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THE INDICATOR OF A DELTA-SUBHARMONIC FUNCTION IN A HALF-PLANE

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Abstract. Delta-subharmonic functions of a completely regular growth in the upper half-plane have been introduced in the joint work of the authors, published in Reports of the Russian Academy of Sciences (2001). In this work, criteria whether a delta-subharmonic function in the upper half-plane belongs to a class of functions of a completely regular growth have been obtained on the basis of the theory of Fourier coefficients of delta-subharmonic functions in the half-plane developed in the beginning of this century by the first author of the present article. The present paper is a natural continuation of this research. The concept of the indicator of a delta-subharmonic function of a completely regular growth in the upper half-plane is introduced. It is proved that the indicator of a delta-subharmonic function of a completely regular growth in the upper half-plane belongs to a class $L_p[0, \pi]$ $(1 < p \leqslant 2)$. The proof is based on the lemma about Polya peaks and the Hausdorff-Young theorem.

Keywords: delta-subharmonic functions of a completely regular growth in the upper half-plane, Fourier coefficients, the indicator, Polya peaks, Hausdorff-Young theorem.

1. Introduction

The theory of entire functions of a completely regular growth (c.r.g) with respect to the function $\gamma(r)$, close to a power one, initiated in the late 30ies of the XX century by B. Ya. Levin [1] and A. Pfluger [2], [3] independently of each other, takes a prominent place in complex analysis. Research on this theory continues, simultaneously expanding its application range from the theory of characteristic functions of probability laws and analytical theory of differential equations to the theory of boundary value problems, the representation of analytic functions by exponential series and the theory of subharmonic functions. A.F. Grishin [4] transferred the Levin-Pfluger theory to subharmonic functions in a complex plane. Using the method of Fourier series, A.A. Kondratyuk [5], [6], [7] generalized the Levin-Pfluger theory of entire functions of a c.r.g to meromorphic functions on an arbitrary γ-type. He made these generalizations in two directions: 1) functions growth was measured with respect to an arbitrary growth function $\gamma(r)$; 2) classes of meromorphic functions in a complex plane of c.r.g. functions were introduced and studied. Note that the Levin-Pfluger theory is included into the Kondratyuk theory as a special case when $\gamma(r) = r^{\rho(r)}$, where $\rho(r)$ is a proximate order [8], $\lim_{r\to\infty} \rho(r) = \rho > 0$.

In particular, the theory of Fourier coefficients allowed Kondratyuk to introduce the concept of an indicator of a meromorphic function of a c.r.g., which in the case of entire functions coincides with the classical definition of the indicator in the sense of Phragmén and Lindelöf.

At the same time, the theory of c.r.g. functions in the upper half-plane of a complex variable $\mathbb{C}_+ = \{z : \Im z > 0\}$ has been developed. In the 60ies of the XX century, A.F. Grishin [9], [10] and N. Govorov [11], [12] developed the Levin-Pfluger theory of functions of a finite order

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in the half-plane independently of each other. Meanwhile, if the Govorov theory is related to analytic functions of a c.r.g. with respect to the function $\gamma(r) = r^\rho$ ($\rho > 0$ is a fixed number), the Grishin theory covers subharmonic functions of a c.r.g. in the half-plane with respect to $\gamma(r) = r^{\rho(r)}$, where $\rho \geq 0$, including the classes of subharmonic functions of the zero order.

The theory of Fourier coefficients for delta-subharmonic functions in a half-plane developed by the first author of the present article at the beginning of this century [13], allowed K.G. Malyutin and N. Sadik to introduce the concept of delta-subharmonic functions of a c.r.g. in a half-plane with respect to a rather arbitrary growth function in their joint work [14]. In this work, as well as in works by A.A. Kondratyuk, generalizations are made in two directions: 1) the growth of functions is measured with respect to an arbitrary function growth function $\gamma(r)$; 2) classes of delta-subharmonic functions of a c.r.g. in the half-plane are introduced. This paper clarifies and supplemented some results of the work [14].

