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NONLINEAR HYPERBOLIC EQUATIONS WITH CHARACTERISTIC RING OF DIMENSION 3

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Abstract. The paper provides a method of classification of the Darboux integrable nonlinear hyperbolic equations $u_{xy} = f(u, u_x, u_y)$ based on investigation of the characteristic pairs of the Lie rings. Constructive conditions on the right-hand side f of the equation with the characteristic ring of the dimension three are obtained. These equations possess second-order integrals. In particular, a list of equations satisfying the constructive conditions is given for the equation $u_{xy} = \varphi(u)\psi(u_x)h(u_y)$. Formulae of x - and y-integrals are given for these equations.

Keywords: integrals, characteristic ring, vector fields

1. Introduction

The object of this paper is to study the equation with two independent variables of the form

$$u_{xy} = f(u, u_x, u_y). \tag{1}$$

It is known that there are two types of integrable equations (1). The first type includes the wave equation $u_{xy} = 0$, as well as the Liouville equation $u_{xy} = e^u$ and its numerous analogues. The most famous equation of the second type is the sine-Gordon equation $u_{xy} = \sin u$.

Equations of the first type were investigated by classics of mathematics of 18-19 century such as Darboux, Euler, Lagrange, Liouville, Laplace, Lee, Jacobi, Goursat [1]–[3] and are referred to as Darboux integrable equations. In 1967, a new fundamental method for integrating non-linear evolution partial differential equations was discovered by Gardner, Greene, Kruskal and Miura, namely the inverse scattering method (see [4]–[6]). The latter led to the development of the theory of exact integration of equations of the second type.

The method related to the characteristic ring is used to solve the classification problem in the present paper. Ideas of this algebraic approach were suggested in the classical works by Darboux, Goursat, Vessiot and others (see [1]–[3], [7]–[8]) however, the final formulation is relatively recent (see [9]–[13]).

The constructive conditions on the right-hand side of equations (1) with the Lie characteristic ring of the dimension three are obtained. These equations possess the x- and y-integrals of the second order (see [14]).

2. The Lie ring of dimension 3

Let us consider equation (1), possessing the the x-integral of the second order $w = w(u, u_x, u_{xx})$.

Let us introduce the notation:

$$u_1 = u_x, \overline{u}_1 = u_y, u_2 = u_{xx}, \overline{u}_2 = u_{yy}, u_3 = u_{xxx}, \overline{u}_3 = u_{yyy}, \dots .$$

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Let us change from the variables $\overline{u}_1$,u,u_1,u_2,u_3,... to the variables $\overline{u}_1$, u, u_1, w, w_1, w_2, ..., w_n, ... On a set of locally analytic functions depending on a finite number of variables $\overline{u}_1$, u, u_1, w, w_1, w_2, ..., w_n, ... the operator of total differentiation with respect to the variable y takes the form

$$\overline{D} = \overline{u}_2\frac{\partial}{\partial \overline{u}_1} + \overline{u}_1\frac{\partial}{\partial u} + f\frac{\partial}{\partial u_1} = \overline{u}_2X_1 + X_2.$$

The operators X_1 and X_2 generate the Lie ring.

We have

$$[X_1, X_2] = \frac{\partial}{\partial u} + f_{u_1}\frac{\partial}{\partial u_1} = X_3.$$

We can see that the fields X_1, X_2, X_3 generate the Lie ring basis.

It follows from expressions for X_1, X_2, X_3 that

$$\frac{\partial}{\partial u_1} = \frac{1}{f - \overline{u}_1 f_{\overline{u}_1}}\left(X_2 - \overline{u}_1X_3\right).$$

Then,

$$\begin{gathered}[X_1, X_3] = \frac{f_{\overline{u}_1\overline{u}_1}}{f - \overline{u}_1 f_{\overline{u}_1}}\left(X_2 - \overline{u}_1X_3\right),\\ [X_2, X_3] = \frac{\overline{u}_1 f_{u\overline{u}_1} + ff_{u_1\overline{u}_1} - f_u - f_{u_1}f_{\overline{u}_1}}{f - \overline{u}_1 f_{\overline{u}_1}}\left(X_2 - \overline{u}_1X_3\right).\end{gathered}$$

