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ESTIMATES OF SOLUTIONS OF AN ANISOTROPIC DOUBLY NONLINEAR PARABOLIC EQUATION

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Abstract. The first mixed problem with the Dirihlet homogeneous boundary-value condition and a finite initial function is considered for a certain class of second-order anisotropic doubly nonlinear parabolic equations in a cylindrical domain $D = (0, \infty) \times \Omega$. Upper estimates characterizing the dependence of the decay rate of the solution to the problem on geometry of an unbounded domain $\Omega \subset \mathbb{R}_n$, $n \geq 3$ are established when $t \to \infty$. Existence of strong solutions is proved by the method of Galerkin's approximations. The method of their construction for the modelling isotropic equation has been earlier offered by F.Kh. Mukminov, E.R. Andriyanova. The estimate of the admissible decay rate of the solution on an unbounded domain has been obtained on the basis of Galerkin's approximations. It proves the accuracy of the upper estimate.

Keywords: anisotropic equation, doubly nonlinear parabolic equations, existence of strong solution, decay rate of solution.

1. Introduction

Let Ω be an unbounded domain of the space $\mathbb{R}_n = \{\mathbf{x} = (x_1, x_2, ..., x_n)\}$, $n \geq 3$. The first mixed problem

$$(|u|^{k-2}u)_t = \sum_{\alpha=1}^{n} (a_\alpha(u_{x_\alpha}^2) u_{x_\alpha})_{x_\alpha}, \quad k \geq 2, \quad (t, \mathbf{x}) \in D; \tag{1}$$

$$u(t, \mathbf{x})\Big|_S = 0, \quad S = \{t > 0\} \times \partial\Omega; \tag{2}$$

$$u(0, \mathbf{x}) = \varphi(\mathbf{x}), \quad \varphi(\mathbf{x}) \in L_k(\Omega), \quad \varphi_{x_\alpha}(\mathbf{x}) \in L_{p_\alpha}(\Omega), \quad \alpha = \overline{1, n} \tag{3}$$

is considered for an anisotropic quasilinear parabolic second-order equation in a cylindric domain $D = \{t > 0\} \times \Omega$. It is assumed that nonnegative functions $a_\alpha(s)$, $s \geq 0$, $\alpha = \overline{1, n}$ obey the conditions: $a_\alpha(0) = 0$, $a_\alpha(s) \in C^1(0, \infty)$,

$$\overline{a} s^{(p_\alpha - 2)/2} \leqslant a_\alpha(s) \leqslant \widehat{a} s^{(p_\alpha - 2)/2}, \tag{4}$$

$$\frac{p_1}{2} a_\alpha(s) \leqslant a_\alpha(s) + a'_\alpha(s)s \leqslant \widehat{b} a_\alpha(s), \tag{5}$$

with the positive constants $\widehat{a} \geq \overline{a}$, $2\widehat{b} \geq p_1 > k$ $(p_1 \leqslant p_2 \leqslant \ldots \leqslant p_n)$. For example, $a_\alpha(s) = s^{(p_\alpha - 2)/2}$, $\alpha = \overline{1, n}$, $\widehat{b} = p_n$.

The present paper is devoted to investigation of the stabilization rate of solution to the problem (1)–(3) with a finite initial function $\varphi(\mathbf{x})$ when $t \to \infty$.

Investigation of the decay rate for large time values of solutions to mixed problems for parabolic equations in unbounded domains with the initial function limited in one of L_p-norms was initiated by the works [1], [2]. In a wide class of unbounded domains, A.K. Gushchin obtained exact estimates of solutions to the second mixed problem for a linear second-order

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parabolic equation in a divergent form in terms of a simple geometric characteristics $v(r) = \operatorname{mes}\Omega(r)$, $\Omega(r) = \{\mathbf{x} \in \Omega \mid |\mathbf{x}| < r\}$.

