ISSN 2304-0122 Ufa Mathematical Journal. Volume 4. 2 (2012). Pp. 13-27.

UDC 517.925

SEPARATION OF AN EQUATION IN THE SYSTEM OF TWO SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

YU.YU. BAGDERINA

Abstract. We consider projectable type systems of two second-order ordinary differential equations with cubic nonlinearity of the right-hand side in first derivatives. For such systems we obtain criteria of reducibility by local transformation to a system with a separating equation in one of the unknown functions. Applications of the criteria and construction of the corresponding transformation is illustrated by a number of examples.
Keywords: second-order equation, decoupling of equations, separation of an equation, submersive system

1. Introduction

The problem of separation of equations in a system, like the problem of linearization of differential equations corresponds to a particular case of the problem of equivalence. Let us call two systems of second-order ordinary differential equations

$$x'' = f(t, x, y, x', y'), \qquad y'' = g(t, x, y, x', y') \tag{1}$$

equivalent, if there is a reversible point substitution of variables

$$\tilde{t} = \theta(t, x, y), \qquad \tilde{x} = \varphi(t, x, y), \qquad \tilde{y} = \psi(t, x, y), \qquad \Delta = \frac{\partial(\theta, \varphi, \psi)}{\partial(t, x, y)} \neq 0, \tag{2}$$

when one system changes to the other. Here we use the following table of symbols $x' = dx/dt$, $x'' = d^2x/dt^2$, $y' = dy/dt$, $y'' = d^2y/dt^2$ for derivatives. As it was shown in S.Lie's papers [1], many particular methods of solution of differential equations are equivalent to finding such a substitution of variables (2), which would reduce the given equation to one of already available equations. Thus, in some cases the problem of linear integration of an ordinary differential equation is considered to be solved, if the equation can be linearized. In the case of the system of equations (1) the problem is reduced to a more simple one, if as a result of the transformation (2) we obtain a system, in which an equation is separated relative to one of the functions, for instance, $\tilde{x}(\tilde{t})$:

$$\tilde{x}'' = \tilde{f}(\tilde{t}, \tilde{x}, \tilde{x}'), \qquad \tilde{y}'' = \tilde{g}(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}'). \tag{3}$$

Thereby the system integration is reduced to solution of the first equation (3) relative to $\tilde{x}(\tilde{t})$ and then, when the function $\tilde{x}(\tilde{t})$ is available - to integration of the second equation (3) relative to $\tilde{y}(\tilde{t})$.

In some rare cases equations of the system (1) can be completely divided into:

$$\tilde{x}'' = \tilde{f}(\tilde{t}, \tilde{x}, \tilde{x}'), \qquad \tilde{y}'' = \tilde{g}(\tilde{t}, \tilde{y}, \tilde{y}')$$

Yu.Yu. Bagderina, Separation of an equation in the system of two second-order ordinary differential equations.

The work is supported by RFFR (grants 10-01-00186-a, 11-01-91330-NNIO-a), MK-8247.2010.1, FTP (state contract 02.740.11.0612).

Submitted on October 24, 2011.

after the transformation (2), and then they are integrated independently from each other. The problem of separation of equations in systems of second-order ordinary differential equations was studied in [2, 3, 4]. Meanwhile there were considered transformations, effecting only dependent variables and not changing t. The problem of separation of equations on the system (in such a class of transformations) was studied in [5, 6].

The system (1), which is linearized to the form $\tilde{x}''=0$, $\tilde{y}''=0$, can be an example of a system with separated equations. An available criterion of linearization [7, 8] in terms of relative invariants γ_i, σ_k of the system (1) can be formulated as follows.

Theorem 1. *The system* (1) *is linearized to the form* $\tilde{x}''=0$, $\tilde{y}''=0$ *by the transformation* (2) *if and only if there is*

$$\gamma_i = 0, \quad i=0,1,2, \qquad \sigma_k = 0, \quad k=0,\dots,4, \tag{4}$$

for it, where

$$\begin{aligned}\gamma_0 &= \tfrac{1}{2}D(f_{y'}) - \tfrac{1}{4}f_{y'}(f_{x'}+g_{y'}) - f_y,\\ \gamma_1 &= \tfrac{1}{4}D(f_{x'}-g_{y'}) + \tfrac{1}{8}(g_{y'}^2 - f_{x'}^2) + \tfrac{1}{2}(g_y - f_x),\\ \gamma_2 &= -\tfrac{1}{2}D(g_{x'}) + \tfrac{1}{4}g_{x'}(f_{x'}+g_{y'}) + g_x,\end{aligned} \tag{5}$$

$D=\partial_t + x'\partial_x + y'\partial_y + f\partial_{x'} + g\partial_{y'}$ *is an operator of differentiation according to the system* (1) *and*

$$\sigma_0 = f_{y'y'y'}, \qquad \sigma_1 = \tfrac{1}{4}(3f_{x'y'y'} - g_{y'y'y'}), \qquad \sigma_2 = \tfrac{1}{2}(f_{x'x'y'} - g_{x'y'y'}),$$
$$\sigma_3 = \tfrac{1}{4}(f_{x'x'x'} - 3g_{x'x'y'}), \qquad \sigma_4 = -g_{x'x'x'}.$$

The system (1), satisfying the conditions $\sigma_0=0, \dots \sigma_4=0$, possesses the form

$$\begin{aligned}x'' &= K_1 + 2L_1x' + 2M_1y' + P_1x'^2 + 2S_1x'y' + Q_1y'^2\\ &\quad + x'(V_1x'^2 + 2V_0x'y' + V_2y'^2),\\ y'' &= K_2 + 2L_2y' + 2M_2x' + P_2y'^2 + 2S_2x'y' + Q_2x'^2\\ &\quad + y'(V_1x'^2 + 2V_0x'y' + V_2y'^2)\end{aligned} \tag{6}$$

with the coefficients K_j, L_j, M_j, P_j, Q_j, S_j, V_0, V_j, $j=1,2$, depending on t, x, y, and can be associated with project connection in three-dimensional space [7]. For the system (6) according to the formulae (5) we obtain

$$\begin{aligned}\gamma_0 &= a_0x'^3 - b_0x'^2y' + a_2x'^2 + a_1x'y' + (2a_5 - a_4)x' + a_3y' + a_7,\\ \gamma_1 &= -a_0x'^2y' + b_0x'y'^2 + \tfrac{1}{2}(b_1x'^2 + (b_2-a_2)x'y' - a_1y'^2\\ &\quad + (b_6 + b_5 - 2b_4)x' + (2a_4 - a_5 - a_6)y' + a_8 - b_8),\\ \gamma_2 &= a_0x'y'^2 - b_0y'^3 - b_1x'y' - b_2y'^2 - b_3x' + (b_4 - 2b_5)y' - b_7,\end{aligned} \tag{7}$$

and the condition of linearization (4) obtains the form of 15 relations

$$\begin{gathered}a_0=0, \quad a_1=0, \quad a_2=0, \quad a_3=0, \quad a_4-2a_5=0, \quad 3a_5-a_6=0,\\ b_0=0, \quad b_1=0, \quad b_2=0, \quad b_3=0, \quad b_4-2b_5=0, \quad 3b_5-b_6=0,\\ a_7=0, \quad b_7=0, \quad a_8-b_8=0.\end{gathered} \tag{8}$$

The explicit form of relative invariants a_j, b_j, $j=0,\dots,8$, of the system (6) is presented in [9]. In particular,

$$\begin{aligned}a_5 &= S_{1t} - L_{1y} - M_1S_2 + M_2Q_1 + \tfrac{3}{2}K_1V_0 + \tfrac{1}{2}K_2V_2,\\ b_5 &= S_{2t} - L_{2x} - M_2S_1 + M_1Q_2 + \tfrac{3}{2}K_2V_0 + \tfrac{1}{2}K_1V_1,\\ b_8 &= L_{2t} - K_{2y} - L_2^2 - M_1M_2 + K_2P_2 + K_1S_2.\end{aligned} \tag{9}$$

In this paper we obtain the criterion of separation of an equation in systems of the form (6). The criterion of complete separation of equations in the system (6) can be found in [10]. The class of equations (6) is closed relative to arbitrary nonsingular substitution of variables (2). Such a substitution transforms the system (6) into the system of the same form

$$\begin{aligned} \tilde{x}'' &= \tilde{K}_1 + 2\tilde{L}_1\tilde{x}' + 2\tilde{M}_1\tilde{y}' + \tilde{P}_1\tilde{x}'^2 + 2\tilde{S}_1\tilde{x}'\tilde{y}' + \tilde{Q}_1\tilde{y}'^2 \\ &\quad + \tilde{x}'(\tilde{V}_1\tilde{x}'^2 + 2\tilde{V}_0\tilde{x}'\tilde{y}' + \tilde{V}_2\tilde{y}'^2), \\ \tilde{y}'' &= \tilde{K}_2 + 2\tilde{L}_2\tilde{y}' + 2\tilde{M}_2\tilde{x}' + \tilde{P}_2\tilde{y}'^2 + 2\tilde{S}_2\tilde{x}'\tilde{y}' + \tilde{Q}_2\tilde{x}'^2 \\ &\quad + \tilde{y}'(\tilde{V}_1\tilde{x}'^2 + 2\tilde{V}_0\tilde{x}'\tilde{y}' + \tilde{V}_2\tilde{y}'^2) \end{aligned} \tag{10}$$

with some coefficients $\tilde{K}_j$, $\tilde{L}_j$, $\tilde{M}_j$, $\tilde{P}_j$, $\tilde{Q}_j$, $\tilde{S}_j$, $\tilde{V}_0$, $\tilde{V}_j$, $j = 1, 2$, operating as functions $\tilde{t}$, $\tilde{x}$, $\tilde{y}$. In the system (10) the first equation is separated if its coefficients satisfy the correlations

$$\tilde{K}_1 = P, \quad \tilde{L}_1 = \frac{3}{2}Q, \quad \tilde{M}_1 = 0, \quad \tilde{P}_1 = 3R, \quad \tilde{S}_1 = 0, \quad \tilde{Q}_1 = 0, \quad \tilde{V}_1 = S, \quad \tilde{V}_0 = 0, \quad \tilde{V}_2 = 0$$

with some functions P, Q, R, S, depending on $\tilde{t}$, $\tilde{x}$, and its equations possess the form

$$\tilde{x}'' = P(\tilde{t}, \tilde{x}) + 3Q(\tilde{t}, \tilde{x})\tilde{x}' + 3R(\tilde{t}, \tilde{x})\tilde{x}'^2 + S(\tilde{t}, \tilde{x})\tilde{x}'^3, \tag{11}$$

$$\tilde{y}'' = \tilde{K}_2 + 2\tilde{L}_2\tilde{y}' + 2\tilde{M}_2\tilde{x}' + \tilde{P}_2\tilde{y}'^2 + 2\tilde{S}_2\tilde{x}'\tilde{y}' + \tilde{Q}_2\tilde{x}'^2 + S(\tilde{t}, \tilde{x})\tilde{x}'^2\tilde{y}'. \tag{12}$$

In §2 there is a case of transformation (2) considered, where $\theta = \theta(t)$. The case of arbitrary transformation (2) (with $\theta_x \neq 0$ or $\theta_y \neq 0$) is studied in §4. In §3, 5 the application of the obtained criteria of separation is demonstrated on the example of a normal form of the system with two degrees of freedom and the system, which can be interpreted as geodesic equations in the space with Riemannian metric.