Now let us turn from the variables $\overline{u}_1$, $u = v$, $u_1 = v_1$, w, w_1, w_2, ..., w_n, ... to the variables $\overline{u}_1, u, u_1, u_2, u_3, \ldots$. The following relations hold:

$$\begin{aligned}\frac{\partial}{\partial u} &= \frac{\partial}{\partial v} + w_u\frac{\partial}{\partial w} + w_{1u}\frac{\partial}{\partial w_1} + \ldots,\\ \frac{\partial}{\partial u_1} &= \frac{\partial}{\partial v_1} + w_{u_1}\frac{\partial}{\partial w} + w_{1u_1}\frac{\partial}{\partial w_1} + \ldots,\\ \frac{\partial}{\partial u_2} &= w_{u_2}\frac{\partial}{\partial w} + w_{1u_2}\frac{\partial}{\partial w_1} + \ldots, \quad \ldots.\end{aligned}$$

Let us denote by Z the field X in the new coordinate system where the operator $\overline{D}$ is rewritten in the form

$$\overline{D} = \overline{u}_2\frac{\partial}{\partial \overline{u}_1} + \overline{u}_1\frac{\partial}{\partial u} + f\frac{\partial}{\partial u_1} + Df\frac{\partial}{\partial u_2} + \ldots = \overline{u}_2Z_1 + Z_2.$$

We have

$$Z_3 = [Z_1, Z_2] = \frac{\partial}{\partial u} + f_{\overline{u}_1}\frac{\partial}{\partial u_1} + (Df)_{\overline{u}_1}\frac{\partial}{\partial u_2} + \ldots.$$

Commutators of the vector fields Z_1, Z_2, Z_3 have the form

$$\begin{gathered}[Z_1, Z_3] = \frac{f_{\overline{u}_1\overline{u}_1}}{f - \overline{u}_1 f_{\overline{u}_1}}\left(Z_2 - \overline{u}_1Z_3\right),\\ [Z_2, Z_3] = \frac{\overline{u}_1 f_{u\overline{u}_1} + ff_{u_1\overline{u}_1} - f_u - f_{u_1}f_{\overline{u}_1}}{f - \overline{u}_1 f_{\overline{u}_1}}\left(Z_2 - \overline{u}_1Z_3\right).\end{gathered} \tag{2}$$

Since the operators D and $\overline{D}$ commute

$$\begin{gathered}[D, \overline{D}] = \overline{D}f \cdot Z_1 + \overline{u}_2[D, Z_1] + [D, Z_2] =\\ = (f_u\overline{u}_1 + ff_{u_1} + \overline{u}_2 f_{\overline{u}_1})Z_1 + \overline{u}_2[D, Z_1] + [D, Z_2],\end{gathered}$$

then,

$$[D, Z_1] = -f_{\overline{u}_1}Z_1, \quad [D, Z_2] = -(f_u\overline{u}_1 + ff_{u_1})Z_1,$$

and using the Jacobi identity, we have

$$\begin{aligned}
[D, Z_3] &= -f_{\overline{u}_1} Z_3 - (f_u + f_{u_1} f_{\overline{u}_1}) Z_1, \\
[D, [Z_1, Z_3]] &= Z_3(f_{\overline{u}_1}) Z_1 - 2 f_{\overline{u}_1} A (Z_2 - \overline{u}_1 Z_3) - \\
&\quad - Z_1(f_{\overline{u}_1}) Z_3 - Z_1(f_u + f_{u_1} f_{\overline{u}_1}) Z_1, \\
[D, [Z_2, Z_3]] &= Z_3(f_u \overline{u}_1 + f f_{u_1}) Z_1 - (f_u \overline{u}_1 + f f_{u_1}) A (Z_2 - \overline{u}_1 Z_3) - \\
&\quad - Z_2(f_{\overline{u}_1}) Z_3 - f_{\overline{u}_1} B (Z_2 - \overline{u}_1 Z_3) - Z_2(f_u + f_{u_1} f_{\overline{u}_1}) Z_1 + (f_u + f_{u_1} f_{\overline{u}_1}) Z_3,
\end{aligned}$$

where $A = \frac{f_{\overline{u}_1 \overline{u}_1}}{f - \overline{u}_1 f_{\overline{u}_1}}$, $\quad B = \frac{\overline{u}_1 f_{u \overline{u}_1} + f f_{u_1 \overline{u}_1} - f_u - f_{u_1} f_{\overline{u}_1}}{f - \overline{u}_1 f_{\overline{u}_1}}$.

