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DIFFERENTIAL PROPERTIES OF ZEROS OF EIGENFUNCTIONS OF THE STURM-LIOUVILLE PROBLEM

A.YU. TRYNIN

Abstract. Differential relations for functionals that set the k-th zero of the n-th eigenfunction in correspondence with the summable potential q of the Sturm-Liouville problem are obtained in terms of the Gato differentials.

Keywords: Differential properties, eigenfunction, Sturm-Liouville problem

1. Introduction

Various properties of eigenfunctions and eigenvalues of the Sturm-Liouville equation with a nonsmooth potential have been the subject of research at the leading research schools of spectral theory of differential operators for many decades. The range of these problems is studied well enough at present. Without claiming the review of publications on this subject to be complete, we mention a number of famous works of this scientific field, published recently.

In [1], for a fixed summable potential we obtain the asymptotic formulae for eigenfunctions and eigenvalues of the classical Sturm-Liouville problem with the modern interpretation of the Liouville-Steklov method.

The works [2], [3] are devoted to to investigation of asymptotics of eigenfunctions and eigenvalues of the Sturm-Liouville operator with a singular potential being a generalized function of the first order, $q(x) = u'(x)$, where $u \in L_2[0, \pi]$.

The works [4], [5] should be attributed to investigations where estimates of the considered parameters of the Sturm-Liouville operators are uniform with respect to the potential q in the ball of the Sobolev space.

In the fundamental papers [6], [7], [8], an analogue of the Sturm oscillation theory of distribution of zeros of eigenfunctions on a spatial network, or graphs is constructed.

Let us assume that $q \in L[0, \pi]$, and $\lambda_n = \lambda_n[q]$ is an $n-$th eigenvalue of the Sturm-Liouville

$$\begin{cases} \hat{y}'' + [\lambda - q]\hat{y} = 0, \\ \sin\alpha\hat{y}'(0) + \cos\alpha\hat{y}(0) = 0, \\ \sin\beta\hat{y}'(\pi) + \cos\beta\hat{y}(\pi) = 0, \end{cases} \tag{1}$$

where α, $\beta \in \mathbf{R}$, and $\hat{y}(x, q, \lambda_n) \equiv \hat{y}_n(x)$ is an orthosimilar eigenfunction of this problem $\|\hat{y}(\cdot, q, \lambda_n)\|_{L_2[0,\pi]} = 1$ corresponding to it. Let us enumerate zeroes of the function $\hat{y}_n$ so that $0 \leq x_{0,n} < x_{1,n} < \cdots < x_{n,n} \leq \pi$. Let us fix some $n \in \mathbb{N}$ и $0 \leq k \leq n$, $k \in \mathbb{Z}$. Denote by $x_{k,n}[q]$ the functional, which sets the $k+1$th zero from the left of the nth eigenfunction $\hat{y}(x, q, \lambda_n[q])$ in correspondence to the potential q. Let us agree to use the notation

$$D\phi[q, w] = \lim_{t\to 0} \frac{\phi(q + tw) - \phi(q)}{t}$$

for the Gateaux differential of the functional $\phi : L[0, \pi] \to \mathbb{R}$ with the increment $w \in L[0, \pi]$.

In [9], some differential correlations are obtained in case of boundary conditions of the first kind in terms of the Gateaux differential for node points of the Sturm-Liouville problem.

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However, they contain derivatives of eigenfunctions with respect to the variable x, as well to the spectral parameter.

Theorem 1 ([9]). *Let q, $w \in L^2[0,\pi]$ then the Gateaux differential of the functional $x_{k,n}[q]$ ($n \in \mathbb{N}$ and $0 \le k \le n$ when $\alpha = \beta = 0$ in (1)) with the increment w satisfies the relation*

$$Dx_{k,n}[q,w] = \frac{1}{\left[y'(x_{k,n}, q, \lambda_n)\right]^2} \int_0^{x_{k,n}} w(\tau) y^2(\tau, q, \lambda_n)\, d\tau$$

$$-\frac{\dot{y}(x_{k,n}, q, \lambda_n)}{\dot{y}(\pi, q, \lambda_n) y'(x_{k,n}, q, \lambda_n) y'(\pi, q, \lambda_n)} \int_0^{\pi} w(\tau) y^2(\tau, q, \lambda_n)\, d\tau,$$

where

$$y'(x,q,\lambda) = \frac{\partial}{\partial x} y(x,q,\lambda), \quad \dot{y}(x,q,\lambda) = \frac{\partial}{\partial \lambda} y(x,q,\lambda).$$

This correlation was used by the author for investigating properties of the inverse nodal Strum-Liouville problem with the potential from the space $L^2[0,\pi]$.

In [10], differential relations are obtained in terms of the Gateaux differentials for nodal points of the regular Sturm-Liouville with a summable potential and boundary-value conditions of the third kind, where one should certainly eliminate conditions of the first kind ($\alpha \ne \pi l$ and $\beta \ne \pi m$, $l, m \in \mathbb{Z}$). In particular, they gave a possibility to demonstrate the lack of stability of the representation problem for a function continuous in the interval $[0,\pi]$ by means of the Lagrange interpolation processes constructed with respect to eigenfunctions of the Sturm-Liouville problem. These processes were suggested for investigation for the first time by G.I. Natanson in [11]. At present, results of the work [10] have been extended to the case of arbitrary boundary-value conditions of the third kind, i.e. $\alpha, \beta \in \mathbb{R}$.