As it was noticed by the reviewer of the paper, the problem of separation of equations in the system (6) can be solved in a more general formulation. Namely, we could find conditions of separation in the system (6) of the equation of the form

$$\tilde{x}'' = P(\tilde{t}, \tilde{x}, \tilde{y}) + 3Q(\tilde{t}, \tilde{x}, \tilde{y})\tilde{x}' + 3R(\tilde{t}, \tilde{x}, \tilde{y})\tilde{x}'^2 + S(\tilde{t}, \tilde{x}, \tilde{y})\tilde{x}'^3, \tag{13}$$

which differs from the equation (11) fact, that the variable $\tilde{y}$ is entered into it as a parameter. The corresponding criterion will also include as a particular case criteria of reducibility of the system (6) to the form (11), (12), obtained in the given paper. The problem of separation in the system (6) of the equation of the form (13) is not studied here. Its solution is a more complicated task, as it is reduced to research of consistency of a redenoted system of 15 equations relative to functions θ, φ, ψ, where the subsystem of 9 equations relative to the functions θ, φ is not separated (see subsystems (16), (17) and (46), (47)) below.

2. Criterion of separation of equations on the system (6). An example of a particular case transformation

Let us find conditions when the system (6) of a nonsingular point substitution of variables

$$\tilde{t} = \theta(t), \qquad \tilde{x} = \varphi(t, x, y), \qquad \tilde{y} = \psi(t, x, y) \tag{14}$$

can be transformed into a system of the form (11), (12). It is assumed, that $\varphi_x \neq 0$, $\varphi_y \neq 0$. In other words, if the system (6) is reduced to the form (11), (12) by the transformation (14), where $\varphi_x = 0$ ($\varphi_y = 0$), it implies that the first (the second) equation in the system (6) has been separated already.

Substitution of the transformation (14) in the equations (11), (12) results in the system of the second-order ordinary differential equations relative to $x(t)$, $y(t)$ with the same type of dependence on x', y', like in the equations (6). Setting its coefficients equal with degrees x', y' to the corresponding coefficients of the equations (6), we obtain 15 correlations, which under

$\varphi_x \neq 0$, $\varphi_y \neq 0$ can be solved relative to all derivatives of the second order of the function ψ:

$$\begin{array}{l}
\psi_{xx} = -P_1\psi_x - Q_2\psi_y + \tilde{P}_2\psi_x^2 + 2\tilde{S}_2\varphi_x\psi_x + \tilde{Q}_2\varphi_x^2 + S(\varphi_x\psi_t + 2\varphi_t\psi_x)\varphi_x/\theta', \\
\psi_{xy} = -S_1\psi_x - S_2\psi_y + \tilde{P}_2\psi_x\psi_y + \tilde{S}_2(\varphi_x\psi_y + \varphi_y\psi_x) + \tilde{Q}_2\varphi_x\varphi_y \\
\qquad +S(\varphi_x\varphi_y\psi_t + \varphi_t\varphi_y\psi_x + \varphi_t\varphi_x\psi_y)/\theta', \\
\psi_{yy} = -Q_1\psi_x - P_2\psi_y + \tilde{P}_2\psi_y^2 + 2\tilde{S}_2\varphi_y\psi_y + \tilde{Q}_2\varphi_y^2 + S(\varphi_y\psi_t + 2\varphi_t\psi_y)\varphi_y/\theta', \\
\psi_{tx} = -L_1\psi_x - M_2\psi_y + \psi_x\theta''/(2\theta') + \theta'(\tilde{L}_2\psi_x + \tilde{M}_2\varphi_x) + \tilde{P}_2\psi_t\psi_x \\
\qquad +\tilde{S}_2(\varphi_x\psi_t + \varphi_t\psi_x) + \tilde{Q}_2\varphi_t\varphi_x + S(\varphi_t\psi_x + 2\varphi_x\psi_t)\varphi_t/(2\theta'), \\
\psi_{ty} = -M_1\psi_x - L_2\psi_y + \psi_y\theta''/(2\theta') + \theta'(\tilde{L}_2\psi_y + \tilde{M}_2\varphi_y) + \tilde{P}_2\psi_t\psi_y \\
\qquad +\tilde{S}_2(\varphi_y\psi_t + \varphi_t\psi_y) + \tilde{Q}_2\varphi_t\varphi_y + S(\varphi_t\psi_y + 2\varphi_y\psi_t)\varphi_t/(2\theta'), \\
\psi_{tt} = -K_1\psi_x - K_2\psi_y + \psi_t\theta''/\theta' + \tilde{K}_2\theta'^2 + 2\theta'(\tilde{L}_2\psi_t + \tilde{M}_2\varphi_t) + \tilde{P}_2\psi_t^2 \\
\qquad +2\tilde{S}_2\varphi_t\psi_t + \tilde{Q}_2\varphi_t^2 + S\varphi_t^2\psi_t/\theta'
\end{array} \tag{15}$$

and derivatives of the function φ:

$$\begin{array}{l}
\varphi_{xx} = -P_1\varphi_x - Q_2\varphi_y + 3(R + S\varphi_t/\theta')\varphi_x^2, \\
\varphi_{xy} = -S_1\varphi_x - S_2\varphi_y + 3(R + S\varphi_t/\theta')\varphi_x\varphi_y, \\
\varphi_{yy} = -Q_1\varphi_x - P_2\varphi_y + 3(R + S\varphi_t/\theta')\varphi_y^2,
\end{array} \tag{16}$$

$$\begin{array}{l}
\varphi_{tx} = -L_1\varphi_x - M_2\varphi_y + \varphi_x\theta''/(2\theta') + 3/2(Q\theta' + 2R\varphi_t + S\varphi_t^2/\theta')\varphi_x, \\
\varphi_{ty} = -M_1\varphi_x - L_2\varphi_y + \varphi_y\theta''/(2\theta') + 3/2(Q\theta' + 2R\varphi_t + S\varphi_t^2/\theta')\varphi_y, \\
\varphi_{tt} = -K_1\varphi_x - K_2\varphi_y + P\theta'^2 + (3Q\theta' + \theta''/\theta')\varphi_t + 3R\varphi_t^2 + S\varphi_t^3/\theta'.
\end{array}$$

The remaining three correlations possess the form

$$V_1 = S\varphi_x^2/\theta', \qquad V_0 = S\varphi_x\varphi_y/\theta', \qquad V_2 = S\varphi_y^2/\theta'. \tag{17}$$

Therefore, the system (6) is transformed into the system with the separating equation (11), (12) by means of substitution of variables (14) if and only if the redetermined system of equations (15)–(17) is combined relative to the functions θ, φ, ψ.

The equations (16), (17) are separated from the system (15)–(17), as they contain only the functions P, Q, R, S, depending on θ, φ, and do not contain the functions $\tilde{K}_2$, $\tilde{L}_2$, $\tilde{M}_2$, $\tilde{P}_2$, $\tilde{S}_2$, $\tilde{Q}_2$, depending on θ, φ, ψ. Their solution determines the functions θ, φ in the transformation (14). Any function such that substitution of variables (14) is nonsingular can be used as ψ. From the six equations (15) we can determine the coefficients $\tilde{K}_2$, $\tilde{L}_2$, $\tilde{M}_2$, $\tilde{P}_2$, $\tilde{S}_2$, $\tilde{Q}_2$ of the equation (12). No limits are imposed on this type of coefficients. Therefore, the equations (15) are combined, the same way as the system (15)–(17) in general, if the subsystem of equations (16), (17) is combined. The system (16), (17) is redetermined, and the study of its compatibility is based on the system of Pfuff equations. A detailed description of the theory of such equations can be found, for instance, in [11].