The correlations (2) are satisfied if ad only if

$$[D, [Z_1, Z_3]] = [D, A(Z_2 - \overline{u}_1 Z_3)], \quad [D, [Z_2, Z_3]] = [D, B(Z_2 - \overline{u}_1 Z_3)].$$

Similar reasoning holds for the y-characteristic Lie ring.

The following statement is true.

Theorem 1. *Equation (1) has x- and y-integrals of the second order if the following relations hold:*

$$A_{u_1} = 0, \quad A_u u_1 + A_{\overline{u}_1} f = -2 f_{\overline{u}_1} A, \tag{3}$$

$$B_{u_1} = 0, \quad B_u u_1 + B_{\overline{u}_1} f = -(f_u \overline{u}_1 + f f_{u_1}) A - f_{\overline{u}_1} B, \tag{4}$$

where $A = \frac{f_{\overline{u}_1 \overline{u}_1}}{f - \overline{u}_1 f_{\overline{u}_1}}$, $\quad B = \frac{\overline{u}_1 f_{u \overline{u}_1} + f f_{u_1 \overline{u}_1} - f_u - f_{u_1} f_{\overline{u}_1}}{f - \overline{u}_1 f_{\overline{u}_1}}$.

$$\overline{A}_{\overline{u}_1} = 0, \quad \overline{A}_u \overline{u}_1 + \overline{A}_{u_1} f = -2 f_{u_1} \overline{A}, \tag{5}$$

$$\overline{B}_{\overline{u}_1} = 0, \quad \overline{B}_u \overline{u}_1 + \overline{B}_{u_1} f = -(f_u u_1 + f f_{\overline{u}_1}) \overline{A} - f_{u_1} \overline{B}, \tag{6}$$

where $\overline{A} = \frac{f_{u_1 u_1}}{f - u_1 f_{u_1}}$, $\quad \overline{B} = \frac{u_1 f_{u u_1} + f f_{u_1 \overline{u}_1} - f_u - f_{u_1} f_{\overline{u}_1}}{f - u_1 f_{u_1}}$.

Proof. The relation $[D, [Z_1, Z_3]] = [D, A(Z_2 - \overline{u}_1 Z_3)]$ is equivalent to the following system of equations

$$\begin{aligned}
Z_3(f_{\overline{u}_1}) - Z_1(f_u + f_{u_1} f_{\overline{u}_1}) + A f_{u_1} (f - \overline{u}_1 f_{\overline{u}_1}) &= 0, \\
DA + 2 f_{\overline{u}_1} A &= 0, \\
DA \cdot \overline{u}_1 + 2 A f_{\overline{u}_1} \overline{u}_1 - Z_1(f_{\overline{u}_1}) + A(f - \overline{u}_1 f_{\overline{u}_1}) &= 0,
\end{aligned} \tag{7}$$

and $[D, [Z_2, Z_3]] = [D, B(Z_2 - \overline{u}_1 Z_3)]$ is rewritten as follows:

$$\begin{aligned}
Z_3(f_u \overline{u}_1 + f f_{u_1}) - Z_2(f_u + f_{u_1} f_{\overline{u}_1}) + B f_{u_1} (f - \overline{u}_1 f_{\overline{u}_1}) &= 0, \\
DB + B f_{\overline{u}_1} + A(f_u \overline{u}_1 + f f_{u_1}) &= 0, \\
DB \cdot \overline{u}_1 + Bf + f_u + f_{u_1} f_{\overline{u}_1} + A \overline{u}_1 (f_u \overline{u}_1 + f f_{u_1}) - Z_2(f_{\overline{u}_1}) &= 0.
\end{aligned} \tag{8}$$

The system of equations (7), (8) is equivalent to the system

$$\begin{aligned}
DA + 2 f_{\overline{u}_1} A &= 0, \\
DB + B f_{\overline{u}_1} + A(f_u \overline{u}_1 + f f_{u_1}) &= 0.
\end{aligned}$$

Whence, (3) and (4) hold. Likewise, considering the y-characteristic Lie ring, one obtains (5) and (6). The theorem is proved.

3. Classification of equations

Let us consider classes of equations, satisfying the conditions (3) – (6).

Let us assume that the right-hand side Equation (1) depends only on u. According to (3) – (6), one has

$$A = 0, \quad B = -\frac{f'}{f}, \quad \overline{A} = 0, \quad \overline{B} = -\frac{f'}{f}, \quad B'u_1 = 0, \quad \frac{f'}{f} = C_1$$

for the equation $u_{xy} = f(u)$. Then, $f = C_2 e^{C_1 u}$, where C_1, C_2 are constants, $C_2 \neq 0$.

