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AN EQUIVALENT INTEGRAL NORM IN A DUAL SPACE

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Abstract. In the present paper, the problem of describing a dual space in terms of the Hilbert transform is considered. We establish the necessary and sufficient conditions for the space $\widetilde{B}_2(G,\mu)$ to possess an integral norm equivalent to the initial one. We find the form of this norm. Using the general result of this work, we specify the recent result of the author and R.S. Yulmukhametov. The method suggested in the paper is based on the theory of orthosimilar systems. This method can be used to solve the problem of describing a dual space in terms of the Fourier– Lapalace transform and in terms of others complete system of functions.

Keywords: Hilbert transform, reproducing kernel, orthosimilar system, wavelet transform, integral frames

1. Introduction and formulation of the problem

Let G be a simply connected domain in the complex plane $\mathbb{C}$, and the measure μ is a certain Borel measure on G. The Bergman weighted space $B_2(G,\mu)$ consists the functions, analytic in the domain G and summable with the squared module with respect to the measure μ:

$$\|f\|^2_{B_2(G,\mu)} = \int_G |f(z)|^2\, d\mu(z) < \infty.$$

Let us impose the following conditions on the measure μ:

1. The system of functions
$$\left\{1/(z-\xi)^2\right\}_{\xi\in\mathbb{C}\setminus\overline{G}}$$
belongs to the space $B_2(G,\mu)$ and is complete there.
2. The space $B_2(G,\mu)$ is a functional Hilbert space with the scalar product:
$$(f,g)_{B_2(G,\mu)} \stackrel{def}{=} \int_G f(z)\cdot\overline{g(z)}\, d\mu(z).$$
Functionality is understood as follows: for any point $z_0\in G$ the functional $f\to f(z_0)$ is a linear and continuous functional over $B_2(G,\mu)$.

If we take the flat Lebesgue measure as the measure μ, and a domain with a Jordan boundary in $\mathbb{C}$ as G then, the space $B_2(G,\mu)$ satisfies the conditions 1 and 2 (see [1]).

Let set the function

$$\widetilde{f}(\xi) \stackrel{def}{=} f^*\left(\frac{1}{(z-\xi)^2}\right) \quad \xi\in\mathbb{C}\setminus\overline{G}.$$

in correspondence with every linear continuous functional f^* on $B_2(G,\mu)$ generated by the function $f\in B_2(G,\mu)$.

Definition 1. *The function $\widetilde{f}$ is termed as the Hilbert transformation of the functional f^*.*

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Since the system of functions $\{\frac{1}{(z-\xi)^2}, \xi \in \mathbb{C}\backslash\overline{G}\}$ is complete in the space $B_2(G,\mu)$, the mapping $f^* \to \widetilde{f}$ is injective. The set of functions $\widetilde{f}$ generates the space

$$\{\widetilde{f} : \widetilde{f}(\xi) = (\tfrac{1}{(z-\xi)^2}, f(z))_{B_2(G,\mu)}\} \stackrel{ob}{=} \widetilde{B}_2(G,\mu),$$

where the induced structure of the Hilbert space, i.e.

$$(\widetilde{f}, \widetilde{g})_{\widetilde{B}_2(G,\mu)} \stackrel{def}{=} (g, f)_{B_2(G,\mu)}$$

is considered, and

$$\|\widetilde{f}\|_{\widetilde{B}_2(G,\mu)} = \|f\|_{B_2(G,\mu)}. \qquad (1)$$

The following question arises. When is it possible to introduce in the space $\widetilde{B}_2(G,\mu)$ an integral norm of the form

$$\|\widetilde{f}\|_\nu = \sqrt{\int_{\mathbb{C}\backslash G} |\widetilde{f}(\xi)|^2 \, d\nu(\xi)},$$

where ν is a nonnegative Borrel measure on $\mathbb{C}\backslash\overline{G}$, equivalent to the induced norm $\|\widetilde{f}\|_{\widetilde{B}_2(G,\mu)}$? Specifically, is there a nonnegative Borrel measure ν in $\mathbb{C}\backslash\overline{G}$ and constants $A_1, A_2 > 0$ such that the correlations

$$A_1\|\widetilde{f}\|_{\widetilde{B}_2(G,\mu)} \le \|\widetilde{f}\|_\nu \le A_2\|\widetilde{f}\|_{\widetilde{B}_2(G,\mu)} \quad , \quad \forall \widetilde{f} \in \widetilde{B}_2(G,\mu) \qquad (2)$$

hold?

Such questions have been considered earlier by many mathematicians.

The problem of describing a space conjugate to the Hilbert spaces of analytical functions in terms of the Laplace transform was investigated, e.g., in the works [2], [3], [4],[5],[6], [7] and others.