It is easy to see, that if one of the functions V_0, V_1, V_2 equals to zero, then the equations (17) can be combined only when $S = 0$, V_0, V_1, $V_2 = 0$. While studying compatibility of the system (16), (17) this case together with the case when the coefficients V_0, V_1, V_2 in the system (6) differ from zero, are considered separately. Together with (9) the following symbols are applied here

$$
\begin{aligned}
\alpha_0 &= a_{5x} - b_{6y} + Q_1 b_3 - S_1 b_4 - S_2 a_5 + Q_2 a_3 + 3(M_2 V_{2t} - M_1 V_{1t}) \\
&\quad +3/2((3L_1 - L_2)V_{0t} - V_1 M_{1t} + V_0(L_{1t} - L_{2t}) + V_2 M_{2t}), \\
\alpha_1 &= a_{3x} - a_{4y} + (S_2 - P_1)a_3 + (2S_1 - P_2)a_4 - S_1 a_6 + Q_1(b_5 - b_6) \\
&\quad +3/2(L_1 + L_2)V_{2t}, \\
\alpha_2 &= a_{7x} - a_{8y} + Q_1 b_7 + S_1(a_8 - b_8) - P_1 a_7 - M_1 b_6 \\
&\quad +L_1 a_4 - L_2 a_5 + M_2 a_3 + 3/2(K_1 V_{0t} + K_2 V_{2t}), \\
\alpha_3 &= b_{5t} - b_{6t} + 3(a_{8x} - b_{8x}) + 6(Q_2 a_7 - S_1 b_7) + 2M_1 b_3 + 2L_1(b_6 - b_5) \\
&\quad +M_2(a_5 - a_4 - 2a_6), \\
\alpha_4 &= a_{4t} - 3a_{7x} + 3S_1(b_8 - a_8) + 3(P_1 - S_2)a_7 \\
&\quad -2M_2 a_3 + (L_2 - 3L_1)a_4 + M_1(b_6 - b_5), \\
\alpha_5 &= a_{5t} - a_{4t} + M_1(b_5 - b_4) - L_2(a_4 + 2a_5) + 9/2(K_1 V_{0t} + K_2 V_{2t}) \\
&\quad +3[a_{7x} - a_{8y} + Q_1 b_7 + S_1(a_8 - b_8) - P_1 a_7 - M_1 b_6 + L_1 a_4 + M_2 a_3 \\
&\quad +V_0(K_{1t}/2 + K_1 L_1 + K_2 M_1) + V_2(K_{2t}/2 + K_1 M_2 + K_2 L_2)], \\
\alpha_6 &= a_{3t} - 3a_{7y} + 3Q_1(b_8 - a_8) + 3(S_1 - P_2)a_7 - M_1(a_4 + a_6) \\
&\quad -(L_1 + L_2)a_3, \\
\alpha_7 &= (\beta_4 + \beta_5 - \alpha_3)_y - \alpha_{4x} - Q_1\beta_6 + S_1(\beta_4 + \beta_5) + S_2\alpha_4 - Q_2\alpha_6, \\
\alpha_8 &= \alpha_{6x} - (\alpha_4 + \alpha_5)_y + Q_1(\alpha_3 - \beta_5) + S_1\beta_6 + (S_1 - P_2)(\alpha_4 + \alpha_5) \\
&\quad +(S_2 - P_1)\alpha_6, \\
\alpha_9 &= b_{6x} + P_1 b_6 - S_1 b_3 + 2Q_2 a_5, \\
\alpha_{10} &= a_{5x} + S_1(b_6 - b_4) + S_2 a_5 + Q_2 a_3, \\
\alpha_{11} &= a_{4y} + S_2 a_3 + P_2 a_4 + Q_1(b_6 - b_5), \\
\alpha_{12} &= a_{3y} + (2P_2 - S_1)a_3 + Q_1(2a_4 - a_6),
\end{aligned}
\tag{18}
$$

and also β_i, $i = 0, \ldots, 12$, formulae for which calculation it is necessary in the expressions for α_i to roles of the following pairs of variables: (x, y), (a_j, b_j), (α_k, β_k) and indexes (1,2) of the coefficients of the system (6). In particular, we obtain

$$
\begin{aligned}
\beta_7 &= (\alpha_4 + \alpha_5 - \beta_3)_x - \beta_{4y} - Q_2\alpha_6 + S_2(\alpha_4 + \alpha_5) + S_1\beta_4 - Q_1\beta_6, \\
\beta_{12} &= b_{3x} + (2P_1 - S_2)b_3 + Q_2(2b_4 - b_6),
\end{aligned}
$$

and etc. According to the same rule the values b_j are obtained from a_j (see (9)). The following criteria of separation of the equation hold in the system (6) as the result of the transformation of the form (14).

Theorem 2. *The system of two second-order ordinary differential equations*

$$
\begin{aligned}
x'' &= K_1 + 2L_1 x' + 2M_1 y' + P_1 x'^2 + 2S_1 x'y' + Q_1 y'^2, \\
y'' &= K_2 + 2L_2 y' + 2M_2 x' + P_2 y'^2 + 2S_2 x'y' + Q_2 x'^2
\end{aligned}
\tag{19}
$$

is reduced to the form

$$\tilde{x}'' = p(\tilde{t}, \tilde{x}) + 2q(\tilde{t}, \tilde{x})\tilde{x}' + r(\tilde{t}, \tilde{x})\tilde{x}'^2, \tag{20}$$

$$\tilde{y}'' = \tilde{K}_2 + 2\tilde{L}_2\tilde{y}' + 2\tilde{M}_2\tilde{x}' + \tilde{P}_2\tilde{y}'^2 + 2\tilde{S}_2\tilde{x}'\tilde{y}' + \tilde{Q}_2\tilde{x}'^2 \tag{21}$$

by the transformation (14)
if and only if its coefficients satisfy the correlations

$$B_{i1}B_{j2} - B_{j1}B_{i2} = 0, \tag{22}$$

$$B_{j2}^2 A_{k1} - B_{j1}B_{j2}A_{k2} + B_{j1}^2 A_{k3} = 0, \tag{23}$$

$$(A_{k1}A_{l3} - A_{l1}A_{k3})^2 + (A_{k2}A_{l1} - A_{l2}A_{k1})(A_{k2}A_{l3} - A_{l2}A_{k3}) = 0, \tag{24}$$

$$\det\begin{pmatrix} A_{k1} & A_{k2} & A_{k3} \\ A_{l1} & A_{l2} & A_{l3} \\ A_{m1} & A_{m2} & A_{m3} \end{pmatrix} = 0, \qquad i, j = 1, \ldots, 36, \quad k, l, m = 1, \ldots, 65, \tag{25}$$

where

$$\begin{array}{lllll} B_{11}=a_1, & B_{21}=a_2, & B_{31}=a_4-a_5, & B_{41}=\alpha_0, & B_{51}=\alpha_1,\\ B_{12}=-b_2, & B_{22}=-b_1, & B_{32}=b_5-b_4, & B_{42}=-\beta_1, & B_{52}=-\beta_0,\\ B_{71}=\alpha_5, & B_{72}=-\beta_5, & B_{61}=\alpha_2, & B_{62}=-\beta_2, & \end{array} \qquad (26)$$

$$\begin{array}{l} B_{81}=\alpha_7+2M_2\alpha_{11}+2(L_1-L_2)\alpha_{10}-2M_1\alpha_9,\\ B_{82}=\beta_8+2M_2\beta_{10}+2(L_1-L_2)\beta_{11}-2M_1\beta_{12},\\ B_{91}=\alpha_8+2M_1\alpha_{10}+2(L_2-L_1)\alpha_{11}-2M_2\alpha_{12},\\ B_{92}=\beta_7+2M_1\beta_{11}+2(L_2-L_1)\beta_{10}-2M_2\beta_9, \end{array} \qquad (27)$$

$$\begin{array}{ll} B_{9+j,1}=(B_{j1})_t-L_1B_{j1}-M_1B_{j2}, & B_{9+j,2}=(B_{j2})_t-M_2B_{j1}-L_2B_{j2},\\ B_{18+j,1}=(B_{j1})_x-P_1B_{j1}-S_1B_{j2}, & B_{18+j,2}=(B_{j2})_x-Q_2B_{j1}-S_2B_{j2},\\ B_{27+j,1}=(B_{j1})_y-S_1B_{j1}-Q_1B_{j2}, & B_{27+j,2}=(B_{j2})_y-S_2B_{j1}-P_2B_{j2},\\ j=1,\ldots,9, & \end{array} \qquad (28)$$

$$\begin{array}{lllll} A_{11}=a_5, & A_{21}=a_3, & A_{31}=a_7, & A_{41}=\alpha_4, & A_{51}=-\alpha_6,\\ A_{12}=b_4-b_6, & A_{22}=a_6-a_4, & A_{32}=b_8-a_8, & A_{42}=\alpha_3, & A_{52}=\beta_3,\\ A_{13}=-b_3, & A_{23}=-b_5, & A_{33}=-b_7, & A_{43}=-\beta_6, & A_{53}=\beta_4, \end{array} \qquad (29)$$

$$\begin{array}{l} A_{n+k,1}=(A_{k1})_t-2L_1A_{k1}-M_1A_{k2},\, A_{n+k,3}=(A_{k3})_t-M_2A_{k2}-2L_2A_{k3},\\ A_{n+k,2}=(A_{k2})_t-2M_2A_{k1}-(L_1+L_2)A_{k2}-2M_1A_{k3},\\ A_{2n+k,1}=(A_{k1})_x-2P_1A_{k1}-S_1A_{k2},\, A_{2n+k,3}=(A_{k3})_x-Q_2A_{k2}-2S_2A_{k3},\\ A_{2n+k,2}=(A_{k2})_x-2Q_2A_{k1}-(P_1+S_2)A_{k2}-2S_1A_{k3},\\ A_{3n+k,1}=(A_{k1})_y-2S_1A_{k1}-Q_1A_{k2},\, A_{3n+k,3}=(A_{k3})_y-S_2A_{k2}-2P_2A_{k3},\\ A_{3n+k,2}=(A_{k2})_y-2S_2A_{k1}-(S_1+P_2)A_{k2}-2Q_1A_{k3}, \end{array} \qquad (30)$$

where 1) $n=5$, $k=1,\ldots,5$*; 2)* $n=15$, $k=6,\ldots,20$.