Thus, the Liouville equation $u_{xy} = e^u$ is obtained, and it has second-order integrals

$$w = u_2 - \frac{1}{2}u_1^2, \quad \overline{w} = \overline{u}_2 - \frac{1}{2}\overline{u}_1^2.$$

Let us consider the equation

$$u_{xy} = \varphi(u)\psi(u_1).$$

The formulae (3) – (6) are transformed as follows when $f = \varphi(u)\psi(u_1)$:

$$A = 0, \quad B = -\frac{\varphi'}{\varphi}, \quad \overline{A} = \frac{\psi''}{\psi - u_1\psi'}, \quad \overline{B} = -\frac{\varphi'}{\varphi},$$
$$B_u u_1 = 0, \quad \overline{A}_{u_1}\psi = -2\psi'\overline{A}, \quad \overline{B}_u \overline{u}_1 = -\varphi'\psi u_1 \overline{A} - \varphi\psi'\overline{B}.$$

Solution of the latter system of equations defines the functions φ and ψ

$$\varphi = C_2 e^{C_1 u}, \quad \psi = C_4\sqrt{u_1^2 + C_3},$$

where C_i are constants, $i = 1, 2, 3, 4$ and $C_2 \neq 0$, $C_3 \neq 0$, $C_4 \neq 0$.

Dilation of functions u, u_1 and independent variables x, y yields the equation of the form

$$u_{xy} = e^u\sqrt{u_1^2 - 4}$$

with the integrals

$$w = u_2 - \frac{1}{2}u_1^2 - \frac{1}{2}e^{2u}, \quad \overline{w} = \frac{\overline{u}_2 - \overline{u}_1^2 + 4}{\sqrt{\overline{u}_1^2 - 4}}.$$

Let us consider the equation (1) with the right-hand side $f = \varphi(u)\psi(u_1)h(\overline{u}_1)$. Then, the relations (3) – (6) take the form

$$\begin{aligned} A &= \frac{h''}{h - \overline{u}_1 h'}, \quad B = -\frac{\varphi'}{\varphi} = -(\ln\varphi)', \\ \overline{A} &= \frac{\psi''}{\psi - \overline{u}_1\psi'}, \quad \overline{B} = -\frac{\varphi'}{\varphi} = -(\ln\varphi)', \\ A_{\overline{u}_1}h = -2h'A, &\quad B_u u_1 = -\psi h(\varphi'\overline{u}_1 + \varphi^2\psi'h)A - \varphi\psi h'B, \\ \overline{A}_{u_1}\psi = -2\psi'\overline{A}, &\quad \overline{B}_u\overline{u}_1 = -\psi h(\varphi' u_1 + \varphi^2\psi h')\overline{A} - \varphi\psi' h\overline{B}. \end{aligned} \tag{9}$$

It follows from $A = \frac{h''}{h - \overline{u}_1 h'}$ and $A_{\overline{u}_1}h = -2h'A$ that the function h satisfies the equation $h' = \frac{C_1\overline{u}_1}{h} + \gamma_1$, where C_1, γ_1 are constants. Likewise, $\overline{A} = \frac{\psi''}{\psi - \overline{u}_1\psi'}$ and $\overline{A}_{u_1}\psi = -2\psi'\overline{A}$ entail that the function ψ is such that the equation

$$\psi' = \frac{C_2 u_1}{\psi} + \gamma_2,$$

holds, where C_2, γ_2 are constants. Let us define the functions A and $\overline{A}$ as well

$$A = \frac{C_1}{h^2}, \quad \overline{A} = \frac{C_2}{\psi^2}.$$

Let us rewrite the sixth and the eighth equations of the system (9) in the form

$$\left(C_1C_2\varphi^2-(\ln\varphi)''\right)u_1=\left(\gamma_1\varphi'-C_1\gamma_2\varphi^2\right)\psi,$$
$$\left(C_1C_2\varphi^2-(\ln\varphi)''\right)\overline{u}_1=\left(\gamma_2\varphi'-C_2\gamma_1\varphi^2\right)h.$$

Since $h-\overline{u}_1h'\neq 0$ and $\psi-u_1\psi'\neq 0$ then,

$$C_1C_2\varphi^2-(\ln\varphi)''=0,\quad \gamma_1\varphi'-C_1\gamma_2\varphi^2=0,\quad \gamma_2\varphi'-C_2\gamma_1\varphi^2. \tag{10}$$