2. Meromorphic functions of a completely regular growth

Meromorphic functions of a completely regular growth with respect to a rather arbitrary growth function were introduced by A.A. Kondratyuk. The main tool of his research is the method of Fourier series for entire and meromorphic functions developed by Rubel and Taylor [15] , which is very effective in studying functions of an infinite order and functions growing irregularly in a neighborhood of infinity.

A strictly positive, continuous, increasing and unbounded function $\gamma(r)$, defined on the semi-axis $[0, +\infty)$ is called a *growth function.*

The values

$$p[\gamma] = \limsup_{r\to\infty} \frac{\ln \gamma(r)}{\ln r}, \quad p_*[\gamma] = \liminf_{r\to\infty} \frac{\ln \gamma(r)}{\ln r}$$

are said to be the *order and the lower order* of the growth function γ.

In what follows, $\gamma(r)$ will always denote a growth function, usually a fixed one. Moreover, following the Titchmarsh, will use the following names and symbols. If a number independent of the main variables is found in some reasoning, it is called a constant. We use the letters A, B to denote absolutely positive constants, not necessarily the same ones. One come across a statement such as "$|f(z)| < A\gamma(Br)$ hence, $3|f(z)| < A\gamma(Br)$," which should not cause confusion.

Definition 1. *A meromorphic function $f(z)$ is said to be a function of a finite γ-type, if there are positive constants A and B such that $T(r, f) \leqslant A\gamma(Br)$ for all $r > 0$.*

Here $T(r, f)$ is the Nevanlinna characteristic of the function $f(z)$.

Let us denote a class of given meromorphic functions when the function γ is fixed by $M(\gamma(r))$. By $E(\gamma(r))$ we denote a class of entire functions of a finite γ-type.

Let us assume that the condition

$$\gamma(2r) \leqslant K\gamma(r) \tag{1}$$

is satisfied for some $K > 0$ and all $r > 0$.

A. A. Kondratyuk introduced the notion of a meromorphic function of a c.r.g. provided that (1) is satisfied

Definition 2. *The function $f \in M(\gamma(r))$ is said to be a meromorphic function of a c.r.g. if the limit*

$$\lim_{r\to\infty} \frac{1}{\gamma(r)} \int_\eta^\varphi \ln |f(re^{i\theta})| \, d\theta \, .$$

exists for all η and φ from $[0, 2\pi]$.

Let us denote a class of meromorphic functions of a c.r.g. by $M^o(\gamma(r))$. By $E^o(\gamma(r))$ we denote a subclass of entire functions from $M^o(\gamma(r))$.

Let us denote the *Fourier coefficients of the function* f by

$$c_k(r, f) = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-ik\theta} \ln |f(re^{i\theta})| \, d\theta, \quad k \in \mathbb{Z}.$$

Theorem 1 (Kondratyuk)**.** *Let f be a meromorphic function, $f(0) = 1$. The following statements are equivalent*:

(i) $f \in M^o(\gamma(r))$;

(ii) $f \in M(\gamma(r))$ *and for every* $k \in \mathbb{Z}$ *there exists the limit*

$$\lim_{r\to\infty} \frac{c_k(r, f)}{\gamma(r)} = c_k\,;$$

(iii) $N(r, f) = O(\gamma(r))$, $r \to \infty$, *and for every function* ψ *from* χ *there exists the finite limit*

$$\lim_{r\to\infty} \frac{1}{\gamma(r)} \int\limits_0^{2\pi} \psi(\theta) \ln |f(re^{i\theta})| \, d\theta\,,$$

where χ *is any of the spaces* $C[0, 2\pi]$, $L_p[0, 2\pi]$, $p > 1$.

A.A. Kondratyuk introduced the notion of an indicator of a meromorphic function of a c.r.g.