2. Main results

In the present paper some differential relations are obtained in terms of the Gateaux differentials for nodal point of the regular Sturm-Liouville problem with arbitrary boundary-value conditions of the third kind.

Theorem 2. *Let q, $w \in L[0,\pi]$ then, the Gateaux differential of the functional $x_{k,n}[q]$ ($n \in \mathbb{N}$ and $0 \le k \le n$) with the increment w satisfies the relation*

$$Dx_{k,n}[q,w] = \frac{1}{\left[\hat{y}'(x_{k,n}, q, \lambda_n)\right]^2} \int_0^{\pi} w(\tau) \hat{y}^2(\tau, q, \lambda_n) \beta_{k,n}(\tau)\, d\tau, \tag{2}$$

where

$$\beta_{k,n}(\tau) = \begin{cases} 1 - \alpha_{k,n}, & \text{if } \tau \in [0, x_{k,n}], \\ -\alpha_{k,n}, & \text{if } \tau \in (x_{k,n}, \pi], \end{cases} \qquad \alpha_{k,n} = \int_0^{x_{k,n}} \hat{y}^2(\tau, q, \lambda_n)\, d\tau.$$

Whence, one can readily obtain the following theorem useful in investigation of stability of representation of a continuous function by means of the Lagrange-Sturm-Liouville interpolation process.

Theorem 3. *Whatever summable potential $q \in L[0,\pi]$ is taken, for any $\xi \in (0,\pi)$, for all $n \in \mathbb{N}$ and $0 \le k \le n$ such that $x_{0,n}[q] \not\equiv 0$ or $x_{n,n}[q] \not\equiv \pi$, the Gateaux differential of the functional $x_{k,n}[q]$ with the increment*

$$w(x) = \begin{cases} 0, & \text{if } x \in [0,\xi], \\ 1, & \text{if } x \in (\xi,\pi] \end{cases} \tag{3}$$

is negative, i.e. $Dx_{k,n}[q,w] < 0$.

Remark. In case when at least one boundary condition takes the form of the Dirichlet conditions: $\alpha = 2\pi l$, or $\beta = 2\pi l$, $l \in \mathbb{Z}$, i.e. $x_{0,n}[q] \equiv 0$, or $x_{n,n}[q] \equiv \pi$, the corresponding Gateaux differential for any q, $w \in L[0,\pi]$

$$Dx_{0,n}[q,w] = 0 \text{ or } Dx_{n,n}[q,w] = 0.$$

Remark. Statement of Theorem 3 agrees with the Sturm theorem, comparison theorem, and the well-known oscillation theorem (see, e.g., [12, гл. 1, § 3, Theorem 3.1 – Theorem 3.3]). Although the increment w of the form (3) is not everywhere positive or almost everywhere in the interval $[0,\pi]$.

3. Proof of the main results

Proof of Theorem 2. First, let s investigate the case $\alpha \neq 2\pi l$, $l \in \mathbb{Z}$, $\beta \neq 2\pi m$, $m \in \mathbb{Z}$.

Let us consider the functional $y(x,q,\lambda)$, which sets the value at the point $x \in [0,\pi]$ of solution to the Cauchy problem

$$\begin{cases} y'' + \big(\lambda - q(x)\big)y = 0, \\ y(0,\lambda) = 1, \\ y'(0,\lambda) = h = -\operatorname{ctg}\alpha. \end{cases} \tag{4}$$

in correspondence with an element of the set $\Omega = [0,\pi] \times L[0,\pi] \times \mathbb{R}$. Let us choose and fix arbitrary entire numbers $n \in \mathbb{N}$ and $k \in [0,n]$.

The Gateaux differential of the functional $y(x,q,\lambda)$ with the increment $w \in L[0,\pi]$ on the surface of the set Ω, defined by the equation $y(x_{k,n},q,\lambda_n) = 0$, equals to zero:

$$\begin{aligned} Dy(x,q,\lambda)[q,w]\Big|_{y(x_{k,n},q,\lambda_n)=0} &= y'(x_{k,n},q,\lambda_n)Dx_{k,n}[q,w] \\ + Dy(x_{k,n},q,\lambda_n)[q,w] &+ \dot{y}(x_{k,n},q,\lambda)D\lambda_n[q,w]\Big|_{y(x_{k,n},q,\lambda_n)=0} = 0. \end{aligned} \tag{5}$$

Here and in what follows it is assumed that

$$y'(x,q,\lambda) = \frac{\partial}{\partial x}y(x,q,\lambda), \quad \dot{y}(x_{k,n},q,\lambda) = \frac{\partial}{\partial \lambda}y(x,q,\lambda).$$

The total Gateaux differential of the functional $y(x,q,\lambda)$ when $x = \pi$, $\lambda = \lambda_n$ and the increment $w \in L[0,\pi]$ is

$$Dy(\pi,q,\lambda)[q,w]\Big|_{\lambda=\lambda_n} = Dy(\pi,q,\lambda_n)[q,w] + \dot{y}(\pi,q,\lambda_n)D\lambda_n[q,w]. \tag{6}$$

Likewise, the total Gateaux differential is obtained for the functional $y'(x,q,\lambda)$ when $x = \pi$, $\lambda = \lambda_n$ and the increment $w \in L[0,\pi]$:

$$Dy'(\pi,q,\lambda)[q,w]\Big|_{\lambda=\lambda_n} = Dy'(\pi,q,\lambda_n)[q,w] + \dot{y}'(\pi,q,\lambda_n)D\lambda_n[q,w]. \tag{7}$$