Theorem 3. *The system of two second-order ordinary equations* (6) *with the coefficients* V_0, V_1, V_2*, different from zero, is reduced to the form* (11), (12) *by the transformation* (14) *if and only if its coefficients satisfy the following correlations*

$$V_0^2=V_1V_2, \qquad a_0=0, \qquad b_0=0, \qquad (31)$$

$$V_1B_{j1}+V_0B_{j2}=0, \qquad V_0B_{j1}+V_2B_{j2}=0, \qquad j=1,\ldots,13, \qquad (32)$$

$$V_1A_{k1}+V_0A_{k2}+V_2A_{k3}=0, \qquad k=1,\ldots,6, \qquad (33)$$

where B_{ji}, A_{kl} *are determined by the formulae* (26), (29),

$$A_{61}=\alpha_8, \qquad A_{62}=\alpha_7+\beta_7, \qquad A_{63}=\beta_8, \qquad (34)$$

$$\begin{array}{l} B_{81}=V_{2x},\ B_{91}=V_{1y}-2V_{0x},\ B_{10,1}=V_{2y},\ B_{11,1}=\varepsilon+V_{0t},\ B_{12,1}=V_{2t},\\ B_{82}=-V_{1y},\ B_{92}=V_{1x},\ B_{10,2}=V_{2x}-2V_{0y},\ B_{11,2}=-V_{1t},\ B_{12,2}=\varepsilon-V_{0t}, \end{array} \qquad (35)$$

where $\varepsilon=M_1V_1+(L_2-L_1)V_0-M_2V_2$ *and*

$$B_{13,1}=\alpha_8, \qquad B_{13,2}=\beta_7 \qquad \text{for } \varepsilon=0, \qquad (36)$$

$$\begin{array}{l} B_{13,1}=\alpha_8\varepsilon_x+\alpha_7\varepsilon_y+\varepsilon[P_1\alpha_8+S_1(\alpha_7+\beta_7)+Q_1\beta_8-\alpha_{7y}-\alpha_{8x}],\\ B_{13,2}=\beta_8\varepsilon_y+\beta_7\varepsilon_x+\varepsilon[P_2\beta_8+S_2(\alpha_7+\beta_7)+Q_2\alpha_8-\beta_{7x}-\beta_{8y}] \text{ for } \varepsilon\neq 0. \end{array} \qquad (37)$$

The conditions of Theorem 2 are obtained by means of standard (analogous to the way it was made in [9]) study of compatibility of the system (16), which results in equations

$$B_{j1}\varphi_x+B_{j2}\varphi_y=0, \qquad (38)$$

linear by φ_x, φ_y with coefficients (26)–(28), and equations

$$A_{k1}\varphi_x^2+A_{k2}\varphi_x\varphi_y+A_{k3}\varphi_y^2=0 \qquad (39)$$

of the second degree by φ_x, φ_y with the coefficients (29), (30). The equalities (22) correspond the condition of compatibility of the system (38), and (24), (25) — of the system (39). The equalities (23) denote the condition of compatibility of the equations (38) with the equations (39).

Remark. The system (38), (39) can have two solutions $\varphi_x/\varphi_y = \phi_1(t,x,y)$, $\varphi_x/\varphi_y = \phi_2(t,x,y)$ such that $\partial(\phi_1,\phi_2)/\partial(x,y) \neq 0$. It denotes, that the use of corresponding solutions as $\tilde{x}$, $\tilde{y}$ results in the system, which equations are completely separated. It is necessary and sufficient for this for the correlations (38) all $B_{ji} = 0$, and for (39) $\mathrm{rank}\|A_{kl}\| = 1$, and, if any line (A_{k1}, A_{k2}, A_{k3}) differs from zero, then $A_{k2}^2 - 4A_{k1}A_{k3} \neq 0$. The same remark also holds for the case of the transformation (2) with $\theta_x \neq 0$ or $\theta_y \neq 0$ (the corresponding statements on separation of equations in the system of two second-order ordinary differential equations are given in [10]).

Theorem 3 is proved similarly. The first condition (31) and the equalities

$$V_0\varphi_x = V_1\varphi_y, \quad V_2\varphi_x = V_0\varphi_y \tag{40}$$

serve as an algebraic corollary of the equations (17). The study of compatibility of the equations (16), (17) results in conditions $a_0 = 0$, $b_0 = 0$, correlations (38) with the coefficients (26), (35), (36) or (37) and the correlations (39) with the coefficients (29), (34), that subject to (40) provides conditions (32), (33) of Theorem 3.

3. Examples of systems with a separating equation

Example 1. Let us consider the family of equations

$$\begin{aligned} x'' &= P_1(x,y)x'^2 + 2S_1(x,y)x'y' + Q_1(x,y)y'^2, \\ y'' &= P_2(x,y)y'^2 + 2S_2(x,y)x'y' + Q_2(x,y)x'^2, \end{aligned} \tag{41}$$

with the form of dependence on the first derivatives, analogous to that of geodesic equations in the space with Riemannian metric

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \qquad i,j,k, = 1,2. \tag{42}$$

If we assume $(x^1, x^2) = (x, y)$, then Christoffel symbols Γ^i_{jk} are connected with the coefficients of the system (41) by the correlations

$$\Gamma^1_{11} = -P_1, \quad \Gamma^1_{12} = -S_1, \quad \Gamma^1_{22} = -Q_1, \quad \Gamma^2_{11} = -Q_2, \quad \Gamma^2_{12} = -S_2, \quad \Gamma^2_{22} = -P_2.$$

It is known, that the equations (41) referred to the parameter x take the form

$$\frac{d^2y}{dx^2} = Q_2 + (2S_2 - P_1)\frac{dy}{dx} + (P_2 - 2S_1)\left(\frac{dy}{dx}\right)^2 - Q_1\left(\frac{dy}{dx}\right)^3,$$

i.e. separation of the equation in the system (41) as a result of the transformation (2) with $\theta_x \neq 0$

$$\tilde{t} = x, \qquad \tilde{x} = y, \qquad \tilde{y} = t$$

holds for any system (41). Let us find conditions, when an equation is separated in the system (41) as a result of the transformation of the form (14).

If we calculate invariants (13) by the formulae (5) we can determine, that for the system (41) four invariants a_1, a_2, b_1, b_2 form 18 a_j, b_j, $j = 0, \dots, 8$ differ from zero. In case of the system (42) they coincide with components of the curvature tensor [10]

$$a_1 = R^1_{221}, \qquad a_2 = R^1_{121}, \qquad b_1 = R^2_{112}, \qquad b_2 = R^2_{212}.$$

It is easy to see, that all values (29), (30) are equal to zero, and, accordingly, for the system (41) all the conditions (23)–(25) of Theorem 2 are satisfied identically. The condition (22) is satisfied, if the rank of the next matrix (formed from non-nil lines of matrix B) is lower than 1:

$$\tilde{B} = \begin{pmatrix} a_1 & b_2 \\ a_2 & b_1 \\ a_{1x} - P_1 a_1 + S_1 b_2 & b_{2x} - S_2 b_2 + Q_2 a_1 \\ a_{2x} - P_1 a_2 + S_1 b_1 & b_{1x} - S_2 b_1 + Q_2 a_2 \\ a_{1y} - S_1 a_1 + Q_1 b_2 & b_{2y} - P_2 b_2 + S_2 a_1 \\ a_{2y} - S_1 a_2 + Q_1 b_1 & b_{1y} - P_2 b_1 + S_2 a_2 \end{pmatrix}.$$

The separation of an equation in the system (41) is possible, if $\mathrm{rank}\tilde{B} = 1$. If $\mathrm{rank}\tilde{B} = 0$, then the equations (41) are completely separated and, due to Theorem 1, reduced to the form $\tilde{x}'' = 0$, $\tilde{y}'' = 0$. It is demonstrated in [9, 10] , that in particular case of the system (42) the equality $a_1 b_1 - a_2 b_2 = 0$ is possible, only if $a_1 = 0$, $a_2 = 0$, $b_1 = 0$, $b_2 = 0$.

Example 2. The system, which in the case $\Gamma_t = 0$ describes plane motion of a particle under gyroscopic forces, has the form

$$x'' = 2\Gamma\, y' - U_x, \qquad y'' = -2\Gamma\, x' - U_y, \qquad \Gamma \neq 0. \tag{43}$$

The functions Γ, U of variables t, x, y are assumed to be real, for the second derivatives of the function U the following table of symbols is used $V = U_{xy}$, $W = U_{xx} - U_{yy}$. Let us consider an example of a nonlinear system (43). The following coefficients in invariants (13) differ from zero for this system : $a_3 = -\Gamma_y$, $b_3 = \Gamma_x$, $a_4 = -\Gamma_x$, $b_4 = \Gamma_y$, $a_7 = U_{xy} + \Gamma_t$, $b_7 = U_{xy} - \Gamma_t$, $a_8 = U_{xx} + \Gamma^2$, $b_8 = U_{yy} + \Gamma^2$. It results from conditions (23), when $k = 1, 2$, $j = 3$ possessing the form $-\Gamma_x(\Gamma_x^2 + \Gamma_y^2) = 0$, $-\Gamma_y(\Gamma_x^2 + \Gamma_y^2) = 0$, that $\Gamma_x = 0$, $\Gamma_y = 0$.

Assume in the system (43) $\Gamma = \Gamma(t)$. Then all the coefficients (26), (28) are equal to zero, and the first several nonlinear coefficients (29), (30) are equal to

$$\begin{aligned} &A_{31} = \Gamma' + U_{xy}, \quad A_{32} = U_{yy} - U_{xx}, \quad A_{33} = \Gamma' - U_{xy}, \quad A_{41} = -A_{43} = -3U_{xxy}, \\ &A_{42} = 3(U_{xyy} - U_{xxx}), \quad A_{51} = -A_{53} = -3U_{xyy}, \quad A_{52} = 3(U_{yyy} - U_{xxy}), \\ &A_{81} = \Gamma'' + \Gamma(U_{xx} - U_{yy}) + U_{txy}, \quad A_{82} = 4\Gamma U_{xy} + U_{tyy} - U_{txx}, \\ &A_{83} = \Gamma'' + \Gamma(U_{yy} - U_{xx}) - U_{txy}. \end{aligned}$$

Together with them the correlations (24) take the form

$$\begin{aligned} &(4V_x^2 + W_x^2)\Gamma'^2 = (WV_x - VW_x)^2, \quad (4V_y^2 + W_y^2)\Gamma'^2 = (WV_y - VW_y)^2, \\ &(W_xV_{xx} - V_xW_{xx})^2 = 0, \quad (W_xV_{xy} - V_xW_{xy})^2 = 0, \\ &(W_yV_{xy} - V_yW_{xy})^2 = 0, \quad (W_yV_{yy} - V_yW_{yy})^2 = 0, \quad (V_xW_y - V_yW_x)^2 = 0, \\ &(\Gamma''^2 + 4\Gamma^2\Gamma'^2)(4V^2 + W^2) - 2\Gamma'\Gamma''(4VV_t + WW_t) + 8\Gamma\Gamma'^2(WV_t - VW_t) \\ &+\Gamma'^2(4V_t^2 + W_t^2) = (\Gamma(4V^2 + W^2) + WV_t - VW_t)^2, \end{aligned}$$

that results in $W = f_1(t)V + f_0(t)$, $f_0^2 = (f_1^2 + 4)\Gamma'^2$, $f_1' = (f_1^2 + 4)\Gamma$. Assume $\Gamma = \gamma'/2$ with some function $\gamma(t)$. Then $f_1(t) = 2\,\mathrm{tg}\gamma$, $f_0(t) = \pm\gamma''/\mathrm{cos}\gamma$ and the function $U(t, x, y)$ should satisfy the equation

$$U_{xx} - U_{yy} = 2\,\mathrm{tg}\gamma\, U_{xy} \pm \frac{\gamma''}{\mathrm{cos}\gamma}. \tag{44}$$