Results of this works find their applications in solving interpolation problems, problems arising in the theory of convolution equations. The problem of describing a conjugate spa=ce in terms of the Cauchy and Hilbert transformation is less investigated, the works [8], [6], [1] are devoted to it.

In [9], the necessary and sufficient condition, that allows one to introduce an equivalent integral norm (Theorem 3) in the space $\widetilde{B}_2(G,\mu)$ is obtained, and corollaries of the theorem are given.

In the present article we establish the necessary and sufficient conditions that give a possibility to introduce in the space $\widetilde{B}_2(G,\mu)$ an equivalent integral norm of a special form. Under these conditions, the explicit form of the norm in the space $\widetilde{B}_2(G,\mu)$ is written out.

Note that the method suggested in the present paper can be used in solving problems on describing the conjugate space in terms of the Fourier-Laplace transform, as well as in the case of Hilbert spaces of analytical functions in the domains $\mathbb{C}^n$. The method is based on the theory of orthosimilar decomposition systems introduced by T.P. Lukashenko in the paper [10].

2. Formulation of the main result and an auxiliary statement

In what follows, for the sake of brevity, we write B_2, $\widetilde{B}_2$ instead of $B_2(G,\mu)$ and $\widetilde{B}_2(G,\mu)$, respectively.

Let us assume that there is an operator $\mathcal{A} : B_2 \to B_2$ that transforms the family of functions

$$\left\{\frac{1/(z-\xi)^2}{\|1/(z-\xi)^2\|_{B_2}}\right\}_{\xi\in\mathbb{C}\backslash\overline{G}}$$

to the family of functions

$$\left\{\frac{K_{B_2}(z,t)}{\sqrt{K_{B_2}(t,t)}}\right\}_{t\in G}.$$

Then, $\mathcal{A}$ defines the mapping $\rho : \mathbb{C}\backslash\overline{G} \to G$, $\xi \to \rho(\xi)$ by the rule:

$$\mathcal{A}\left(\frac{1/(z-\xi)^2}{\|1/(z-\xi)^2\|_{B_2}}\right) = \frac{K_{B_2}(z,\rho(\xi))}{\sqrt{K_{B_2}(\rho(\xi),\rho(\xi))}}. \tag{3}$$

Theorem 1. *The following conditions are equivalent:*

1. *There is a linear continuous bijective unitary operator $\mathcal{A}$, carrying out the isometry of the space B_2 into itself, which maps the system of functions*

$$\left\{\frac{1/(z-\xi)^2}{\|1/(z-\xi)^2\|_{B_2}}\right\}_{\xi\in\mathbb{C}\backslash\overline{G}}$$

to the system

$$\left\{\frac{K_{B_2}(z,t)}{\sqrt{K_{B_2}(t,t)}}\right\}_{t\in G}.$$

2. *There is a homeomorphism ρ of the domain $\mathbb{C}\backslash\overline{G}$ to the domain G such that the norm in the space $\widetilde{B}_2$ has the form:*

$$\|g\|_{\widetilde{B}_2} = \sqrt{\int_{\mathbb{C}\backslash\overline{G}} |g(\xi)|^2 \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}}\, d\mu(\rho(\xi))}, \quad g \in \widetilde{B}_2. \tag{4}$$

3. *There is a homeomorphism ρ of the domain $\mathbb{C}\backslash\overline{G}$ to the domain G such that the norm in the space $\widetilde{B}_2$, introduced by a formula in the form:*

$$\|g\|_1 = \sqrt{\int_{\mathbb{C}\backslash\overline{G}} |g(\xi)|^2 \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}}\, d\mu(\rho(\xi))}, \quad g \in \widetilde{B}_2, \tag{5}$$

is equivalent to the initial one, i.e.

$$A_1\|g\|_{\widetilde{B}_2} \le \|g\|_1 \le A_2\|g\|_{\widetilde{B}_2}, \quad \forall g \in \widetilde{B}_2,$$

where $A_1, A_2 > 0$ are constants.

First, let us prove the auxiliary statements.

Lemma 1. *Let H be a function Hilbert space, consisting of functions analytical in the domain G, and $K_H(z,t)$, $z,t \in G$ be a reproducing kernel of the space H. The system of functions*

$$\left\{\frac{K_H(z,t)}{\sqrt{K_H(t,t)}}\right\}_{t\in G}$$

is an orthosimilar decomposition system in the space H with the measure $K_H(t,t)\,d\mu(t)$ If and only if the space H coincides with the space B_2.

Proof. It is readily obtained from Theorem 1 of the work [9].

Likewise, on the basis of Theorem 2 from [9] the following is proved.