Then, by virtue of (1), (6) and (7), assuming that $H = \operatorname{ctg}\beta$, one has

$$\begin{aligned} D\big(y'(\pi,q,\lambda) + Hy(\pi,q,\lambda)\big)[q,w]\Big|_{\lambda=\lambda_n} &= D\big(y'(\pi,q,\lambda_n) + Hy(\pi,q,\lambda_n)\big)[q,w] \\ &+ \big(\dot{y}'(\pi,q,\lambda_n) + H\dot{y}(\pi,q,\lambda_n)\big)D\lambda_n[q,w] = 0. \end{aligned} \tag{8}$$

Let us calculate the partial Gateaux differential $Dy(x,q,\lambda)[q,w]$ for fixed x and λ and the increment $w \in L[0,\pi]$. Substituting in the equation of the Cauchy problem (4) q by $q + tw$, we obtain

$$y'' + [\lambda - q]y = twy. \tag{9}$$

Let us use the notation

$$\Phi(x,\lambda,\tau) = \begin{vmatrix} \varphi(x,\lambda) & \psi(x,\lambda) \\ \varphi(\tau,\lambda) & \psi(\tau,\lambda) \end{vmatrix},$$

where $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of the sine and cosine type of the equation of the Cauchy problem (4) (i.e. solutions with the initial conditions $\varphi(0,\lambda)=\psi'(0,\lambda)=0$, $\varphi'(0,\lambda)=\psi(0,\lambda)=1$). Then, the following identity holds for solution of the Cauchy problem (4) and the problem with the same initial conditions and Equation (9):

$$y(x,q+tw,\lambda)-y(x,q,\lambda)\equiv t\int_0^x \Phi(x,\lambda,\tau)w(\tau)y(\tau,q+tw,\lambda)\,d\tau.$$

Let us divide both sides of the resulting equality by t and pass to the limit when $t\to 0$

$$\begin{aligned} Dy(x,q,\lambda)[q,w] &= \lim_{t\to 0}\frac{y(x,q+tw,\lambda)-y(x,q,\lambda)}{t} \\ &= \int_0^x \Phi(x,\lambda,\tau)w(\tau)y(\tau,q,\lambda)\,d\tau. \end{aligned} \tag{10}$$

It is possible to pass to the limit under the integrand because every $y(x,q+tw,\lambda)$ is a function continuous on $[0,\pi]$, and the uniform convergence $y(x,q+tw,\lambda)$ to $y(x,q,\lambda)$ for $t\to 0$ follows, e.g., from the theorem on differentiability with respect to parameters of solution to the Cauchy problem [14, Ch.4, §24, Theorem 16]. It follows from (8) and (10) that

$$\begin{aligned} D\lambda_n[q,w] &= -\big(\dot y'(\pi,q,\lambda_n)+H\dot y(\pi,q,\lambda_n)\big)^{-1}\big(Dy'(\pi,q,\lambda_n)[q,w]+HDy(\pi,q,\lambda_n)[q,w]\big) \\ &= -\big(\dot y'(\pi,q,\lambda_n)+H\dot y(\pi,q,\lambda_n)\big)^{-1}\Big\{\int_0^\pi \Phi'_x(\pi,\lambda_n,\tau)w(\tau)y(\tau,q,\lambda_n)\,d\tau \\ &\qquad +H\int_0^\pi \Phi(\pi,\lambda_n,\tau)w(\tau)y(\tau,q,\lambda_n)\,d\tau\Big\} \\ &= -\big(\dot y'(\pi,q,\lambda_n)+H\dot y(\pi,q,\lambda_n)\big)^{-1}\int_0^\pi \Phi_1(\lambda_n,\tau)w(\tau)y(\tau,q,\lambda_n)\,d\tau, \end{aligned} \tag{11}$$

where

$$\Phi_1(\lambda_n,\tau)=\begin{vmatrix} \varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n) & \psi'(\pi,\lambda_n)+H\psi(\pi,\lambda_n) \\ \varphi(\tau,\lambda_n) & \psi(\tau,\lambda_n) \end{vmatrix}. \tag{12}$$

Since $y(x,q,\lambda_n)=\psi(x,\lambda_n)+h\varphi(x,\lambda_n)$, (1) provides the equality

$$-h\big(\varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n)\big)=\psi'(\pi,\lambda_n)+H\psi(\pi,\lambda_n),$$

and the representation

$$\begin{aligned} \Phi_1(\lambda_n,\tau) &= \big(\varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n)\big)\begin{vmatrix} 1 & -h \\ \varphi(\tau,\lambda_n) & \psi(\tau,\lambda_n) \end{vmatrix} \\ &= \big(\varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n)\big)y(\tau,q,\lambda_n). \end{aligned} \tag{13}$$

Since the equation of the Cauchy problem (4) does not contain first derivatives then, by virtue of the Liouville formula [14, Ch. 3, §18, (15)], the Wronskian determinant W of the fundamental system φ, ψ $W\equiv const=-1$.