Consequently, separation of an equation in the system (43) is possible, if

$$\begin{gathered} \Gamma = \frac{\gamma'(t)}{2}, \qquad U = \frac{\gamma''}{4}(m(x^2 - y^2)\mathrm{cos}\gamma - 2xy\,\mathrm{sin}\gamma) \\ +V_{+1}(t, y\mathrm{cos}\gamma + x(\mathrm{sin}\gamma + 1)) + +V_{-1}(t, y\mathrm{cos}\gamma + x(\mathrm{sin}\gamma - 1)), \end{gathered} \tag{45}$$

where m is equal either to $+1$ or to -1. In case of the functions (45) all the conditions (24), (25) of Theorem 2 become identical, and all quadratic equations (39) possess a general root $\varphi_x/\varphi_y = (\mathrm{sin}\gamma + m)/\mathrm{cos}\gamma$. The solution of the last equation is $\varphi = \phi(t, z)$, where $z = y\,\mathrm{cos}\gamma + x(\mathrm{sin}\gamma + m)$. Its substitution to (16) results in the system of equations, when

$\theta = t$ possessing partial solution $\phi = (\sin\gamma + m)^{-1/2} z$. Therefore, in the system (43) with the coefficients (45) there is the equation

$$\tilde{x}'' + \frac{\gamma'^2}{4}\tilde{x} + 2m\sqrt{\sin\gamma + m}\,\frac{\partial V_m}{\partial z} = 0$$

separated relative to the function $\tilde{x}(t) = (\sin\gamma + m)^{-1/2}(y\cos\gamma + x(\sin\gamma + m))$.

4. Criterion of separation of an equation in the system (6). General case

Let us find conditions, when the system (6) can be transformed into the system with a separating equation (11), (12) by substitution of the variables (2) with $\theta_x \neq 0$ or $\theta_y \neq 0$. Assume $\theta_x \neq 0$ for more precision. Substitution of the transformation (2) in the system (11), (12) results in the system of the second-order ordinary differential equations with the same form of dependence on x', y', like in the equations (6). If we equate its coefficients with the degrees x', y' with corresponding coefficients of the equations (6), we obtain 15 correlations, which with $\theta_x \neq 0$ can be solved relative to all derivatives of the second order of the functions φ, ψ and three derivatives of the function θ. Likewise in the case of transformation (14), considered in §2, we obtain separation of a subsystem of nine equations

$$\begin{aligned}
&\theta_{yy} = V_2\theta_t - Q_1\theta_x + (2F_2 - P_2)\theta_y - S\varphi_y^2,\\
&\theta_{ty} = F_2\theta_t - M_1\theta_x + (F_1 - L_2)\theta_y - S\varphi_t\varphi_y,\\
&\theta_{tt} = 2F_1\theta_t - K_1\theta_x - K_2\theta_y - S\varphi_t^2,
\end{aligned} \tag{46}$$

$$\begin{aligned}
&\varphi_{xx} = V_1\varphi_t + (2F_0 - P_1)\varphi_x - Q_2\varphi_y + P\theta_x^2 + 3Q\theta_x\varphi_x + 3R\varphi_x^2,\ \ \varphi_{xy} = V_0\varphi_t\\
&+(F_2 - S_1)\varphi_x + (F_0 - S_2)\varphi_y + P\theta_x\theta_y + 3/2Q(\theta_x\varphi_y + \theta_y\varphi_x) + 3R\varphi_x\varphi_y,\\
&\varphi_{yy} = V_2\varphi_t - Q_1\varphi_x + (2F_2 - P_2)\varphi_y + P\theta_y^2 + 3Q\theta_y\varphi_y + 3R\varphi_y^2,\\
&\varphi_{tx} = F_0\varphi_t + (F_1 - L_1)\varphi_x - M_2\varphi_y + P\theta_t\theta_x + 3/2Q(\theta_t\varphi_x + \theta_x\varphi_t) + 3R\varphi_t\varphi_x,\\
&\varphi_{ty} = F_2\varphi_t - M_1\varphi_x + (F_1 - L_2)\varphi_y + P\theta_t\theta_y + 3/2Q(\theta_t\varphi_y + \theta_y\varphi_t) + 3R\varphi_t\varphi_y,\\
&\varphi_{tt} = 2F_1\varphi_t - K_1\varphi_x - K_2\varphi_y + P\theta_t^2 + 3Q\theta_t\varphi_t + 3R\varphi_t^2,
\end{aligned} \tag{47}$$

which compatibility results in compatibility of all 15 equations relative to the functions θ, φ, ψ. The following table of symbols is used here

$$\begin{aligned}
&F_0 = (\theta_{xx} - V_1\theta_t + P_1\theta_x + Q_2\theta_y + S\varphi_x^2)/(2\theta_x),\\
&F_1 = (\theta_{tx} - F_0\theta_t + L_1\theta_x + M_2\theta_y + S\varphi_t\varphi_x)/\theta_x,\\
&F_2 = (\theta_{xy} - V_0\theta_t + S_1\theta_x + (S_2 - F_0)\theta_y + S\varphi_x\varphi_y)/\theta_x.
\end{aligned}$$

The proof of the below criterion of separation of an equation in the system (6) is carried out by analogy with the proof of Theorems 2, 3 and results in equalities, equivalent to (38), (39). The role of derivatives φ_x, φ_y in them is carried out by minors $M_{31} = \theta_x\varphi_y - \theta_y\varphi_x$, $M_{33} = \theta_t\varphi_x - \theta_x\varphi_t$ of Jacobi matrix of the transformation (2). They cannot be equal to zero simultaneously, otherwise, we obtain $M_{32} = 0$ from the identity $\theta_t M_{31} - \theta_x M_{32} + \theta_y M_{33} = 0$ when $\theta_x \neq 0$. Then it results from the expansion $\Delta = \psi_t M_{31} - \psi_x M_{32} + \psi_y M_{33}$ that Jacobi transformation (2) is equal to zero, that contradicts supposition on its nonsingularity.

Theorem 4. *The system of two second-order ordinary differential equations* (6) *is reduced to the form* (11), (12) *by the transformation* (2) *with* $\theta_x \neq 0$ *if and only if the system of algebraic and differential equations is compatible relative to* T, Y*:*

$$\begin{aligned}
\Phi_1 \equiv\ & a_3 - a_1T + (a_6 - a_5 - a_4)Y - b_0T^2 + (a_2 + b_2)TY\\
& +(b_6 - b_5 - b_4)Y^2 - a_0T^2Y - b_1TY^2 + b_3Y^3 = 0,\\
\Phi_2 \equiv\ & a_7 + (a_4 - 2a_5)T + (b_8 - a_8)Y + a_2T^2 + (b_6 + b_5 - 2b_4)TY\\
& -b_7Y^2 - a_0T^3 - b_1T^2Y + b_3TY^2 = 0,
\end{aligned} \tag{48}$$

$$\Delta_t\Phi_1 = 0, \quad \Delta_t\Phi_2 = 0, \quad \Delta_y\Phi_1 = 0, \quad \Delta_y\Phi_2 = 0, \tag{49}$$

$$T_y - Y_t = YT_x - TY_x, \tag{50}$$

$$B_{i1}B_{j2} - B_{j1}B_{i2} = 0, \tag{51}$$

$$B_{j2}^2 A_{k1} - B_{j1}B_{j2}A_{k2} + B_{j1}^2 A_{k3} = 0, \tag{52}$$

$$(A_{k1}A_{l3} - A_{l1}A_{k3})^2 + (A_{k2}A_{l1} - A_{l2}A_{k1})(A_{k2}A_{l3} - A_{l2}A_{k3}) = 0, \tag{53}$$

$$\det\begin{pmatrix} A_{k1} & A_{k2} & A_{k3} \\ A_{l1} & A_{l2} & A_{l3} \\ A_{m1} & A_{m2} & A_{m3} \end{pmatrix} = 0, \quad i,j = 1,\ \ldots,10, \quad k,l,m = 1,\ \ldots,15, \tag{54}$$

where $\Delta_t = \partial_t - T\partial_x + \lambda_0\partial_T + \lambda_1\partial_Y + (\lambda_{0x} + T_x\lambda_{0T} + Y_x\lambda_{0Y} + T_x^2)\partial_{T_x}$
$+(\lambda_{1x} + T_x\lambda_{1T} + Y_x\lambda_{1Y} + T_xY_x)\partial_{Y_x}$,
$\Delta_y = \partial_y - Y\partial_x + \lambda_1\partial_T + \lambda_2\partial_Y + (\lambda_{1x} + T_x\lambda_{1T} + Y_x\lambda_{1Y} + T_xY_x)\partial_{T_x}$
$+(\lambda_{2x} + T_x\lambda_{2T} + Y_x\lambda_{2Y} + Y_x^2)\partial_{Y_x}$,
$B_{11} = \Phi_{1Y}, \qquad B_{12} = -\Phi_{1T}, \qquad B_{21} = -\Phi_{2Y}, \qquad B_{22} = \Phi_{2T}$,
$B_{2+j,1} = \Delta_t B_{j1} + (\lambda_{1Y} + 2T_x)B_{j1} - \lambda_{0Y}B_{j2}$,
$B_{2+j,2} = \Delta_t B_{j2} - (\lambda_{1T} + Y_x)B_{j1} + (\lambda_{0T} + 3T_x)B_{j2}$,

