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ON SPECTRAL PROPERTIES OF A DIFFERENTIAL OPERATOR WITH SUMMABLE COEFFICIENTS WITH A RETARDED ARGUMENT

S.I. MITROKHIN

Abstract. The paper considers spectral properties of differential operators of the sixth order with a retarded argument. It is supposed that coefficients of the operator are summable functions on a segment. One can study 36 kinds of boundary-valued conditions simultaneously by one method. The asymptotics of eigenvalues of the differential operator is also calculated.

Keywords: differential operator, summable coefficients, retarded argument, asymptotics of eigenvalues, asymptotics of solutions.

The present article is devoted to investigation of spectral properties of a differential operator, given by a sixth-order differential equation with a retarded argument of the following form:

$$y^{(6)}(x) + r(x) \cdot y''(x-\tau) + p(x) \cdot y'(x-\tau) + q(x) \cdot y(x-\tau) = \lambda \cdot a^6 \cdot y(x), \tag{1}$$

where $0 \leqslant x \leqslant \pi$, $a > 0$, τ is the retardation, $\tau > 0$, with the initial conditions of the form

$$y(x-\tau) = y(0) \cdot \varphi(x-\tau), \quad x \leqslant \tau, \quad \varphi(0) = 1, \tag{2}$$

with the boundary-value conditions (separated, irregular) in the following form:

$$y^{(m_1)}(0) = y^{(m_2)}(0) = y^{(m_3)}(0) = y^{(m_4)}(0) = y^{(m_5)}(0) = y^{(n_1)}(\pi) = 0, \tag{3}$$

where $m_1 < m_2 < m_3 < m_4 < m_5$; $m_k, n_1 \in \{0, 1, 2, 3, 4, 5\}$, $k = 1, 2, 3, 4, 5$.

Coefficients of the differential equation (1) are supposed to be summable function on the interval $[0; \pi]$, i. e. conditions of the Riemann-Lebesgue theorem hold for them:

$$r(x) \in L_1[0;\pi], \quad p(x) \in L_1[0;\pi], \quad q(x) \in L_1[0,\pi] \Leftrightarrow \left(\int_0^x r(t)dt\right)'_x = r(x),$$

$$\left(\int_0^x p(t)dt\right)'_x = p(x), \quad \left(\int_0^x q(t)dt\right)'_x = q(x) \text{ almost everywhere in the interval } [0,\pi]. \tag{4}$$

Note that it follows form the initial conditions (2) that

$$y'(x-\tau) = y(0) \cdot \varphi'(x-\tau), \quad y''(x-\tau) = y(0) \cdot \varphi''(x-\tau), \quad x \leqslant \tau, \quad \varphi(0) = 1. \tag{5}$$

Therefore, we assume that $\varphi(x) \in D^2[-\tau; 0]$.

In the differential equation (1), the number λ is a spectral parameter, $\rho(x) = a^6 = \text{const}$ ($\forall x \in [0;\pi]$) is the weight function. The purpose of the present article is to find the asymptotics of solutions to a differential equation for large values of the spectral parameter λ, as well as to find the asymptotics of eigenvalues of the differential operator (1)–(2)–(3) in case if the summability conditions (4) are satisfied.

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Differential equations of the (1)–(2) type with a retarded argument (mostly of the second order) in case of continuous, smooth or infinitely differentiable coefficients have been studied for a long time (see [1]–[5]).

The Sturm-Liouville boundary-value problem (with obtaining the asymptotics of eigenvalues in the main approximation) for a differential operator of the second order (with a retarded argument) with separated boundary-value problems of the general form in case of a smooth potential $q(x)$ has been thoroughly investigated in the monograph [6, Ch 3]. However, the resulting asymptotics is not enough for calculating the first regularized trace of the considered operator.

In the work [7], more exact asymptotics of solutions and eigenvalues (as compared to [6, Ch. 3]) of a second-order differential operator with separated boundary-value conditions in case of sufficiently smooth (infinitely differentiable) coefficients $q(x)$ and $\varphi(x)$ were obtained. As a result, regularized traces of the considered differential operator were calculated.

The work [8] contains the solution of the inverse problem for determining the second-order differential operator with a retarded argument with separated boundary-value conditions in the general form with respect to two spectrums in case of an analytic potential $q(x)$, if $\varphi(x) \equiv 0$ when $x \leqslant 0$.

We suggest a methodology for investigating spectral properties of differential operators of order higher than two with a retarded argument in the form (1)–(2) with the boundary-value conditions (3) in case of summability of coefficients (i. e. satisfaction of the conditions (4)) and of the initial function $\varphi(x)$, satisfying the condition $\varphi(x) \in D^2[-\tau;0]$. If the condition $r(x) \equiv 0$, $p(x) \equiv 0$ $\forall x \in [0;\pi]$ is satisfied then our methodology is reliable even in the case $\varphi(x) \in L_1[0;\pi]$.

In case of ordinary differential operators (of the Sturm-Liouville type of an arbitrary even order, with non-retarding argument), this methodology is described by the author in the monograph [9, Ch 5]. An example of investigating the fourth-order operator is given in [10].

Let us find the asymptotics of the solution to the differential equation (1) when the summability conditions (4) are met.

Let us assume that $\lambda = s^6$ (λ is a spectral parameter), $s = \sqrt[6]{\lambda}$ is one of the six branches of the root, fixed by the condition $\sqrt[6]{1} = +1$. Let w_k $(k = 1, 2, \dots, 6)$ be various roots in the sixth power of one:

$$w_k^6 = 1, \quad w_k = e^{\frac{2\pi i}{6}(k-1)} \quad (k = 1, 2, \dots, 6);$$
$$w_1 = -w_4 = 1, \quad w_2 = -w_5 = \frac{1+\sqrt{3}i}{2}, \quad w_3 = -w_6 = \frac{-1+\sqrt{3}i}{2}. \tag{6}$$

One can readily prove that the following equations hold for the numbers w_k $(k = 1, 2, \dots, 6)$ from (6):

$$\sum_{k=1}^{6} w_k^m = 0, \ m = 1, 2, 3, 4, 5; \quad \sum_{k=1}^{6} w_k^m = 6, \ m = 0, \ m = 6. \tag{7}$$

In view of the property (7), we prove the following statement by the method of variation of the arbitrary constants.

Theorem 1. *Solution $y(x,s)$ to the differential equation* (1) *is a solution of the following integral Volterra equation:*

$$y(x,s) = \sum_{k=1}^{6} C_k \cdot e^{aw_k sx} - \frac{1}{6a^5 s^5} \cdot \sum_{k=1}^{6} w_k \cdot e^{aw_k sx} \cdot \int_0^x e^{-aw_k st} \cdot F(t-\tau, s) \cdot dt_{ak}, \tag{8}$$

where C_k $(k = 1, 2, \dots, 6)$ are arbitrary constants, and the following notation is introduced:

$$F(x-\tau, s) = r(x) \cdot y''(x-\tau, s) + p(x) \cdot y'(x-\tau, s) + q(x) \cdot y(x-\tau, s). \tag{9}$$

Meanwhile, by virtue of the properties (4), *the following formulae hold:*

$$y'(x,s) = \sum_{k=1}^{6} C_k \cdot aw_k s \cdot e^{aw_k sx} - \\ - \frac{1}{6a^5 s^5} \cdot \sum_{k=1}^{6} w_k \cdot aw_k s \cdot e^{aw_k sx} \cdot \left(\int_0^x \dots \right)_{ak} + \varphi_1(x,s), \quad (10)$$

where $\varphi_1(x,s) \overset{(8)}{=} -\frac{1}{6a^5s^5} \cdot \sum_{k=1}^{6} w_k \cdot e^{aw_k sx} \cdot e^{-aw_k sx} \cdot F(x-\tau,s) = \frac{1}{6a^5s^5} \cdot F(x-\tau,s) \cdot \sum_{k=1}^{6} w_k = 0$ *by virtue of the properties* (7) *for* $m = 1$;

$$y''(x,s) = \sum_{k=1}^{6} C_k \cdot (aw_k s)^2 \cdot e^{aw_k sx} - \\ - \frac{1}{6a^5 s^5} \cdot \sum_{k=1}^{6} w_k \cdot (aw_k s)^2 \cdot e^{aw_k sx} \cdot \left(\int_0^x \dots \right)_{ak} - \varphi_2(x,s), \quad (11)$$

where $\varphi_2(x,s) \overset{(10)}{=} \frac{1}{6a^5s^5} \cdot \sum_{k=1}^{6} w_k \cdot aw_k s \cdot e^{aw_k sx} \cdot e^{-aw_k sx} \cdot F(x-\tau,s) = \frac{1}{6a^4s^4} \cdot F(x-\tau,s) \cdot \sum_{k=1}^{6} w_k^2 = 0$ *by virtue of* (7) *when* $m = 2$.

The formulae (10)–(11) give us a possibility to give another proof of Theorem 1.
Differentiating the formula (11) several times with respect to the variable x, we obtain:

$$y^{(m)}(x,s) = \sum_{k=1}^{6} C_k \cdot (aw_k s)^m \cdot e^{aw_k sx} - \\ - \frac{1}{6a^5 s^5} \cdot \sum_{k=1}^{6} w_k \cdot (aw_k s)^m \cdot e^{aw_k sx} \left(\int_0^x \dots \right)_{ak} - \varphi_m(x,s), \quad (12)$$

where $\varphi_m(x,s) \overset{(11)}{=} \frac{1}{6a^5s^5} \cdot \sum_{k=1}^{6} w_k \cdot (aw_k s)^{m-1} \cdot e^{aw_k sx} \cdot e^{-aw_k sx} \cdot F(x-\tau,s) = \frac{1}{6 \cdot a^{6-m} s^{6-m}} \cdot F(x-\tau,s) \cdot \sum_{k=1}^{6} w_k^m = 0$ by virtue of the condition (7) for $m = 3, 4, 5$.

