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INTEGRATING THE KORTEWEG-DE VRIES EQUATION WITH A SPECIAL FREE TERM IN THE CLASS OF PERIODIC FUNCTIONS

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Abstract. In the present paper we use the method of the inverse spectral problem to integrate the Korteweg-de Vries equation with a free term independent of the spatial variable in the class of periodic functions.

Keywords: Sturm-Liouville operator, inverse spectral problem, the Dubrovin-Trubowitz system of equations, the Korteweg-de Vries equation.

1. Introduction

In [1–8] and other works, the Korteweg-de Vries (KdV) equation is investigated in the class of periodic functions.

In the present paper the KdV equation with a free term independent of the space variable is investigated, namely the following equation is considered:

$$q_t = q_{xxx} - 6qq_x + f(t), \quad t > 0, \quad x \in R^1 \tag{1}$$

with the initial condition

$$q(x,t)|_{t=0} = q_0(x), \tag{2}$$

where $f(t)$ is a real continuous function. Find the real function $q(x,t)$, which is π-periodic with respect to the variable x:

$$q(x+\pi,t) \equiv q(x,t), \quad t > 0, \quad x \in R^1 \tag{3}$$

and satisfies the smoothness conditions:

$$q(x,t) \in C_x^3(t>0) \cap C_t^1(t>0) \cap C(t \geqslant 0). \tag{4}$$

Note that the KdV equation with a self-consistent source in the class of rapidly decreasing functions was considered in [9–13] and others, and the nonlinear equations with a self-consistent source in the class of periodic functions are studied in the works [14–16] in various formulations.

The aim of the present paper is to provide a procedure for constructing a solution $q(x,t)$ of the problem (1)–(4) in the framework of the inverse spectral problem for the Sturm-Liouville operator.

2. Preliminaries

For the sake of completeness, the present section provides some basic facts related to the inverse spectral problem for the Sturm-Liouville operator with a periodic potential (see [17–19]).

Let us consider the following Sturm-Liouville operator on the whole line

$$Ly \equiv -y'' + q(x)y = \lambda\, y, \quad x \in R^1, \tag{5}$$

where $q(x)$ is a real continuous π-periodic function.

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Let us denote by $c(x,\lambda)$ и $s(x,\lambda)$ solutions to Equation (5), satisfying the initial conditions $c(0,\lambda)=1$, $c'(0,\lambda)=0$ and $s(0,\lambda)=0$, $s'(0,\lambda)=1$. The function $\Delta(\lambda)=c(\pi,\,\lambda)+s'(\pi,\,\lambda)$ is termed as the Lyapunov function or the Hill discriminant.

The spectrum of the operator (5) is purely continuous and coincides with the following set

$$E=\{\lambda\in R^1:\ \ -2\leqslant\Delta(\lambda)\leqslant 2\ \}=[\lambda_0,\lambda_1]\cup[\lambda_2,\ \lambda_3]\cup...\cup[\lambda_{2n},\lambda_{2n+1}]\cup\dots .$$

Intervals $(-\infty,\ \lambda_0)$, $(\lambda_{2n-1},\ \lambda_{2n})$, $n=1,\ 2,\ ...$ are called lacunas. Here λ_0, λ_{4k-1}, λ_{4k} are eigenvalues of the periodic problem $(y(0)=y(\pi),\ y'(0)=y'(\pi))$, a λ_{4k+1}, λ_{4k+2} are eigenvalues of the antiperiodic problem $(y(0)=-y(\pi),\ y'(0)=-y'(\pi))$ for Equation (5).

Let ξ_n, $n=1,\ 2,\ ...$ be roots of the equation $s(\pi,\lambda)=0$. Note that ξ_n, $n=1,\ 2,\ ...$ coincide with eigenvalues of the Dirichlet problem $(y(0)=0,\ y(\pi)=0)$ for the equation (5). Moreover, the following inclusions hold: $\xi_n\in[\lambda_{2n-1},\lambda_{2n}]$, $n=1,\ 2,\ ...$.