Lemma 2. *Let H be a functional Hilbert space consisting of functions of the variable $\xi \in \mathbb{C}\backslash\overline{G}$. The system of functions*

$$\left\{\frac{1/(\xi-t)^2}{\|1/(\xi-t)^2\|_H}\right\}_{t\in G}$$

is an orthosimilar decomposition system in the space H *with the measure* $\|\frac{1}{(\xi-t)^2}\|_H^2 d\mu(t)$ *if and only if the space* H *coincides with the space* $\widetilde{B}_2$.

Using Theorem 3 of the work [9], one can prove

Lemma 3. *In order introduce in the space* $\widetilde{B}_2$ *a norm equivalent to the initial one*

$$\|\widetilde{f}\|_\nu = \sqrt{\int_{\mathbb{C}\setminus G} |\widetilde{f}(\xi)|^2\, d\nu(\xi)}$$

where ν *is a nonnegative Borrel measure on* $\mathbb{C}\setminus\overline{G}$ *It is necessary and sufficient that there should exist a linear continuous self-adjoint operator* S *giving the automorphism of the Banach space* B_2 *such that the system of functions* $\{\frac{S_z(1/(z-\xi)^2)}{\|1/(z-\xi)^2\|}\}_{\xi\in\mathbb{C}\setminus\overline{G}}$ *is an orthosimilar decomposition system with the measure* $\|1/(z-\xi)^2\|^2\, d\nu(\xi)$ *in the space* B_2. *Any element* $f \in B_2$ *can be represented in the form:*

$$f(z) = \int_{\mathbb{C}\setminus\overline{G}} \left(f(\tau), \tfrac{S_\tau 1/(\tau-\xi)^2}{\|1/(\tau-\xi)^2\|}\right)_{B_2} \times$$
$$\times \tfrac{S_z(1/(z-\xi)^2)}{\|1/(z-\xi)^2\|} \|1/(z-\xi)^2\|^2\, d\nu(\xi), \quad z \in \mathbb{C}\setminus G. \tag{6}$$

Let us formulate several other auxiliary lemmas, which will be required for proving the main result of the paper.

Lemma 4. *Let us assume that* H_1 *and* H_2 *are Hilbert spaces with a reproducing kernel, consisting of functions of the variables* $t \in M$, *where* M *is a certain set. Let the system of functions* $\{e_\omega(t)\}_{\omega\in\Omega}$ *be contained in the space* H_1 *as well as in the space* H_2 *and, moreover, be an orthosimilar decomposition system with the measure* μ *in the space* H_1, *and in the space* H_2. *Then, the space* H_1 *coincides with the space* H_2.

Lemma 5. *Let* $\{\xi_k\}_{k=1}^n$ *be a set of* n *various points belonging to the domain* G, *and* $g(z)$ *be a function of the form*

$$g(z) = \sum_{k=1}^n c_k K_{B_2}(z, \xi_k),$$

where c_k, $k = 1, \ldots, n$ *are some constants.*
Then, the condition $g(z) \equiv 0$ *entails that* $c_k = 0$, $k = 1, \ldots, n$.

Lemma 6. *Let us assume that* $\{\xi_k\}_{k=1}^n$ *is a set of* n *various points belonging to the domain* $\mathbb{C}\setminus\overline{G}$ *and the function* $g(z)$ *has the form*

$$g(z) = \sum_{k=1}^n c_k \frac{1}{(z-\xi_k)^2},$$

where c_k, $k = 1, \ldots, n$ *are some constants.*
Then, the condition $g(z) \equiv 0$ *entails that* $c_k = 0$, $k = 1, \ldots, n$.

Let us assume that there is a linear continuous bijective operator $\mathcal{A}$, carrying out the isometry of the space B_2, which maps the system of functions

$$\left\{\frac{1/(z-\xi)^2}{\|1/(z-\xi)^2\|_{B_2}}\right\}_{\xi\in\mathbb{C}\setminus\overline{G}}$$

to the system

$$\left\{\frac{K_{B_2}(z,t)}{\sqrt{K_{B_2}(t,t)}}\right\}_{t\in G}.$$

Then, the operator $\mathcal{A}$ generates the mapping $t = \rho(\xi)$, $\rho : \mathbb{C}\backslash\overline{G} \to G$ by the rule

$$\mathcal{A}\frac{1/(z-\xi)^2}{\|1/(z-\xi)^2\|_{B_2}} = K_{B_2}(z, \rho(\xi)),\ \xi \in \mathbb{C}\backslash\overline{G}.$$

Lemma 7. *The mapping $\rho(z)$ maps the domain $\mathbb{C}\backslash\overline{G}$ to the domain G homeomorphically.*