Differentiating the formula (12) once more for $m = 5$, we substitute the resulting expression and the formulae (8)–(12) into the differential equation (1) and obtain:

$$
\begin{aligned}
&y^{(6)}(x,s) + F(x-\tau, s) - \lambda \cdot a^6 \cdot y(x,s) \overset{(12),(8)}{=\!=} \sum_{k=1}^{6} C_k \cdot (aw_k s)^6 \cdot e^{aw_k sx} - \\
&- \frac{1}{6a^5 s^5} \cdot \sum_{k=1}^{6} w_k \cdot (aw_k s)^6 \cdot e^{aw_k sx} \cdot \left(\int_0^x \ldots \right)_{ak} - \frac{1}{6a^5 s^5} \cdot \sum_{k=1}^{6} w_k \cdot (aw_k s)^5 \cdot \\
&\cdot e^{aw_k sx} \cdot e^{-aw_k sx} \cdot F(x-\tau, s) + F(x-\tau, s) - s^6 \cdot a^6 \cdot \sum_{k=1}^{6} C_k \cdot e^{aw_k sx} + \\
&+ s^6 \cdot a^6 \cdot \frac{1}{6a^5 s^5} \cdot \sum_{k=1}^{6} w_k \cdot e^{aw_k sx} \cdot \left(\int_0^x \ldots \right)_{ak} = \sum_{k=1}^{6} C_k \cdot e^{aw_k sx} \cdot [(aw_k s)^6 - \\
&- s^6 \cdot a^6] + \frac{1}{6a^5 s^5} \cdot \sum_{k=1}^{6} w_k \cdot e^{aw_k sx} \cdot \left(\int_0^x \ldots \right)_{ak} \cdot [s^6 a^6 - (aw_k s)^6] + \\
&+ F(x-\tau, s) - \frac{1}{6a^5 s^5} \cdot F(x-\tau, s) \cdot a^5 s^5 \cdot \sum_{k=1}^{6} w_k^6 = 0
\end{aligned}
$$

almost everywhere in the interval $[0;\pi]$ (13)

By virtue of the equalities (7) and the property (6), we have $w_k^6 = 1$ $(k = 1, 2, \ldots, 6)$.

The equality (13) demonstrates that the function $y(x,s)$ from (8)–(12) is a solution to the differential equation (1) indeed.

Deduction of the formulae for asymptotics of the solution $y(x,s)$ to the differential equation (1) for $|s| \to +\infty$ (for large values of the spectral parameter λ) depends on the value of retardation τ. Since the number τ (retardation, $\tau > 0$) is constant then, there is a natural number $k_0 + 1$ $(k_0 \in \mathbb{N} \cup \{0\})$ such that the inequality

$$0 < \tau < 2\tau < \cdots < k_0 \cdot \tau \leqslant \pi < (k_0 + 1) \cdot \tau \tag{14}$$

holds.

Depending on the value of this number k_0, solutions $y(x,s)$ to the differential equation (1)–(2) will be written out differently: If the argument of the function $y(t-\tau, s)$ from (8)–(9) is less than zero, this function should be substituted by $y(0) \cdot \varphi(t-\tau)$ by virtue of the initial condition (2), and the functions $y'(t-\tau, s)$ and $y''(t-\tau, s)$ should be substituted by the functions $y(0) \cdot \varphi'(t-\tau, s)$ and $y(0) \cdot \varphi''(t-\tau, s)$, respectively due to the property (5).

Therefore, let us consider the cases $k_0 = 0$, $k_0 = 1$, $k_0 = 2$, $k_0 \geqslant 3$, $k_0 \in \mathbb{N}$ sequentially.

Let us consider the first case: let $\tau > \pi$ $(k_0 = 0)$.

In this case, arguments of functions $y(t-\tau, s)$, $y'(t-\tau, s)$ and $y''(t-\tau, s)$ in the formulae (8)–(9) are negative $(0 \leqslant x \leqslant \pi$ by virtue of (1), $0 \leqslant t \leqslant x \leqslant \pi$, $-\tau \leqslant t - \tau \leqslant \pi - \tau < 0)$ therefore, these functions should be substituted using the formulae (2) and (5):

$$y(x,s) = \sum_{k=1}^{6} C_k \cdot e^{aw_k sx} - \frac{1}{6a^5 s^5} \cdot \sum_{k=1}^{6} w_k \cdot e^{aw_k sx}.$$

$$\cdot\left[\int_0^x r(t)\cdot e^{-aw_kst}\cdot y(0)\cdot\varphi''(t-\tau)\cdot dt_{ark}+\right.$$
$$+\int_0^x p(t)\cdot e^{-aw_kst}\cdot y(0)\cdot\varphi'(t-\tau)\cdot dt_{apk}+$$
$$\left.+\int_0^x q(t)\cdot e^{-aw_kst}\cdot y(0)\cdot\varphi(t-\tau)dt_{aqk}\right]. \quad (15)$$

Taking into account that $y(0)=\sum\limits_{k=1}^{6}C_k$ due to the formula (8), we substitute this value $y(0)$ into (15), regroup the addends and and arrive to the following conclusion.

Theorem 2. *The general solution $y(x,s)$ of the differential equation (1)–(2) in the case $\tau\in(\pi;+\infty)$ ($k_0=0$ in (14)) provided that the summability condition (4) is obtained in the following explicit form:*

$$y(x,s)=\sum_{k=1}^{6}C_k\cdot y_k(x,s);\quad y^{(m)}(x,s)=\sum_{k=1}^{6}C_k\cdot y_k^{(m)}(x,s),\quad m=1,2,3,4,5, \quad (16)$$

and the fundamental system of solutions $\{y_k(x,s)\}_{k=1}^{6}$ is represented in the form

$$y_k(x,s)=e^{aw_ksx}-\frac{1}{6a^5s^5}\cdot\sum_{k_1=1}^{6}w_{k_1}\cdot e^{aw_{k_1}sx}\cdot\left[\int_0^x r(t)\cdot e^{-aw_{k_1}st}\cdot\varphi''(t-\tau)\cdot\right.$$
$$\cdot\ dt_{rk_1}+\int_0^x p(t)\cdot e^{-aw_{k_1}st}\cdot\varphi'(t-\tau)\cdot dt_{pk_1}+$$
$$\left.+\int_0^x q(t)\cdot e^{-awk_1st}\cdot\varphi(t-\tau)\cdot dt_{qk_1}\right],\quad k=1,2,\ldots,6; \quad (17)$$

$$\frac{y_k^{(m)}(x,s)}{(as)^m}=w_k^m\cdot e^{aw_ksx}-\frac{1}{6a^5s^5}\cdot\sum_{k_1=1}^{6}w_{k_1}\cdot w_{k_1}^m\cdot e^{aw_{k_1}sx}\cdot$$
$$\cdot\left[\left(\int_0^x\ldots\right)_{rk_1}+\left(\int_0^x\ldots\right)_{pk_1}+\left(\int_0^x\ldots\right)_{qk_1}\right],$$
$$k=1,2,\ldots,6;\quad m=1,2,3,4,5. \quad (18)$$

Note that the functions $y_k(x,s)$ $(k=1,2,\ldots,6)$ from (16)–(18) satisfy the following initial-value problems:

$$y_k(0,s)=1;\quad y_k^{(m)}(0,s)=w_k^m\cdot a^m\cdot s^m;$$
$$y(0,s)=\sum_{k=1}^{6}C_k;\quad y^{(m)}(0,s)=\sum_{k=1}^{6}C_k\cdot w_k^m\cdot a^m\cdot s^m,$$
$$k=1,2,\ldots,6;\ m=1,2,3,4,5. \quad (19)$$

Let us point out again that solutions are found in the explicit form in the formulae (16)–(18) unlike differential equations without retardation (see [10], [9, Ch 5]): solutions there are written out in the form of asymptotic series without a gap.

Let us consider the second case now: $\tau \in \left(\frac{\pi}{2};\pi\right]$ (i. e. $k_0 = 1$ in the formula (14)). In this case, arguments of the function $y(t-\tau,s)$, $y'(t-\tau,s)$ and $y''(t-\tau,s)$ in the formula (8)–(9) are not always negative, and the formulae (2), (5) cannot be used for them so far.

In order to find asymptotic solutions in this case, let us use the Picard method of successive approximations: find $y(t-\tau,s)$ from (8), $y'(t-\tau,s)$ from (10), $y''(t-\tau,s)$ from (11) and substitute them into the formula (8)–(9). We obtain

$$y(x,s) = \sum_{k=1}^{6} C_k \cdot e^{aw_k sx} - \frac{1}{6a^5s^5} \cdot \sum_{k_1=1}^{6} w_{k_1} \cdot e^{aw_{k_1}sx} \cdot \int_0^x r(t) \cdot e^{-aw_{k_1}st} \cdot$$
$$\cdot\ \Phi_1(t,\tau) \cdot dt_{ark_1} - \frac{1}{6a^5s^5} \cdot \sum_{k_1=1}^{6} w_{k_1} \cdot e^{aw_{k_1}sx} \int_0^x p(t) \cdot e^{-aw_{k_1}st} \cdot$$
$$\cdot\ \Phi_2(t,\tau) \cdot dt_{apk_1} - \frac{1}{6a^5s^5} \cdot \sum_{k_1=1}^{6} w_{k_1} \cdot e^{aw_{k_1}sx} \cdot \int_0^x q(t) \cdot e^{-aw_{k_1}st} \cdot$$
$$\cdot\ \Phi_3(t,\tau) \cdot dt_{aqk_1}, \tag{20}$$

where the following notation is introduced:

$$\Phi_1(t,\tau) = \sum_{k=1}^{6} C_k \cdot (aw_k s)^2 \cdot e^{aw_k s(t-\tau)} - \frac{1}{6a^5s^5} \cdot \sum_{k=1}^{6} w_k \cdot (aw_k s)^2 \cdot$$
$$\cdot\, e^{aw_k s(t-\tau)} \cdot \left[\left(\int_0^{t-\tau} \ldots \right)_{ark} + \left(\int_0^{t-\tau} \ldots \right)_{apk} + \left(\int_0^{t-\tau} \ldots \right)_{aqk} \right], \tag{21}$$

$$\Phi_2(t,\tau) = \sum_{k=1}^{6} C_k \cdot (aw_k s) \cdot e^{aw_k s(t-\tau)} -$$
$$- \frac{1}{6a^5s^5} \cdot \sum_{k=1}^{6} w_k \cdot (aw_k s) \cdot e^{aw_k s(t-\tau)} \cdot \psi(t,\tau);$$
$$\Phi_3(t,\tau) = \sum_{k=1}^{6} C_k \cdot e^{aw_k s(t-\tau)} - \frac{1}{6a^5s^5} \cdot \sum_{k=1}^{6} w_k \cdot e^{aw_k s(t-\tau)} \cdot \psi(t,\tau), \tag{22}$$

$$\psi(t,\tau)=\left(\int\limits_0^{t-\tau}\dots\right)_{ark}+\left(\int\limits_0^{t-\tau}\dots\right)_{apk}+\left(\int\limits_0^{t-\tau}\dots\right)_{aqk}=$$

$$=\int\limits_0^{t-\tau} r(\xi)\cdot e^{-aw_k s\xi}\cdot y''(\xi-\tau,s)\cdot d\xi_{ark}+$$

$$+\int\limits_0^{t-\tau} p(\xi)\cdot e^{-aw_k s\xi}\cdot y'(\xi-\tau,s)\cdot d\xi_{apk}+$$

$$+\int\limits_0^{t-\tau} q(\xi)\cdot e^{-aw_k s\xi}\cdot y(\xi-\tau,s)d\xi_{aqk}. \tag{23}$$

