in the brackets to what the density is equivalent

ρ_i	(3.2)	(3.3)	(3.4)	(3.5)	(3.6)	(3.7)	(3.8)	(3.9)
ρ_1	0, $(\sim u)$	0, $(\sim u)$	0, (~ 0)	0, (~ 0)	0, $(\sim u^2)$	1	1	1
ρ_3	~ 0	~ 0	1	~ 0	~ 0	~ 0	~ 0	~ 0
ρ_5	1	1	2	2	2	3	3	3
ρ_7	2	2	3	3	3	4	4	4
ρ_9	~ 0	~ 0	4	~ 0	~ 0	~ 0	~ 0	~ 0
ρ_{11}	4	4	5	5	5	6	6	6
ρ_i	(3.10)	(3.11)	(3.12)	(3.13)	(3.14)	(3.15)	(3.16)	
ρ_1	2	2	1	2	2	2	2	
ρ_3	~ 0	~ 0	~ 0	~ 0	~ 0	~ 0	~ 0	
ρ_5	4	4	3	4	4	4	4	
ρ_7	5	5	4	5	5	5	5	
ρ_9	~ 0	~ 0	3	~ 0	~ 0	~ 0	~ 0	
ρ_{11}	7	7	6	7	7	7	7	

(3.10)→(3.9): $\tilde u = \ln u_1, \quad \mu_1 = \lambda_2, \quad \mu_2 = -\lambda_1^2.$
(3.16)→(3.9): $\tilde u = \ln(f+u_1) - \ln(2\,\omega\lambda_2^{-1}), \quad c = \dfrac{1}{4}\,\lambda_1^2\lambda_2^2.$
(3.16)→(3.8): $\tilde u = \ln(f+u_1) + \dfrac{1}{2}\ln(-4\,\omega\lambda_1^{-1}), \quad c = \dfrac{1}{4}\,\lambda_1\lambda_2^2.$
(3.16)→(3.6): $\tilde u = \dfrac{u_2}{2f} - \dfrac{f\omega'}{\omega} + \dfrac{1}{2\omega}(3\sqrt{c}-\omega')u_1.$
(3.14)→(3.8): $\tilde u = \ln(f+u_1) + \dfrac{1}{2}\ln\varphi$. At that, $\omega = -\lambda_1\,\varphi + 4\,\lambda_2^2\,\varphi^{-2},\ c = -108\,\lambda_1^2\lambda_2^2.$
(3.11)→(3.8): $\tilde u = -\dfrac{1}{2}\ln u_1, \quad \mu_1 = \lambda_1, \quad \mu_2 = -\lambda_2^2.$
(3.7)→(3.6): $\tilde u = u_1.$
(3.9)→(3.2): $\tilde u = -u_2 - u_1^2 \pm 3\,\lambda_1 e^u u_1 - \lambda_1^2 e^{2\,u} + \lambda_2 e^{-u}.$
(3.12)→(3.6): $\tilde u = \sqrt{u_1} - \mu$. At that, equation (3.12) should involve an additional term $5\,\mu^4 u_1$.
(3.8)→(3.3): $\tilde u = 2\,u_2 - u_1^2 \pm 6\,\lambda_2 e^{-2\,u}u_1 + \lambda_1 e^{2\,u} - \lambda_2^2 e^{-4\,u}.$
(3.6)→(3.2): $\tilde u = -u_1 - u^2.$
(3.6)→(3.3): $\tilde u = 2\,u_1 - u^2.$
(3.4)→(3.2): $\tilde u = u_1.$
(3.5)→(3.3): $\tilde u = u_1.$

Moreover, there also exist the substitutions for special values of the parameters involved in the equations.

Example 2. (3.8)→(3.6): $\tilde u = u_1 + \sqrt{-\lambda_1}\,e^u \pm \lambda_2\,e^{2\,u},\ \lambda_1\lambda_2 = 0$. In each of the cases $\lambda_2 = 0$ or $\lambda_1 = 0$, the logarithmic substitution $u \to -\ln u$ or $u \to -\dfrac{1}{2}\ln u$ leads us to a first order linear equation for u. That is, the function u can be expressed in terms of $\widetilde u$ by one quadrature.

If here we let $\lambda_2 = 0,\ \lambda_1 \to -\lambda_1^2$, then we obtain substitution **(3.9)→(3.6)** with $\lambda_2 = 0$.

Example 3. (3.13)→(3.8): $\tilde u = \ln(f+u_1),\ \lambda_2 = 0,\ \lambda_1 = \mu$. Here one can also express u in terms of $\tilde u$ by one quadrature. Indeed, as one can check easily, third degree curve (3.17) has the parametric representation

$$u_1 = \frac{1}{3}\left(2\,e^v + e^{-2\,v}\right), \qquad f = \frac{1}{3}\left(e^v - e^{-2\,v}\right),$$

at that, $v = \tilde{u}$. Thus, we have $u = \dfrac{1}{3}\displaystyle\int \left(2\,e^{\tilde{u}} + e^{-2\,\tilde{u}}\right) dx$.