Proof Let us denote

$$\varphi(z, w) \overset{def}{=} \frac{K_{B_2}(z, w)}{\sqrt{K_{B_2}(w, w)}}, \quad \psi(z, \xi) \overset{def}{=} \frac{1/(z-\xi)^2}{\|1/(z-\xi)^2\|_{B_2}}. \tag{7}$$

Let us define the mappings $\mathcal{Z}$ and $\mathcal{L}$.

$$\mathcal{Z} : G \to B_2, \quad z \overset{\mathcal{Z}}{\to} \varphi(\tau, z), \quad z \in G,$$

$$\mathcal{L} : \mathbb{C}\backslash\overline{G} \to B_2, \quad \xi \overset{\mathcal{L}}{\to} \psi(\tau, \xi), \quad \xi \in \mathbb{C}\backslash\overline{G}.$$

Then, the mapping ρ can be represented in the form:

$$\rho = \mathcal{Z}^{-1} \circ \mathcal{A} \circ \mathcal{L},$$

(see the diagram)

$$\begin{array}{ccc} B_2 & \xrightarrow{\mathcal{A}} & B_2 \\ {\scriptstyle\mathcal{L}}\big\uparrow & & \big\downarrow{\scriptstyle\mathcal{Z}^{-1}} \\ \mathbb{C}\backslash\overline{G} & \xrightarrow{\rho} & G \end{array}$$

The operator $\mathcal{A}$ is bijective and continuous as the isometry of the space B_2. One can readily demonstrate that the mapping $\mathcal{Z}$ transforms points of the domain G bijectively to the family of functions

$$\{\varphi(z, w)\}_{w\in G} \subset B_2,$$

and the mapping $\mathcal{L}$ transforms points of the domain $\mathbb{C}\backslash\overline{G}$ bijectively to the family of functions

$$\{\psi(z, \xi)\}_{\xi\in\mathbb{C}\backslash\overline{G}} \subset B_2.$$

Moreover, the mapping $\mathcal{Z}$ is homeomorphic, i.e. the following two conditions hold for any fixed point $z_0 \in G$:

1. If the sequence of points $\{\eta_k\}_{k\geq 0} \in G$ is such that

   $$|\eta_k - z_0| \to 0, \quad k \to \infty,$$

   then

   $$\|\varphi(z, \eta_k) - \varphi(z, z_0)\|_{B_2} \to 0, \quad k \to \infty.$$

2. If the sequence of points $\{\eta_k\}_{k\geq 0} \in G$ is such that

   $$\|\varphi(z, \eta_k) - \varphi(z, z_0)\|_{B_2} \to 0, \quad k \to \infty,$$

   then,

   $$|\eta_k - z_0| \to 0, \quad k \to \infty.$$

The proof of this fact is rather standard and is omitted here. Let us consider only the theorem used in the proof (see e.g., [13]).

Let us assume that Ω is a bounded domain in the complex plane and $H(\Omega)$ is a functional Hilbert space, consisting of functions of the variable $\eta \in \Omega$, the function $K_H(\eta, z)$ is a reproducing kernel of the space $H(\Omega)$.

Definition 2. *The reproducing kernel $K_H(\eta, z)$ of the space $H(\Omega)$ is said to be locally bounded if the function $K_H(\eta, z)$ is bounded on $K_1 \times K_2$ for any pair $\{K_1, K_2\}$ of compact subset Ω.*

Theorem A *A functional Hilbert space $H(\Omega)$ consists of analytical functions in the domain Ω if and only if the reproducing kernel $K_H(\eta, z)$ is locally bounded and is a function analytical with respect to the variable $\eta \in \Omega$ and anti-analytical with respect to the variable $z \in \Omega$.*

The fact that the mapping $\mathcal{L}$ maps the domain $\mathbb{C}\backslash\overline{G}$ homeomorphically to the family of functions $\{\psi(\tau,\xi)\}_{\xi\in\mathbb{C}\backslash\overline{G}}$ is proved likewise.

The mapping ρ is represented in the form:

$$\rho = \mathcal{Z}^{-1} \circ \mathcal{A} \circ \mathcal{L}.$$

Since $\mathcal{Z}^{-1}$, $\mathcal{A}$, $\mathcal{L}$ are homeomorphisms then, ρ maps the domain $\mathbb{C}\backslash\overline{G}$ to the domain G homeomorphically.