In the integrals involved in the formulae (20)–(23), we have: $0\leqslant x\leqslant\pi$, $0\leqslant t\leqslant x\leqslant\pi$, $0\leqslant\xi\leqslant t-\tau$. Therefore, $0\leqslant\xi\leqslant t-\tau\leqslant\pi-\tau$, $-\tau\leqslant\xi-\tau\leqslant\pi-2\tau<0$ (because we are considering the case $k_0=1$ in (14), i. e. $\frac{\pi}{2}<\tau\leqslant\pi$), i. e. we obtain: the arguments of the function $y(\xi-\tau,s)$, $y'(\xi-\tau,s)$ and $y''(\xi-\tau,s)$ in the integrals, involved in the formulae (20)–(23) are negative and hence, the initial conditions (2) and (5), where $y(0)=\sum\limits_{k=1}^{6}C_k$, can be applied to them by virtue of the formula (8).

Substituting the expression $y(0)=\sum\limits_{k=1}^{6}C_k$ into the formulae (20)–(23) and carrying out the necessary calculations and transformations, we arrive to the conclusion, that the following theorem holds.

Theorem 3. *The general solution $y(x,s)$ of the differential equation* (1)–(2) *in the case $\tau\in\left(\frac{\pi}{2};\pi\right]$ ($k_0=1$ in* (14)) *is derived in the following explicit form*:

$$y(x,s)=\sum_{k=1}^{6}C_k\cdot g_k(x,s);\quad y^{(m)}(x,s)=\sum_{k=1}^{6}\cdot g_k^{(m)}(x,s),\quad m=1,2,3,4,5, \tag{24}$$

and the following formulae hold true for the fundamental system:

$$y_k(x,s)=e^{aw_k sx}-\frac{1}{6a^4s^4}\cdot\Phi_{3k}(x,s,\tau)-\frac{1}{6a^4s^4}\cdot\Phi_{4k}(x,s,\tau)-\frac{1}{6a^5s^5}\cdot$$
$$\cdot\Phi_{5k}(x,s,\tau)+\frac{1}{36a^8s^8}\cdot\psi_{8k}(x,s,\tau)+\frac{1}{36a^9s^9}\cdot\psi_{9k}(x,s,\tau)+$$
$$+\frac{1}{36a^{10}s^{10}}\cdot\psi_{10,k}(x,s,\tau),\quad k=1,2,\dots,6, \tag{25}$$

$$\frac{y^{(m)}(x,s)}{(as)^m}=w_k^m\cdot e^{aw_k sx}-\frac{1}{6a^3s^3}\cdot\Phi_{3k}^m(x,s,\tau)-\frac{1}{6a^4s^4}\cdot\Phi_{4k}^m(x,s,\tau)-$$
$$-\frac{1}{6a^5s^5}\cdot\Phi_{5k}^m(x,s,\tau)+\frac{1}{36a^8s^8}\cdot\psi_{8k}^m(x,s,\tau)+\frac{1}{36a^9s^9}\cdot\psi_{9k}^m(x,s,\tau)+$$
$$+\frac{1}{36a^{10}s^{10}}\cdot\psi_{10,k}^m(x,s,\tau),\quad k=1,2,\dots,6;\quad m=1,2,3,4,5 \tag{26}$$

where the following notation is introduced:

$$\Phi_{3k}(x,s,\tau)=w_k^2\cdot e^{-aw_ks\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}\cdot e^{aw_{k_1}sx}\cdot\int_0^x r(t)\cdot e^{a(w_k-w_{k_1})st}\cdot dt_{rkk_1};$$

$$\Phi_{3k}^m(x,s,\tau)=w_k^2\cdot e^{-aw_ks\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}\cdot w_{k_1}^m\cdot e^{aw_{k_1}sx}\cdot\left(\int_0^x\dots\right)_{rkk_1};$$

$$\Phi_{4k}(x,s,\tau)=w_k\cdot e^{-aw_ks\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}\cdot\ e^{aw_{k_1}sx}\cdot\int_0^x p(t)\cdot e^{a(w_k-w_{k_1})st}\cdot dt_{pkk_1};$$

$$\Phi_{4k}^m(x,s,\tau)=w_k\cdot e^{-aw_ks\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}\cdot w_{k_1}^m\cdot e^{aw_{k_1}sx}\left(\int_0^x\dots\right)_{pkk_1};$$

$$\Phi_{5k}(x,s,\tau)=e^{-aw_ks\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}\cdot e^{aw_{k_1}sx}\cdot\int_0^x q(t)\cdot e^{a(w_k-w_{k_1})st}\cdot dt_{qkk_1};$$

$$\Phi_{5k}^m(x,s,\tau)=e^{-aw_ks\tau}\cdot\sum_{k_1=1}^{6}\ w_{k_1}\cdot w_{k_1}^m\cdot e^{aw_{k_1}sx}\cdot\left(\int_0^x\dots\right)_{qkk_1};$$

$$\psi_{8k}(x,s,\tau)=\sum_{k_1=1}^{6}w_{k_1}^3\ \cdot\ e^{-aw_{k_1}s\tau}\cdot\left(\sum_{k_2=1}^{6}w_{k_2}\cdot e^{aw_{k_2}sx}\cdot\right.$$
$$\left.\cdot\left[R_8(x,s,\tau,\varphi'')+P_8(x,s,\tau,\varphi')+Q_8(x,s,\tau,\varphi)\right]\right),$$

$$R_8(x,s,\tau\varphi'')=\int_0^x r(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int_0^{t-\tau}r(\xi)\cdot e^{-aw_{k_1}s\xi}\cdot\varphi''(\xi-\tau)d\xi\right)\cdot dt;$$

$$P_8(x,s,\tau\varphi')=\int_0^x r(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int_0^{t-\tau}p(\xi)\cdot e^{-aw_{k_1}s\xi}\cdot\varphi'(\xi-\tau)d\xi\right)\cdot dt;$$

$$\psi_{8k}^m(x,s,\tau)=\sum_{k_1=1}^{6}w_{k_1}^3\ \cdot\ e^{-aw_{k_1}s\tau}\cdot\left(\sum_{k_2=1}^{6}w_{k_2}\cdot w_{k_2}^m\cdot e^{aw_{k_2}sx}\cdot\right.$$
$$\left.\cdot\left[R_8(x,s,\tau,\varphi'')+P_8(x,s,\tau,\varphi')+Q_8(x,s,\tau,\varphi)\right]\right);$$

$$k=1,2,\dots,6,\quad m=1,2,3,4,5;$$

$$\psi_{8k}(x,s,\tau)=\psi_{81}(x,s,\tau);\quad \psi_{8k}^m(x,s,\tau)=\psi_{81}^m(x,s,\tau),$$
$$k=1,2,\dots,6;\quad m=1,2,3,4,5;$$

$$\psi_{9k}(x,s,\tau)=\sum_{k_1=1}^{6}w_{k_1}^2\ \cdot\ e^{-aw_{k_1}s\tau}\cdot\left(\sum_{k_2=1}^{6}w_{k_2}\cdot e^{aw_{k_2}sx}\cdot\left[R_9(x,s,\tau,\varphi'')+\right.\right.$$
$$\left.\left.+\ P_9(x,s,\tau,\varphi')+Q_9(x,s,\tau,\varphi)\right]\right);$$

$$R_9(x,s,\tau,\varphi'') = \int\limits_0^x p(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int\limits_0^{t-\tau} r(\xi)\cdot e^{-aw_{k_1}s\xi}\cdot\varphi''(\xi-\tau)d\xi\right)\cdot dt;$$

$$P_9(x,s,\tau\varphi') = \int\limits_0^x p(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int\limits_0^{t-\tau} p(\xi)\cdot e^{-aw_{k_1}s\xi}\cdot\varphi'(\xi-\tau)d\xi\right)\cdot dt;$$

$$Q_9(x,s,\tau\varphi) = \int\limits_0^x p(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int\limits_0^{t-\tau} q(\xi)\cdot e^{-aw_{k_1}s\xi}\cdot\varphi(\xi-\tau)d\xi\right)\cdot dt;$$

$$\psi_{9k}^m(x,s,\tau) = \sum_{k_1=1}^{6} w_{k_1}^2\ \cdot\ e^{-aw_{k_1}s\tau}\cdot\left(\sum_{k_2=1}^{6} w_{k_2}\cdot w_{k_2}^m\cdot e^{aw_{k_2}sx}\cdot[R_9(x,s,\tau,\varphi'')+\right.$$

$$\left.+\ P_9(x,s,\tau,\varphi')+Q_9(x,s,\tau,\varphi)]\right);\quad \psi_{9k}(x,s,\tau)=\psi_{91}(x,s,\tau);$$

$$\psi_{9k}^m(x,s,\tau) = \psi_{91}^m(x,s,\tau)$$

$$\psi_{10,k}^m(x,s,\tau) = \sum_{k_1=1}^{6} w_{k_1}\ \cdot\ e^{-aw_{k_1}s\tau}\cdot\left(\sum_{k_2=1}^{6} w_{k_2}\cdot e^{aw_{k_2}sx}\cdot[R_{10}(x,s,\tau,\varphi'')+\right.$$

$$\left.+\ P_{10}(x,s,\tau,\varphi')+Q_{10}(x,s,\tau,\varphi)]\right);\quad \psi_{10,k}(x,s,\tau)=\psi_{10,1}(x,s,\tau);$$

$$R_{10}(x,s,\tau,\varphi'') = \int\limits_0^x q(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int\limits_0^{t-\tau} r(\xi)\cdot e^{-aw_{k_1}s\xi}\cdot\varphi''(\xi-\tau)d\xi\right)\cdot dt;$$

$$P_{10}(x,s,\tau,\varphi') = \int\limits_0^x q(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int\limits_0^{t-\tau} p(\xi)\cdot e^{-aw_{k_1}s\xi}\cdot\varphi'(\xi-\tau)d\xi\right)\cdot dt;$$

$$Q_{10}(x,s,\tau,\varphi) = \int\limits_0^x q(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int\limits_0^{t-\tau} q(\xi)\cdot e^{-aw_{k_1}s\xi}\cdot\varphi(\xi-\tau)d\xi\right)\cdot dt;$$

$$\psi_{10,k}^m(x,s,\tau) = \sum_{k_1=1}^{6} w_{k_1}\ \cdot\ e^{-aw_{k_1}s\tau}\cdot\left(\sum_{k_2=1}^{6} w_{k_2}\cdot w_{k_2}^m\cdot\right.$$

$$\left.\cdot\ e^{aw_{k_2}sx}\cdot[R_{10}(x,s,\tau,\varphi'')+P_{10}(x,s,\tau,\varphi')+Q_{10}(x,s,\tau,\varphi)]\right);$$

$$\psi_{10,k}^m(x,s,\tau) = \psi_{10,1}^m(x,s,\tau);\quad k=1,2,\ldots,6;\quad m=1,2,3,4,5. \tag{27}$$