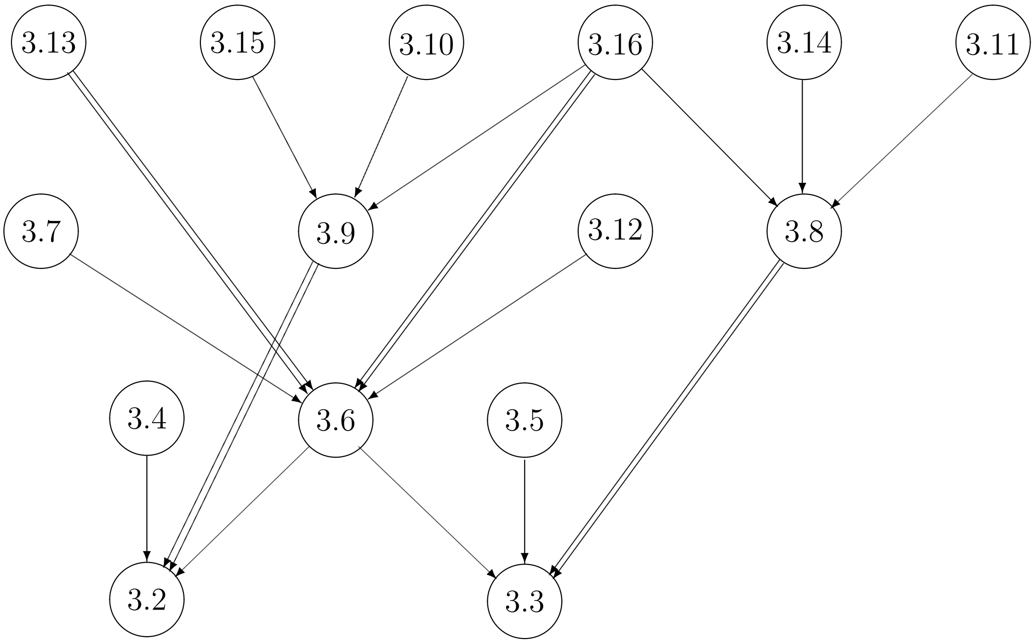

Figure 2. The graph of the substitutions for fifth order S-integrable equations

Appendix 1. Discrete symmetries of Weierstrass function ω

Equations (3.13) – (3.16) can be written in various ways. We note first that in the paper [9] these equations are written in terms of the function $R = f + u_1$ satisfying the equation $2R^3 - 3u_1R^2 + 1 = 0$.

Moreover, there exist transformations preserving the form of equation (3.18). Indeed, consider the functions ω and $\tilde{\omega}$ satisfying the following equations of the form (3.18),

$$\omega'^2 = 4\,\omega^3 + c, \qquad \text{(A1.1)}$$

$$\tilde{\omega}'^2 = 4\,\tilde{\omega}^3 + k, \qquad \text{(A1.2)}$$

where $c\,k \neq 0$. It is easy to check that the simplest transformations

$$\omega = a\frac{a - \tilde{\omega}}{a + 2\,\tilde{\omega}}, \quad k = c = \frac{1}{2}\,a^3; \qquad (\boldsymbol{T}_1)$$

$$\omega = \tilde{\omega} + \frac{k}{\tilde{\omega}^2}, \quad c = -27\,k; \qquad (\boldsymbol{T}_2)$$

$$\omega = \frac{c + \sqrt{c}\,\tilde{\omega}'}{2\,\tilde{\omega}^2}, \quad k = c \qquad (\boldsymbol{T}_3)$$

map a solution of (A1.2) to that of (A1.1).

The transformation $\boldsymbol{T}_1$ is invertible, and $\tilde{\omega}$ is expressed in terms of ω by the same formula. It is also possible to invert the transformation $\boldsymbol{T}_2$, but the problem is reduced to solving a cubic equation. To invert the transformation $\boldsymbol{T}_3$ one has to solve the Riccatti equation. We note that formula ($\boldsymbol{T}_2$) helps to find the parametrization of cubic equation (3.66).

The superpositions of the elementary transformations $\boldsymbol{T}_i$ lead to new transformations preserving the form of equation (3.18). For instance,

$$\boldsymbol{T}_1 * \boldsymbol{T}_2 : \quad \omega = \frac{3}{2}\,a + \frac{27\,a^2\tilde{\omega}^2}{2(2\tilde{\omega} + a)(\tilde{\omega} - a)^2}, \quad k = \frac{a^3}{2}, \ c = -\frac{27}{2}\,a^3;$$

$$\boldsymbol{T}_2 * \boldsymbol{T}_2 : \quad \omega = \tilde{\omega} + \frac{k}{\tilde{\omega}^2} - \frac{27\,k\,\tilde{\omega}^4}{(\tilde{\omega}^2 + k)^2}, \quad c = 729\,k;$$

$$\boldsymbol{T}_3 * \boldsymbol{T}_1 : \quad \omega = \frac{a(2\tilde{\omega} + a)^2 - 3\sqrt{2a^3}\,\tilde{\omega}'}{4(\tilde{\omega} - a)^2}, \quad c = \frac{a^3}{2};$$

$$\boldsymbol{T}_3 * \boldsymbol{T}_2 : \quad \omega = \frac{c\,\tilde{\omega}^4 + \sqrt{c}\,\tilde{\omega}(\tilde{\omega}^3 - 2\,k)}{2(\tilde{\omega}^3 + k)^2}, \quad c = -27\,k.$$

Moreover, $\boldsymbol{T}_1 * \boldsymbol{T}_1$ is the identity transformation, and $\boldsymbol{T}_3 * \boldsymbol{T}_3$ differs from $\boldsymbol{T}_3$ just by the sign of the root $\sqrt{c}$. Thus, the, equations (3.13) – (3.16) can be written in an infinite number of ways different from the first impression.