Calculating the determinant (12) when $\tau=\pi$ one obtains $\Phi_1(\lambda_n,\pi)=-W=1$. It follows from (13) when $\tau=\pi$ that

$$\big(\varphi'(\pi,\lambda_n)+H\varphi(\pi,\lambda_n)\big)=\frac{1}{y(\pi,q,\lambda_n)}.$$

Calculation of the Gateaux differential (11) and invoking (13), one obtains

$$\begin{aligned} D\lambda_n[q,w] &= -\big(\dot y'(\pi,q,\lambda_n)+H\dot y(\pi,q,\lambda_n)\big)^{-1} \\ &\quad\times\frac{1}{y(\pi,q,\lambda_n)}\int_0^\pi w(\tau)y^2(\tau,q,\lambda_n)\,d\tau. \end{aligned} \tag{14}$$

By virtue of (5), (10) and (14),

$$Dx_{k,n}[q,w]=-\frac{1}{y'(x_{k,n},q,\lambda_n)}\bigg\{\int_0^{x_{k,n}}\varPhi(x_{k,n},\lambda_n,\tau)w(\tau)y(\tau,q,\lambda_n)\,d\tau$$

$$-\big(\dot y'(\pi,q,\lambda_n)+H\dot y(\pi,q,\lambda_n)\big)^{-1}\frac{\dot y(x_{k,n},q,\lambda_n)}{y(\pi,q,\lambda_n)}\int_0^{\pi}w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\bigg\}. \tag{15}$$

Note that $\varPhi(x_{k,n},\lambda_n,\tau)$ is a solution of the Cauchy problem with the same differential equation, as in the problem (4) and the initial conditions $y(x_{k,n},q,\lambda_n)=0$, $y'(x_{k,n},q,\lambda_n)=W=-1$. According to the Picard theorem,

$$\varPhi(x_{k,n},\lambda_n,\tau)=-\frac{y(\tau,q,\lambda_n)}{y'(x_{k,n},q,\lambda_n)}.$$

Then, (15) provides

$$Dx_{k,n}[q,w]=\frac{1}{\big[y'(x_{k,n},q,\lambda_n)\big]^2}\bigg\{\int_0^{x_{k,n}}w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\bigg\}$$

$$+\big(\dot y'(\pi,q,\lambda_n)+H\dot y(\pi,q,\lambda_n)\big)^{-1}\frac{\dot y(x_{k,n},q,\lambda_n)}{y'(x_{k,n},q,\lambda_n)y(\pi,q,\lambda_n)}\bigg\{\int_0^{\pi}w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\bigg\}. \tag{16}$$

Invoking (1), one can rewrite this equality in the form

$$Dx_{k,n}[q,w]=\frac{1}{\big[y'(x_{k,n},q,\lambda_n)\big]^2}\bigg\{\int_0^{x_{k,n}}w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\bigg\}$$

$$-\big(\dot y'(\pi,q,\lambda_n)+H\dot y(\pi,q,\lambda_n)\big)^{-1}\frac{H\dot y(x_{k,n},q,\lambda_n)}{y'(x_{k,n},q,\lambda_n)y'(\pi,q,\lambda_n)}\bigg\{\int_0^{\pi}w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\bigg\}.$$

Passing to the limit formally when $h\to\infty$, $H\to\infty$ one obtains from this formula a generalization (in case of the space $L[0,\pi]$) of the result of [9], formulated in Theorem 1.

Then, let us differentiate the equation of the Cauchy problem (4) with respect to λ (for a fixed potential $q_\lambda\equiv q$ with respect to the variable λ solution of the Cauchy problem (4) is an entire function, see [12]), multiply it by $y(x,q,\lambda)$ and integrate within the limits from x to π:

$$\int_x^{\pi}y\frac{\partial y''}{\partial\lambda}\,d\tau+\int_x^{\pi}y\big[\lambda-q(\tau)\big]\frac{\partial y}{\partial\lambda}\,d\tau+\int_x^{\pi}y^2\,d\tau=0.$$

Double integration by parts of the first integral provides

$$\Big(y\frac{\partial y'}{\partial\lambda}-y'\frac{\partial y}{\partial\lambda}\Big)\Big|_x^{\pi}+\int_x^{\pi}\Big\{y''+\big[\lambda-q(\tau)\big]y\Big\}\frac{\partial y}{\partial\lambda}\,d\tau$$

$$+\int_x^{\pi}y^2\,d\tau=0. \tag{17}$$

Since y is a solution to the equation of the problem (4), the second integral in (17) vanishes. Let us substitute $x=x_{k,n}$, $\lambda=\lambda_n$ into the resulting identity (17). Taking into account that such substitution turns the solution of the initial problem (4) $y(x,q,\lambda_n)$ into the eigenfunction of the Sturm-Liouville problem (1), and $y(x_{k,n},q,\lambda_n)=0$, the boundary-value conditions (1) provide

$$y(\pi,q,\lambda_n)\Big(\dot y'(\pi,q,\lambda_n)+H\dot y(\pi,q,\lambda_n)\Big)+$$

$$y'(x_{k,n},q,\lambda_n)\dot y(x_{k,n},q,\lambda_n)+\int_{x_{k,n}}^{\pi}y^2(\tau,q,\lambda_n)\,d\tau=0. \tag{18}$$

Let us transform (16)

$$Dx_{k,n}[q,w] = \frac{1}{\big[y'(x_{k,n},q,\lambda_n)\big]^2}\left\{\int_0^{x_{k,n}} w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\right\}$$

$$-\frac{\dot y(x_{k,n},q,\lambda_n)}{y'(x_{k,n},q,\lambda_n)\big[y'(x_{k,n},q,\lambda_n)\dot y(x_{k,n},q,\lambda_n)+\int_{x_{k,n}}^{\pi} y^2(\tau,q,\lambda_n)\,d\tau\big]}$$

$$\times\left\{\int_0^{\pi} w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\right\}.$$