Note that the following initial conditions hold for the fundamental system $\{g_k(x,s)\}_{k=1}^6$ from (24)–(27):

$$g_k(0,s)=1;\quad g_k^{(m)}(0,s)=w_k^m\cdot a^m\cdot s^m;\quad y(0,s)\overset{(24)}{=}\sum_{k=1}^6 C_k,$$

$$y^{(m)}(0,s)\overset{(24)}{=}\sum_{k=1}^6 C_k\cdot w_k^m\cdot a^m\cdot s^m,\quad k=1,2,\dots,6;\quad m=1,2,3,4,5. \tag{28}$$

Let us consider the third case $\tau\in\left(\frac{\pi}{3};\frac{\pi}{2}\right]$ (i. e. $k_0=2$ in (14)).

In this case one should substitute into (20)–(23) the values of functions $y(\xi-\tau,s)$, $y'(\xi-\tau,s)$ and $y''(\xi-\tau,s)$ from (8)–(11) correspondingly $\Bigg($for example, $y''(\xi-\tau,s)\overset{(11)}{=}\sum\limits_{k=1}^6 C_k\cdot(aw_ks)^2\cdot e^{aw_ks(\xi-\tau)}-\frac{1}{6a^5s^6}\cdot\sum\limits_{k=1}^6 w_k\cdot(aw_ks)^2\cdot e^{aw_k(\xi-\tau)}\cdot\Bigg[\int\limits_0^{\xi-\tau} r(\theta)\cdot e^{-aw_ks\theta}\cdot$ $\cdot y''(\theta-\tau,s)\cdot d\theta_{ark}+\left(\int\limits_0^{\xi-\tau}\dots\right)_{apk}+\left(\int\limits_0^{\xi-\tau}\dots\right)_{aqk}\Bigg]\Bigg)$, and then use the initial conditions (2) and (5):

$$y(\theta-\tau,s)=\sum_{k=1}^6 C_k\cdot\varphi(\theta-\tau),\quad y^{(m)}(\theta-\tau,s)=\sum_{k=1}^6 C_k\cdot\varphi^{(m)}(\theta-\tau),\quad m=1,2$$

for the functions $y(\theta-\tau,s)$, $y'(\theta-\tau,s)$ and $y''(\theta-\tau,s)$. (Note that the argument $\theta-\tau$ in the case $\tau\in\left(\frac{\pi}{3};\frac{\pi}{2}\right]$ is negative: $0\leqslant x\leqslant\pi$, $0\leqslant t\leqslant x\leqslant\pi$, $-\tau\leqslant t-\tau\leqslant x-\tau\leqslant\pi-\tau$; $-\tau\leqslant\xi-\tau\leqslant t-\tau\leqslant\pi-2\tau$; $-\tau\leqslant\theta-\tau\leqslant\xi-2\tau\leqslant\pi-3\tau<0$.)

Making the necessary calculations, we obtain the following result.

Theorem 4. *The general solution $y(x,s)$ of the differential equation (1) in the case $\tau\in\left(\frac{\pi}{3};\frac{\pi}{2}\right]$ ($k_0=2$) has the following form:*

$$y(x,s)=\sum_{k=1}^6 C_k\cdot h_k(x,s);\quad y^{(m)}(x,s)\sum_{k=1}^6 C_k\cdot h_k^{(m)}(x,s),\quad m=1,2,3,4,5,6, \tag{29}$$

and the following asymptotic formulae hold when $|s|\to+\infty$:

$$\begin{aligned}h_k(x,s)=e^{aw_ksx}&-\frac{1}{6a^3s^3}\cdot\Phi_{3k}(x,s,\tau)-\frac{1}{6a^4s^4}\cdot\ \Phi_{4k}(x,s,\tau)-\\&-\frac{1}{6a^5s^5}\cdot\Phi_{5k}(x,s,\tau)+\frac{1}{36a^6s^6}\cdot\Phi_{6k}(x,s,\tau)+\\&+\frac{1}{36a^7s^7}\cdot\Phi_{7k}(x,s,\tau)+\frac{1}{36a^8s^8}\cdot\Phi_{8k}(x,s,\tau)+\underline{O}\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^9}\right),\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{h^{(m)}(x,s)}{(as)^m}=w_k^m\cdot e^{aw_ksx}&-\frac{1}{6a^3s^3}\cdot\Phi_{3k}^m(x,s,\tau)-\frac{1}{6a^4s^4}\cdot\Phi_{4k}^m(x,s,\tau)-\\&-\frac{1}{6a^5s^5}\cdot\Phi_{5k}^m(x,s,\tau)+\frac{1}{36a^6s^6}\cdot\Phi_{6k}^m(x,s,\tau)+\frac{1}{36a^7s^7}\cdot\Phi_{7k}^m(x,s,\tau)+\\&+\frac{1}{36a^8s^8}\cdot\Phi_{8k}^m(x,s,\tau)+\underline{O}\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^9}\right),\\&k=1,2\dots,6;\quad m=1,2,3,4,5.\end{aligned} \tag{31}$$

Here the functions $\Phi_{3k}(x,s,\tau)$, $\Phi_{4k}(x,s,\tau)$, $\Phi_{5k}(x,s,\tau)$, $\Phi^m_{3k}(x,s,\tau)$, $\Phi^m_{4k}(x,s,\tau)$, $\Phi^m_{5k}(x,s,\tau)$ *are defined in the formulae* (27) *of Theorem 3, and the following formulae hold for the remaining expansion coefficients* (30)–(31)*:*

$$\Phi_{6k}(x,s,\tau)=w_k^2\cdot e^{-aw_k s\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}^3\cdot e^{-aw_{k_1}s\tau}\left[\sum_{k_2=1}^{6}w_{k_2}\cdot e^{aw_{k_2}sx}\cdot\right.$$

$$\left.\cdot\int_0^x r(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int_0^{t-\tau}r(\xi)\cdot e^{a(w_k-w_{k_1})s\xi}\cdot d\xi\right)\cdot dt_{rkk_1rk_1rk_2}\right];$$

$$\Phi^m_{6k}(x,s,\tau)=w_k^2\cdot e^{-aw_k s\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}^3\cdot e^{-aw_{k_1}s\tau}\cdot\left[\sum_{k_2=1}^{6}w_{k_2}\cdot w_{k_2}^m\cdot\right.$$

$$\left.\cdot\, e^{aw_{k_2}sx}\left(\int_0^x\dots\right)_{rkk_1rk_1k_2}\right];$$

$$\Phi_{7k}(x,s,\tau)=w_k\cdot e^{-aw_k s\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}^3\cdot e^{-aw_{k_1}s\tau}\cdot\left[\sum_{k_2=1}^{6}w_{k_2}\cdot e^{aw_{k_2}sx}\cdot\right.$$

$$\left.\cdot\;\Phi_{7k_1}(x,s,\tau)\right]+w_k^2\cdot e^{-aw_k s\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}^2\cdot e^{-aw_{k_1}s\tau}\cdot$$

$$\cdot\left[\sum_{k_2=1}^{6}w_{k_2}\cdot e^{aw_{k_2}sx}\cdot\Phi_{7k_2}(x,s,\tau)\right];$$

$$\Phi^m_{7k}(x,s,\tau)=w_k\cdot e^{-aw_k s\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}^3\cdot e^{-aw_{k_1}s\tau}\cdot\left[\sum_{k_2=1}^{6}w_{k_2}\cdot w_{k_2}^m\cdot e^{aw_{k_2}sx}\cdot\right.$$

$$\left.\cdot\,\Phi_{7k_1}(x,s,\tau)\right]+w_k^2\cdot e^{-aw_k s\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}^2\cdot e^{-aw_{k_1}s\tau}\cdot$$

$$\cdot\left[\sum_{k_2=1}^{6}w_{k_2}\cdot w_{k_2}^m\cdot e^{aw_{k_2}sx}\cdot\Phi_{7k_2}(x,s,\tau)\right];$$

$$\Phi_{7k_1}(x,s,\tau)=\int_0^x r(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int_0^{t-\tau}p(\xi)\cdot e^{a(w_k-w_{k_1})s\xi}\cdot d\xi\right)\cdot dt_{pkk_1rk_1k_2};$$

$$\Phi_{7k_2}(x,s,\tau)=\int_0^x p(t)\cdot e^{a(w_{k_1}-w_{k_2})st}\cdot\left(\int_0^{t-\tau}r(\xi)\cdot e^{a(w_k-w_{k_1})s\xi}\cdot d\xi\right)\cdot dt_{rkk_1pk_1k_2};$$

$$\Phi_{8k}(x,s,\tau)=e^{-aw_k s\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}^3\cdot e^{-aw_{k_1}s\tau}\cdot\left[\sum_{k_2=1}^{6}w_{k_2}\cdot e^{aw_{k_2}sx}\cdot\right.$$

$$\left.\cdot\,\Phi_{8k_1}(x,s,\tau)\right]+w_k\cdot e^{-aw_k s\tau}\cdot\sum_{k_1=1}^{6}w_{k_1}^2\cdot e^{-aw_{k_1}s\tau}\cdot$$