Appendix 2. Explicit integration of functions depending on u_1 and f

To check the integrability conditions of equations (3.13) – (3.16) we need the table of the integrals of rational expression $R(u_1, f)$, where the function f is defined by equation (3.17). These integrals can be found by the rational parametrization

$$f = \frac{w^3 - 1}{3\,w^2}, \qquad u_1 = \frac{2\,w^3 + 1}{3\,w^2} \tag{A2.1}$$

of curve (3.17). The parametrization allows one to convert the integral of an irrational function $R(u_1, f)$ of the variable u_1 into that of a rational function of the variable w,

$$\int R(u_1, f)\,du_1 = \int R\left(\frac{2\,w^3 + 1}{3\,w^2}, \frac{w^3 - 1}{3\,w^2}\right)\left(\frac{2\,w^3 + 1}{3\,w^2}\right)' dw.$$

The answer can be written in terms of original variables u_1 and f by the formula $w = f + u_1$ implied by (A2.1).

To check the integrability conditions we have made use of the integrals

$$\int u_1^n\, f^m\, du_1, \quad n = 0, 1, 2; \quad -5 \leqslant m \leqslant 11.$$

For instance,

$$\int f\,du_1 = \frac{1}{2}u_1^2 - f^2, \qquad \int u_1^2 f\,du_1 = \frac{1}{20}\big(8f^4 + 14f^3 u_1 + f^2 u_1^2 - u_1\big),$$

$$\int u_1 f\,du_1 = \frac{1}{9}\big(2f^3 + f^2 u_1 + 2f u_1^2 - 2\ln(f + u_1)\big),$$

$$\int \frac{du_1}{f} = 2\ln(f + u_1), \qquad \int \frac{u_1\,du_1}{f} = 2f + u_1, \qquad \int \frac{u_1\,du_1}{f^2} = 2\ln(f + u_1) + 2\ln f.$$

To calculate iterated integrals, one should add to the table also the integrals of logarithms, for instance,

$$\int \ln(f + u_1)\,du_1 = u_1 \ln(f + u_1) - \frac{u_1}{2} - f.$$

In the proof of Theorem 2 we have used around two tens of such formulas.

To check any of given formulas, it is sufficient to differentiate it, exclude $f' = \dfrac{u_1 - f}{2\,f}$, and lower the degree of u_1, if needed, by the identities

$$u_1^3 = 1 + 3u_1 f^2 + 2f^3, \quad u_1^4 = u_1(1 + 3u_1 f^2 + 2f^3), \ldots$$

implied by (3.17).

Appendix 3. On recurrent formulas for canonical densities

Here we discuss the way of obtaining recurrent formulas like (2.2) and (3.26). The original idea of this method is contained in the work [55], where a simple method for deducing recurrent formulas for the conservation laws of Lax equations was suggested. In the work [10] this approach was applied for the linearization of evolution equations and systems.

For the sake of completeness of the content, we first describe briefly the essence of Zakharov-Shabat method.

Suppose equation (0.1) has a Lax representation,

$$\frac{dL}{dt} = [A,\, L] \Longleftrightarrow u_t = u_n + F(x, u, u_1, \dots, u_{n-1}),$$

where by square brackets we denote the commutator of linear operators. For simplicity we assume that $A = A(\partial_x, \mu, u)$ and $L = L(\partial_x, \mu, u)$ are scalar differential operators independent of ∂_t, μ is a spectral parameter, u is a solution to equation (0.1).

The Lax representation ensures the compatibility of the linear system

$$L\psi = 0, \qquad \psi_t = A\psi. \tag{A3.1}$$

We introduce notations for the logarithmic derivatives of the function ψ,

$$(\ln\psi)_x = R, \qquad (\ln\psi)_t = T.$$

It is obvious that the functions R and T are related by the identity

$$R_t = T_x, \tag{A3.2}$$

and $R\,dx + T\,dt = d\ln\psi$. This is why up to a multiplicative constant we have

$$\psi = \exp\left(\int R\,dx + T\,dt\right), \tag{A3.3}$$

where the integral in the exponent is a curvilinear integral with variable upper limit (x, t).