Or

$$Dx_{k,n}[q,w] = \frac{1}{\big[y'(x_{k,n},q,\lambda_n)\big]^2}\int_0^{\pi} w(\tau)y^2(\tau,q,\lambda_n)\beta_{k,n}(\tau)\,d\tau, \tag{19}$$

where

$$\beta_{k,n}(\tau) = \begin{cases} 1-\alpha_{k,n}, & \text{если } \tau\in[0,x_{k,n}],\\ -\alpha_{k,n}, & \text{если } \tau\in(x_{k,n},\pi],\end{cases}$$

$$\alpha_{k,n} = \frac{\dot y(x_{k,n},q,\lambda_n)y'(x_{k,n},q,\lambda_n)}{y'(x_{k,n},q,\lambda_n)\dot y(x_{k,n},q,\lambda_n)+\int_{x_{k,n}}^{\pi} y^2(\tau,q,\lambda_n)\,d\tau}.$$

Note that by virtue of the oscillation theorem (see, e.g., [12, Ch. 1, § 3, Theorem 3.3]) $\dot y(x_{k,n},q,\lambda_n)y'(x_{k,n},q,\lambda_n)>0$ and $\int_{x_{k,n}}^{\pi} y^2(\tau,q,\lambda_n)\,d\tau>0$ therefore, $\alpha_{k,n}\in(0,1)$ for any $n\in\mathbb{N}$ and $0\le k\le n$. The value $\alpha_{k,n}$ is independent of the increment choice $w\in L[0,\pi]$. Let us calculate $\alpha_{k,n}$. For this purpose, assume that $w\equiv 1$. Obviously, $Dx_{k,n}[q,1]=0$ for all $n\in\mathbb{N}$ and $0\le k\le n$. Then, (19) provides

$$(1-\alpha_{k,n})\int_0^{x_{k,n}} y^2(\tau,q,\lambda_n)\,d\tau-\alpha_{k,n}\int_{x_{k,n}}^{\pi} y^2(\tau,q,\lambda_n)\,d\tau=0 \tag{20}$$

or

$$\alpha_{k,n} = \frac{1}{\|y(\cdot,q,\lambda_n)\|^2_{L_2[0,\pi]}}\int_0^{x_{k,n}} y^2(\tau,q,\lambda_n)\,d\tau.$$

Since the relation соотношения (19) is invariant with respect to multiplication of the functional $y(x,q,\lambda)$ by a nonzero constant, the formula (19) can be written in the form (2).

In the case $\alpha\ne 2\pi l$, $l\in\mathbb{Z}$, $\beta\ne 2\pi m$, $m\in\mathbb{Z}$, statement of Theorem 2 is proved.

Let us investigate the case $\alpha=2\pi l$, $l\in\mathbb{Z}$, $\beta=2\pi m$, $m\in\mathbb{Z}$.

Consider the functional $y(x,q,\lambda)$, which sets the value of solution to the Cauchy problem

$$\begin{cases} y''+\big(\lambda-q(x)\big)y=0,\\ y(0,\lambda)=0,\\ y'(0,\lambda)=1\end{cases} \tag{21}$$

at the point $x\in[0,\pi]$ in correspondence with an element of the set $\Omega=[0,\pi]\times L[0,\pi]\times\mathbb{R}$. If $k=0$ or $k=n$, the relation (2) is verified by a direct substitution $x_{k,n}=x_{k,n}[q]$. Let us choose and fix arbitrary entire numbers $n\in\mathbb{N}$ and $k\in[1,n-1]$.

The Gateaux differential of the functional $y(x,q,\lambda)$ with the increment $w\in L[0,\pi]$ on the surface of the set Ω, defined by the equation $y(x_{k,n},q,\lambda_n)=0$, equals to zero:

$$Dy(x,q,\lambda)[q,w]\Big|_{y(x_{k,n},q,\lambda_n)=0} = y'(x_{k,n},q,\lambda_n)Dx_{k,n}[q,w]$$

$$+Dy(x_{k,n},q,\lambda_n)[q,w]+\dot y(x_{k,n},q,\lambda)D\lambda_n[q,w]\Big|_{y(x_{k,n},q,\lambda_n)=0}=0. \tag{22}$$

Total Gateaux differential of the functional $y(x,q,\lambda)$ with $x=\pi$, $\lambda=\lambda_n$ and the increment $w\in L[0,\pi]$ is

$$Dy(\pi,q,\lambda)[q,w]\Big|_{\lambda=\lambda_n}=Dy(\pi,q,\lambda_n)[q,w]+\dot y(\pi,q,\lambda_n)D\lambda_n[q,w]=0. \tag{23}$$

Let us calculate a partial Gateaux differential $Dy(x,q,\lambda)[q,w]$ for fixed x and λ and the increment $w\in L[0,\pi]$. Substituting q by $q+tw$ in the equation of the Cauchy problem (21), one obtains

$$y''+[\lambda-q]y=twy. \tag{24}$$

Let us use the notation

$$\varPhi(x,\lambda,\tau)=\begin{vmatrix}\varphi(x,\lambda) & \psi(x,\lambda)\\ \varphi(\tau,\lambda) & \psi(\tau,\lambda)\end{vmatrix},$$

where $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of the sine and cosine type to the Cauchy problem equation (21). Then, the following identity holds for solution of the Cauchy problem (21) and the problem with the same initial conditions and the equation (24):

$$y(x,q+tw,\lambda)-y(x,q,\lambda)\equiv t\int_0^x\varPhi(x,\lambda,\tau)w(\tau)y(\tau,q+tw,\lambda)\,d\tau.$$