$$\cdot\left[\sum_{k_2=1}^{6}w_{k_2}\cdot e^{aw_{k_2}sx}\cdot\Phi_{8k_2}(x,s,\tau)\right]+w_k^2\cdot e^{-aw_k s\tau}\cdot$$

$$\cdot \sum_{k_1=1}^{6} w_{k_1} \cdot e^{-aw_{k_1}s\tau} \cdot \left[\sum_{k_2=1}^{6} w_{k_2} \cdot e^{aw_{k_2}sx} \cdot \Phi_{8k_3}(x,s,\tau)\right];$$

$$\Phi_{8k}^{m}(x,s,\tau) = e^{-aw_k s\tau} \cdot \sum_{k_1=1}^{6} w_{k_1}^3 \cdot e^{-aw_{k_1}s\tau} \cdot \left[\sum_{k_2=1}^{6} w_{k_2} \cdot w_{k_2}^m \cdot e^{aw_{k_2}sx} \cdot \right.$$

$$\left. \cdot \Phi_{8k_1}(x,s,\tau)\right] + w_k \cdot e^{-aw_k s\tau} \cdot \sum_{k_1=1}^{6} w_{k_1}^2 \cdot e^{-aw_{k_1}s\tau} \cdot$$

$$\cdot \left[\sum_{k_2=1}^{6} w_{k_2} \cdot w_{k_2}^m \cdot e^{aw_k sx} \cdot \Phi_{8k_2}(x,s,\tau)\right] +$$

$$+ w_k^2 \cdot e^{-aw_k s\tau} \cdot \sum_{k_1=1}^{6} w_{k_1} \cdot e^{-aw_{k_1}s\tau} \cdot$$

$$\cdot \left[\sum_{k_2=1}^{6} w_{k_2} \cdot w_{k_2}^m \cdot e^{aw_{k_2}sx} \cdot \Phi_{8k_3}(x,s,\tau)\right];$$

$$\Phi_{8k_1}(x,s,\tau) = \int_0^x r(t) \cdot e^{a(w_{k_1}-w_{k_2})st} \cdot \left(\int_0^{t-\tau} q(\xi) \cdot e^{a(w_k-w_{k_1})s\xi} \cdot d\xi\right) \cdot dt_{qkk_1rk_1k_2};$$

$$\Phi_{8k_2}(x,s,\tau) = \int_0^x p(t) \cdot e^{a(w_{k_1}-w_{k_2})st} \cdot \left(\int_0^{t-\tau} p(\xi) \cdot e^{a(w_k-w_{k_1})s\xi} \cdot d\xi\right) \cdot dt_{pkk_1pk_1k_2};$$

$$\Phi_{8k_3}(x,s,\tau) = \int_0^x q(t) \cdot e^{a(w_{k_1}-w_{k_2})st} \cdot \left(\int_0^{t-\tau} r(\xi) \cdot e^{a(w_k-w_{k_1})s\xi} \cdot d\xi\right) \cdot dt_{rkk_1qk_1k_2};$$

$$k = 1,2,\ldots,6; \quad m = 1,2,3,4,5. \tag{32}$$

The values $\Phi_{3k}(x,s,\tau),\ldots,\Phi_{6k}(x,s,\tau)$ and $\Phi_{3k}^m(x,s,\tau),\ldots,\ \Phi_{6k}^m(x,s,\tau)$ in (29)–(31) are important for calculating the asymptotics of eigenvalues of the necessary order of accuracy in calculating the first regularized trace of differential operators, connected with the differential equation (1)–(2).

Estimates of the remaining series in the formulae (30)–(31) are made similarly to the estimates provided in the monographs [11, Chapter 1] and [12, Chapter 1].

In the formulae (30)–(31), the values $\underline{O}\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^9}\right)$ represent a sum of double iterated integrals independent of the function $\varphi(x)$ $\Bigg($for example, $H_1(x,s) = \frac{1}{36a^9s^9} \cdot w_k \cdot e^{-aw_k s\tau} \cdot \sum\limits_{k_1=1}^{6} w_{k_1} \cdot e^{-aw_{k_1}s\tau}\left[\sum\limits_{k_2=1}^{6} w_{k_2} \cdot e^{aw_{k_2}sx} \int\limits_0^x q(t) \cdot e^{a(w_{k_1}-w_{k_2})st} \cdot \left(\int\limits_0^{t-\tau} p(\xi) \cdot e^{a(w_{k_2}-w_{k_1})s\xi} \cdot d\xi\right) \cdot dt_{pkk_1qk_1k_2}\right]$, etc$\Bigg)$, and triple iterated integrals depending on the function $\varphi(x)$ by virtue of the initial conditions (2) and (5) $\Bigg($for example, $H_2(x,s) = -\frac{1}{216a^9s^9} \cdot e^{-aw_k s\tau} \cdot \sum\limits_{k_1=1}^{6} w_{k_1}^3 \cdot \left[\sum\limits_{k_2=1}^{6} w_{k_2}^3 \cdot e^{-aw_{k_2}s\tau} \cdot \left(\sum\limits_{k_3=1}^{6} w_{k_3} \cdot e^{aw_{k_3}sx} \cdot \int\limits_0^x r(t) \cdot e^{a(w_{k_2}-w_{k_3})st}\left[\int\limits_0^{t-\tau} r(\xi) \cdot e^{a(w_{k_1}-w_{k_2})s\xi} \cdot \left(\int\limits_0^{t-\tau} r(\theta) \cdot e^{-aw_{k_1}s\theta} \cdot \varphi''(\theta-\tau) \cdot d\theta\right) d\xi\right] \cdot dt_{rk_2k_3rk_1k_2rk_1\varphi''}\right)\right] + \underline{O}\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s^{10}|}\right)\Bigg)$.

Thus, let us give the summary of the intermediate results. Asymptotics of solutions of the differential equation (1)–(2) in the case $\tau \in (\pi; +\infty)$ (if $k_0 = 0$ in the formula (14)) is completely obtained in the formulae (16)—(19). When $\tau \in \left(\frac{\pi}{2}; \pi\right]$ ($k_0 = 1$), the asymptotics of solutions is completely obtained in the formulae (24)–(28). Finally, when$\tau \in \left(\frac{\pi}{3}; \frac{\pi}{2}\right]$ ($k_0 = 2$) asymptotics of solutions to the differential equation (1)–(2) is completely obtained in the formulae (29)–(32).

Let us consider the last case: $\tau \in \left(0; \frac{\pi}{3}\right]$ (i. e. $k_0 \geqslant 3$ in the formula (14)). In this case, the following statement holds true.

Theorem 5. *The general solution of the differential equation* (1)–(2) *in the case* $\tau \in \left(0; \frac{\pi}{3}\right]$ ($k_0 \geqslant 3$, $k_0 \in \mathbb{N}$) *has the following form:*

$$y(x,s) = \sum_{k=1}^{6} C_k \cdot y_k(x,s); \quad y_k^{(m)}(x,s) = \sum_{k=1}^{6} C_k \cdot y_k^{(m)}(x,s), \quad m = 1,2,3,4,5, \tag{33}$$

and the following formulae hold:

$$y_k(x,s) = e^{aw_k sx} - \frac{1}{6a^3s^3} \cdot \Phi_{3k}(x,s,\tau) - \frac{\Phi_{4k}(x,s,\tau)}{6a^4s^4} - \frac{\Phi_{5k}(x,s,\tau)}{6a^5s^5} + \\ + \frac{\Phi_{6k}(x,s,\tau)}{36a^6s^6} + \frac{\Phi_{7k}(x,s,\tau)}{36a^7s^7} + \frac{\Phi_{8k}(x,s,\tau)}{36a^8s^8} + \underline{O}_1\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^9}\right), \tag{34}$$

$$\frac{y^{(m)}(x,s)}{(as)^m} = w_k^m \cdot e^{aw_k sx} - \frac{\Phi_{3k}^m(x,s,\tau)}{6a^3s^3} - \frac{\Phi_{4k}^m(x,s,\tau)}{6a^4s^4} - \frac{\Phi_{5k}^m(x,s,\tau)}{6a^5s^5} + \\ + \frac{\Phi_{6k}^m(x,s,\tau)}{36a^6s^6} + \frac{\Phi_{7k}^m(x,s,\tau)}{36a^7s^7} + \frac{\Phi_{8k}^m(x,s,\tau)}{36a^8s^8} + \underline{O}_2\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^9}\right), \tag{35}$$

while the functions $\Phi_{nk}(x,s,\tau)$, $\Phi_{nk}^m(x,s,\tau)$ (for $n = 3,4,5$) are defined in the formulae (27) of Theorem 3, functions $\Phi_{nk}(x,s,\tau)$, $\Phi_{nk}^m(x,s,\tau)$ ($n = 6,7,8$) are defined in the formulae (32) of Theorem 4, the value $\underline{O}_{1,2}\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^9}\right)$ of the formulae (33)–(35) differs from the value $\underline{O}\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^9}\right)$ of the formulae (29)–(31) as follows: double iterated integrals of the $H_1(x,s)$ type, independent of the function $\varphi(x)$, remain unaltered, and triple iterated integrals of the $H_2(x,s)$ type, depending on the function $\varphi(x)$ are rewritten in the following form: $H_{2,1}(x,s) = -\frac{1}{216a^9s^9} \cdot e^{-aw_k s\tau} \cdot \sum\limits_{k_1=1}^{6} w_{k_1}^3 \cdot \left[\sum\limits_{k_2=1}^{6} w_{k_2}^3 \cdot e^{-aw_{k_2}s\tau} \cdot \left(\sum\limits_{k_3=1}^{6} w_{k_3} \cdot \right.\right.$ $e^{aw_{k_3}sx} \cdot \int\limits_0^x r(t) \cdot e^{a(w_{k_2}-w_{k_3})st} \cdot \left[\int\limits_0^{t-\tau} r(\xi) \cdot e^{a(w_{k_1}-w_2)s\xi} \cdot \left(\int\limits_0^{\xi-\tau} r(\theta) \cdot e^{a(w_k-w_{k_1})s\theta} \cdot \right.\right.$ $\left.\left.\left.\left. d\theta\right) d\xi\right] \cdot dt_{rkk_1rk_1k_2rk_2k_3}\right)\right] + \underline{O}_3\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^{10}}\right)$ etc, and the value $\underline{O}\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^{12}}\right)$ when $\tau \in \left(\frac{\pi}{4}; \frac{\pi}{3}\right]$ ($k_0 = 3$ in (14)) represents a sum of various quadruple iterated integrals of the $H_2(x,s)$ type depending on the function $\varphi(x)$. Likewise, the value $\underline{O}\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^{15}}\right)$ when $\tau \in \left(\frac{\pi}{5}; \frac{\pi}{4}\right]$ ($k_0 = 4$ in (14)) represents a sum of quintuple iterated of the $H_2(x,s)$ type depending on the function $\varphi(x)$. The value $\underline{O}\left(\frac{e^{|\mathrm{Im}s|\cdot x}}{|s|^{3N}}\right)$ when $\tau \in \left(\frac{\pi}{N}; \frac{\pi}{N-1}\right]$ ($k_0 = N-1$ in (14)) represents a sum of various N-tuple iterated integrals of the $H_2(x,s)$ type depending on $\varphi(x)$.