Since $\psi_x = \psi R$, $\psi_t = \psi T$, the operator formulas

$$\psi^{-1}\left(\frac{d}{dx}\right)^n \psi = \left(\frac{d}{dx} + R\right)^n, \qquad \psi^{-1}\left(\frac{d}{dt}\right)^n \psi = \left(\frac{d}{dt} + T\right)^n, \quad n = 0, 1, 2, \dots$$

hold true. By these formulas we have

$$\psi^{-1}L(\partial_x, \mu, u)\psi = L(\partial_x + R, \mu, u), \quad \psi^{-1}A(\partial_x, \mu, u)\psi = A(\partial_x + R, \mu, u).$$

Hence, equation (A3.1) can be rewritten in terms of the functions R and T,

$$L(\partial_x + R, \mu, u)(1) = 0, \tag{A3.4}$$

$$T = A(\partial_x + R, \mu, u)(1). \tag{A3.5}$$

These two equations are nonlinear in R. Their solutions are often sought as Laurent series in the parameter μ. Due to (A3.2), the coefficients of these series are the densities of the conservation laws.

Example 4. For Korteweg-de Vries equation $u_t = u_{xxx} - 6uu_x$ the associated linear system can be written as

$$\psi_{xx} - u\psi - \mu^2\psi = 0, \tag{A3.6}$$

$$\psi_t = 4\psi_{xxx} - 6u\psi_x - 3u_x\psi. \tag{A3.7}$$

Formulas (A3.4),(A3.5) lead us to the equations for R and T,

$$R_x + R^2 - u - \mu^2 = 0, \tag{A3.8}$$

$$T = 4(\partial_x + R)^2(R) - 6u\,R - 3u_x. \tag{A3.9}$$

Equation (A3.9) can be simplified by (A3.8) that yields

$$T = (4\mu^2 - 2u)R + u_x. \tag{A3.10}$$

If we substitute the series

$$R = \mu + \sum_{n=0}^{\infty} \rho_n \mu^{-n} \tag{A3.11}$$

into equation (A3.8) and equate the coefficients at the equal powers of μ to zero, we obtain the recurrent formula

$$\rho_{n+1} = \frac{1}{2}\left(u\delta_{n0} - \sum_{i=1}^{n-1} \rho_i \rho_{n-i} - \frac{d}{dx}\rho_n\right), \quad n = 0, 1, 2, \ldots, \tag{A3.12}$$

where δ_{n0} is the Kronecker delta. We note that the scale transformation $\rho_n \to \rho_n(-2)^{-n}$ reduces the formula to the form provided in the monograph [19]. Let us write down first elements of the sequence ρ_n,

$$\rho_0 = 0, \quad \rho_1 = \frac{1}{2}u, \quad \rho_2 = -\frac{1}{4}u_1, \quad \rho_3 = \frac{1}{8}(u_2 - u^2).$$

Next, we substitute series (A3.11) into equation (A3.10) to obtain the expansion

$$T = 4\mu^3 + \sum_{n=1}^{\infty} \theta_n \mu^{-n}, \tag{A3.13}$$

where

$$\theta_n = 4\rho_{n+2} - 2u\rho_n, \quad n > 0. \tag{A3.14}$$

Since the parameter μ is arbitrary, formula (A3.2) defines an infinite sequence of the conservation laws

$$\frac{d}{dt}\rho_n = \frac{d}{dx}\theta_n, \quad n = 1, 2, \ldots, \tag{A3.15}$$

To obtain the canonical densities ρ_n, it is sufficient to have (A3.2) and one of equations (A3.8) or (A3.9).

If we use equation (A3.8), we arrive again at recurrent formula (A3.12), but we lose formula (A3.14). The fluxes θ_n associated with the densities ρ_n can be found from (A3.15) by inverting the total derivative operator $\frac{d}{dx}$ (the algorithm was discussed in Remark 4 on page 113).

For further reasoning it is more important to understand how to get the canonical densities from equations (A3.9) and (A3.2). Since equation (A3.9) does not involve the parameter, we introduce the parameter apriori and we can choose the structure of the expansion for R as we wish. If, for instance, we assume that R is the Taylor series

$$R = \sum_{n=0}^{\infty} \rho_n \mu^n,$$

where μ is the parameter, then

$$T = \sum_{n=0}^{\infty} \theta_n \mu^n,$$

where the coefficients θ_n are determined by equation (A3.9). It is easy to check that

$$\theta_n = 4\sum_0^n \rho_i\rho_j\rho_k - 3\,u_1\delta_{n0} - 6\,u\rho_n + 4\frac{d^2}{dx^2}\rho_n + 6\frac{d}{dx}\sum_0^n \rho_i\rho_j,$$

where we have used the notations for the sums introduced on the page 127. Since in the left and right hand sides of this formula the unknown functions θ_n and ρ_n appear simultaneously, it does not help for calculating the conservation laws.