Works of V.I. Ushakov [3], [4], F.Kh. Mukminov [5], [6], A.F. Tedeev [7] – [9], I.M. Bikkulov, F.Kh. Mukminov [10], L.M. Kojevnikova, F.Kh. Mukminov [11], [12], L.M. Kojevnikova [13], [14], L.M. Kojevnikova, R.Kh. Karimova [15] and others were devoted to investigation of behaviour of solutions to mixed problems for linear and quasi-linear parabolic equations of the second and higher orders when $t \to \infty$. Surveys of the corresponding results are available in [11], [13], [15].

In the isotropic case, i.e. when all p_α are equal to each other and are qual to p, $p \geq 2$, the problem (1)–(3) when $k = 2$ was investigated in [15]. The anisotropic case for mixed problems is little investigated. Decay rate estimates for solutions to the cauchy problem for a degenerate parabolic equation with an anisotropic p-Laplacian and a double nonlinearity are obtained by S.P. Degtyarev, A.F. Tedeev in [16].

For the sake of simplicity we limit our consideration by domains located along the distinguished axis Ox_s, $s \in \overline{2, n-1}$ (the domain Ω lies in the half-space $\mathbb{R}_n^+[s] = \{\mathbf{x} \in \mathbb{R}_n \mid x_s > 0\}$, the cut $\gamma_r = \{\mathbf{x} \in \Omega \mid x_s = r\}$ is not empty and bounded for any $r > 0$). In what follows, we use the notation: $\Omega_a^b = \{\mathbf{x} \in \Omega \mid a < x_s < b\}$, while the values $a = 0$, $b = \infty$ are omitted.

To study the decay of solution to the problem (1) – (3) when $x_s \to \infty$, we will use a geometric characteristics, which is t be defined as follows. Let us assume that

$$\nu_\alpha(r) = \inf\left\{ \|g_{x_\alpha}\|_{L_{p_\alpha}(\gamma_r)} \;\middle|\; g(\mathbf{x}) \in C_0^\infty(\Omega),\ \|g\|_{L_{p_\alpha}(\gamma_r)} = 1 \right\}, \quad r > 0, \tag{6}$$

$\nu(r) = \min\{\nu_1(r), \nu_n(r)\}$. We consider that the domain Ω satisfies the condition

$$\int\limits_1^\infty \nu(r)dr = \infty. \tag{7}$$

It is assumed that the initial function has a bounded support so that

$$\operatorname{supp}\varphi \subset \Omega^{R_0}, \quad R_0 > 0. \tag{8}$$

Theorem 1. *Let $k \geq 2$ and the conditions* (7), (8) *are satisfied. Then, there are positive numbers $\kappa(p_s, k)$, $\mathcal{M}(p_s, k)$ such that the generalized solution $u(t, \mathbf{x})$ to the problem* (1)–(3) *for all $t \geq 0$, $r \geq 2R_0$ satisfies the estimate*

$$\|u(t)\|_{L_k(\Omega_r)} \leqslant \mathcal{M} \exp\left(-\kappa \int\limits_1^r \nu(\rho)d\rho \right) \|\varphi\|_{L_k(\Omega)}. \tag{9}$$

Results on decay of solution to the problem (1)–(3) for $t \to \infty$ are obtained on the basis of the estimate (9).

The admissible stabilization rate of solution to an isotropic quasi-linear parabolic equation of a higher order for $k = 2$ was investigated by A.F. Tedeev [17] for the first mixed problem and by N. Alikakos, R. Rostmanian [18] for the Cauchy problem. The lower estimate for solution to the problem (1)–(3) is obtained in the following theorem.