Let us divide both sides of the resulting identity by t and pass to the limit when $t\to 0$

$$\begin{aligned}Dy(x,q,\lambda)[q,w]&=\lim_{t\to0}\frac{y(x,q+tw,\lambda)-y(x,q,\lambda)}{t}\\&=\int_0^x\varPhi(x,\lambda,\tau)w(\tau)y(\tau,q,\lambda)\,d\tau.\end{aligned} \tag{25}$$

The limit is possible under the integral for the same reasons as in (10) when $\alpha\neq 2\pi l$, $l\in\mathbb{Z}$, $\beta\neq 2\pi m$, $m\in\mathbb{Z}$. Then, (23) and (25) entail

$$\begin{aligned}D\lambda_n[q,w]&=-\big(\dot y(\pi,q,\lambda_n)\big)^{-1}\big(Dy(\pi,q,\lambda_n)[q,w]\big)\\&=-\big(\dot y(\pi,q,\lambda_n)\big)^{-1}\left\{\int_0^\pi\varPhi(\pi,\lambda_n,\tau)w(\tau)y(\tau,q,\lambda_n)\,d\tau\right\}\\&=-\big(\dot y(\pi,q,\lambda_n)\big)^{-1}\int_0^\pi\varPhi_1(\lambda_n,\tau)w(\tau)y(\tau,q,\lambda_n)\,d\tau,\end{aligned} \tag{26}$$

where

$$\begin{aligned}\varPhi_1(\lambda_n,\tau)&=\begin{vmatrix}\varphi(\pi,\lambda_n) & \psi(\pi,\lambda_n)\\ \varphi(\tau,\lambda_n) & \psi(\tau,\lambda_n)\end{vmatrix}\\&=-\psi(\pi,\lambda_n)\varphi(\tau,\lambda_n)\equiv-\psi(\pi,\lambda_n)y(\tau,q,\lambda_n).\end{aligned} \tag{27}$$

This follows from the fact that $\varphi(\pi,\lambda_n)=0$ and $y(\tau,q,\lambda_n)\equiv\varphi(\tau,\lambda_n)$.

Since $\varPhi_1'(\lambda_n,\pi)=W=-1$ then, invoking the relation $\varphi(\pi,\lambda_n)=0$, one has

$$-\varphi'(\pi,\lambda_n)\psi(\pi,\lambda_n)=-1.$$

The identity $y(\tau,q,\lambda_n)\equiv\varphi(\tau,\lambda_n)$ provides

$$\psi(\pi,\lambda_n)=\frac{1}{\varphi'(\pi,\lambda_n)}=\frac{1}{y'(\pi,q,\lambda_n)}.$$

Proceeding to calculate the Gateaux differential (26) and taking into account (27), we obtain the representation

$$D\lambda_n[q,w]=\big(\dot y(\pi,q,\lambda_n)y'(\pi,q,\lambda_n)\big)^{-1}\int_0^\pi w(\tau)y^2(\tau,q,\lambda_n)\,d\tau. \tag{28}$$

By virtue of (22), (25) and (28), one has

$$Dx_{k,n}[q,w] = -\frac{1}{y'(x_{k,n},q,\lambda_n)}\left\{\int_0^{x_{k,n}} \Phi(x_{k,n},\lambda_n,\tau)w(\tau)y(\tau,q,\lambda_n)\,d\tau \right.$$
$$\left. + \frac{\dot y(x_{k,n},q,\lambda_n)}{y'(\pi,q,\lambda_n)\dot y(\pi,q,\lambda_n)}\int_0^{\pi} w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\right\}. \tag{29}$$

Note that $\Phi(x_{k,n},\lambda_n,\tau)$ is a solution to the Cauchy problem with the same differential equation as in the problem (21) and the initial conditions $y(x_{k,n},q,\lambda_n)=0$, $y'(x_{k,n},q,\lambda_n)=W=-1$. According to the Picard theorem $\Phi(x_{k,n},\lambda_n,\tau) = -\frac{y(\tau,q,\lambda_n)}{y'(x_{k,n},q,\lambda_n)}$. Then, (29) provides

$$Dx_{k,n}[q,w] = \frac{1}{\big[y'(x_{k,n},q,\lambda_n)\big]^2}\left\{\int_0^{x_{k,n}} w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\right\}$$
$$-\frac{\dot y(x_{k,n},q,\lambda_n)}{y'(\pi,q,\lambda_n)y'(x_{k,n},q,\lambda_n)\dot y(\pi,q,\lambda_n)}\left\{\int_0^{\pi} w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\right\}. \tag{30}$$

This formula entails that the result of [9], given in the statement of Theorem 1, admits a generalization to the case $q,w\in L[0,\pi]$.

Further, Let us differentiate the Cauchy problem equation (21) with respect to λ, multiply it by $y(x,q,\lambda)$ and integrate within the limits from x to π:

$$\int_x^{\pi} y\frac{\partial y''}{\partial\lambda}\,d\tau + \int_x^{\pi} y\big[\lambda - q(\tau)\big]\frac{\partial y}{\partial\lambda}\,d\tau + \int_x^{\pi} y^2\,d\tau = 0.$$

Double integration of the first integral by parts provides

$$\left(y\frac{\partial y'}{\partial\lambda} - y'\frac{\partial y}{\partial\lambda}\right)\Big|_x^{\pi} + \int_x^{\pi}\Big\{y'' + \big[\lambda - q(\tau)\big]y\Big\}\frac{\partial y}{\partial\lambda}\,d\tau$$
$$+\int_x^{\pi} y^2\,d\tau = 0. \tag{31}$$