$$\begin{aligned}
&B_{4+j,1} = \Delta_y B_{j1} + (\lambda_{2Y} + 3Y_x)B_{j1} - (\lambda_{1Y} + T_x)B_{j2},\\
&B_{4+j,2} = \Delta_y B_{j2} - \lambda_{2T}B_{j1} + (\lambda_{1T} + 2Y_x)B_{j2}, \quad j = 1,2,\\
&B_{71} = \lambda_0 + TT_x - T_t,\ B_{72} = B_{81} = \lambda_1 + TY_x - Y_t,\ B_{82} = \lambda_2 + YY_x - Y_y,\\
&B_{91} = 2\alpha_4 - \lambda_{0TY}\alpha_6 + (\lambda_{0TT} - 2\lambda_{1TY})\alpha_7 + \lambda_{1TT}\alpha_8,\\
&B_{92} = 2\beta_4 - \lambda_{0TY}\beta_6 + (\lambda_{0TT} - 2\lambda_{1TY})\beta_7 + \lambda_{1TT}\beta_8,\\
&B_{10,1} = 2\alpha_5 - \lambda_{0TY}\alpha_7 + (\lambda_{0TT} - 2\lambda_{1TY})\alpha_8 + \lambda_{1TT}\alpha_9,\\
&B_{10,2} = 2\beta_5 - \lambda_{0TY}\beta_7 + (\lambda_{0TT} - 2\lambda_{1TY})\beta_8 + \lambda_{1TT}\beta_9,
\end{aligned} \tag{55}$$

$$\begin{aligned}
&A_{11} = b_3, \quad A_{12} = b_1, \quad A_{13} = -a_0,\\
&A_{21} = b_4 + b_5 - b_6 + b_1T - 2b_3Y, \quad A_{22} = a_2 + b_2 - 2a_0T - b_1Y, \quad A_{23} = b_0,\\
&A_{31} = b_7, \quad A_{32} = b_6 + b_5 - 2b_4 - b_1T + 2b_3Y, \quad A_{33} = -a_2 + 2a_0T + b_1Y,\\
&A_{3+k,1} = \Delta_t A_{k1} + 2(\lambda_{1Y} + 2T_x)A_{k1} - \lambda_{0Y}A_{k2},\\
&A_{3+k,2} = \Delta_t A_{k2} - 2(\lambda_{1T} + Y_x)A_{k1} + (\lambda_{0T} + \lambda_{1Y} + 5T_x)A_{k2} - 2\lambda_{0Y}A_{k3},\\
&A_{3+k,3} = \Delta_t A_{k3} - (\lambda_{1T} + Y_x)A_{k2} + 2(\lambda_{0T} + 3T_x)A_{k3},\\
&A_{6+k,1} = \Delta_y A_{k1} + 2(\lambda_{2Y} + 3Y_x)A_{k1} - (\lambda_{1Y} + T_x)A_{k2},\\
&A_{6+k,2} = \Delta_y A_{k2} - 2\lambda_{2T}A_{k1} + (\lambda_{1T} + \lambda_{2Y} + 5Y_x)A_{k2} - 2(\lambda_{1Y} + T_x)A_{k3},\\
&A_{6+k,3} = \Delta_y A_{k3} - \lambda_{2T}A_{k2} + 2(\lambda_{1T} + 2Y_x)A_{k3}, \quad k = 1,2,3,\\
&A_{10,1} = \alpha_2, \quad A_{10,2} = \beta_2 - \alpha_1, \quad A_{10,3} = -\beta_1,\\
&A_{11,1} = \alpha_3, \quad A_{11,2} = \beta_3 + \alpha_2, \quad A_{11,3} = \beta_2,\\
&A_{12,1} = \alpha_5, \quad A_{12,2} = \beta_5 - \alpha_4, \quad A_{12,3} = -\beta_4,\\
&A_{13,1} = \alpha_7, \quad A_{13,2} = \beta_7 - \alpha_6, \quad A_{13,3} = -\beta_6,\\
&A_{14,1} = \alpha_8, \quad A_{14,2} = \beta_8 - \alpha_7, \quad A_{14,3} = -\beta_7,\\
&A_{15,1} = \alpha_9, \quad A_{15,2} = \beta_9 - \alpha_8, \quad A_{15,3} = -\beta_8.
\end{aligned}$$

In (55) the following table of symbols is used

$$
\begin{aligned}
&\lambda_0 = -K_1 + 2L_1T - P_1T^2 + V_1T^3 + (-K_2 + 2M_2T - Q_2T^2)Y,\\
&\lambda_1 = S_1T - M_1 + (L_1 - L_2)Y - V_0T^2 + (S_2 - P_1)TY + M_2Y^2 + V_1T^2Y - Q_2TY^2,\\
&\lambda_2 = -Q_1 + (2S_1 - P_2)Y + (2S_2 - P_1)Y^2 - Q_2Y^3 + (V_2 - 2V_0Y + V_1Y^2)T,\\
&\beta_1 = 2a_0(\lambda_{1T} + \lambda_{2Y} + 4Y_x) + 2\gamma_1Y + 3(-a_{0y} - S_1a_0 - V_2b_1 + V_0a_2) - b_{0x} - P_2a_0\\
&\quad +(S_2 - P_1)b_0 - V_1a_1 + V_0b_2, \qquad \gamma_1 = a_{0x} + P_1a_0 - Q_2b_0 + V_0b_1 - V_1a_2,\\
&\alpha_1 = 2a_0(\lambda_{0T} + \lambda_{1Y} + 4T_x) + 2\gamma_1T + 3(a_{0t} + b_{1y} + L_1a_0 - Q_2a_1 + S_1b_1 - 2V_2b_3)\\
&\quad -2a_{2x} - b_{2x} - L_2a_0 - 4M_2b_0 - P_1(2a_2 + b_2) + S_2(a_2 + 2b_2)\\
&\quad +V_0(8b_4 - b_5 - 5b_6) + V_1(7a_5 - 2a_4 - a_6),\\
&\beta_2 = b_1(\lambda_{1T} + \lambda_{2Y} + 4Y_x) + \gamma_2Y + 3a_{0t} - a_{2x} + 3(L_1 + L_2)a_0 - 2M_2b_0 - Q_2a_1\\
&\quad +P_2b_1 - (P_1 + S_2)a_2 + V_0(2b_4 - b_5 - b_6) + 2V_1(2a_5 - a_4),\\
&\alpha_2 = b_1(\lambda_{0T} + \lambda_{1Y} + 4T_x) + \gamma_2T + (b_4 + b_5 - b_6)_x - 3b_{3y} - 2L_2b_1 + 3(P_2 - S_1)b_3\\
&+M_2(a_2 + b_2) + (P_1 - 3S_2)(b_4 + b_5 - b_6) + 2Q_2(a_6 - a_5 - a_4) + 2V_0b_7 + V_1(a_8 - b_8),\\
&\gamma_2 = b_{1x} - 2M_2a_0 + (P_1 - S_2)b_1 + Q_2(a_2 + b_2) - 2V_0b_3 + V_1(2b_4 - b_5 - b_6),\\
&\beta_3 = 2b_3(\lambda_{1T} + \lambda_{2Y} + 4Y_x) + 2\gamma_3Y + 3(b_{3y} - b_{1t} - b_{4x} + 2K_2a_0 + (L_2 - L_1)b_1\\
&\quad -M_2b_2 + S_1b_3 + S_2(b_4 + b_5 - b_6) + V_1(b_8 - a_8)) + 2b_{6x} - 5M_2a_2 - P_2b_3\\
&\quad +Q_2(2a_4 + 5a_5 - 3a_6) + P_1(2b_6 - 3b_4) - 4V_0b_7,\\
&\gamma_3 = b_{3x} + M_2b_1 + (P_1 - 2S_2)b_3 + Q_2(b_4 + b_5 - b_6) - V_1b_7,\\
&\alpha_3 = 2b_3(\lambda_{0T} + \lambda_{1Y} + 4T_x) + 2\gamma_3T - 3(b_{3t} + K_2b_1 + L_1b_3) + b_{7x} + 5L_2b_3\\
&\quad +M_2(3b_6 - b_5 - 4b_4) + (P_1 - S_2)b_7 + Q_2(b_8 - a_8),\\
&\beta_4 = \Delta_t\beta_1 + \Delta_y\beta_2 + (\lambda_{0T} - \lambda_{1Y} - T_x)\beta_1 - (\lambda_{1T} + Y_x)\alpha_1 + (2\lambda_{1T} - \lambda_{2Y} - 2Y_x)\beta_2\\
&\quad -\lambda_{2T}(\alpha_2 + \beta_3),\\
&\alpha_4 = \Delta_t\alpha_1 + \Delta_y\alpha_2 - \lambda_{0Y}\beta_1 - 2T_x\alpha_1 - (\lambda_{1Y} + T_x)\beta_2 + (\lambda_{1T} - Y_x)\alpha_2 - \lambda_{2T}\alpha_3,\\
&\beta_5 = \Delta_t\beta_2 - \Delta_y\beta_3 + \lambda_{0Y}\beta_1 + (\lambda_{1Y} - T_x)\beta_2 - (\lambda_{1T} + Y_x)\alpha_2 + 2Y_x\beta_3 + \lambda_{2T}\alpha_3,\\
&\alpha_5 = \Delta_t\alpha_2 - \Delta_y\alpha_3 + \lambda_{0Y}(\alpha_1 - \beta_2) + (2\lambda_{1Y} - \lambda_{0T} - 2T_x)\alpha_2 + (\lambda_{1Y} + T_x)\beta_3\\
&\quad +(\lambda_{1T} - \lambda_{2Y} + Y_x)\alpha_3,\\
&\beta_6 = 2(b_0 + a_0Y)(2\lambda_{2Y} - \lambda_{1T} + 4Y_x) + 2a_0(T\lambda_{2T} - \lambda_2) + 2(b_2 - b_1Y)\lambda_{2T}\\
&\quad +2(-b_{0y} + (b_{0x} - a_{0y})Y + a_{0x}Y^2),\\
&\alpha_6 = 2b_0(\lambda_{0T} + \lambda_{1Y} + 4T_x) + \alpha_1Y + 3V_1\Phi_1 - \lambda_{1TT}\Phi_{1Y}\\
&\quad +(3b_{0x} + a_{0y} + (P_2 - S_1)a_0 + (3S_2 + P_1)b_0 + 3V_1a_1 + V_2b_1 - V_0(a_2 + 3b_2))T\\
&\quad +3(b_{0t} + a_{1x} + L_2b_0 - Q_1b_1 + S_2a_1 - 2V_1a_3) - a_{2y} - 2b_{2y} - L_1b_0 - 4M_1a_0\\
&\quad -P_2(a_2 + 2b_2) + S_1(2a_2 + b_2) + V_0(8a_4 - a_5 - 5a_6) + V_2(7b_5 - 2b_4 - b_6),\\
&\beta_7 = 2(b_0 + a_0Y)(\lambda_{0T} - 2\lambda_{1Y} - T_x) + 2a_0(\lambda_1 - T\lambda_{1T} - TY_x)\\
&\quad +2(b_1Y - b_2)(\lambda_{1T} + Y_x) + 2(b_{0t} - b_{0x}T + a_{0t}Y - a_{0x}TY),\\
&\alpha_7 = (a_0T - a_2 - 2b_2 + 3b_1Y)(\lambda_{1Y} + 2T_x) + \gamma_1T^2 + 3\gamma_2TY - 3\gamma_3Y^2 + 3V_0\Phi_{2Y}\\
&\quad +(b_4 + b_5 - b_6 + b_1T - 3b_3Y)(2Y_x - \lambda_{1T}) + V_1(T\Phi_{2T} - \Phi_2) + 5M_2\Phi_{1T}\\
&\quad +3Q_2(2\Phi_1 - 2T\Phi_{1T} - Y\Phi_{1Y}) - 3b_3Y\lambda_{2Y} + b_1(T\lambda_{1T} + 3Y\lambda_{1Y} - \lambda_1)\\
&\quad +3b_0\lambda_{0Y} - b_7\lambda_{2T} + a_0(T\lambda_{0T} + 6Y\lambda_{0Y} - 2\lambda_0) - (a_{0t} + a_{2x} + 2b_{2x})T\\
&\quad +(3(b_4 + b_5 - b_6)_x - 3b_{3y} + 4K_2a_0 + 2(L_2 - L_1)b_1 - 2M_2(a_2 + b_2)\\
&+2V_1(a_8 - b_8) + 4V_0b_7)Y + (a_{2t} - b_{2t})/4 + 3/4((4a_4 + a_5 - 3a_6)_x + 3b_{5y} - b_{6y})\\
&\quad -K_1a_0 + 2K_2b_0 + 2M_2a_1 - M_1b_1 + 3Q_1b_3 - 3Q_2a_3 + V_1a_7 - 2V_2b_7,\\
&\beta_8 = (3a_0T - a_2 + b_1Y)(\lambda_{1Y} - 2T_x) + (b_6 + 3b_5 - 3b_4 - 3b_1T + b_3Y)(\lambda_{1T} + 2Y_x)\\
&\quad -3\gamma_1T^2 - 3\gamma_2TY + \gamma_3Y^2 + 3M_2\Phi_{1T} + Q_2(Y\Phi_{1Y} - \Phi_1) + 5V_0\Phi_{2Y}\\
&\quad +3V_1(2\Phi_2 - T\Phi_{2T} - 2Y\Phi_{2Y}) + b_0\lambda_{0Y} - 3a_0T\lambda_{0T} + b_1(\lambda_1 - 3T\lambda_{1T} - Y\lambda_{1Y})\\
&\quad -3b_7\lambda_{2T} + b_3(6T\lambda_{2T} + Y\lambda_{2Y} - 2\lambda_2) + ((b_6 + 3b_5 - 3b_4)_x - b_{3y})Y\\
&\quad +(3a_{2x} - 3a_{0t} + 4M_2b_0 + 2Q_2a_1 + 2S_1b_1 + 2V_0(2b_4 - b_5 - b_6) - 4V_2b_3)T\\
&\quad +3/4(a_{2t} - b_{2t} + (4a_4 - 5a_5 - a_6)_x + 3b_{5y} - b_{6y})/4 + 3K_1a_0 - 2K_2b_0\\
&\quad +M_1b_1 + Q_2a_3 - Q_1b_3 - 3V_1a_7 + 2V_2b_7 + 2V_0(a_8 - b_8),\\
&\alpha_8 = 2(b_7 - b_3T)(2\lambda_{1T} - \lambda_{2Y} + Y_x) + 2b_3(\lambda_1 - Y\lambda_{1Y} - YT_x)\\
&\quad +2(2b_5 - b_4 - b_1T)(\lambda_{1Y} + T_x) + 2(-b_{7y} + b_{3y}T + b_{7x}Y - b_{3x}TY),
\end{aligned}
$$