Theorems 2–5 give a possibility to determine the asymptotics of the solution $y(x,s)$ to the differential equation (1) with a retarded argument τ, with the initial conditions (2) completely if the conditions (4) of summability of coefficients $r(x)$, $p(x)$ to $q(x)$ are satisfied.

The asymptotics of a fundamental system of solutions of an arbitrary accuracy order in case of a classical differential Sturm-Liouville operator of the second order with a summable potential was calculated in the works [13]–[14] for the first time.

In case of a functionally differential operator of the Sturm-Liouville type of the second order with a summable potential, asymptotics of the fundamental system of solutions was obtained in [15].

Let us consider the boundary-value conditions (3). Let us start with the case $\tau > \pi$ ($k_0 = 0$).

The general solution of the differential equation (1)–(2) when $\tau > \pi$ is derived in Theorem 2 and is described by the formulae (16)–(18), and the initial conditions (19) hold.

Theorem 6. *Equation for eigenvalues of the differential operator* (1)–(2) *with the boundary value conditions* (3) *has the following form*:

$$f(s) = \begin{vmatrix} w_1^{m_1} & w_2^{m_1} & w_3^{m_1} & w_4^{m_1} & w_5^{m_1} & w_6^{m_1} \\ w_1^{m_2} & w_2^{m_2} & w_3^{m_2} & w_4^{m_2} & w_5^{m_2} & w_6^{m_2} \\ w_1^{m_3} & w_2^{m_3} & w_3^{m_3} & w_4^{m_3} & w_5^{m_3} & w_6^{m_3} \\ w_1^{m_4} & w_2^{m_4} & w_3^{m_4} & w_4^{m_4} & w_5^{m_4} & w_6^{m_4} \\ w_1^{m_5} & w_2^{m_5} & w_3^{m_5} & w_4^{m_5} & w_5^{m_5} & w_6^{m_5} \\ y_1^{(n_1)}(\pi, s) & y_2^{(n_1)}(\pi, s) & y_3^{(n_1)}(\pi, s) & y_4^{(n_1)}(\pi, s) & y_5^{(n_1)}(\pi, s) & y_6^{(n_1)}(\pi, s) \end{vmatrix} = 0. \tag{36}$$

Proof. The first five conditions, involved in the boundary value conditions (3), provide

$$y^{(m_n)}(0) \overset{(3)}{=} 0 \overset{(16)}{\Leftrightarrow} \sum_{k=1}^{6} C_k \cdot y_k^{(m_n)}(0, s) = 0 \overset{(19)}{\Leftrightarrow}$$

$$\Leftrightarrow \sum_{k=1}^{6} C_k \cdot w_k^{m_n} \cdot a^{m_n} \cdot s^{m_n} = 0, \quad n = 1, 2, 3, 4, 5, \tag{37}$$

and $a > 0$ from (1). One can readily verify that the number $s = 0$ ($\lambda = 0$) is not an eigenvalue of the differential operator (1)–(2)–(3). Therefore, it follows from (37) that $\sum_{k=1}^{6} C_k \cdot w_k^{m_n} = 0$, $n = 1, 2, 3, 4, 5$.

The sixth equation, involved in the boundary-value conditions (3), yields:

$$y^{(n_1)}(\pi, s) \overset{(3)}{=} 0 \overset{(16)}{\Leftrightarrow} \sum_{k=1}^{6} C_k \cdot y_k^{(n_1)}(\pi, s) = 0, \quad n_1 \in \{0, 1, 2, 3, 4, 5\}. \tag{38}$$

The system (37)–(38) represents a homogeneous system of six linear equations with six unknowns $C_1, C_2, \ldots, C_6$. It has nonzero solutions $\left(\sum_{k=1}^{6} C_k^2 \neq 0\right)$ if and only if its determinant vanishes. The determinant of the system (37)–(38) is written in the formula (36) of Theorem 6. Therefore, Theorem 6 is proved. ■

Thus, in order to find eigenvalues λ_k ($\lambda_k = s_k^6$, $k = 1, 2, 3, \ldots$) of the differential operator (1)–(2)–(3), one has to learn how to obtain the roots s_k of the equation $f(s) = 0$ from (36). We will look for the asymptotics of the roots s_k of the equation (36) by means of the methodology, described in the monographs [16, Chapter 12].

Equation $f(s) = 0$ from (35) can be rewritten in the following form, expanding the operator in the sixth row and multiplying it by (-1):

$$f(s) = \delta_{61} \cdot y_1^{(n_1)}(\pi, s) - \delta_{62} \cdot y_2^{(n_1)}(\pi, s) + \delta_{63} \cdot y_3^{(n_1)}(\pi, s) -$$

$$- \delta_{64} \cdot y_4^{(n_1)}(\pi, s) + \delta_{65} \cdot y_5^{(n_1)}(\pi, s) - \delta_{66} \cdot y_6^{(n_1)}(\pi, s) = 0, \tag{39}$$

where δ_{6k} ($k = 1, 2, \ldots, 6$) are algebraic minors to elements of the sixth row.

Let us introduce the notation $z = w_2$ for calculating the determinants δ_{6k} $(k = 1, 2, \ldots, 6)$. The equality (6) provides:

$$w_1 = 1 = z^0; \quad w_2 = z = e^{\frac{2\pi i}{6}}, \quad w_3 = z^2, \quad w_4 = z^3, \quad w_5 = z^4, \quad w_6 = z^5. \tag{40}$$

Using properties of the determinants and the formulae (40), we obtain:

$$\delta_{66} = \begin{vmatrix} w_1^{m_1} & w_2^{m_1} & \ldots & w_5^{m_1} \\ w_1^{m_2} & w_2^{m_2} & \ldots & w_5^{m_2} \\ \ldots & \ldots & \ldots & \ldots \\ w_1^{m_5} & w_2^{m_5} & \ldots & w_5^{m_5} \end{vmatrix} = \begin{vmatrix} 1 & z^{m_1} & z^{2m_1} & z^{3m_1} & z^{4m_1} \\ 1 & z^{m_2} & z^{2m_2} & z^{3m_2} & z^{4m_2} \\ 1 & z^{m_3} & z^{2m_3} & z^{3m_3} & z^{4m_3} \\ 1 & z^{m_4} & z^{2m_4} & z^{3m_4} & z^{4m_4} \\ 1 & z^{m_5} & z^{2m_5} & z^{3m_5} & z^{4m_5} \end{vmatrix} =$$

$$= \mathrm{detWandermond's}(z^{m_1}, z^{m_2}, z^{m_3}, z^{m_4}, z^{m_5}) = \prod_{\substack{k>n \\ k,n\in\{1,2,3,4,5\}}} (z^{m_k} - z^{m_n}) \neq 0; \tag{41}$$

$$\delta_{61} = \begin{vmatrix} w_2^{m_1} & w_3^{m_1} & \ldots & w_6^{m_1} \\ w_2^{m_2} & w_3^{m_2} & \ldots & w_6^{m_2} \\ \ldots & \ldots & \ldots & \ldots \\ w_2^{m_5} & w_3^{m_5} & \ldots & w_6^{m_5} \end{vmatrix} = \begin{vmatrix} z^{m_1} & z^{2m_1} & \ldots & z^{5m_1} \\ z^{m_2} & z^{2m_2} & \ldots & z^{5m_2} \\ \ldots & \ldots & \ldots & \ldots \\ z^{m_5} & z^{2m_5} & \ldots & z^{5m_5} \end{vmatrix} = z^M \cdot \delta_{66}, \tag{42}$$

where

$$M = m_1 + m_2 + m_3 + m_4 + m_5 = \sum_{k=1}^{5} m_k; \tag{43}$$

$$\delta_{62} = \begin{vmatrix} w_1^{m_1} & w_3^{m_1} & \ldots & w_6^{m_1} \\ w_1^{m_2} & w_3^{m_2} & \ldots & w_6^{m_2} \\ \ldots & \ldots & \ldots & \ldots \\ w_1^{m_5} & w_3^{m_5} & \ldots & w_6^{m_5} \end{vmatrix} = (1 = z^0 = z^6) =$$

$$= \begin{vmatrix} 1 & z^{2m_1} & \ldots & z^{5m_1} \\ 1 & z^{2m_2} & \ldots & z^{5m_2} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & z^{2m_5} & \ldots & z^{5m_5} \end{vmatrix} = \begin{vmatrix} z^{2m_1} & \ldots & z^{5m_1} & z^{6m_1} \\ z^{2m_2} & \ldots & z^{5m_2} & z^{6m_2} \\ \ldots & \ldots & \ldots & \ldots \\ z^{2m_5} & \ldots & z^{5m_5} & z^{6m_5} \end{vmatrix} = z^{2M} \cdot \delta_{66}. \tag{44}$$

Likewise, it is proved that

$$\delta_{63} = z^{3M} \cdot \delta_{66}; \quad \delta_{64} = z^{4M} \cdot \delta_{66}; \quad \delta_{65} = z^{5M} \cdot \delta_{66}; \quad \delta_{66} = z^{6M} \cdot \delta_{66} \neq 0. \tag{45}$$

Using the formulae (41)–(45), we can rewrite Equation (39) in the form

$$f(s) = z^M \cdot \delta_{66} \Big\{ y_1^{(n_1)}(\pi, s) - z^M \cdot y_2^{(n_1)}(\pi, s) + z^{2M} \cdot y_3^{(n_1)}(\pi, s) -$$
$$- z^{3M} \cdot y_4^{(n_1)}(\pi, s) + z^{4M} \cdot y_5^{(n_1)}(\pi, s) - z^{5M} \cdot y_6^{(n_1)}(\pi, s) \Big\} = 0, \tag{46}$$

and divide it by $z^M \cdot \delta_{66}$ because $z^M \neq 0$, $\delta_{66} \neq 0$.