The numbers ξ_n, $n=1,\ 2,\ ...$ together with the signs $\sigma_n=sign\{s'^2(\pi,\xi_n)-1\}$, $n=1,\ 2,\ ...$ are said to be spectral parameters of the problem (5). Spectral parameters ξ_n, σ_n, $n=1,\ 2,\ ...$ and the boundaries λ_n, $n=0,\ 1,\ 2,\ ...$ of the spectrum are said to be spectral data of the operator (5). Reconstruction of the coefficient $q(x)$ with respect to the spectral data is said to be an inverse spectral problem for the operator (5).

The spectrum of the Sturm-Liouville operator with the coefficient $q(x+\tau)$ is independent of the real parameter τ, and spectral parameters depend on τ: $\xi_n(\tau)$, $\sigma_n(\tau)$, $n=1,\ 2,\ ....$ Spectral parameters satisfy the following system of the Dubrovin-Trubowitz equations:

$$\frac{d\xi_n}{d\tau}=2\sigma_n(\tau)\cdot\sqrt{(\xi_n-\lambda_{2n-1})(\lambda_{2n}-\xi_n)}\times$$

$$\times\sqrt{(\xi_n-\lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\xi_n-\lambda_{2k-1})(\xi_n-\lambda_{2k})}{(\xi_k-\xi_n)^2}},\quad n\geqslant 1.$$

The Dubrovin-Trubowitz system of equations and the following trace formula

$$q(\tau)=\lambda_0+\sum_{k=1}^{\infty}(\lambda_{2k-1}+\lambda_{2k}-2\xi_k(\tau))$$

provide a method for solving the inverse problem.

3. Main theorem

The main result of the present work is the following theorem.

Theorem 1. *Let $q(x,t)$ be a solution to the problem (1)–(4). Then, the boundaries $\lambda_n(t)$, $n\geqslant 0$ of the spectrum of the following operator*

$$L(t)y\equiv -y''+q(x,t)y=\lambda y,\ \ x\in R^1 \tag{6}$$

satisfy the equations

$$\dot\lambda_n(t)=f(t),\quad n\geqslant 0, \tag{7}$$

and the spectral parameters $\xi_n(t)$, $n\geqslant 1$ satisfy the analogue of the system of Dubrovin-Trubowitz equations:

$$\dot\xi_n(t)=4\sigma_n(t)[q(0,t)+2\xi_n(t)]\cdot\sqrt{(\xi_n(t)-\lambda_{2n-1}(t))(\lambda_{2n}(t)-\xi_n(t))}\times$$

$$\times\sqrt{(\xi_n(t)-\lambda_0(t))\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\xi_n(t)-\lambda_{2k-1}(t))(\xi_n(t)-\lambda_{2k}(t))}{(\xi_k(t)-\xi_n(t))^2}}+f(t),\quad n\geqslant 1, \tag{8}$$

where the sign $\sigma_n(t)$ changes in each collision of the point $\xi_n(t)$ with the boundary of its lacuna $[\lambda_{2n-1}(t),\ \lambda_{2n}(t)]$. Moreover, the following initial conditions hold:

$$\xi_n(t)|_{t=0} = \xi_n^0, \quad \sigma_n(t)|_{t=0} = \sigma_n^0\,, \quad n \geqslant 1,$$

where ξ_n^0, σ_n^0, $n \geqslant 1$ are spectral parameters of the Sturm-Liouville operator with the coefficient $q_0(x)$.

Proof. Let us denote by $y_n(x,t)$, $n = 1,\ 2,\ ...$ orthonormal eigenfunctions of the Dirichlet problem ($y(0) = 0,\ \ y(\pi) = 0$) for the equation (6), corresponding to eigenvalues $\xi_n(t)$, $n = 1,\ 2,\ ...$. Differentiating the identity $(L(t)y_n, y_n) = \xi_n$ with respect to t and using symmetry of the operator $L(t)$, one has

$$\dot{\xi}_n = (L(t)\dot{y}_n + q_t y_n, y_n) + (L(t)y_n, \dot{y}_n) = (\dot{y}_n, L(t)y_n) + (L(t)y_n, \dot{y}_n) + (q_t y_n, y_n) =$$

$$= \xi_n((y_n, y_n))^{\dot{}} + (q_t y_n, y_n) = \int_0^\pi q_t(x,t) y_n^2(x,t)dx. \tag{9}$$

Here $(\cdot\ ,\ \cdot)$ is a scalar product of the space $L_2(0,\ \pi)$.