3. Proof of Theorem 1

Let us demonstrate that Condition 1 entails Condition 2. According to Lemma 1, any function $f \in B_2$ can be represented in the form:

$$f(z) = \int_G \big(f(\tau), \tfrac{K_{B_2}(\tau,t)}{\sqrt{K_{B_2}(t,t)}}\big)_{B_2} \cdot \tfrac{K_{B_2}(z,t)}{\sqrt{K_{B_2}(t,t)}}\, K_{B_2}(t,t)d\mu(t), z \in G. \tag{8}$$

Condition 1 entails that the operator $\mathcal{A}$ generates a homomorphism $\rho : \mathbb{C}\backslash\overline{G} \to G$, $\xi \to \rho(\xi)$ (see the relation (3) and Lemma (7)). Let us substitute the variables $t = \rho(\xi)$ in the integral (8). Then,

$$f(z) = \int_{\mathbb{C}\backslash\overline{G}} \big(f(\tau), \tfrac{K_{B_2}(\tau,\rho(\xi))}{\sqrt{K_{B_2}(\rho(\xi),\rho(\xi))}}\big)_{B_2} \times$$
$$\times \tfrac{K_{B_2}(z,\rho(\xi))}{\sqrt{K_{B_2}(\rho(\xi),\rho(\xi))}}\, K_{B_2}(\rho(\xi),\rho(\xi))d\mu(\rho(\xi)), z \in G. \tag{9}$$

According to Condition 1

$$\frac{K_{B_2}(z,\rho(\xi))}{\sqrt{K_{B_2}(\rho(\xi),\rho(\xi))}} = \mathcal{A}\Big(\frac{1/(z-\xi)^2}{\|1/(z-\xi)^2\|_{B_2}}\Big), \quad \xi \in \mathbb{C}\backslash\overline{G}.$$

Hence, the equality (9) can be written as follows:

$$f(z) = \int_{\mathbb{C}\backslash\overline{G}} \big(f(\tau), \mathcal{A}_\tau \tfrac{1/(\tau-\xi)^2}{\|1/(\tau-\xi)^2\|_{B_2}}\big)_{B_2} \times$$
$$\times \mathcal{A}_z \tfrac{1/(z-\xi)^2}{\|1/(z-\xi)^2\|_{B_2}}\, K_{B_2}(\rho(\xi),\rho(\xi))d\mu(\rho(\xi)), z \in G. \tag{10}$$

Whence,

$$f(z) = \int_{\mathbb{C}\backslash\overline{G}} \big(f(\tau), \mathcal{A}_\tau \tfrac{1}{(\tau-\xi)^2}\big)_{B_2} \cdot \mathcal{A}_z \tfrac{1}{(z-\xi)^2}\, \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}} d\mu(\rho(\xi)), z \in G.$$

Let us assume that $\mathcal{A}^*$ is an operator conjugate to the operator $\mathcal{A}$. It is also an isometry of the space B_2. Then, $\mathcal{A}^* \circ \mathcal{A} = I$, where I is an identical operator in the space B_2. Using the theorem from ([14], стр.128), one can demonstrate that

$$\mathcal{A}^* f(z) = \int_{\mathbb{C}\backslash\overline{G}} \big(f(\tau), \mathcal{A}_\tau \tfrac{1}{(\tau-\xi)^2}\big)_{B_2} \cdot \mathcal{A}^* \circ \mathcal{A} \tfrac{1}{(z-\xi)^2}\, \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|_{B_2^2}} d\mu(\rho(\xi)) =$$
$$= \int_{\mathbb{C}\backslash\overline{G}} \big(f(\tau), \mathcal{A}_\tau \tfrac{1}{(\tau-\xi)^2}\big)_{B_2} \cdot \tfrac{1}{(z-\xi)^2}\, \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}} d\mu(\rho(\xi)),$$
$$= \int_{\mathbb{C}\backslash\overline{G}} \big(\mathcal{A}^*_\tau f(\tau), \tfrac{1}{(\tau-\xi)^2}\big)_{B_2} \cdot \tfrac{1}{(z-\xi)^2}\, \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}} d\mu(\rho(\xi)), z \in G. \tag{11}$$

Since the operator $\mathcal{A}^*$ is an isometry, this operator affects the whole space B_2, $\mathcal{A}^* f(z) \stackrel{ob}{=} g(z)$. Therefore,

$$g(z) = \int_{\mathbb{C}\setminus\overline{G}} \big(g(\tau), \tfrac{1}{(\tau-\xi)^2}\big)_{B_2} \cdot \tfrac{1}{(z-\xi)^2} \cdot \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}} d\mu(\rho(\xi)),$$

$$g \in B_2,\ z \in G. \qquad (12)$$

Thus, the system of functions $\big\{\frac{1}{(z-\xi)^2}\big\}_{\xi\in\mathbb{C}\setminus\overline{G}}$ is an orthosimilar decomposition system in the space B_2 with the measure $\frac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}} d\mu(\rho(\xi))$.