Substituting the formulae (17)–(18) into Equation (46), we obtain:

$$f(s)=1\cdot\left[w_1^{n_1}\cdot e^{aw_1s\pi}-\frac{A_5^{n_1}(\pi,s)}{6a^5s^5}\right]-z^M\cdot\left[w_2^{n_1}\cdot e^{aw_2s\pi}-\frac{A_5^{n_1}(\pi,s)}{6a^5s^5}\right]+$$
$$+z^{2M}\cdot\left[w_3^{n_1}\cdot e^{aw_3s\pi}-\frac{A_5^{n_1}(\pi,s)}{6a^5s^5}\right]-z^{3M}\cdot\left[w_4^{n_1}\cdot e^{aw_4s\pi}-\frac{A_5^{n_1}(\pi,s)}{6a^5s^5}\right]+$$
$$+z^{4M}\cdot\left[w_5^{n_1}\cdot e^{aw_5s\pi}-\frac{A_5^{n_1}(\pi,s)}{6a^5s^5}\right]-z^{5M}\cdot\left[w_6^{n_1}\cdot e^{aw_6s\pi}-\frac{A_5^{n_1}(\pi,s)}{6a^5s^5}\right]=0, \quad (47)$$

where the following notation is introduced:

$$A_5^{n_1}(\pi,s)=\sum_{k_1=1}^{6}w_{k_1}^{n_1+1}\cdot e^{aw_{k_1}sx}\cdot\left[\int_0^x r(t)\cdot e^{-aw_{k_1}st}\cdot\varphi''(t-\tau)\cdot dt_{rk_1}+\right.$$
$$+\int_0^x p(t)\cdot e^{-aw_{k_1}st}\cdot\varphi'(t-\tau)\cdot dt_{pk_1}+$$
$$\left.+\int_0^x q(t)\cdot e^{-aw_{k_1}st}\varphi(t-\tau)\cdot dt_{qk_1}\right],$$
$$n_1\in\{0,1,2,3,4,5\}. \quad (48)$$

It follows from Equation (47)–(48) that

$$f(s)=f_0(s)-\frac{1}{6a^5s^5}\cdot f_5(s)=0, \quad (49)$$

where

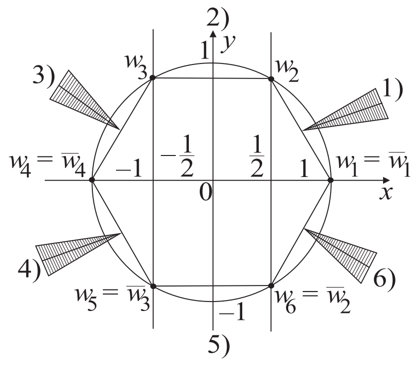
$$f_0(s)=1\cdot w_1^{n_1}-z^M\cdot w_2^{n_1}+z^{2M}\cdot w_3^{n_1}-z^{3M}\cdot w_4^{n_1}+z^{4M}\cdot w_5^{n_1}-z^{5M}\cdot w_6^{n_1}, \quad (50)$$
$$f_5(s)=A_5^{n_1}(\pi,s)\cdot[1-z^M+z^{2M}-z^{3M}+z^{4M}-z^{5M}]. \quad (51)$$

The main approximation of Equation (49) has the form

$$f_0(s)=0, \quad (52)$$

where $f_0(s)$ is defined in the formula (50).

The indicator diagram (see [16, Chapter 12]) for Equations (49) and (52) has the following form: The indicator diagram (53) represents a regular hexagon with vertices at



(53)

the points w_k $(k=1,2,\ldots,6)$, because they divide a unit circle into six equal parts $\left(w_k=e^{\frac{2\pi i}{6}(k-1)},\ k=1,2,\ldots;\ w_1=1\right)$.

It follows from [16, Chapter 12] that eigenvalues can be obtained only in the shaded sectors, illustrated on Figure (53), of an infinitely small angle, and bisectrices of these sectors are mean perpendiculars to the sides of the hexagon.

Let us investigate the sixth sector in more detail. It follows from Chapter 12 of the monograph [16] that roots of the function $f(s)$ from (36), (39), (46)–(51) coincide asymptotically with roots of the function $g_6(s)$. We keep only exponentials with the indices $\overline{w}_1 = w_1$ and $\overline{w}_6 = w_2$ (only these two numbers belong to the sector boundary) in Equation (49)–(51) for the latter function. Therefore, the following fact holds true: equation in eigenfunctions of the sixth sector has the following form:

$$g_6(s) = \left[w_1^{n_1} \cdot e^{aw_1 s\pi} - z^N \cdot w_2^{n_1} \cdot e^{aw_2 s\pi}\right] - \frac{1}{6a^5 s^5} \cdot \left[1 - z^M\right] \cdot$$
$$\cdot \left[w_1 \cdot w_1^{n_1} \cdot e^{aw_1 s\pi} \left(\int_0^\pi \ldots\right)_{a_1} + w_2 \cdot w_2^{n_1} \cdot e^{aw_2 s\pi} \cdot \left(\int_0^\pi \ldots\right)_{a_2} + \overline{\overline{o}}(1)\right] = 0, \tag{54}$$

where we used the formula (48) for $A_5^{n_1}(\pi, s)$, and

$$\left(\int_0^\pi \ldots\right)_{a_1} \overset{(48)}{=} \int_0^\pi r(t) \cdot e^{-aw_1 st} \cdot \varphi''(t-\tau) \cdot dt_{r_1} + \int_0^\pi p(t) \cdot e^{-aw_1 st} \cdot \varphi'(t-\tau) \cdot dt_{p_1} +$$
$$+ \int_0^\pi g(t) \cdot e^{-aw_1 st} \cdot \varphi(t-\tau) \cdot dt_{q_1} = \int_0^\pi \Phi(t) \cdot e^{-aw_1 st} \cdot dt_{a_1},$$
$$\Phi(t) = r(t) \cdot \varphi''(t-\tau) + p(t) \cdot \varphi'(t-\tau) + q(t) \cdot \varphi(t-\tau), \tag{55}$$

$$\left(\int_0^\pi \ldots\right)_{a_2} = \int_0^\pi \Phi(t) \cdot e^{-aw_2 st} \cdot dt_{a_2} = \left(\int_0^\pi \ldots\right)_{r_2} + \left(\int_0^\pi \ldots\right)_{p_2} + \left(\int_0^\pi \ldots\right)_{q_2}. \tag{56}$$

The main approximation in Equation $g_6(s) = 0$ from (54) has the following form:

$$g_{06}(s) = w_1^{n_1} \cdot e^{aw_1 s\pi} - z^M \cdot w_2^{n_1} \cdot e^{aw_2 s\pi} = 0 \Leftrightarrow e^{a(w_1 - w_2)s\pi} = \frac{w_2^{n_1} \cdot z^M}{w_1^{n_1}} =$$
$$= e^{\frac{2\pi i}{6}\cdot(M+n_1)} \Leftrightarrow a(w_1 - w_2)s\pi = 2\pi i k + \frac{2\pi i}{6} \cdot (M + n_1) \Leftrightarrow$$
$$\Leftrightarrow S_{k,6,\text{main}} = \frac{2i}{a(w_1 - w_2)} \cdot \widetilde{k}, \quad \widetilde{k} = k + \frac{M + n_1}{6}, \quad k = 1, 2, 3, \ldots \tag{57}$$

On the basis of the formula (57), taking into account the methodology of the monograph [16, Chapter 12], we make the conclusion that the following theorem holds.

Theorem 7. *Asymptotics of eigenvalues of the differential operator* (1)–(2)–(3) *in the sixth sector of the indicator diagram* (53) *has the following form*:

$$s_{k,6} = \frac{2i\widetilde{k}}{a(w_1 - w_2)} + \frac{2i \cdot d_{5k,6}}{a(w_1 - w_2) \cdot \widetilde{k}^5} + \underline{O}\left(\frac{1}{\widetilde{k}^{10}}\right), \quad \widetilde{k} = k + \frac{M + n_1}{6}, \tag{58}$$

and

$$M \overset{(43)}{=} \sum_{k=1}^{5} m_k, \quad m_k \in \{0, 1, 2, 3, 4, 5\}, \quad k = 1, 2, 3, \ldots$$

Proof. The fact that asymptotics of eigenvalues of the differential operator (1)–(2)–(3) (invoking (53)) should be sought for in the form (58) follows from [16, Chapter 12] and [17]. In case of an indicator diagram of the form (53) expansion in fractional $\widetilde{k}$ is impossible and there is no "decomposition" effect of "multiple in the main" eigenvalues demonstrated by the author in [18].

Therefore, to prove Theorem 7 we have to find a formula for determining the coefficient $d_{5k,6}$ from (58) in the explicit form.

Using the Taylor formulae and (58) for $s_{k,6}$, we have:

$$e^{a(w_1-w_2)s\pi}\Big|_{s_{k,6}} = e^{a(w_1-w_2)\pi\cdot\frac{2i\widetilde{k}}{a(w_1-w_2)}}\cdot e^{a(w_1-w_2)\pi\cdot\left[\frac{2id_{5k,6}}{a(w_1-w_2)\widetilde{k}^5}+\dots\right]} =$$

$$= 1\cdot e^{\frac{2\pi i}{6}\cdot(M+n_1)}\cdot\left[1+\frac{2\pi i d_{5k,6}}{\widetilde{k}^5}+\underline{O}\left(\frac{1}{\widetilde{k}^{10}}\right)\right], \tag{59}$$

$$\frac{1}{s_{k,6}} = \frac{a(w_1-w_2)}{2i\widetilde{k}}+\underline{O}\left(\frac{1}{\widetilde{k}^6}\right); \quad \frac{1}{s_{k,6}^5} = \frac{a^5(w_1-w_2)^5}{2^5 i^5\widetilde{k}^5}+\underline{O}\left(\frac{1}{\widetilde{k}^{11}}\right), \tag{60}$$

$$\left(\int_0^\pi \dots\right)_{a_1,a_2}\Bigg|_{s_{k,6}} = \int_0^\pi \Phi(t)\cdot e^{-aw_{1,2}st}\cdot dt\Bigg|_{s_{k,6}} = \int_0^\pi \Phi(t)\cdot e^{\frac{-2iw_{1,2}\widetilde{k}t}{w_1-w_2}}\cdot$$

$$\cdot\ dt_{p_1,p_2}+\underline{O}\left(\frac{1}{\widetilde{k}^5}\right),$$

$$\Phi(t) = r(t)\cdot\varphi''(t-\tau)+p(t)\cdot\varphi'(t-\tau)+q(t)\cdot\varphi(t-\tau). \tag{61}$$