Using (1), one rewrites the equality (9) in the form

$$\dot{\xi}_n = \int_0^\pi q_{xxx} y_n^2 dx - 6\int_0^\pi q_x y_n (q y_n) dx + f(t). \tag{10}$$

Applying the formulae for integration by parts in (10), one obtains

$$\dot{\xi}_n = \int_0^\pi y_n^2 dq_{xx} - 6\int_0^\pi (\xi_n y_n^2 + y_n'' y_n) dq\ + f(t) =$$

$$= y_n^2\, q_{xx}|_0^\pi - \int_0^\pi 2y_n y_n' dq_x - 6(\xi_n y_n^2 + y_n'' y_n)\, q|_0^\pi +$$

$$+6\int_0^\pi (2\xi_n y_n y_n' + y_n''' y_n + y_n'' y_n') q dx + f(t) =$$

$$= -2y_n y_n'\, q_x|_0^\pi + 2\int_0^\pi ((y_n')^2 + y_n'' y_n) dq +$$

$$+6\int_0^\pi \{2\xi_n y_n' \cdot (q y_n) + y_n''' \cdot (q y_n)\} dx + 6\int_0^\pi y_n'' y_n' q dx + f(t) =$$

$$= 2((y_n')^2 + y_n y_n'')\, q|_0^\pi - 2\int_0^\pi \{3y_n' y_n'' + y_n y_n'''\} q dx +$$

$$+12\xi_n \int_0^\pi \{\xi_n y_n y_n' + y_n' y_n''\} dx + 6\int_0^\pi \{\xi_n y_n''' y_n + y_n'' y_n'''\} dx + 6\int_0^\pi y_n'' y_n' q dx + f(t) =$$

$$= 2(y_n')^2\, q|_0^\pi + 4\int_0^\pi \{\xi_n y_n''' y_n + y_n'' y_n'''\} dx + 6\xi_n^2\, y_n^2\Big|_0^\pi + 6\xi_n\, (y_n')^2\Big|_0^\pi + f(t) =$$

$$= 2(y_n')^2\,(q+3\xi_n)|_0^\pi + 4\xi_n\int_0^\pi y_n dy_n'' + 2\,(y_n'')^2\big|_0^\pi + f(t) =$$

$$= 2(y_n')^2\,(q+3\xi_n)|_0^\pi + 4\xi_n y_n\, y_n''|_0^\pi - 4\xi_n\int_0^\pi y_n' y_n'' dx + f(t) =$$

$$= 2(y_n')^2\,(q+3\xi_n)|_0^\pi - 2\xi_n\,(y_n')^2\big|_0^\pi + f(t).$$

Thus,

$$\dot{\xi}_n = 2[(y_n'(\pi,t))^2 - (y_n'(0,t))^2]\cdot[q(0,t)+2\xi_n(t)] + f(t). \tag{11}$$

Let us denote by $c(x,\lambda,t)$ и $s(x,\lambda,t)$ solutions of Equation (6), satisfying the initial conditions $c(0,\lambda,t)=1$, $c'(0,\lambda,t)=0$ and $s(0,\lambda,t)=0$, $s'(0,\lambda,t)=1$. In this case, the Lyapunov function is defined by the equality $\Delta(\lambda,t)=c(\pi,\lambda,t)+s'(\pi,\lambda,t)$.

Assuming that $\lambda=\xi_n(t)$ in the identity

$$\int_0^\pi s^2(x,\lambda,t)dx = s'(\pi,\lambda,t)\frac{\partial\, s(\pi,\lambda,t)}{\partial\lambda} - s(\pi,\lambda,t)\frac{\partial^2 s(\pi,\lambda,t)}{\partial\lambda\,\partial\, x}$$

and using the equality $s(\pi,\xi_n(t),t)=0$, one obtains

$$c_n^2(t) \equiv \int_0^\pi s^2(x,\xi_n(t),t)dx = s'(\pi,\xi_n(t),t)\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}.$$