The situation changes if we postulate the expansion of the function R as the Laurent series

$$R = \mu^{-1} + \sum_{n=0}^{\infty} \rho_n \mu^n. \tag{A3.16}$$

In this case by equation (A3.9) we obtain the expansion for T

$$T = 4\mu^{-3} + \theta_{-2}\mu^{-2} + \theta_{-1}\mu^{-1} + \sum_{n=0}^{\infty} \theta_n \mu^n, \tag{A3.17}$$

and the recurrent formula

$$\rho_{n+2} = \frac{1}{2}\, u\rho_n + \frac{1}{4}u_1\delta_{n,0} - \sum_{0}^{n+1} \rho_i\rho_j + \frac{1}{12}\theta_n - \frac{1}{3}\sum_{0}^{n} \rho_i\rho_j\rho_k + \frac{1}{12}\theta_{-2}\delta_{n,-2} -$$
$$- \frac{d}{dx}\left(\rho_{n+1} + \frac{1}{2}\sum_{0}^{n} \rho_i\rho_j + \frac{1}{3}\frac{d}{dx}\rho_n\right) + \frac{1}{12}\left(6\,u + \theta_{-1}\right)\delta_{n,-1}, \tag{A3.18}$$

where $n = -2, -1, 0, \dots$ Let us consider the corresponding series of conservation laws (A3.15), where $n = -2, -1, 0, 1, 2, \dots$. If the conservation laws with the indices $i \leqslant n+1$ are known, we find ρ_{n+2} by (A3.18), and then θ_{n+2} by (A3.15), and so forth. While finding θ_{n+2}, we have to invert the operator $\dfrac{d}{dx}$. Under the assumption that the densities and fluxes of conservation laws (A3.15) are explicitly independent of t, this procedure is absolutely algorithmic (see page 113). At that, the function θ_{n+2} is determined uniquely up to an integration constant.

The beginning of this recurrence is as follows. According to (A3.16), we have $\rho_{-2} = \rho_{-1} = 0$, and this is why by (A3.15) we obtain that the corresponding fluxes are constant, $\theta_{-1} = 12c_{-1}$, $\theta_{-2} = 12c_{-2}$. Then we find $\rho_0 = c_{-2}$. Next two densities read as

$$\rho_1 = \frac{1}{2}u + c_{-1}, \quad \rho_2 = \frac{1}{12}\theta_0 - \frac{c_{-2}}{2}u - \frac{1}{4}u_1 - \frac{c_{-2}^3}{3} - 2c_{-1}c_{-2}.$$

To determine θ_0, we again have to employ equation (A3.15) as $n = 0$ that implies $\theta_0 = c_0$.

It is important to note that the constants c_i appearing in finding the fluxes θ_i are not essential since they can be eliminated by the change of the parameter μ,

$$\mu \to \mu + \sum_{i=2}^{\infty} k_i \mu^i. \tag{A3.20}$$

Consider now an arbitrary evolution equation with one spatial variable

$$u_t = K(x, u, u_x, \dots, u_n), \qquad n > 1. \tag{A3.20}$$

In case (0.1) we have $K = u_n + F(x, u, u_x, \dots, u_{n-1})$. Denote by K_* the Fréchet derivative of the function K,

$$K_* = \sum_{i=0}^{n} \frac{\partial K}{\partial u_i}\frac{d^i}{dx^i}.$$

The formal series

$$L = \sum_{k=-\infty}^{1} f_k \frac{d^k}{dx^k},$$

whose coefficients depend on $x, u, u_x, \dots$ that satisfies the equation

$$L_t = \left[K_*,\, L\right], \tag{A3.21}$$

is called a formal symmetry (formal recurrence operator) of equation (A3.20). It is known that the equation possessing generalized symmetries or conservation laws has a formal symmetry [4, 7, 9].

Equation (A3.21) ensures the compatibility of the following pair of linear equations,

$$L\psi = \lambda\psi, \qquad \psi_t = K_*\psi, \tag{A3.23}$$

where λ is a spectral parameter. To this system one can apply the procedure of obtaining canonical densities described above. Since the operator L is not known apriori, we employ equation (A3.5),

$$T = \sum_{i=0}^{n} \frac{\partial K}{\partial u_i}\left(\frac{d}{dx} + R\right)^i (1). \tag{A3.25}$$

Let

$$R = \rho_{-1}\mu^{-1} + \sum_{k=0}^{\infty} \rho_k \mu^k, \tag{A3.27}$$

then

$$T = \mu^{-n} + \sum_{i=1}^{n-1} \theta_{-i}\mu^{-i} + \sum_{k=0}^{\infty} \theta_k \mu^k. \tag{A3.28}$$

Indeed, the minimal degree of μ in the right hand side of identity (A3.25) is contained in the term

$$\frac{\partial K}{\partial u_n}\left(\frac{d}{dx} + R\right)^n (1) = \frac{\partial K}{\partial u_n} R^n + \dots = \frac{\partial K}{\partial u_n}(\rho_{-1})^n \mu^{-n} + \dots,$$

and hence the series for T should begin with the term $\theta_{-n}\mu^{-n}$. Since $n > 1$, then $\rho_{-n} = 0$ and θ_{-n}=const$\neq 0$. By scaling the parameter μ we convert θ_{-n} into one and obtain (A3.28).

Substituting expansions (A3.27), (A3.28) into (A3.25) and equating the terms at μ^{-n} in the equation (A3.25), we obtain the first density

$$\rho_{-1} = \left(\frac{\partial K}{\partial u_n}\right)^{-1/n}.$$

The formulas for several next canonical densities can be found in [9].

Let us consider now equation (0.2) and adduce the deduction of recurrent formula (2.2) for canonical densities following of the above scheme.