Theorem 2. *Let $2 \leqslant k < p_1$ and the conditions* (7), (8) *are satisfied. Then, there is a positive number $C(\varphi, k, p_1, \widehat{a}, \widehat{b})$ such that the generalized solution $u(t, \mathbf{x})$ to the problem* (1)–(3) *for all $t \geq 0$ obeys the estimate*

$$\|u(t)\|_{L_k(\Omega)} \geq \|\varphi\|_{L_k(\Omega)} \left(C(\varphi)t + 1\right)^{-1/(p_1 - k)}. \tag{10}$$

Let us define the function

$$\mu_1(r) = \inf\left\{ \|g_{x_1}\|_{L_{p_1}(\Omega^r)} \;\middle|\; g(\mathbf{x}) \in C_0^\infty(\Omega), \|g\|_{L_k(\Omega^r)} = 1 \right\}, \quad r > 0. \tag{11}$$

We will investigate decay in the domains where the following condition is satisfied:
$$\lim_{r\to\infty}\mu_1(r)=0. \tag{12}$$
It is demonstrated that if the condition is not met, the maximum decay rate of solution is reached, i.e. the estimate
$$\|u(t)\|_{L_k(\Omega)}\leqslant Mt^{-1/(p_1-k)},\quad t>0 \tag{13}$$
holds (see Corollary 2).

Let $r(t)$ be an arbitrary positive function, satisfying the inequality
$$(\mu_1^{p_1}(r(t))t)^{-1/(p_1-k)}\exp\left(\kappa\int\limits_1^{r(t)}\nu(\rho)d\rho\right)\geq 1,\quad t>0. \tag{14}$$
Existence of such function follows from (12).

Theorem 3. *Let $2\leqslant k<p_1$ and the conditions (7), (8), (12) are met. Then, there is a positive number $M(p_s,p_1,\|\varphi\|_{L_k(\Omega)})$ such that the estimate*
$$\|u(t)\|_{L_k(\Omega)}\leqslant M\left(t\mu_1^{p_1}(r(t))\right)^{-1/(p_1-k)},\quad t>0 \tag{15}$$
holds for solution $u(t,\mathbf{x})$ of the problem (1)–(3)

If the conditions
$$\int\limits_1^r\nu(\rho)d\rho\geq b\ln r, \tag{16}$$
$$\mu_1(r)\geq Cr^{-a} \tag{17}$$
are satisfied for $r>1$ with positive constants a,b,C, then, one can assume
$$r(t)=t^{1/(ap_1+\kappa b(p_1-k))},\quad t>0,$$
and the estimate (15) takes the form
$$\|u(t)\|_{L_k(\Omega)}\leqslant Mt^{-\kappa/\left(\frac{a}{b}p_1+\kappa(p_1-k)\right)},\quad t>0. \tag{18}$$
If the condition
$$\lim_{r\to\infty}\frac{1}{\ln r}\int\limits_1^r\nu(\rho)d\rho=\infty, \tag{19}$$
is satisfied instead of the inequality (16) then, one can choose
$$r(t)=t^{\varepsilon/(ap_1)},\quad t>0,\quad \varepsilon\in(0,1), \tag{20}$$
and the estimate (15) takes the form
$$\|u(t)\|_{L_k(\Omega)}\leqslant Mt^{-(1-\varepsilon)/(p_1-k)},\quad \varepsilon\in(0,1),\quad t>0. \tag{21}$$
The choice of the function $r(t)$ by the formula (20) is satisfactory, because the estimate (21) has the exponent close to the exponent $1/(p_1-k)$ of the lower estimate (10).

Let us consider the rotation domain
$$\Omega(f)[s]=\{\mathbf{x}\in\mathbb{R}_n\mid x_s>0,\ |\mathbf{x}'_s|<f(x_s)\},\quad s\in\overline{2,n-1}, \tag{22}$$
$\mathbf{x}'_s=(x_1,\dots,x_{s-1},x_{s+1},\dots,x_n)$, with a positive function $f(x_s)<\infty$. There is only one requirement for the function f, namely the set $\Omega(f)[s]$ should be a domain.