Substituting the formulae (58)–(61) into Equation (54)–(56), we obtain:

$$\left[w_1^{n_1}\cdot e^{\frac{2\pi i}{6}\cdot(M+n_1)} - w_2^{n_1}\cdot z^M\right] + \frac{2\pi i d_{5k,6}}{\widetilde{k}^5}\cdot e^{\frac{2\pi i}{6}\cdot(M+n_1)} -$$

$$-\frac{1}{6a^5}\cdot\frac{a^5(w_1-w_2)^5\cdot i}{2^5\cdot i^5\cdot\widetilde{k}^5\cdot i}\cdot\left(1+\underline{O}\left(\frac{1}{\widetilde{k}^6}\right)\right)\cdot[1-z^M]\cdot$$

$$\cdot\ B_2(\pi)+\underline{O}\left(\frac{1}{\widetilde{k}^{10}}\right) = 0, \tag{62}$$

where

$$B_2(\pi) = w_1\cdot w_1^{n_1}\cdot e^{\frac{2\pi i}{6}\cdot(M+n_1)}\cdot\left(\int_0^\pi\dots\right)_{p_1} + w_2\cdot w_2^{n_1}\cdot\left(\int_0^\pi\dots\right)_{p_2}. \tag{63}$$

Coefficient in the formula (62)–(63) vanishes when $\widetilde{k}^0$: $w_1^{n_1}\cdot e^{\frac{2\pi i}{6}\cdot(M+n_1)} - w_2^{n_1}\cdot z^M = = 1^{n_1}\cdot e^{\frac{2\pi i}{6}\cdot M}\cdot e^{\frac{2\pi i}{6}\cdot n_1} - \left(e^{\frac{2\pi i}{6}}\right)^{n_1}\cdot\left(e^{\frac{2\pi i}{6}}\right)^M = 0$, which verifies the asymptotics of eigenvalues in the form (58).

Equating coefficients of $\widetilde{k}^{-5}$ in (62)–(63), we have:

$$2\pi i d_{5k,6}\cdot e^{\frac{2\pi i}{6}(M+n_1)} + \frac{i(w_1-w_2)^5}{6\cdot 32}\cdot[1-z^M]\cdot B_2(\pi) = 0,$$

whence, we conclude that the formula

$$d_{5k,6} = \frac{-(w_1-w_2)^5}{12\pi\cdot 32}\cdot e^{\frac{-2\pi i}{6}(M+n_1)}\cdot[1-z^M]\cdot B_2(\pi) \tag{64}$$

holds.

In view of the formulae (6), we have:

$$w_1 - w_2 = 1 - e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{6}} \cdot \left[e^{\frac{-\pi i}{6}} - e^{\frac{\pi i}{6}}\right] = (-2i) \cdot e^{\frac{\pi i}{6}} \sin\left(\frac{\pi}{6}\right);$$

$$1 - z^M = 1 - e^{\frac{2\pi i}{6}\cdot M} = e^{\frac{\pi i}{6}\cdot M}\left[e^{\frac{-\pi i}{6}\cdot M} - e^{\frac{\pi i}{6}\cdot M}\right] = (-2i)\cdot e^{\frac{\pi i}{6}\cdot M}\sin\left(\frac{\pi M}{6}\right); \tag{65}$$

$$B_2(\pi) = 1 \cdot 1^{n_1} \cdot e^{\frac{2\pi i}{6}\cdot M} \cdot e^{\frac{2\pi i}{6}\cdot n_1} \cdot \int_0^\pi \Phi(t) \cdot e^{\frac{-2iw_1\widetilde{k}t}{w_1-w_2}} \cdot dt_{p_1} + e^{\frac{2\pi i}{6}} \cdot$$

$$\cdot\ e^{\frac{2\pi i}{6}\cdot n_1} \int_0^\pi \Phi(t) \cdot e^{\frac{-2iw_2\widetilde{k}t}{w_1-w_2}} \cdot dt_{p_2} = e^{\frac{2\pi i}{6}\cdot n_1} \cdot e^{\frac{\pi i}{6}} \cdot e^{\frac{2\pi}{6}\cdot M} \cdot$$

$$\cdot \left[e^{\frac{-\pi i}{6}} \cdot e^{\frac{\pi i}{6}\cdot M} \cdot \int_0^\pi \Phi(t) \cdot e^{\frac{-2i\widetilde{k}t}{w_1-w_2}\cdot\left(\frac{w_1+w_2}{2}+\frac{w_1-w_2}{2}\right)} \cdot dt_{p_1} + \right.$$

$$\left. + e^{\frac{\pi i}{6}} \cdot {}^{\frac{-\pi i}{6}\cdot M} \cdot \int_0^\pi \Phi(t) \cdot e^{\frac{-2i\widetilde{k}t}{w_1-w_2}\cdot\left(\frac{w_1+w_2}{2}-\frac{w_1-w_2}{2}\right)} \cdot dt_{p_2}\right] =$$

$$= e^{\frac{2\pi i}{6}\cdot n_1} \cdot e^{\frac{\pi i}{6}} \cdot e^{\frac{\pi i}{6}\cdot M} \cdot 2 \cdot \left(\int_0^\pi \dots\right)_{V_1}, \tag{66}$$

where

$$\left(\int_0^\pi \dots\right)_{V_1} = \int_0^\pi \Phi(t) \cdot e^{-\sqrt{3}\widetilde{k}t} \cdot \cos\left(\widetilde{k}t + \frac{\pi}{6} - \frac{\pi M}{6}\right) \cdot dt_{V_1}, \tag{67}$$

because $\frac{w_1+w_2}{w_1-w_2} = \sqrt{3}\cdot i$.

Substituting the formulae (65)–(67) into the formula (64), we obtain: $d_{5k,6} = -\frac{1}{12\pi\cdot 32}\cdot(-2i)^5\cdot e^{\frac{\pi i}{6}\cdot 5}\left(\frac{1}{2}\right)^5 \cdot e^{\frac{-2\pi i}{6}\cdot M}\cdot e^{\frac{-2\pi i}{6}\cdot n_1}\cdot(-2i)\cdot e^{\frac{\pi i}{6}\cdot M}\cdot\sin\left(\frac{\pi M}{6}\right)\cdot e^{\frac{2\pi i}{6}\cdot n_1}\cdot e^{\frac{\pi i}{6}}\cdot e^{\frac{\pi i}{6}\cdot M}\cdot 2\cdot\left(\int_0^\pi\dots\right)_{V_1} = -\frac{1}{192\pi}\cdot\sin\left(\frac{\pi M}{6}\right)\cdot\left(\int_0^\pi\dots\right)_{V_1}$, i. e. we have:

$$d_{5k,6} = -\frac{1}{192\pi}\cdot\sin\left(\frac{\pi M}{6}\right)\cdot\int_0^\pi \Phi(t)\cdot e^{-\sqrt{3}\cdot\widetilde{k}t}\cdot\cos\left(\widetilde{k}t + \frac{\pi}{6} - \frac{\pi M}{6}\right)\cdot dt_{V_1}, \tag{68}$$

where $M = \sum_{k=1}^{5} m_k$, $\Phi(t) = r(t)\cdot\varphi''(t-\tau) + p(t)\cdot\varphi'(t-\tau) + q(t)\cdot\varphi(t-\tau)$, $\tau > \pi$, $\widetilde{k} = k + \frac{M+n_1}{6}$, $k = 1, 2, 3, \dots$.

Derivation of the formula (68) completes the proof. of Theorem 7. ■

Note that asymptotics of eigenvalues of the differential operator (1)–(2)–(3) is obtained likewise in the remaining sectors of the indicator diagram (53). In this case the following relations hold:

$$s_{k,1} = s_{k,6}\cdot e^{\frac{4\pi i}{6}};\quad s_{k,2} = s_{k,1}\cdot e^{\frac{2\pi i}{6}} = s_{k,6}\cdot e^{\frac{2\pi i}{6}};\dots;$$

$$s_{k,m} = s_{k,6}\cdot e^{\frac{2\pi i m}{6}},\quad m = 1,2,3,4,5,6. \tag{69}$$

The formulae (58) and (68) allow us to find the asymptotics of eigenfunctions of the differential operator (1)–(2)–(3) (if $\tau > \pi$) similarly to the works [13]–[15].

Similarly to the above, one can find asymptotics of eigenvalues of the differential operator (1)–(2)–(3) in cases if $\tau \in \left(\frac{\pi}{2};\pi\right]$, $\tau \in \left(\frac{\pi}{3};\frac{\pi}{2}\right]$ and $\tau \in \left(0;\frac{\pi}{3}\right]$ as well. Meanwhile, in view of the formulae (24)–(27) when $\tau \in \left(\frac{\pi}{2};\pi\right]$ and (29)–(32) when $\tau \in \left(\frac{\pi}{3};\frac{\pi}{2}\right]$, asymptotics of eigenvalues is sought for in the form other than the formula (58):

$$s_{k,6} = \frac{2i\widetilde{k}}{a(w_1-w_2)} + \frac{2i}{a(w_1-w_2)} \cdot \left[\frac{d_{3k,6}}{\widetilde{k}^3} + \frac{d_{4k,6}}{\widetilde{k}^4} + \frac{d_{5k,6}}{\widetilde{k}^5} + \underline{O}\left(\frac{1}{\widetilde{k}^6}\right)\right], \tag{70}$$

where $\widetilde{k} = k + \frac{M+n_1}{6}$; then the indicator diagram has the form (53), and the main approximations of the asymptotics coincide. In the remaining sectors of the indicator diagram, the relations (69) hold.

Finally, we would like to mention that the spectrum of the operator (1)–(2) with boundary value conditions

$$y^{(m_1)}(0) = y^{(m_2)}(0) = y^{(m_3)}(0) = y^{(m_4)}(0) = y^{(n_1)}(\pi) = y^{(n_2)}(\pi) = 0$$

is investigated likewise, but by means of significantly more complicated calculations.

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Sergey Ivanovich Mitrokhin,
Research Computer Center, Lomonosov Moscow State University,
Leninskie gory, 6,
141075, Moscow, Russia
E-mail: mitrokhin-sergey@yandex.ru

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