1st step. We write linearization for equation (0.2),

$$\left[\left(\frac{d}{dx}\right)^3 + \frac{\partial F}{\partial u_2}\left(\frac{d}{dx}\right)^2 + \frac{\partial F}{\partial u_1}\frac{d}{dx} + \frac{\partial F}{\partial u} - \frac{d}{dt}\right]\psi = 0.$$

2nd step. By the substitutions

$$\psi = \exp\left(\int R\,dx + T\,dt\right), \quad \text{where} \quad R_t = T_x,$$

we obtain the equation with "extended derivatives",

$$\left[\left(\frac{d}{dx} + R\right)^3 + \frac{\partial F}{\partial u_2}\left(\frac{d}{dx} + R\right)^2 + \frac{\partial F}{\partial u_1}\left(\frac{d}{dx} + R\right) + \frac{\partial F}{\partial u} - \left(\frac{d}{dt} + T\right)\right](1) = 0,$$

which is equivalent to the relation

$$T = \left(\frac{d^2}{dx^2} + \frac{d}{dx}R + R\frac{d}{dx} + R^2\right)(R) + \frac{\partial F}{\partial u_2}\left(\frac{d}{dx} + R\right)(R) + \frac{\partial F}{\partial u_1}R + \frac{\partial F}{\partial u}. \tag{A3.29}$$

3rd step. We choose an appropriate expansion for R. The simplest choice is to let

$$R = \mu^{-1} + \sum_{n=0}^{\infty} \rho_n \mu^n. \tag{A3.30}$$

Remark 12. Our several attempts to find the expansions with the poles of higher order gave nothing new. If, for instance, we assume for equation (0.2) $R = \mu^{-2} + \sum_{n=-1}^{\infty} \rho_n \mu^n$, then after checking several conditions (A3.15) we obtain $\rho_{2n+1} = 0,\ \forall n$. It is equivalent to that R is expanded w.r.t. the parameter $\xi = \mu^2$. Similar results were obtained for some other equations and systems as well (see [11])

Having chosen expansion (A3.30), we should accept

$$T = \mu^{-3} + \theta_{-2}\mu^{-2} + \theta_{-1}\mu^{-1} + \sum_{n=0}^{\infty} \theta_n \mu^n, \tag{A3.31}$$

in order to cancel the terms with μ^{-3} in equation (A3.29).

For expansion (A3.30) we have $\rho_{-1} = 1, \rho_{-2} = 0$, which implies that θ_{-2} and θ_{-1} are constants. Since additive integration constants in the fluxes are eliminated by the transformation of parameter (A3.20), we let $\theta_{-2} = \theta_{-1} = 0$.

Now, as one can easily make sure, substituting expansions (A3.30) and (A3.31) into equation (A3.29), we arrive at formula (2.2) with indicated there ρ_0 and ρ_1.

BIBLIOGRAPHY

1. *What is Integrability?* Ed. V.E. Zakharov. Springer series in Nonlinear Dynamics. 1991.
2. *Integrability.* Ed. A.V. Mikhailov. Lecture Notes in Physics. Springer. 2009. V. 767.
3. A.C. Newell. *Solitons in mathematics and physics.* SIAM. 1985. 260 pp.
4. N.Kh. Ibragimov, A.B. Shabat. *Infinite Lie-Bäklund algebras* // Funkts. anal. i ego prilozh.. 1980. V. 14, No. 4. P. 79–80. [Funct. anal. appl. 1980. V. 14, No. 4, P. 313–315.]
5. S.I. Svinolupov, V.V. Sokolov. *Evolution equations with nontrivial conservative laws* // Funkts. anal. i ego prilozh. 1982. V. 16, No. 4. P. 86–87. [Funct. anal. appl. 1982. V. 16, No. 4. P. 317-319.]
6. S.I. Svinolupov, V.V. Sokolov. *On conservations laws for the equations possessing nontrivial Lie-Bäcklund algebra* // Integrable systems: collection of the papers. Ed. A.B. Shabat. BB AS USSR. Ufa. 1982. P. 53–67. (in Russian)
7. V.V. Sokolov and A.B. Shabat. *Classification of Integrable Evolution Equations* // Soviet Scientific Reviews, Section C. 1984. V. 4. P. 221–280.
8. A.V. Mikhailov, A.B. Shabat, R.I. Yamilov. *The symmetry approach to the classification of non-linear equations. Complete lists of integrable systems* // Uspekhi matem. nauk. 1987. V. 42, No. 4. P. 3–53. [Russ. Math. Surv. 1987. V. 42, No. 4. P. 1–63.]
9. A.V. Mikhailov, V.V. Sokolov, A.B. Shabat. *The symmetry approach to classification of integrable equations* // What is Integrability? Ed. V.E. Zakharov. Springer series in Nonlinear Dynamics. 1991. P. 115–184.
10. H.H. Chen, Y.C. Lee and C.S. Liu. *Integrability of nonlinear Hamiltonian systems by inverse scattering method* // Phys. Scr. 1979. V. 20, No. 3–4. P. 490–492.
11. A.G. Meshkov. *Necessary conditions of the integrability* // Inverse Problems. 1994. V.10. 635–653.
12. V.G. Drinfel'd, V.V. Sokolov. *Lie algebras and equations of Korteweg-de Vries type* // Itogi nauki i tekhniki. Sovrem. probl. matem. Noveiishie dostizheniya. Moscow, VINITI. 1984. V. 24. P. 81–180. [J. Soviet Math. 1985. V. 30, No. 2. P. 1975-2036.]
13. S.I. Svinolupov. *Second-order evolution equations with symmetries* // Uspekhi matem. nauk. 1985. V. 40, No.5. P. 263–264. [Russ. Math. Surv. 1985. V. 40, No.5. P. 241–242.]
14. S.I. Svinolupov, V.V. Sokolov. *Weak nonlocalities in evolution equations* // Matem. zametki. 1990, No. 6. P. 91–97. [Math. notes Acad. Sci. USSR. 1990, No. 6. P. 1234-1239.]