Definition 3. *If* $f \in M^o(\gamma(r))$, *the function*

$$h(\theta, f) = \sum_{k=-\infty}^{+\infty} c_k e^{ik\theta}$$

is termed as an indicator of the function f.

He demonstrated that for any growth function $\gamma(r)$, satisfying the condition (1) and for any function $f \in E^o(\gamma(r))$

$$h(\theta, f) = \limsup_{r\to\infty} \frac{\ln |f(re^{i\theta})|}{\gamma(r)}\,.$$

A. A. Kondratyuk developed the theory of meromorphic functions of a c.r.g., similar to that by Levin-Pfluger. One of advantages of the latter is the fact that if $f \in E^o(r^{\rho(r)})$, f is an entire function of a c.r.g. in the Levin-Pfluger sense. Thus, the Levin-Pfluger theory is included into the Kondratyuk theory.

3. Delta-subharmonic functions of a completely regular growth in the half-plane

3.1. Delta-subharmonic functions of a finite γ-type in the half-plane в полуплоскости. We will use the terminology of [13]. Let $J\delta = JS - JS$ be a class of δ-subharmonic functions in $\mathbb{C}_+$. For a fixed measure λ, let

$$d\lambda_k(\zeta) = \frac{\sin k\varphi}{\sin\varphi} \tau^{k-1} \, d\lambda(\zeta) \; (\zeta = \tau e^{i\varphi}), \lambda_k(r) = \lambda_k\Big(\overline{C(0, r)}\Big)\,,$$

where the function $\dfrac{\sin k\varphi}{\sin\varphi}$ for $\varphi = 0, \pi$ is defined by continuity. In particular, $\lambda(r) = \lambda(\overline{C(0, r)})$.

The Fourier coefficients of a function $v \in J\delta$ are defined by the formulae:

$$c_k(r, v) = \frac{2}{\pi} \int\limits_0^{\pi} v(re^{i\theta}) \sin k\theta \, d\theta, \quad k \in \mathbb{N}\,.$$

Let $v = v_+ - v_-$ and λ be a full measure of the functions v. Let $\lambda = \lambda_+ - \lambda_-$ be the Jordanian measure decomposition λ. We set

$$m(r,v) := \frac{1}{r}\int_0^\pi v_+(re^{i\varphi})\sin\varphi\, d\varphi,\ N(r,r_0,v) := N(r,v) := \int_{r_0}^r \frac{\lambda_-(t)}{t^3}\, dt\,,$$

$$T(r,r_0,v) := T(r,v) := m(r,v) + N(r,v) + m(r_0,-v)\,,$$

where $r_0 > 0$ is an arbitrary fixed number, $r_0 < r$; we may take $r_0 = 1$ as well.

Further, let us assume that the growth function satisfies the condition:

$$\liminf_{r\to\infty} \frac{\gamma(r)}{r} > 0\,. \tag{2}$$

Definition 4. *The function $v \in J\delta$ is said to be a function of finite γ-type, if there are constants $A, B > 0$ such that*

$$T(r,v) \leqslant \frac{A}{r}\gamma(Br), \quad r > r_0\,.$$

We denote the corresponding class of δ-subharmonic functions of finite γ-type by $J\delta(\gamma(r))$. Denote by $JS(\gamma(r))$ the corresponding class of subharmonic functions of finite γ-type.

Remark. If the condition (2) is not met, we use another characteristics for describing the growth of functions

$$T(r,v) := m(r,v) + N\left(r, \frac{r}{2}, v\right) + m\left(\frac{r}{2}, -v\right)\,.$$

All the statements hold true in this case.

Definition 5. *The positive measure λ has a finite γ-density if there are positive constants A and B such that*

$$N(r,\lambda) := \int_{r_0}^r \frac{\lambda(t)}{t^3}\, dt \leqslant \frac{A}{r}\gamma(Br)$$

for all $r > r_0$.