$$\beta_9 = -2b_7(\lambda_{1T} + \lambda_{2Y} + 4Y_x) + \beta_3 T + 3Q_2\Phi_2 + \lambda_{1YY}\Phi_{2T} + (b_{3t} - 3b_{7x} + K_2 b_1 \\ -(L_1 + L_2)b_3 + M_2(4b_4 - 5b_5 - b_6) + (5S_2 - P_1)b_7 + 3Q_2(a_8 - b_8))Y \\ +3(b_{8x} - a_{8x} - b_{7y} + K_1 b_1 + K_2 b_2 + L_2(b_4 - 2b_5) - 2Q_2 a_7 + S_2(a_8 - b_8)) \\ +(b_6 + 3b_5 - 3b_4)_t - K_2 a_2 - 4M_1 b_3 + L_1(3b_4 - 2b_6) + M_2(3a_6 + 7a_5 - 8a_4) \\ +(P_2 + S_1)b_7,$$

$$\alpha_9 = 2(b_7 - b_3 T)(\lambda_{1Y} - 2\lambda_{0T} - 4T_x) + 2b_3(Y\lambda_{0Y} - \lambda_0) \\ +2(b_4 - 2b_5 + b_1 T)\lambda_{0Y} + 2(b_{7t} - (b_{3t} + b_{7x})T + b_{3x}T^2). \tag{56}$$

Proof. Let us find conditions of compatibility of the system (46), (47). If we denote $T = \theta_t/\theta_x$, $Y = \theta_y/\theta_x$, then the condition of equality of the derivatives θ_{tyy}, θ_{yyt} and θ_{tty}, θ_{tyt}, calculated my means of differentiation of expressions (46) by t, y, takes the form (48). Differentiation of (47) by t, x, y and comparison of mixed third-order derivatives of the function φ results in two correlations

$$B_{i1}M_{31} + B_{i2}M_{33} = 0, \tag{57}$$

where B_{ij}, $i, j = 1, 2$ are introduced in (55), and six correlations

$$(G_0 + b_1 + \Omega_1\theta_x^2 + \Omega_2\theta_x\varphi_x + \Omega_3\varphi_x^2)M_{31} - a_0 M_{33} = 0, \tag{58}$$

$$b_3 M_{31} - (G_0 + \Omega_1\theta_x^2 + \Omega_2\theta_x\varphi_x + \Omega_3\varphi_x^2)M_{33} = 0, \tag{59}$$

$$(G_1 - 2b_4 - 2(G_0 + b_1)T + 2b_3 Y)M_{31} + (G_2 - a_2 + 2a_0 T + (b_1 - 2G_0)Y)M_{33} = 0, \tag{60}$$

$$(3G_2 + a_2 + 2b_2 - 4a_0 T - (4G_0 + b_1)Y + 2(\Omega_1\theta_x^2 + \Omega_2\theta_x\varphi_x + \Omega_3\varphi_x^2)Y)M_{31} \\ +(2\Omega_2 - PS + 3\Omega_3\varphi_x/\theta_x)M_{31}^2 + 2b_0 M_{33} = 0, \tag{61}$$

$$(3G_2 - a_2 + (b_1 - 4G_0)Y + 2(\Omega_1\theta_x^2 + \Omega_2\theta_x\varphi_x + \Omega_3\varphi_x^2)Y)M_{33} \\ +(2\Omega_2 - PS + 3\Omega_3\varphi_x/\theta_x)M_{31}M_{33} + 2(b_6 - b_5 - b_4 - b_1 T + 2b_3 Y)M_{31} = 0, \tag{62}$$

$$(3G_1 - 4b_6 - 4G_0 T - 4b_3 Y + 2(\Omega_1\theta_x^2 + \Omega_2\theta_x\varphi_x + \Omega_3\varphi_x^2)T)M_{33} \\ -(2\Omega_2 - PS + 3\Omega_3\varphi_x/\theta_x)M_{33}^2 - 2b_7 M_{31} = 0, \tag{63}$$

where $\Omega_1 = 3/2Q_\theta - P_\varphi - 9/4Q^2 + 3PR$, $\Omega_2 = 3R_\theta - 3/2Q_\varphi + 2PS$, $\Omega_3 = S_\theta + 3/2QS$,

$$G_0 = F_{0x} - F_0^2 + P_1 F_0 + Q_2 F_2 - V_{1t} - (L_1 + F_1)V_1 - M_2 V_0, \\ G_1 = F_{1x} + (L_1 - F_1)F_0 + M_2 F_2 + b_6 - (K_1 V_1 + K_2 V_0)/2 + G_0 T + b_3 Y, \\ G_2 = F_{2x} + (S_1 - F_2)F_0 + S_2 F_2 - V_{0t} - M_1 V_1 - (L_2 + F_1)V_0 + a_0 T + G_0 Y.$$

Given by (55) equalities

$$A_{k1}M_{31}^2 + A_{k2}M_{31}M_{33} + A_{k3}M_{33}^2 = 0, \tag{64}$$

where A_{kl}, $k, l = 1, 2, 3$, serve as their algebraic corollary .