15. A.S. Fokas. *A symmetry approach to exactly solvable evolution equations* // J. Math. Phys. 1980. V. 21, No. 6. P. 1318–1325.
16. R.H. Heredero, V.V. Sokolov, and S.I. Svinolupov. *Toward the classification of third order integrable evolution equations* // J. Phys. A. 1994. V. 13. P. 4557–4568.
17. R.H. Heredero. *Classification of fully nonlinear integrable evolution equations of third order* // J. Nonlin. Math. Phys. 2005. V. 12, No. 4. P. 567–585.
18. V.G. Drinfel'd, S.I. Svinolupov, V.V. Sokolov. *Classification of fifth order evolution equations possessing infinite series of conservation laws* // Dokl. AN USSR. 1985. V. A10. P. 7–10. (in Russian)
19. V.E. Zakharov, S.V. Manakov, S.P. Novikov, L.P. Pitaevskii. *Theory of solitons. Method of inverse problem.* Moscow. "Nauka". 1980. 320 pp. (in Russian)
20. V.V. Sokolov. *On the symmetries of evolution equations* // Uspekhi matem. nauk. 1988. V. 43, No. 5. P. 165–204. [Russ. Math. Surv. 1988. V. 43, No. 5. P. 165–204.]
21. Sanders Jan and Jing Ping Wang. *On the Integrability of homogeneous scalar evolution equations* // J. Diff. Eqs. 1998. V. 147. P. 410–434.
22. P. Olver, Jing Ping Wang. *Classification of integrable one-component systems on associative algebras* // Proc. London Math. Soc. 2000. V. 81, No. 3. P. 566–586.
23. M. Gürses, A. Karasu and V.V. Sokolov. *On construction of recursion operator from Lax representation* // J. Math. Phys. 1999. V. 40, No.12. P. 6473–6490.
24. V.V. Sokolov and V. Wolf. *A symmetry test for quasilinear coupled systems* // Inverse Problems. 1999. V. 15. P. L5–L11.
25. A.G. Meshkov, B.B. Mikhalyaev. *Equations of gas dynamics admitting an infinite number of symmetries* // Teoret. i matem. fizika. 1987. V. 72, No. 2. P. 163–171. [Theor. Math. Phys. 1987. V. 72, No. 2. P. 795-801.]
26. S.I. Svinolupov. *On the analogues of the Burgers equation* // Phys. Lett. A. 1989. V. 135, No. 1. P. 32–36.
27. S.I. Svinolupov. *Generalized Schrödinger equations and Jordan pairs* // Commun. Math. Phys. 1992. V. 143, No. 1. P. 559–575.
28. S.I. Svinolupov. *Jordan algebras and generalized Korteweg-de Vries equations* // Teoret. i matem. fizika. 1991. V. 87, No. 3. P. 391–403. [Theor. Math. Phys.
29. S.I. Svinolupov, V.V. Sokolov. *Vector-matrix generalizations of classical integrable equations* // Teoret. i matem. fizika. 1994. V. 100, No. 2. P. 214–218. [Theor. Math. Phys. 1994. V. 100, No. 2. P. 959– 962.]
30. V.V. Sokolov and S.I. Svinolupov. *Deformation of nonassociative algebras and integrable differential equations* // Acta Applicandae Mathematica.1995. V. 41, No. 1–2. P. 323–339.
31. I.V. Habibullin, V.V. Sokolov, R.I. Yamilov. *Multi-component integrable systems and nonassociative structures* // Nonlinear Physics: theory and experiment. Eds: E. Alfinito, M. Boiti, L. Martina, F. Pempinelli. World Scientific Publisher. Singapore. 1996. P. 139–168.
32. P.J. Olver and V.V. Sokolov. *Integrable evolution equations on associative algebras* // Comm. Math. Phys. 1998. V. 193, No. 2. P. 245–268.
33. V.V. Sokolov, V. Wolf. *Classification of integrable polynomial vector evolution equations* // J. Phys. A. 2001. V. 34. P. 11139–11148.
34. A.G. Meshkov and V.V. Sokolov. *Integrable evolution equations on the N-dimensional sphere* // CommMath. Phys. 2002. V. 232, No. 1. P. 1–18.
35. A.G. Meshkov, V.V. Sokolov. *Classification of integrable divergent N-component evolution systems* // Teoret. i matem. fizika. 2004. V. 139, No.2. P. 192–208. [Theor. Math. Phys. 2004. V. 139, No.2. P. 192–208.]
36. M.Ju. Balakhnev, A.G. Meshkov. *Integrable anisotropic evolution equations on a sphere* // SIGMA. 2005, V. 1. Paper 027. 11 p.
37. M.Ju. Balakhnev. *A class of integrable evolutionary vector equations* // Teoret. i matem. fizika. 2005. V. 142, No. 1. P. 13–20. [Theor. Math. Phys. 2005. V. 142, No. 1. P. 8-14.]