Since y is a solution of the Cauchy problem equation (21), the second integral in (31) vanishes. Let us substitute $x=x_{k,n}$, $\lambda=\lambda_n$ into the resulting identity (31). Taking into account that the Cauchy problem solution (21) $y(x,q,\lambda_n)$ turned into an eigenfunction of the Sturm-Liouville problem (1), and $y(x_{k,n},q,\lambda_n)=0$ under such substitution, one obtains

$$-y'(\pi,q,\lambda_n)\dot y(\pi,q,\lambda_n)+$$
$$y'(x_{k,n},q,\lambda_n)\dot y(x_{k,n},q,\lambda_n) + \int_{x_{k,n}}^{\pi} y^2(\tau,q,\lambda_n)\,d\tau = 0$$

from the boundary-values conditions (1). Let us make the transformation (30)

$$Dx_{k,n}[q,w] = \frac{1}{\big[y'(x_{k,n},q,\lambda_n)\big]^2}\left\{\int_0^{x_{k,n}} w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\right\}$$
$$-\frac{\dot y(x_{k,n},q,\lambda_n)}{y'(x_{k,n},q,\lambda_n)\big[y'(x_{k,n},q,\lambda_n)\dot y(x_{k,n},q,\lambda_n) + \int_{x_{k,n}}^{\pi} y^2(\tau,q,\lambda_n)\,d\tau\big]}$$
$$\times\left\{\int_0^{\pi} w(\tau)y^2(\tau,q,\lambda_n)\,d\tau\right\}.$$

Or

$$Dx_{k,n}[q,w] = \frac{1}{\big[y'(x_{k,n},q,\lambda_n)\big]^2}\int_0^{\pi} w(\tau)y^2(\tau,q,\lambda_n)\beta_{k,n}(\tau)\,d\tau, \tag{32}$$

where

$$\beta_{k,n}(\tau) = \begin{cases} 1-\alpha_{k,n}, & \text{если } \tau\in[0,x_{k,n}],\\ -\alpha_{k,n}, & \text{если } \tau\in(x_{k,n},\pi],\end{cases}$$

$$\alpha_{k,n} = \frac{\dot{y}(x_{k,n}, q, \lambda_n) y'(x_{k,n}, q, \lambda_n)}{y'(x_{k,n}, q, \lambda_n)\dot{y}(x_{k,n}, q, \lambda_n) + \int_{x_{k,n}}^{\pi} y^2(\tau, q, \lambda_n)\, d\tau}.$$

Note that by virtue of the oscillation theorem (see, e.g., [12, Ch. 1, § 3, Theorem 3.3]) $\dot{y}(x_{k,n}, q, \lambda_n) y'(x_{k,n}, q, \lambda_n) > 0$ and $\int_{x_{k,n}}^{\pi} y^2(\tau, q, \lambda_n)\, d\tau > 0$. Therefore, $\alpha_{k,n} \in (0, 1)$ for any $n \in \mathbb{N}$ and $1 \le k \le n-1$. The value $\alpha_{k,n}$ is independent of the increment choice $w \in L[0, \pi]$. Let us calculate $\alpha_{k,n}$. To this end assume that $w \equiv 1$. Obviously, $Dx_{k,n}[q, 1] = 0$ for all $n \in \mathbb{N}$ and $0 \le k \le n$. One obtains from (32), that

$$(1 - \alpha_{k,n}) \int_0^{x_{k,n}} y^2(\tau, q, \lambda_n)\, d\tau - \alpha_{k,n} \int_{x_{k,n}}^{\pi} y^2(\tau, q, \lambda_n)\, d\tau = 0$$

or

$$\alpha_{k,n} = \frac{1}{\|y(\cdot, q, \lambda_n)\|^2_{L_2[0,\pi]}} \int_0^{x_{k,n}} y^2(\tau, q, \lambda_n)\, d\tau.$$

Note that the resulting representation $\alpha_{k,n}$, and hence (32) as well, hold true for the values $k = 0$ and $k = n$. The formulae (32) can be written in the form (2).

When $\alpha = 2\pi l$, $l \in \mathbb{Z}$, $\beta = 2\pi m$, $m \in \mathbb{Z}$, Theorem 2 is proved.

Let us consider the case when $\alpha \ne 2\pi l$, $l \in \mathbb{Z}$, $\beta = 2\pi m$, $m \in \mathbb{Z}$. Let denote the functional, setting the value of the Cauchy problem solution (4) at the point $x \in [0, \pi]$ in correspondence with an element f the set $\Omega = [0, \pi] \times L[0, \pi] \times \mathbb{R}$ by $y(x, q, \lambda)$ again. Let us fix the arbitraries $n \in \mathbb{N}$ and the integer $k \in [0, n-1]$. Further proof is similar to investigation of the case $\alpha \ne 2\pi l$, $l \in \mathbb{Z}$, $\beta \ne 2\pi m$, $m \in \mathbb{Z}$. Instead of (8) in (11), one should use the relation (23), and the function $\varPhi_1(\lambda_n, \tau)$ is to be defined as follows. Since $\beta = 2\pi m$, $m \in \mathbb{Z}$ then,

$$\varPhi_1(\lambda_n, \tau) = \begin{vmatrix} \varphi(\pi, \lambda_n) & \psi(\pi, \lambda_n) \\ \varphi(\tau, \lambda_n) & \psi(\tau, \lambda_n) \end{vmatrix} = C y(\tau, q, \lambda_n). \tag{33}$$