If $\gamma_1=0$ then ${}_1\gamma_2=0$ and $\gamma_2\varphi'=0$. Let $\gamma_2=0$. Then, the functions $\varphi(u),\psi(u_1),h(\overline{u}_1)$ satisfy

$$(\ln\varphi)''=C_1C_2\varphi^2,\quad h'=\frac{C_1\overline{u}_1}{h},\quad \psi'=\frac{C_2u_1}{\psi}.$$

If $\gamma_2\neq 0$ then, $C_1=\varphi'=h'=0$ $(f=\widetilde{\psi}(u_1))$.

Let us assume now that $\gamma_1\neq 0$. If $\gamma_2\neq 0$ then,

$$\varphi'=\frac{C_1\gamma_2}{\gamma_1}\varphi^2,\quad \varphi'=\frac{C_2\gamma_1}{\gamma_2}\varphi^2,\quad (\ln\varphi)''=C_1C_2\varphi^2.$$

Whence, $C_1\gamma_2^2=C_2\gamma_1^2$. If $\gamma_2=0$ then, $C_2=\varphi'=0$ $(f=\widetilde{\psi}(u_1)\widetilde{h}(\overline{u}_1))$.

Thus, the following statement holds.

Lemma 1. *The characteristic Lie ring of the equation $u_{xy}=\varphi(u)\psi(u_1)h(\overline{u}_1)$ has the dimension three if and only if the functions $\varphi(u),\psi(u_1),h(\overline{u}_1)$ satisfy one of the following conditions:*

— *either*

$$(\ln\varphi)''=C_1C_2\varphi^2,\quad h'=C_1\frac{\overline{u}_1}{h},\quad \psi'=C_2\frac{u_1}{\psi}; \tag{11}$$

— *or*

$$\varphi'=\frac{C_2C_3}{C_4}\varphi^2,\quad h'=C_1\frac{\overline{u}_1}{h}+C_3,\quad \psi'=C_2\frac{u_1}{\psi}+C_4. \tag{12}$$

Here C_1,C_2,C_3,C_4 are constants.

The substitution $\varphi\to\frac{\varphi}{\sqrt{C_1C_2}},\quad h\to\sqrt{C_1}h,\quad \psi\to\sqrt{C_2}\psi$ reduces the formulae (11) to the form

$$(\ln\varphi)''=\varphi^2,\quad h'=\frac{\overline{u}_1}{h},\quad \psi'=\frac{u_1}{\psi}.$$

The equation $u_{xy}=\varphi(u)\sqrt{u_1^2+\beta_1}\sqrt{\overline{u}_1^2+\beta_2}$, where the function φ is such that $(\ln\varphi)''=\varphi^2$ and β_1,β_2 are constants, has integrals of the second order

$$w=\frac{u_2}{\sqrt{u_1^2+\beta_1}}-\varphi\sqrt{u_1^2+\beta_1},\quad \overline{w}=\frac{\overline{u}_2}{\sqrt{\overline{u}_1^2+\beta_2}}-\varphi\sqrt{\overline{u}_1^2+\beta_2}.$$

The substitution $\varphi\to\frac{C_4\varphi}{C_2C_3},\quad h\to\sqrt{C_1}h,\quad \psi\to\sqrt{C_2}\psi$ reduces the formulae (12) to the form
$(\widetilde{C}_3=\frac{C_3}{\sqrt{C_1}},\ \widetilde{C}_4=\frac{C_4}{\sqrt{C_2}})$

$$\varphi'=\varphi^2,\quad h'=\frac{\overline{u}_1}{h}+\widetilde{C}_3,\quad \psi'=\frac{u_1}{\psi}+\widetilde{C}_4.$$

Second-order integrals for the equation $u_{xy}=-\frac{1}{u+\alpha}\psi(u_1)h(\overline{u}_1)$, where the functions ψ and h satisfy the equations

$$\psi'=\frac{u_1}{\psi}+\widetilde{C}_4\quad \text{и}\quad h'=\frac{\overline{u}_1}{h}+\widetilde{C}_3,$$

respectively, and α is a constant have the form

$$w = \frac{u_2}{\psi} - \frac{\psi}{u}, \quad \overline{w} = \frac{\overline{u}_2}{h} - \frac{h}{u}.$$

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