The following correlation holds for such domains:
$$\nu(r)=\frac{c}{f(r)},\quad r>0, \tag{23}$$

therefore, the condition (19) takes the form

$$\lim_{r\to\infty}\frac{1}{\ln r}\int\limits_1^r\frac{d\rho}{f(\rho)}=\infty. \tag{24}$$

Let us express the estimates (9), (15) via the function $f(x)$ for rotation domains of the form (22). Joining (9), (23), we obtain the following estimate

$$\|u(t)\|_{L_k(\Omega_r)}\leqslant\widetilde{\mathcal{M}}\exp\left(-\widetilde{\kappa}\int\limits_1^r\frac{d\rho}{f(\rho)}\right)\|\varphi\|_{L_k(\Omega)},\quad t\geq 0,\quad r\geq 2R_0. \tag{25}$$

The choice of the function $r(t)$ by the formula (20) is justified for rotation domains, satisfying the condition (24), and the estimate (21) holds. However, one can obtain finer estimates for such domains.

The following estimate is a corollary of Theorem 3 for rotation domains of the form (22) (see Statement 1):

$$\|u(t)\|_{L_k(\Omega(f))}\leqslant\widetilde{M}t^{-1/(p_1-k)}\widetilde{g}(t),\quad t\geq 1, \tag{26}$$

where the function $\widetilde{g}(t)$ grows slower than any power function $t^\gamma,\gamma>0$.

The estimate (26) takes the form (see Example 1 §5)

$$\begin{gathered}\|u(t)\|_{L_k(\Omega(f))}\leqslant\widetilde{M_a}t^{-1/(p_1-k)}(\ln t)^{\chi/(1-a)},\quad t\geq e,\\ \chi=a\frac{p_1}{p_1-k}+a\frac{n-1}{k}+\frac{1}{k}\end{gathered} \tag{27}$$

in the domain $\Omega(f_a)[s]$ with the function $f_a(x)=x^a,\ 0\leqslant a<1,\ x>0$ for solving the problem (1)–(3).

For solving the problem (1)–(3), the estimate (26) takes the following form (see Example 2 §5) in the domain $\Omega(f)[s]$ with the function $f(x)=e,\ 0<x<e,\ f(x)=x/\ln x,\ x\geq e:$

$$\begin{gathered}\|u(t)\|_{L_k(\Omega(f))}\leqslant\widetilde{M}t^{-1/(p_1-k)}(\ln t)^{-\sigma/2}\exp\left(\varrho(\ln t)^{1/2}\right),\quad t\geq e,\\ \sigma=\frac{p_1}{p_1-k}+\frac{n-1}{k},\quad \varrho>0.\end{gathered} \tag{28}$$

2. Auxiliary statements

Let us assume that $\|\cdot\|_{p,Q}$ is a norm in $L_p(Q),\ p>1,\ (\cdot,\cdot)_Q$ is a scalar product in $L_2(Q)$, and the values $p=2,\ Q=\Omega$ are omitted. Let us denote the cylinder by $D_a^b=(a,b)\times\Omega$, the values $a=0$ and $b=\infty$ can be absent.

Let us define the Banach space $\overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega)$ as a complement to the space $C_0^\infty(\Omega)$ by the norm

$$\|u\|_{W^1_{k,\mathbf{p}}(\Omega)}=\sum_{\alpha=1}^n\|u_{x_\alpha}\|_{p_\alpha}+\|u\|_k.$$

The Banach spaces $\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$, $\overset{\circ}{W}{}^{1,1}_{k,\mathbf{p}}(D^T)$ are to be defined as complements of the space $C_0^\infty(D_{-1}^{T+1})$ by the norms

$$\|u\|_{W^{0,1}_{k,\mathbf{p}}(D^T)}=\|u\|_{k,D^T}+\sum_{\alpha=1}^n\|u_{x_\alpha}\|_{p_\alpha,D^T},$$

$$\|u\|_{W^{1,1}_{k,\mathbf{p}}(D^T)}=\|u\|_{k,D^T}+\|u_t\|_{k,D^T}+\sum_{\alpha=1}^n\|u_{x_\alpha}\|_{p_\alpha,D^T},$$

respectively.