There is an analogue of the Parseval identity for orthosimilar decomposition systems (Theorem 1 of the work [10]). According to the theorem, the equality

$$\|f\|^2_{B_2} = \int_{\mathbb{C}\setminus\overline{G}} |(f(\tau), \tfrac{1}{(z-\xi)^2})|^2 \, \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}} d\mu(\rho(\xi)) =$$

$$= \int_{\mathbb{C}\setminus\overline{G}} |\widetilde{f}(\xi)|^2 \, \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}} d\mu(\rho(\xi)) = \|\widetilde{f}\|^2_1, \forall f \in B_2 \qquad (13)$$

holds. Whence, invoking the equality (1), one obtains

$$\|\widetilde{f}\|_{\widetilde{B}_2} = \|f\|_{B_2} = \|\widetilde{f}\|_1, \quad \forall \widetilde{f} \in \widetilde{B}_2.$$

Thus, Condition 1 entails Condition 2.

Obviously, Condition 3 follows from Condition 2.

Let us prove no that Condition 3 entails Condition 1.

According to Condition 3, there is a homeomorphic mapping $\rho : \mathbb{C}\setminus\overline{G} \to G$ such that

$$\|\widetilde{f}\|_1 = \sqrt{\int_{\mathbb{C}\setminus\overline{G}} |\widetilde{f}(\xi)|^2 \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}} \, d\mu(\rho(\xi))}$$

is equivalent to the form $\|\widetilde{f}\|_{\widetilde{B}_2}$ in the Banach space $\widetilde{B}_2$. Applying Lemma 3, we obtain that there is a linear continuous bijective operator S, which governs the automorphism of the Banach space B_2 such that the system of functions

$$\left\{\frac{S_z(1/(z-\xi)^2)}{\|1/(z-\xi)^2\|}\right\}_{\xi\in\mathbb{C}\setminus\overline{G}}$$

is an orthosimilar decomposition system in the space B_2 with the measure

$$\|1/(z-\xi)^2\|^2 \cdot \tfrac{K_{B_2}(\rho(\xi),\rho(\xi))}{\|1/(z-\xi)^2\|^2_{B_2}} \, d\mu(\rho(\xi)) = K_{B_2}(\rho(\xi),\rho(\xi)) \, d\mu(\rho(\xi)).$$

Hence, the representation

$$f(z) = \int_{\mathbb{C}\setminus\overline{G}} (f(\tau), \tfrac{S_\tau(1/(\tau-\xi)^2)}{\|1/(z-\xi)^2\|_{B_2}})_{B_2} \times \qquad (14)$$

$$\times \tfrac{S_z(1/(z-\xi)^2)}{\|1/(z-\xi)^2\|_{B_2}} \, K_{B_2}(\rho(\xi),\rho(\xi)) \, d\mu(\rho(\xi)), z \in G, f \in B_2$$

holds. The equality (14) means that the system of functions $\{S_z\psi(z,\xi)\}_{\xi\in\mathbb{C}\setminus\overline{G}}$

$$\psi(z,\xi) = \frac{1/(z-\xi)^2}{\|1/(z-\xi)^2\|_{B_2}}, \quad \xi \in \mathbb{C}\setminus\overline{G}$$

(see (7)) is an orthosimilar decomposition system in the space B_2 with the measure $K_{B_2}(\rho(\xi),\rho(\xi)) \, d\mu(\rho(\xi))$.

According to Lemma 1, any function $f \in B_2$ can be represented in the form:

$$f(z) = \int_G \big(f(\tau), \tfrac{K_{B_2}(\tau,t)}{\sqrt{K_{B_2}(t,t)}}\big)_{B_2} \cdot \tfrac{K_{B_2}(z,t)}{\sqrt{K_{B_2}(t,t)}} \, K_{B_2}(t,t) d\mu(t), \quad z \in G.$$

Let us substitute variables in the latter integral $t = \rho(\xi)$. Then,

$$f(z) = \int_{\mathbb{C}\setminus\overline{G}} \big(f(\tau), \tfrac{K_{B_2}(\tau,\rho(\xi))}{\sqrt{K_{B_2}(\rho(\xi),\rho(\xi))}}\big)_{B_2} \times$$
$$\times \tfrac{K_{B_2}(z,\rho(\xi))}{\sqrt{K_{B_2}(\rho(\xi),(\rho(\xi))}} K_{B_2}(\rho(\xi),\rho(\xi))d\mu(\rho(\xi)), \quad z \in G. \qquad (15)$$