In particular, it follows that

$$sign\left\{\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}\right\} = sign\{s'(\pi,\xi_n(t),t)\}. \tag{12}$$

Substituting the following expression

$$y_n(x,t) = \frac{1}{c_n(t)} s(x,\xi_n(t),t)$$

into the equality (11), one has

$$\dot{\xi}_n(t) = 2[q(0,t)+2\xi_n(t)]\cdot\frac{\left[s'(\pi,\xi_n(t),t) - \frac{1}{s'(\pi,\xi_n(t),t)}\right]}{\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}} + f(t). \tag{13}$$

Substituting values $x=\pi$ and $\lambda=\xi_n(t)$ into the identity

$$c(x,\lambda,t)s'(x,\lambda,t) - c'(x,\lambda,t)s(x,\lambda,t) = 1,$$

one obtains

$$c(\pi,\xi_n(t),t) = \frac{1}{s'(\pi,\xi_n(t),t)}. \tag{14}$$

By means of (14) and the equality

$$[c(\pi,\lambda,t) - s'(\pi,\lambda,t)]^2 = (\Delta^2(\lambda,t)-4) - 4c'(\pi,\lambda,t)s(\pi,\lambda,t),$$

one arrives at

$$s'(\pi,\xi_n(t),t) - \frac{1}{s'(\pi,\xi_n(t),t)} = \frac{\sigma_n(t)}{sign\{s'(\pi,\xi_n(t),t)\}}\sqrt{\Delta^2(\xi_n(t),t)-4}, \tag{15}$$

where $\sigma_n(t) = sign\{s'^2(\pi,\xi_n(t),t)-1\}$. Using (12) and (15), one deduces

$$\frac{s'(\pi,\xi_n(t),t) - \frac{1}{s'(\pi,\xi_n(t),t)}}{\frac{\partial s(x,\xi_n(t),t)}{\partial\lambda}} = \sigma_n(t)\cdot\sqrt{\frac{\Delta^2(\xi_n(t),t)-4}{\left(\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}\right)^2}}.$$

It follows from the expansions

$$s(\pi,\lambda,t)=\pi\prod_{k=1}^{\infty}\frac{\xi_k(t)-\lambda}{k^2},$$

and

$$\Delta^2(\lambda,t)-4=-4\pi^2(\lambda-\lambda_0(t))\cdot\prod_{k=1}^{\infty}\frac{(\lambda-\lambda_{2k-1}(t))(\lambda-\lambda_{2k}(t))}{k^4},$$

that

$$\frac{s'(\pi,\xi_n(t),t)-\frac{1}{s'(\pi,\xi_n(t),t)}}{\frac{\partial s(x,\xi_n(t),t)}{\partial\lambda}}=2\sigma_n(t)\cdot\sqrt{(\xi_n(t)-\lambda_{2n-1}(t))(\lambda_{2n}(t)-\xi_n(t))}\times$$
$$\times\sqrt{(\xi_n(t)-\lambda_0(t))\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\xi_n(t)-\lambda_{2k-1}(t))(\xi_n(t)-\lambda_{2k}(t))}{(\xi_k(t)-\xi_n(t))^2}}. \tag{16}$$

Then, (12) and (16) entail (8).

It is known that the boundaries $\lambda_n(t)$, $n=0,\ 1,\ 2,\ \ldots$ of the spectrum of the operator (6) coincide either with eigenvalues of the periodic problem, or the antiperiodic problem for the Sturm-Liouville equation (6). Denoting by $v_n(x,t)$ the normalized eigenfunction, corresponding to the eigenvalue $\lambda_n(t)$, of the periodic or antiperiodic problem for the Sturm-Liouville problem (6), acting similarly to the above, one deduces the equalities (7). **Theorem is proved.**

4. Corollaries of the main theorem

Corollary 1. If $q(x+\tau,t)$ is considered instead of $q(x,t)$, the eigenvalues of the periodic and antiperiodic problem depend only on the parameter t, and eigenvalues ξ_n of the Dirichlet problem and the signs σ_n depend on τ and t: $\xi_n=\xi_n(\tau,t)$, $\sigma_n=\sigma_n(\tau,t)=\pm1$, $n\geqslant1$. In this case, the system (8) takes the form