38. A.G. Meshkov. *On symmetry classification of third-order evolutionary systems of divergent type* // Fundam. prikl. matem. 2006. V. 12, No. 7. P.141–161. [J. Math. Sci. 2008. V. 151, No. 4. P. 3167–3181.]
39. M.Ju. Balakhnev, A.G. Meshkov. *Two-field integrable evolutionary systems of the third order and their differential substitutions* // SIGMA. 2008, V. 4. Paper 018. 29 p.
40. M.Ju. Balakhnev, A.G. Meshkov. *On a classification of integrable vectorial evolutionary equations* // J. Nonlin. Math. Phys. 2008. V. 15, No. 2. P. 212–226.
41. A.V. Mikhailov, V.V. Sokolov. *Symmetries of differential equations and the problem of integrability* // Integrability. Ed. A.V. Mikhailov. Lecture Notes in Physics. Springer. 2009. V. 767. P. 19–88.
42. A.V. Zhiber, A.B. Shabat. *Klein-Gordon equations with a nontrivial group* // Dokl. AN SSSR. 1979. V. 247, No. 5. P. 1103–1107. [Sov. Phys. Dokl. 1979. V. 24, No. 8. P. 608-609]
43. A.V. Zhiber, A.B. Shabat. *Systems of equations* $u_x = p(u, v)$, $v_y = q(u, v)$ *that possess symmetries* // Dokl. AN SSSR. V. 277, No. 1. 1984. P. 29–33. [Soviet Math. Dokl. V. 30, No. 1. 1984. P. 23–26.)]
44. A.G. Meshkov. *Symmetries of scaler fields. III. Two-dimensional integrable models* // Teoret. i matem. fizika. 1985. V. 63, No. 3. P. 323–332. [Theor. Math. Phys. 1985. V. 63, No. 3. P. 539–545.]
45. A.G. Meshkov. *Hamiltonian and recursion operators for two-dimensional scalar fields* // Phys. Lett. A. 1992. V. 170, No. 6. P. 405–408.
46. A.V. Zhiber. *Quasilinear hyperbolic equations with an infinite-dimensional symmetry algebra* // Izv. RAN. Ser. matem. V. 58, No. 4. 1994. P. 33–54. [Russ AC SC Izv. Math.. V. 45, No. 1. 1995. P. 33–54.]
47. A.V. Zhiber, V.V. Sokolov. *Exactly integrable hyperbolic equations of Liouville type* // Uspekhi matem. nauk. V. 56, No. 1(337). 2001. P. 63–106. [Russ. Math. Surv. 2001. V. 56, No. 1. 2001. P.61–101.]
48. A.G. Meshkov. *Nonlocal symmetries in two-field divergent evolutionary systems* // Teoret. i matem. fizika. 2008. V. 156, No. 3. P. 351–363. [Theor. Math. Phys. 2008. V. 156, No. 3. P. 1268-1279]
49. A.G. Meshkov. *Vector hyperbolic equations with higher symmetries* // Teoret. i matem. fizika. 2009. V. 161, No. 2. P. 176–190. [Theor. Math. Phys. 2009. V. 161, No. 2. P. 1471–1484.]
50. A.G. Meshkov, V.V. Sokolov. *Hyperbolic equations with symmetries of third order* // Teoret. i matem. fizika. 2011. V. 166, No. 1. P. 51–67. [Theor. Math. Phys. 2011. V. 166, No. 1. P. 43-57.]
51. P.J. Olver. *Applications of Lie Groups to Differential Equations.* Sec. Edition, Graduate Texts in Mathematics. V. 107. Springer–Verlag, New York, 1993. 513 pp.
52. S.I. Svinolupov, V.V. Sokolov, R.I. Yamilov. *On Bäcklund transformations for integrable evolution equation* // Dokl. AN SSSR. 1983. V. 271, No. 4. P. 802–805. [Sov. Math., Dokl. 1983. V. 28. P. 165–168.]
53. V.G. Drinfel'd, V.V. Sokolov. *On equations related to the Korteweg-de Vries equation* // Dokl. AN SSSR 1985. V. 284, No. 1. P. 29–33. [Sov. Math., Dokl. 1985. V. 32. P. 361–365.]
54. H. Bateman and A. Èrdĭi. *Higher transcendental functions. Elliptic and modular functions. Lamé and Mathieu functions.* V. 3. 1955. New York, Toronto, London. Mc Grow-Hill Book Company, Inc. 301 pp.
55. V.E. Zakharov, A.B. Shabat. *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media* // Zh. Eks. Teor. Phys. 1971. V. 61. P. 118–134. [Soviet Physics JETP. 1972. V. 34, No. 1, P. 62-69.]

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