The constant $C = \varphi(\pi, \lambda_n)$ will be defined from the relation $\varPhi_1(\lambda_n, 0) = \varphi(\pi, \lambda_n)$. Thus, the representation of the function

$$\varPhi_1(\lambda_n, \tau) = \varphi(\pi, \lambda_n) y(\tau, q, \lambda_n)$$

is obtained. Furthermore, derivative of the function $\varPhi_1(\lambda_n, \tau)$ at the point π is the value of the Wronskian determinant $W = -1$. Thus,

$$-1 = \varphi(\pi, \lambda_n) y'(\pi, q, \lambda_n).$$

One obtains the equality

$$\varPhi_1(\lambda_n, \tau) = -\frac{1}{y'(\pi, q, \lambda_n)} y(\tau, q, \lambda_n).$$

The latter and (26) provide (28), and hence, the representation of the Gateaux differential of the form (30) for the case $\alpha \ne 2\pi l$, $l \in \mathbb{Z}$, $\beta = 2\pi m$, $m \in \mathbb{Z}$. Then, let us complete the proof of the theorem for $\alpha \ne 2\pi l$, $l \in \mathbb{Z}$, $\beta = 2\pi m$, $m \in \mathbb{Z}$ similarly to the case $\alpha = 2\pi l$, $l \in \mathbb{Z}$, $\beta = 2\pi m$, $m \in \mathbb{Z}$. Note that the resulting representation $\alpha_{k,n}$, and hence, (32) as well hold true for the value $k = n$.

It remains to consider the case $\alpha = 2\pi l$, $l \in \mathbb{Z}$, $\beta \ne 2\pi m$, $m \in \mathbb{Z}$. Denote the functional, setting the value of the solution to the Cauchy problem at the pint $x \in [0, \pi]$ in correspondence to an element of the set $\Omega = [0, \pi] \times L[0, \pi] \times \mathbb{R}$ by $y(x, q, \lambda)$ here. In this case, let s fix the arbitrary $n \in \mathbb{N}$ and the integer $k \in [1, n]$. Further proof is conducted similarly to the case $\alpha \ne 2\pi l$, $l \in \mathbb{Z}$, $\beta \ne 2\pi m$, $m \in \mathbb{Z}$. Since when $\alpha = 2\pi l$, $l \in \mathbb{Z}$, $\beta \ne 2\pi m$, $m \in \mathbb{Z}$ the eigenfunction $y(x, q, \lambda_n) \equiv \varphi(x, \lambda_n)$, i.e. $\varphi'(\pi, \lambda_n) + H\varphi(\pi, \lambda_n) = 0$, and the determinant (12) equals

$$\varPhi_1(\lambda_n, \tau) = -\big(\psi'(\pi, \lambda_n) + H\psi(\pi, \lambda_n)\big)\varphi(\tau, \lambda_n).$$

Calculation of the determinant (12) for $\tau = \pi$ yields $\Phi_1(\lambda_n, \pi) = -W = 1$. Then, (12) and the identity $y(x, q, \lambda_n) \equiv \varphi(x, \lambda_n)$ for $\tau = \pi$ provide

$$\big(\psi'(\pi, \lambda_n) + H\psi(\pi, \lambda_n)\big) = -\frac{1}{y(\pi, q, \lambda_n)}.$$

Hence,

$$\Phi_1(\lambda_n, \tau) = \frac{1}{y(\pi, q, \lambda_n)}\varphi(\tau, \lambda_n) = \frac{y(\tau, q, \lambda_n)}{y(\pi, q, \lambda_n)}.$$

Then, similarly to the case $\alpha \neq 2\pi l$, $l \in \mathbb{Z}$, $\beta \neq 2\pi m$, $m \in \mathbb{Z}$, we obtain the relations (15), (16) and (18). Whence, the representation of the Gateaux differential for the functional $Dx_{k,n}[q, w]$ of the form (19) follows again. Then, we complete the proof of Theorem for $\alpha = 2\pi l$, $l \in \mathbb{Z}$, $\beta \neq 2\pi m$, $m \in \mathbb{Z}$ with arbitrary $n \in \mathbb{N}$ and integer $k \in [1, n]$ similarly to the case $\alpha \neq 2\pi l$, $l \in \mathbb{Z}$, $\beta \neq 2\pi m$, $m \in \mathbb{Z}$. Note again that the resulting representation $\alpha_{k,n}$, and hence, (19) as well, hold true for the value $k = 0$ as well.

Theorem 2 is proved.

□

Proof of Theorem 3. If $w \equiv 1$ then, the integral (19) or (32) for the corresponding values of the parameters α and β splits into two integrals because the function $y^2(x, q, \lambda_n)$ is almost everywhere positive in the interval $[0, \pi]$ and $\alpha_{k,n} \in (0, 1)$. Namely, it is positive in the interval $(0, x_{k,n})$ and negative in the interval $(x_{k,n}, \pi)$. By virtue of (20) (this relation is considered for the corresponding values of the parameters α and β), the constants $\alpha_{k,n}$ have the property that these two integrals are equal to each other in the absolute value. If the function (3) is taken as the increment then, for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $x_{k,n} \in (0, \pi)$, the „positive" part of the integral (32) becomes strictly less than the absolute value of the „negative" part. The most important part is played by the fact that all zeros $x_{k,n}[q]$ are located inside the interval $(0, \pi)$.

Theorem 3 is proved.

□

Remark. Results obtained in [10] now can be extended to the case o arbitrary boundary-value conditions of the third kind using the work [15].

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Aleksandr Yur'evich Trynin,
Saratov State University,
Astrakhanskaya Str., 83,
410012, Saratov, Russia
E-mail: atrynin@gmail.com

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