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APPROXIMATE STUDY OF MULTIPARAMETER BIFURCATIONS IN MODELS OF POPULATION DYNAMICS

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Abstract. The paper presents a new general scheme for qualitative and approximate investigation of basic scenarios of local bifurcations in models of population dynamics. Necessary and sufficient conditions for bifurcation of the equilibrium state and the Andronov-Hoph systems for population dynamics are given.

Keywords: plate deflection, critical forces, bifurcation points, asymptotic formulas, equilibrium state.

1. Introduction

One of the most general models in mathematical biology is the model of population dynamics [1] (or Kolmogorov's model), described by the system of differential equations

$$\begin{cases} x_1' = x_1 g_1(x,\mu), \\ x_2' = x_2 g_2(x,\mu), \\ \dots \\ x_n' = x_n g_n(x,\mu). \end{cases} \tag{1}$$

In this system, the variables $x_1, x_2, \dots, x_n$ denote the number of every separate biological population, $x = (x_1, x_2, \dots, x_n) \in R^n$, $\mu \in R^k$. Functions $g_i(x,\mu)$ are continuous with respect to $x \in K_+ = \{x : \ x_j \geqslant 0\}$ and $\mu \in R^k$ and partial derivatives $g'_{jx_i}(x,\mu)$ when $i \neq j$ preserve the sign in the octant K_+.

The system (1) depends on the scalar or vector parameter μ, whose variation may cause different bifurcations. The so-called local bifurcations [2] occurring in the the neighborhood of the equilibrium points or cycles of the system (1) are of special interest. One of the main issues here is the problem on evolution of bifurcation solutions depending on variation of the parameter: it is important that these solutions should remain in the first octant K_+. These issues have been studied by many authors (see [1] with the bibliography). As a rule, specific models are studied, one-parameter bifurcations are investigated, and computer simulation is based on a direct numerical calculation.

The present paper provides a scheme for a qualitative and approximate investigation of problems on a multi-parameter bifurcation for systems of the form (1). The scheme is based on the operator method described in [3].

2. Investigation scheme

The following investigation scheme for problems on a multi-parameter bifurcation for the system (1) is suggested. Let us limit our consideration by cases $x \in R^2$ and $x \in R^3$.

Local bifurcations of the system (1) are possible in the the neighborhoods of fixed points (equilibrium states) and cycles (of periodic solutions). In this connection note that the system (1) has a zero equilibrium point $x = 0$, while it can also have fixed points on coordinate planes as well.

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Let x^* be a fixed point of the system (1). The vector x^* can have several zero components. For the sake of definiteness, let the fixed point of the system (1) have the form $x^* = (x_1^*, \dots, x_m^*, 0, \dots, 0)$. Assuming that $h = x - x^*$, let us turn from (1) to the system

$$h' = A(\mu)h + a(h, \mu), \tag{2}$$

where $A(\mu)$ is the Jacobian matrix of the right-hand side of the system (1) calculated at the point x^*, $a(h,\mu) = o(\|h\|)$ when $\|h\| \to 0$. The matrix $A(\mu)$ has the following form here:

$$A(\mu) = \begin{bmatrix} x_1^* g'_{1x_1}(x^*,\mu) & x_1^* g'_{2x_1}(x^*,\mu) & \cdots & x_1^* g'_{nx_1}(x^*,\mu) \\ \cdots & \cdots & \cdots & \cdots \\ x_m^* g'_{1x_m}(x^*,\mu) & x_m^* g'_{2x_m}(x^*,\mu) & \cdots & x_m^* g'_{nx_m}(x^*,\mu) \\ 0 & \cdots & g_m(x^*,\mu) & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & g_n(x^*,\mu) \end{bmatrix}.$$

The necessary condition for a local bifurcation in the neighborhood of the point x^* is (see, e.g., [2]) the requirement that the matrix $A(\mu)$ should have a purely imaginary eigenvalue for a certain value $\mu = \mu_0$. Meanwhile, if the matrix $A(\mu_0)$ has a zero eigenvalue, then the value $\mu = \mu_0$ is usually a point of bifurcation for fixed points of the system (1): when the values μ are close to μ_0, a new equilibrium state x^{**} appears in the neighborhood of the equilibrium state x^* for the system (1). If the matrix $A(\mu_0)$ has a pair of eigenvalues of the form $\pm i\omega_0$ for $\omega_0 > 0$ then, the value $\mu = \mu_0$ is usually the Andronov-Hopf bifurcation point of the system (1): a nonstationary periodic solution of a small amplitude to the system (1) appears in the neighborhood of the equilibrium state x^* when the values μ are close to μ_0.

In order to analyze the corresponding bifurcations in the system (1), let us make se of the operator method provided in [3]. Let us enumerate some bifurcation criteria limiting our consideration to the case $n = 2$.

Theorem 1. *Let us assume that $n = 2$ in the system (1) and x^* is its fixed point. Let $\mu = \mu_0$ is the bifurcation point of fixed points. Then, $g_1(x^*, \mu_0) = 0$ or $g_2(x^*, \mu_0) = 0$. If only one of these equalities holds then, the bifurcation has the codimension 1, if both equalities hold, the codimension of the bifurcation equals 2.*

This statements provides the necessary criterion for bifurcation of fixed points. The following theorem provides a sufficient criterion for bifurcation of the codimension 1.