Definition 1. *A generalized solution to the problem* (1)–(3) *is a function* $u(t,\mathbf{x}) \in \overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$*, satisfying the integral identity*

$$\int\limits_{D^T} \left(-|u|^{k-2}uv_t + \sum_{\alpha=1}^{n} a_\alpha(u^2_{x_\alpha})u_{x_\alpha}v_{x_\alpha} \right) d\mathbf{x}dt = \tag{29}$$

$$= -\int\limits_{\Omega} |u(T,\mathbf{x})|^{k-2}u(T,\mathbf{x})v(T,\mathbf{x})d\mathbf{x} + \int\limits_{\Omega} |\varphi(\mathbf{x})|^{k-2}\varphi(\mathbf{x})v(0,\mathbf{x})d\mathbf{x}$$

for any function $v(t,\mathbf{x}) \in \overset{\circ}{W}{}^{1,1}_{k,\mathbf{p}}(D^T)$ *with every* $T>0$.

Definition of the generalized solution is correct, since integrals involved in (29) are finite. Indeed, in view of the Holder inequality, due to (4), we have

$$\sum_{\alpha=1}^{n}\int\limits_{D^T} |a_\alpha(u^2_{x_\alpha})||u_{x_\alpha}||v_{x_\alpha}|d\mathbf{x}dt \leqslant \widehat{a}\sum_{\alpha=1}^{n}\int\limits_{D^T} |u_{x_\alpha}|^{p_\alpha-1}|v_{x_\alpha}|d\mathbf{x}dt \leqslant$$

$$\leqslant \widehat{a}\sum_{\alpha=1}^{n} \|u_{x_\alpha}\|^{p_\alpha-1}_{p_\alpha,D^T}\|v_{x_\alpha}\|_{p_\alpha,D^T},$$

$$\int\limits_{D^T} |u|^{k-1}|v_t|d\mathbf{x}dt \leqslant \|u\|^{k-1}_{k,D^T}\|v_t\|_{k,D^T}$$

for the functions $u(t,\mathbf{x}) \in \overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$, $v(t,\mathbf{x}) \in \overset{\circ}{W}{}^{1,1}_{k,\mathbf{p}}(D^T)$.

The conditions (5) entail the inequalities

$$(p_1-1)a_\alpha(s) \leqslant a_\alpha(s) + 2a'_\alpha(s)s \leqslant \widehat{c}a_\alpha(s), \quad \widehat{c} = 2\widehat{b}-1, \quad s \geq 0, \quad \alpha = \overline{1,n}, \tag{30}$$

which can be written in the form

$$0 \leqslant (a_\alpha(z^2)z)' \leqslant \widehat{c}a_\alpha(z^2), \quad z \in \mathbb{R}, \quad \alpha = \overline{1,n}. \tag{31}$$

Let us assume that $A_\alpha(s) = \int\limits_0^s a_\alpha(\tau)d\tau$ then, using the conditions (5), we deduce the inequalities

$$\frac{p_1}{2}A_\alpha(s) \leqslant a_\alpha(s)s \leqslant \widehat{b}A_\alpha(s), \quad s \geq 0, \quad \alpha = \overline{1,n}. \tag{32}$$

Lemma 1. *Any bounded set of a reflective Banach space is weakly compact (see* [19, Ch.V, §19.7, Theorem 1]*).*