Definition 6. *The positive measure λ in the half-plane is termed as a measure of a finite γ-type, if there are positive constants A and B such that*

$$\lambda(r) \leqslant Ar\gamma(Br) \tag{3}$$

for all $r > 0$.

The following theorem is obtained in [13].

Theorem 2. *Let γ be a growth function and let $v \in J\delta$. Then, the following two statements are equivalent:*

(i) $v \in J\delta(\gamma(r))$;

(ii) *the measure $\lambda_+(v)$ (or $\lambda_-(v)$) has a finite γ-density and*

$$|c_k(r,v)| \leqslant A\gamma(Br), \quad k \in \mathbb{N}\,,$$

for some positive A, B and all $r > 0$.

3.2. Delta-subharmonic functions of a completely regular growth in a half-plane.

Definition 7. *The function $v \in J\delta$ is said to be a function of a completely regular growth with respect to $\gamma(r)$ if there exists the limit*

$$\lim_{r\to\infty} \frac{1}{\gamma(r)} \int_{\eta}^{\varphi} v(re^{i\theta}) \sin\theta \, d\theta \tag{4}$$

for all η and φ from the interval $[0, \pi]$.

We denote the corresponding class of δ-subharmonic functions of a c.r.g. with respect to $\gamma(r)$ by $J\delta(\gamma(r))^o$. The notation $JS(\gamma(r))^o$ will be used to denote a class of proper subharmonic functions of a c.r.g. from $J\delta(\gamma(r))^o$.

Let $\widetilde{L}^\infty[0, \pi]$ be a Banach subspace $L^\infty[0, \pi]$ generated by a family of characteristic functions of all intervals from $[0, \pi]$. According to the Cantor theorem on uniform continuity $C[0, \pi] \subset \widetilde{L}^\infty[0, \pi]$. Denote by $\mathcal{L}[0, \pi]$ any of the spaces $C[0, \pi]$ or $\widetilde{L}^\infty[0, \pi]$. The following theorem is obtained in [14].

Theorem 3. *Let $v \in J\delta$. The following statements are equivalent:*

(i) $v \in J\delta(\gamma(r))^o$;

(ii) $v \in J\delta(\gamma(r))$ *and for all $k \in \mathbb{N}$ there exists the limit*

$$\lim_{r\to\infty} \frac{c_k(r, v)}{\gamma(r)} = c_k\,; \tag{5}$$

(iii) *the measure $\lambda_-(v)$ has a finite γ-density and for any function ψ from $\mathcal{L}[0, \pi]$ there exists the limit*

$$\lim_{r\to\infty} \frac{1}{\gamma(r)} \int_{0}^{\pi} \psi(\theta) v(re^{i\theta}) \sin\theta \, d\theta\,.$$

Here $\lambda(v)=\lambda_+(v) - \lambda_-(v)$ is the full measure corresponding to the function v, and $c_k(r, v)$ are the Fourier coefficients of the function v.

Note that if v belongs the class $JS(\gamma(r))^o$, the restriction on the measure $\lambda_-(v)$ in (iii) is lacking ($\lambda_-(v) \equiv 0$).

3.3. Indicator of the delta-subharmonic function of a completely regular γ-growth.

Following Kondratyuk, we introduce the following definition.

Definition 8. *Let $v \in J\delta(\gamma(r))^o$, and c_k be defined by the equalities* (5). *The function*

$$h(\theta, v) = \sum_{k=1}^{\infty} c_k \sin k\theta$$

is termed as the indicator of the function v.

We will need the following lemma on Pólya peaks [16].