To find the condition of compatibility of the equations (48), (57), (64) with the system (46), (47), let us differentiate them by t (as respect to y) and subtract from them the same equations, differentiated by x and multiplied by T (by Y). It provides correlations (49) and equalities of the form (57), (64) with coefficients B_{ij}, $i = 3, 4, 5, 6$, A_{kl}, $k = 4, \ldots, 9$, denoted by correlations (55). The identity (50) is a corollary of the equality of the derivatives θ_{ty} and θ_{yt}. If we use the table of symbols (56), the equations (46) can be presented in the form

$$T_t - TT_x - \lambda_0 + SM_{33}^2/\theta_x^3 = 0, \\ Y_t - TY_x - \lambda_1 - SM_{31}M_{33}/\theta_x^3 = 0, \quad Y_y - YY_x - \lambda_2 + SM_{31}^2/\theta_x^3 = 0.$$

The exclusion of the summands with S results in

$$(\lambda_0 + TT_x - T_t)M_{31} + (\lambda_1 + TY_x - Y_t)M_{33} = 0, \\ (\lambda_1 + TY_x - Y_t)M_{31} + (\lambda_2 + YY_x - Y_y)M_{33} = 0.$$

To obtain the condition of the mutual equality of mixed derivatives of the forth-order function θ , let us differentiate (60)–(63) by x and substitute forth-order derivatives θ subject to the equations (58), (59), differentiated by t, x, y. In case of the equation (60) it results in identity, and for the equations (61)–(63) it provides correlations

$$\begin{aligned} &\alpha_1 M_{31} + \beta_1 M_{33} - 3(\Gamma_1\theta_x + \Gamma_2\varphi_x)M_{31}^2 = 0,\\ &\alpha_2 M_{31} + \beta_2 M_{33} - 3(\Gamma_1\theta_x + \Gamma_2\varphi_x)M_{31}M_{33} = 0,\\ &\alpha_3 M_{31} + \beta_3 M_{33} + 3(\Gamma_1\theta_x + \Gamma_2\varphi_x)M_{33}^2 = 0, \end{aligned} \tag{65}$$

where $\Gamma_1 = P_{\varphi\varphi} - 2Q_{\theta\varphi} + R_{\theta\theta} + P(2S_\theta - 3R_\varphi) + 3Q(2Q_\varphi - R_\theta) - 3RP_\varphi + SP_\theta$, $\Gamma_2 = Q_{\varphi\varphi} - 2R_{\theta\varphi} + S_{\theta\theta} - PS_\varphi + 3QS_\theta + 3R(Q_\varphi - 2R_\theta) + S(3Q_\theta - 2P_\varphi)$. Algebraic corollary of the equations (65) can be presented as

$$\alpha_2 M_{31}^2 + (\beta_2 - \alpha_1)M_{31}M_{33} - \beta_1 M_{33}^2 = 0, \quad \alpha_3 M_{31}^2 + (\beta_3 + \alpha_2)M_{31}M_{33} + \beta_2 M_{33}^2 = 0.$$

Having differentiated the second equation (65) by y, t and having compiled, correspondingly, a sum with the first equation (65), differentiated by t, and the difference with the third one, differentiated by y, we obtain correlations

$$\begin{aligned} &2\alpha_4 M_{31} + 2\beta_4 M_{33}\\ &\quad +3\Gamma_2[\lambda_{0TY} M_{31}^2 + (2\lambda_{1TY} - \lambda_{0TT})M_{31}M_{33} - \lambda_{1TT}M_{33}^2]M_{31}/\theta_x = 0,\\ &2\alpha_5 M_{31} + 2\beta_5 M_{33}\\ &\quad +3\Gamma_2[\lambda_{0TY} M_{31}^2 + (2\lambda_{1TY} - \lambda_{0TT})M_{31}M_{33} - \lambda_{1TT}M_{33}^2]M_{33}/\theta_x = 0. \end{aligned} \tag{66}$$

To exclude Γ_2 we use correlations

$$\begin{aligned} &\alpha_6 M_{31} + \beta_6 M_{33} + 3\Gamma_2 M_{31}^3/\theta_x = 0, \ \alpha_7 M_{31} + \beta_7 M_{33} + 3\Gamma_2 M_{31}^2 M_{33}/\theta_x = 0,\\ &\alpha_8 M_{31} + \beta_8 M_{33} + 3\Gamma_2 M_{31}M_{33}^2/\theta_x = 0, \ \alpha_9 M_{31} + \beta_9 M_{33} + 3\Gamma_2 M_{33}^3/\theta_x = 0, \end{aligned} \tag{67}$$

which are obtained during differentiating (61), (63) by y, t and substitution of derivatives θ subject to the equations (46), twice differentiated by x. The equalities of the form (64) with the coefficients A_{kl}, $k = 12, 13, 14, 15$ is an algebraic corollary of (66), (67). Substitution of (67) into (66) provides two more correlations of the form (57), which the coefficients B_{91}, B_{92}, $B_{10,1}$, $B_{10,2}$ defined by the formulae (55).

Hence, studying compatibility of the system (47) we have obtained ten equations (57) linear by M_{31}, M_{33} and fifteen second-degree equations (64) by M_{31}, M_{33}. Their conditions of compatibility is supported by the equalities (51)–(54), which are added to the conditions of compatibility (48)–(50) of the equations (46).

The system (46), (47) is compatible in case, when the system of equations (48)–(54) is compatible relative to T, Y. This system is divided into the subsystem of equations (48), (49) and (51)–(54) with $i, j = 1, 2$, $k, l, m = 1, 2, 3$, algebraic relative to T, Y, and the subsystem of differential equations, including equations (50) and the remaining equations (51)–(54). The system (48)–(54) is compatible when the subsystem of algebraic equations is decidable relative to the values T, Y, and their substitution into the remaining equations of the system results in identities. The theorem has been proved.

Let us note, that in most cases to state, that the equation in the system of two ordinary differential equations does not separate, it is sufficient to study compatibility of the algebraic subsystem of equations (48)–(54). If it proves to be compatible, and the substitution of its solution T, Y into the remaining equations (48)–(54) does not result in contradicting equality, the substitution of variables (2), which results in the given system of ordinary differential equations taking the form (11), (12), can be calculated from compatibility of the system of the equations $\theta_t/\theta_x = T$, $\theta_y/\theta_x = Y$, (46), (47), (57)–(64).

To check up whether the system (6) can take the form (11), (12) by the transformation (2), where $\theta_y \neq 0$, it is sufficient to make a substitution $\tilde{x} = y$, $\tilde{y} = x$ in the system (6) and apply Theorem 4 to it.

5. Example of a system with a separating equation

Example 3. Let us continue the study of the system (43) and find, what Γ, U in it result in separation of the equation by transformation (2) with $\theta_x \neq 0$. The first condition (48) and the condition (52) when $k = 1, 2$, $j = 2$ possess the form

$$(Y^2+1)(Y\Gamma_x - \Gamma_y) = 0, \quad \Gamma_x((Y^2-1)\Gamma_x - 2Y\Gamma_y)^2 = 0,$$
$$(2Y\Gamma_x - \Gamma_y)((Y^2-1)\Gamma_x - 2Y\Gamma_y)^2 = 0,$$

this implies $\Gamma_x = 0$, $\Gamma_y = 0$.

If $\Gamma = \Gamma(t)$, then all correlations (53), (54) become identities, and the simplest of the conditions (51) possess the form

$$Y_x\Psi = 0, \qquad (Y_y - YY_x)\Psi = 0, \qquad (Y_t - TY_x + (Y^2+1)\Gamma)\Psi = 0, \tag{68}$$

where $\Psi = U_{xx} - U_{yy} + 2Y(U_{xy} - \Gamma')$. Assume $\Psi \neq 0$ and $\Gamma = \gamma'(t)/2$, then $Y_x = 0$, $Y_y = 0$, $Y_t + (Y^2+1)\gamma'/2 = 0$. The substitution of $Y = -\mathrm{tg}(\gamma/2)$ into the second equality (48)

$$Y^2(\Gamma' - U_{xy}) + Y(U_{yy} - U_{xx}) + \Gamma' + U_{xy} = 0 \tag{69}$$

with precision to substitution $\tilde{\gamma} = \gamma \pm \pi/2$ results in the equation (44) relative to $U(t, x, y)$.

If $\Psi = 0$, then all the conditions of Theorem 4 are reduced to compatible system of equations

$$4V^2 + W^2 = 4\Gamma'^2, \quad 2(V - \Gamma')V_y + WV_x = 0, \quad 2(V - \Gamma')W_y + WW_x = 0, \tag{70}$$

$$\begin{aligned} &\Gamma'V_xT + V\Gamma'' - \Gamma'V_t - \Gamma\Gamma'W = 0, \qquad &&2(V - \Gamma')T_y + WT_x = 4\Gamma\Gamma', \\ &\Gamma'W_xT + W\Gamma'' - \Gamma'W_t + 4\Gamma\Gamma'V = 0, \qquad &&2(V - \Gamma')Y + W = 0, \end{aligned} \tag{71}$$

where $T = \theta_t/\theta_x$, $Y = \theta_y/\theta_x$. It is supposed, that $V \pm \Gamma' \neq 0$, $\Gamma' \neq 0$, otherwise, it results from the first equation (70), that the system (43) is linear. Let the function $U(t, x, y)$ satisfy three correlations (70), then we can obtain θ from the equations (71). Moreover, $A_{31} = V - \Gamma'$, $A_{32} = 0$, $A_{33} = 0$, and it follows from (64) that $M_{31} = 0$, which is equivalent to the equation

$$2(V - \Gamma')\varphi_y + W\varphi_x = 0 \tag{72}$$

relative to φ. The solution θ, φ of this equation and equations (71) is necessary to substitute into the system (46), (47), to completely denote the form of the functions θ, φ in the transformation (2). Whereas the function θ is denoted from the correlations $\theta_t - T\theta_x = 0$, $\theta_y - Y\theta_x = 0$ as a function of one argument, then the system (46) is reduced to one ordinary differential equation relative to this function. The function φ from the equation (72) is denoted as a function of two arguments, and its substitution transforms (47) into a compatible system relative to this function. In the capacity of φ we can take any partial solution of this system, chosen the way for the corresponding transformation (2) to be nonsingular.

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Yulya Yurjevna Bagderina,
Institute of Mathematics with Computer Center of the Ufa Science Center of the Russian Academy of Sciences,
112, Chernyshevsky str.,
Ufa, Russia, 450008
E-mail: yulya@mail.rb.ru