The latter equality means that the system of functions $\{\varphi(z,\rho(\xi))\}_{\xi\in\mathbb{C}\setminus\overline{G}}$

$$\varphi(z,\rho(\xi)) = \frac{K_{B_2}(z,\rho(\xi))}{\sqrt{K_{B_2}(\rho(\xi),(\rho(\xi))}}, \quad \xi \in \mathbb{C}\setminus\overline{G}$$

(see (7)) is an orthosimilar system decomposed in the space B_2 with the measure $K_{B_2}(\rho(\xi),\rho(\xi))\, d\mu(\rho(\xi))$. Let us introduce the notation

$$K_{B_2}(\rho(\xi),\rho(\xi))\, d\mu(\rho(\xi)) \overset{ob}{=} d\mu_1(\xi).$$

Thus, the following representations of the arbitrary function f from the space B_2 hold:

$$f(z) = \int_{\mathbb{C}\setminus\overline{G}} (f(\tau), \varphi(\tau,\rho(\xi)))_{B_2}\varphi(z,\rho(\xi))\, d\mu_1(\xi),\ z \in G.$$

The equality (14) is written as

$$f(z) = \int_{\mathbb{C}\setminus\overline{G}} (f(\tau), S_\tau\psi(\tau,\xi))_{B_2} S_z\psi(z,\xi)\, d\mu_1(\xi),\ z \in G.$$

Whence, using the fact that S is a bijective self-adjoint operator, implementing the automorphism of the space B_2, applying the theorem from ([14], p. 128), one obtains

$$S^{-1}f(z) = \int_{\mathbb{C}\setminus\overline{G}} (S_\tau f(\tau), \psi(\tau,\xi))_{B_2}\psi(z,\xi)\, d\mu_1(\xi),\ z \in G, \forall f \in B_2,$$

$$f(z) = \int_{\mathbb{C}\setminus\overline{G}} (S \circ S_\tau f(\tau), \psi(\tau,\xi))_{B_2}\psi(z,\xi)\, d\mu_1(\xi),\ z \in G, \forall f \in B_2.$$

Let us denote

$$(f,g)_I \overset{def}{=} (S \circ Sf, g)_{B_2}. \qquad (16)$$

Since the operator S is self-adjoint, the value $(f,g)_I$, $f,g \in B_2$ is a scalar product in the Hilbert space H_1, consisting of the same functions as the space B_2.

$$f(z) = \int_{\mathbb{C}\setminus\overline{G}} (f(\tau), \psi(\tau,\xi))_I\psi(z,\xi)\, d\mu_1(\xi),\ z \in G. \qquad (17)$$

The equality (17) means that the system of functions $\{\psi(z,\xi)\}_{\xi\in\mathbb{C}\setminus\overline{G}}$ is an orthosimilar decomposition system in the Hilbert space H_1. Let us define the operator $\mathcal{A}_1$ as follows. Let us assume that $\mathcal{A}_1(\varphi(z,\rho(\xi))) \overset{def}{=} \psi(z,\xi)$ for any $\xi \in \mathbb{C}\setminus\overline{G}$.

Let us define a linear manifold of functions $\mathcal{L}$ as a set of functions $g \in B_2$ such that there is a final set of points $\{\xi_k\}_{k=1}^n \in \mathbb{C}\setminus\overline{G}$ and a set of complex numbers $\{c_k\}_{k=1}^n \in G$, and the function g has the form

$$g(z) = \sum_{k=1}^n c_k\varphi(z,\rho(\xi_k)),\ z \in G. \qquad (18)$$

Thus, $\mathcal{L}$ is a linear envelope of the system of functions

$$\{\varphi(z,\rho(\xi))\}_{\xi\in\mathbb{C}\setminus\overline{G}},\ z \in G.$$

By virtue of Lemma 5, the function $g(z) \in B_2$ is uniquely defined by its coefficients c_k, $k = 1, \dots, n$, т.е. для $f, g \in \mathcal{L}$

$$g(z) = \sum_{k=1}^{n} c_k \varphi(z, \rho(\xi_k)), \quad f(z) = \sum_{k=1}^{n} d_k \varphi(z, \rho(\xi_k)),\ z \in G,$$

the condition

$$g(z) \equiv f(z),\ z \in G$$

entails that

$$c_k = d_k,\ k = 1, \dots, n.$$

Likewise, the operator $\mathcal{A}_1$ is defined on functions from $\mathcal{L}$:

$$\mathcal{A}_1(g)(z) \stackrel{def}{=} \sum_{k=1}^{n} c_k \psi(z, \xi_k),\ z \in G.$$

The set of functions $\mathcal{A}_1(g)$, $g \in \mathcal{L}$ generates a linear manifold $\mathcal{A}_1\mathcal{L}$. According to Lemma 6, the function $\mathcal{A}_1(g)(z)$ из $\mathcal{A}_1\mathcal{L}$ is also bijectively defined by its coefficients c_k, $k = 1, \dots, n$.