Lemma 1. *Let us assume that ψ_1, ψ_2, ψ are positive continuous functions of r on the ray $[r_0, \infty)$ such that the relation $\psi_2(r)/\psi_1(r)$ grows and*

$$\limsup_{r\to\infty} \frac{\psi(r)}{\psi_1(r)} = \infty, \quad \limsup_{r\to\infty} \frac{\psi(r)}{\psi_2(r)} = 0\,.$$

Then, there exists a sequence $\{r_n\}$, $r_n \to \infty$ $(n \to \infty)$ *such that*

$$\frac{\psi(t)}{\psi_1(t)} \leqslant \frac{\psi(r_n)}{\psi_1(r_n)}, \quad r_0 \leqslant t \leqslant r_n,$$
$$\frac{\psi(t)}{\psi_2(t)} \leqslant \frac{\psi(r_n)}{\psi_2(r_n)}, \quad r_n \leqslant t < \infty$$

holds for $r = r_n$.

It follows from (1) that the order $\beta := p[\gamma] < \infty$. Let us formulate this proposition in the form of the following lemma.

Lemma 2. *Let us assume that a strictly positive, continuous, increasing and unbounded function* $\gamma(r)$, *defined on a semi-axis* $[0, +\infty)$, *satisfies the condition* (1). *Then there exist numbers* $p \in \mathbb{N}$ *and* $B > 0$ *such that*

$$\gamma(r) \leqslant Br^p$$

for all $r \in [0, \infty)$.

Proof. Let us choose $p \in \mathbb{N}$ such that $K \leqslant 2^p$.

First, let us prove that

$$\gamma(2^n) \leqslant K^n \gamma(1)$$

for all $n \in \mathbb{N}$. Indeed, it follows from the condition (1) for $n = 1$. Let this inequality hold for a certain $n \in \mathbb{N}$. Then

$$\gamma(2^{n+1}) \leqslant K\gamma(2^n) \leqslant K^{n+1}\gamma(1)\,.$$

Now, let $r \in [2^n, 2^{n+1}]$. We have

$$\gamma(r) \leqslant \gamma(2^{n+1}) \leqslant K^{n+1}\gamma(1) \leqslant (2^{n+1})^p\gamma(1) = (2^n)^p 2^p \gamma(1) \leqslant Br^p\,,$$

where $B = 2^p\gamma(1)$.

Theorem 4. *Let the function* v *belong to the class* $J\delta(\gamma(r))^o$ *and* $\gamma(r)$ *satisfies the condition* (1). *The indicator* $h(\theta, v)$ *belongs to* $L_p[0, \pi]$ $(1 < p \leqslant 2)$.

Proof. Then, $\lim_{r\to\infty} \gamma(r)/r^k = 0$ for all $k > \beta$. The inequality $|c_k(r, v)| \leqslant A\gamma(r)$ and the formula for Fourier's coefficients [14] for $r > r_0$

$$c_k(r, v) = c_k(r_0, v)\left(\frac{r}{r_0}\right)^k + \frac{2r^k}{\pi}\int\limits_{r_0}^{r} \frac{\lambda_k(t)}{t^{2k+1}}\,dt, \quad k \in \mathbb{N}\,,$$

provide

$$c_k(r, v) = -\frac{2r^k}{\pi}\int\limits_{r}^{\infty} \frac{\lambda_k(t)}{t^{2k+1}}\,dt, \quad k > \beta\,. \tag{6}$$

Applying the formula of integration by parts to the integral in (6), we obtain

$$c_k(r, v) = -\frac{1}{\pi k r^k}\iint\limits_{C_+(0,r)} \frac{\sin k\varphi}{\Im\zeta}\tau^k\,d\lambda(\zeta) - \frac{r^k}{\pi k}\iint\limits_{|\zeta|\geq r} \frac{\sin k\varphi}{\tau^k \Im\zeta}\,d\lambda(\zeta),\ \zeta = \tau e^{i\varphi} \tag{7}$$

for all $k > \beta$.