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=4\sigma_n(\tau,t)[q(\tau,t)+2\xi_n]\cdot\sqrt{(\xi_n-\lambda_{2n-1})(\lambda_{2n}-\xi_n)}\times$$
$$\times\sqrt{(\xi_n-\lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\xi_n-\lambda_{2k-1})(\xi_n-\lambda_{2k})}{(\xi_k-\xi_n)^2}}+f(t),\quad n\geqslant1. \tag{17}$$

Using the traces formula

$$q(\tau,t)=\lambda_0+\sum_{k=1}^{\infty}[\lambda_{2k-1}+\lambda_{2k}-2\xi_k(\tau,t)], \tag{18}$$

one can write Equation (17) in the following closed form:

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=4\sigma_n(\tau,t)\left\{\lambda_0+\sum_{k=1}^{\infty}(\lambda_{2k-1}+\lambda_{2k}-2\xi_k)+2\xi_n\right\}\cdot\sqrt{(\xi_n-\lambda_{2n-1})(\lambda_{2n}-\xi_n)}\times$$
$$\times\sqrt{(\xi_n-\lambda_0)\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{(\xi_n-\lambda_{2k-1})(\xi_n-\lambda_{2k})}{(\xi_k-\xi_n)^2}}+f(t),\quad n\geqslant1. \tag{19}$$

Corollary 2. This theorem provides a method for solving the problem (1)-(4). Indeed, denote by $\lambda_n(t)$, $n=0,\ 1,\ 2,\ \ldots$, $\xi_n(\tau,t)$, $\sigma_n(\tau,t)$, $n=1,\ 2,\ \ldots$ spectral data of the problem

$$-y''+q(x+\tau,t)y=\lambda\,y,\qquad x\in R^1.$$

Let us find the spectral data λ_n^0, $n=0,\ 1,\ 2,\ \ldots$, $\xi_n^0(\tau)$, $\sigma_n^0(\tau)$, $n=1,\ 2,\ \ldots$ for the equation

$$-y''+q_0(x+\tau)y=\lambda y,\qquad x\in R^1.$$

Solving Equations (7) with the initial conditions $\lambda_n(t)|_{t=0} = \lambda_n^0$, $n = 0,\ 1,\ 2,\ \dots$, one obtains

$$\lambda_n(t) = \lambda_n^0 + \int_0^t f(s)ds, \quad n = 0,\ 1,\ 2,\ \dots . \tag{20}$$

Then, let us solve the Cauchy problem

$$\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau), \quad \sigma_n(\tau,t)\big|_{t=0} = \sigma_n^0(\tau), \quad n = 1,\ 2,\ \dots$$

for the system of Dubrovin-Trubowitz equations (19). According to the formula of traces (18), one finds the solution $q(\tau,t)$ to the problem (1)–(4).

Remark 1. One can see from the equalities (20) that the spectrum of the Sturm-Liouville operator (6) moves on the axis preserving the initial structure, i.e., lacunas' lengths remain unaltered.

Corollary 3. In [18], the following theorem is proved: for lacunas length of the Sturm-Liouville operator with a π-periodic real coefficient to decrease exponentially, it is necessary and sufficient that this coefficient be analytical. Whence, if the initial function $q_0(x)$ is a real analytical function then, the lengths $\lambda_{2n}^0 - \lambda_{2n-1}^0$ of lacunas, corresponding to the coefficient, decrease exponentially. Since the lengths of lacunas, corresponding to this coefficient $q(x,t)$, are independent of t then, $q(x,t)$ is an analytical function in x.

Corollary 4. In the work [20], the generalization of the inverse Borg theorem is proved: for a π-periodic real potential of the Sturm-Liouville operator to have the period $\frac{\pi}{n}$, it is necessary and sufficient that all lacunas with numbers that cannot be divided by n must vanish. Here $n \geqslant 2$ is a natural number and the lacuna $(\lambda_{2k-1},\ \lambda_{2k})$ has the number k. Therefore, if $q_0(x)$ has the period $\frac{\pi}{n}$, then the solution to the problem (1)-(4) $q(x,t)$ is $\frac{\pi}{n}$-periodic with respect to x.

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