Theorem 2. *Let us assume that exactly one of the following correlations holds: $g_1(x^*, \mu_0) = 0$ or $g_2(x^*, \mu_0) = 0$. Let $g''_{1x_1\mu}(x^*, \mu_0) \neq 0$ or $g'_{2\mu}(x^*, \mu_0) \neq 0$, respectively. Then, $\mu = \mu_0$ is the bifurcation point of fixed points of the system (1).*

Similar statement can be obtained for bifurcation of the codimension 2 as well. The validity of these statements and other theorems given in the paper follows from more general results obtained in [3].

The Andronov-Hopf bifurcation in systems of the form (1) is possible only in the neighbourhood of the fixed point with two or more nonzero coordinates. In particular, it is impossible for a two-dimensional system in the neighbourhood of a point of the form $x^* = (0, x_2)$ or $x^* = (x_1, 0)$. Let us give a sufficient criterion for the Andronov-Hopf bifurcation for the system (1) when $n = 2$.

Theorem 3. *Let $x^* = (x_1^*, x_2^*)$ be a fixed point of the system (1) with positive components. The value μ_0 is the Andronov-Hopf bifurcation point for this system if the following conditions are satisfied:*

$$g'_{1x_1}(x^*, \mu_0) = 0,\ g'_{2x_2}(x^*, \mu_0) = 0,\ g'_{2x_1}(x^*, \mu_0) g'_{1x_2}(x^*, \mu_0) < 0.$$

Similar statement can be obtained for higher-order systems as well.

It is suggested to carry out the approximate investigation of bifurcation processes in systems of the form (1) according to the scheme described in [3]. Namely, at the first stage, it is necessary to turn to the operator equation containing key parameters of the problem. Meanwhile, to investigate the bifurcation scenario, one has to choose the number of parameters corresponding to the codimension of the bifurcation. At the second stage, the method of parameter functionalization is applied to the resulting operator equation. Finally, the Newton-Kantorovich method is used to investigate the functionalized equation.

3. The system "one predator — two prey"

Let us consider the model "predator–two prey" [1], described by the following equations as an illustration:

$$\begin{cases} u_1' = u_1(\alpha_1 - u_1 - 6u_2 - 4v), \\ u_2' = u_2(\alpha_2 - u_2 - u_1 - 10v), \\ v' = -v(1 - 0,25u_1 - 4u_2 + v). \end{cases} \tag{3}$$

Here u_1 and u_2 is the size of population of every prey, v is the size of predators population.

The system (3) contains two parameters α_1 and α_2. Let us consider the Andronov-Hopf bifurcation in the neighbourhood of a nonzero fixed point with the coordinates $x^* = (-11,2 + 8,2\alpha_1 - 4,4\alpha_2;\ 1,2 - 0,7\alpha_1 + 0,4\alpha_2;\ 1 - 0,75\alpha_1 + 0,5\alpha_2)$. The codimension of this bifurcation equals to one therefore, let us consider one parameter as a bifurcational one, namely α_2. Let $\alpha_1 = 3,7$. Cycles appear in this system the parameter α_2 is varied. When $\alpha_2 \approx 3,874$ The Jacobian matrix of the right-hand side of the system (3) has two purely imaginary eigenvalues.

Fig. 1 demonstrates results of numerical investigation of bifurcation in the system (3), obtained by the program elaborated by the authors. This figure represents projections of phase trajectories of the system (3) to the (u_1, u_2) plane for different values of the parameter α_2. The symbol $*$ is used to denote initial points of the corresponding periodic solutions.

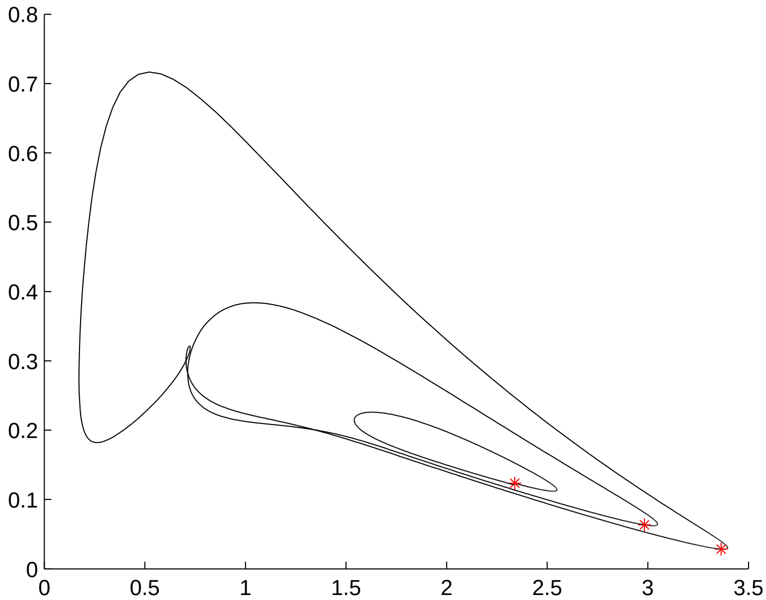

Fig. 1. Family of periodic trajectories of the system (3)

BIBLIOGRAPHY

1. Bazykin A.D. *Nonlinear dynamics of interacting populations.* M.-Ijevsk: Institute of computer stdies, 2003. 368 p. In Russian.
2. Arnold V.I. *Geometric methods in the theory of ordinary differential equations.* M.: NIZ ¡¡Regular and chaotic dynamics¿¿, 2000. 400 p. In Russian.
3. Vyshinskiy A.A., Ibragimova L.S., Murtazina S.A., Yumagulov M.G. *An operator method for approximate investigation of a regular bifurcation in multiparameter dynamical systems* // Ufimsk. Mat. Zh., 2010. Vol.2. 4. Pp. 3–26. In Russian.

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