We set $\tilde{\lambda} = |\lambda|$,

$$N_1(r, v) := \int\limits_{r_0}^{r} \frac{\tilde{\lambda}(t)}{t^3}\,dt\,.$$

It follows from Theorem 2 that the measure $\tilde{\lambda}$ has a finite γ-density. The formula (7) provides the inequality

$$|c_k(r,v)| \leqslant \frac{1}{\pi r^k}\int\limits_0^r t^{k-1}\,d\tilde{\lambda}(t) + \frac{r^k}{\pi}\int\limits_r^\infty \frac{d\tilde{\lambda}(t)}{t^{k+1}}, \quad k>\beta\,.$$

Applying the integration by parts formula in the right-hand side of the inequality, we obtain

$$\begin{aligned}|c_k(r,v)| &\leqslant \frac{(k+1)r^k}{\pi}\int\limits_r^\infty \frac{\tilde{\lambda}(t)}{t^{k+2}}\,dt - \frac{k-1}{r^k\pi}\int\limits_0^r t^{k-2}\tilde{\lambda}(t)\,dt = \\ &= \frac{(k+1)r^k}{\pi}\int\limits_r^\infty \frac{dN_1(t)}{t^{k-1}} - \frac{k-1}{r^k\pi}\int\limits_0^r t^{k+1}\,dN_1(t) = \qquad (8)\\ &= \frac{(k^2-1)}{\pi}\left\{\int\limits_r^\infty \left(\frac{r}{t}\right)^k N_1(t)\,dt + \int\limits_0^r \left(\frac{t}{r}\right)^k N_1(t)\,dt\right\} - \frac{2k}{\pi} r N_1(r)\end{aligned}$$

for all $k>\beta$.

Let $\limsup\limits_{r\to\infty} N_1(r)/r^{\beta-\varepsilon}=\infty$ for all $\varepsilon>0$. Applying Lemma 1 to the functions $\psi(r)=N_1(r)$, $\psi_1(r)=r^{\beta-\varepsilon}$, $\psi_2(r)=r^{\beta+\varepsilon}$, we obtain the sequence $\{r_n\}$, $r_n\to\infty$ $(n\to\infty)$ such that

$$N_1(t)\leqslant\left(\frac{t}{r_n}\right)^{\beta-\varepsilon}, r_0\leqslant t\leqslant r_n; \quad N_1(t)\leqslant\left(\frac{t}{r_n}\right)^{\beta+\varepsilon}, r_n\leqslant t<\infty\,. \qquad (9)$$

Using (9), we obtain

$$\begin{aligned}|c_k(r_n,v)| &\leqslant \frac{2k}{\pi}N(r_n)\left\{\frac{k^2+\beta-\varepsilon k}{(k-\varepsilon)^2-\beta^2}-1\right\}\leqslant \\ &\leqslant \frac{Ak}{\pi}\gamma(r_n)\left\{\frac{k^2+\beta-\varepsilon k}{(k-\varepsilon)^2-\beta^2}-1\right\}, \quad k>\beta\end{aligned}$$

from из (8). The latter inequality entails that

$$|c_k| = \lim_{r\to\infty}\frac{|c_k(r,v)|}{\gamma(r)} = \lim_{n\to\infty}\frac{|c_k(r_n,v)|}{\gamma(r_n)} \leqslant \frac{Ak}{\pi}\left\{\frac{k^2+\beta-\varepsilon k}{(k-\varepsilon)^2-\beta^2}-1\right\}$$

when $k>\beta$. Since $\varepsilon>0$ is an arbitrary number,

$$|c_k|\leqslant\frac{Ak}{\pi}\left\{\frac{\beta^2+\beta}{k^2-\beta^2}\right\}, \quad k>\beta\,.$$

Theorem 4 is proved completely.

Theorem 5. *Let $v\in J\delta(\gamma(r))^o$. Then, there exists the finite limit*

$$\lim_{r\to\infty}\frac{1}{\gamma(r)}\int\limits_0^\pi v(re^{i\theta})\sin k\theta\,d\theta = \int\limits_0^\pi h(\theta,v)\sin k\theta\,d\theta$$

for any $k\in\mathbb{N}$.

This equality is obtained by means of expanding the integrand in the right-hand side into the Fourier series, its integration term by term, and passing to the limit in the left-hand side of the equality.

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