Let us introduce the norm $\|\mathcal{A}_1(g)\|_J \stackrel{def}{=} \|g\|_{B_2}$ on the elements $\mathcal{A}_1\mathcal{L}$.

Let J be a complement of the manifold $\mathcal{A}_1\mathcal{L}$ по норме $\|\cdot\|_J$. The operator $\mathcal{A}_1$ acts linearly and continuously from B_2 to J. J also generates the Hilbert space with the scalar product $(f, g)_J$. One has the equality

$$(\mathcal{A}_1 f, \mathcal{A}_1 g)_J = (f, g)_{B_2}, \quad f, g \in B_2.$$

Applying ([14], стр. 128), one can demonstrate that

$$\mathcal{A}_1 f(z) = \int_{\mathbb{C}\setminus\overline{G}} (f(\tau), \varphi(\tau, \rho(\xi)))_{B_2} \mathcal{A}_1 \varphi(z, \xi)\, d\mu'(\xi) =$$
$$= \int_{\mathbb{C}\setminus\overline{G}} (f(\tau), \varphi(\tau, \rho(\xi)))_{B_2} \psi(z, \xi)\, d\mu'(\xi) =$$
$$= \int_{\mathbb{C}\setminus\overline{G}} (\mathcal{A}_1 f(\tau), \psi(\tau, \xi))_J \psi(z, \xi)\, d\mu'(\xi),\ z \in G \tag{19}$$

for any $\mathcal{A}_1 f \in J$. The latter means that the system of functions $\{\psi(z, \xi)\}_{\xi \in \mathbb{C}\setminus\overline{G}}$ is an orthosimilar decomposition system in the space J. According to Lemma 4, the spaces J and H_1 coincide.

Then, the operator $\mathcal{A}_1$ establishes the isometry of the spaces B_2 and H_1. By construction

$$\mathcal{A}_1 \varphi(z, \rho(\xi)) = \psi_{H_1}(z, \xi), \quad \xi \in \mathbb{C}\setminus\overline{G},$$

where $\psi_{H_1}(z, \xi)$ is the function $\psi(z, \xi)$, considered as an element of the space H_1.

Let us define the linear manifold $\mathcal{N}$ as a set of functions g in the form:

$$g(z) = \sum_{k=1}^{n} c_k \psi_{H_1}(z, \xi_k),\ z \in G,$$

where $\{\xi_k\}_{k=1}^n \in \mathbb{C}\setminus\overline{G}$ is a finite choice of points, $\{c_k\}_{k=1}^n$ is the finite choice of complex numbers. Let us define the operator $\mathcal{B}$ on functions from $\mathcal{N}$. If

$$g(z) = \sum_{k=1}^{n} c_k \psi_{H_1}(z, \xi_k),\ z \in G$$

then,

$$\mathcal{B}g(z) \stackrel{def}{=} \sum_{k=1}^{n} c_k \psi(z, \xi_k),\ z \in G.$$

The operator $\mathcal{B}$ is generated before isometry of the spaces H_1 and B_2.

The operator $\mathcal{A}_1$, establishing the isometry of the spaces B_2 and H_1, has the property:

$$\mathcal{A}_1\varphi(z,\rho(\xi)) = \psi_{H_1}(z,\xi), \quad \xi \in \mathbb{C}\backslash\overline{G}.$$

The operator $\mathcal{B}$, establishing the isometry of the spaces H_1 and B_2, has the property:

$$\mathcal{B}\psi_{H_1}(z,\xi) = \psi(z,\xi),\ \xi \in \mathbb{C}\backslash\overline{G}.$$

Therefore, the operator

$$\mathcal{A} \stackrel{def}{=} \mathcal{B} \circ \mathcal{A}_1$$

is an automorphic isometry of the space B_2 and

$$\mathcal{A}\varphi(z,\rho(\xi)) = \psi(z,\xi), \quad \xi \in \mathbb{C}\backslash\overline{G}.$$

Thus, the operator $\mathcal{A}$, being an automorphic isometry of the space B_2, for which

$$\mathcal{A}\frac{K_{B_2}(z,\rho(\xi))}{\sqrt{K_{B_2}(\rho(\xi),\rho(\xi))}} = \frac{1/(z-\xi)^2}{\|1/(z-\xi)^2\|_{B_2}},\ \xi \in \mathbb{C}\backslash\overline{G}$$

has been constructed.

Thus, it has been proved that Condition 3 entails Condition 1. Theorem 1 is proved.

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