Remark 1. *The spaces* $\overset{\circ}{W}{}^{1}_{k,\mathbf{p}}(\Omega)$, $\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$ *are reflexive separable Banach spaces. Indeed, the space* $\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$ *is a closure of the image of the mapping* $I: v(t,\mathbf{x}) \in C_0^\infty(D^{T+1}_{-1}) \mapsto (v, v_{x_1}, v_{x_2}, ..., v_{x_n}) \in$ $\in L_k(D^T)\oplus L_{p_1}(D^T)\oplus ...\oplus L_{p_n}(D^T)$. *Since the spaces* $L_k(D^T), L_{p_1}(D^T), ..., L_{p_n}(D^T)$ *are reflexive, the subspace* $\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$ *of the reflexive space* $L_k(D^T)\oplus L_{p_1}(D^T)\oplus ...L_{p_n}(D^T)$ *is also reflexive.*

Remark 2. *In what follows, to avoid cumbersome reasoning, instead of such statement as "one can single out a subsequence* u^{M_i}*, converging in* $L_2(\Omega)$ *when* $i\to\infty$ *from the sequence* u^M*", we will say only "the sequence* u^M *convergence selectively in* $L_2(\Omega)$ *when* $M\to\infty$*". Correspondingly, we use the term "weakly selectively converges" etc.*

Lemma 2. *Let us assume that $g^M(t,\mathbf{x})$, $M=\overline{1,\infty}$, $g(t,\mathbf{x})$ are functions from $L_p((0,T)\times Q)$, $1<p<\infty$ such that*

$$\|g^M\|_{p,(0,T)\times Q}\leqslant C,\quad g^M\to g \text{ for } M\to\infty \text{ almost everywhere in } (0,T)\times Q.$$

Then, $g^M\rightharpoonup g$ for $M\to\infty$ weakly in $L_p((0,T)\times Q)$ (see [20, Ch. I, §1.4, Lemma 1.3]*).*

Lemma 3. *Let the system of functions $\psi_i(\mathbf{x})\in C_0^\infty(\Omega)$, $i=\overline{1,\infty}$ be linearly independent and its linear envelope be a dense set everywhere in the space $\overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega)$. Denote by P_L a set of functions $\sum\limits_{i=1}^{L} d_i(t)\psi_i(\mathbf{x})$, where $d_i(t)\in C^\infty[0,T]$. Then, the set $P=\bigcup\limits_{L=1}^{\infty} P_L$ is dense in the space $\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$ (see* [21, Ch. II, §4, Lemma 4.12]*).*

Lemma 4. *Let the sequence $\{u^M(t,\mathbf{x})\}_{M=1}^\infty$ be bounded in the space $\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$, $k\leqslant p_1$. Then, there is a countable dense set $\{t_j\}_{j=1}^\infty\subset[0,T]$ such that $\{u^M(t_j,\mathbf{x})\}_{M=1}^\infty$ selectively weakly converges in the space $L_k(Q)$ for any bounded domain $Q\subset\Omega$ with a smooth boundary when $M\to\infty$ for every fixed t_j, $j=\overline{1,\infty}$.*

Let us carry out the proof by the scheme, suggested by J.-L. Lions [20, Ch. I, §12.2, Theorem 12.1]. The lemma condition provides the inequalities

$$\|u^M\|^k_{k,D^T}+\sum_{\alpha=1}^{n}\|u^M_{x_\alpha}\|^{p_\alpha}_{p_\alpha,D^T}\leqslant C,\quad M=\overline{1,\infty}. \tag{33}$$

Let us consider the set E of points $t\in[0,T]$ such that

$$\lim_{M\to\infty}\left(\|u^M(t)\|_k^k+\sum_{\alpha=1}^{n}\|u^M_{x_\alpha}(t)\|^{p_\alpha}_{p_\alpha}\right)=\infty.$$

The measure E equals 0, because otherwise

$$\lim_{M\to\infty}\int_0^T\left(\|u^M(t)\|_k^k+\sum_{\alpha=1}^{n}\|u^M_{x_\alpha}(t)\|^{p_\alpha}_{p_\alpha}\right)dt\geq$$

$$\geq\lim_{M\to\infty}\int_E\left(\|u^M(t)\|_k^k+\sum_{\alpha=1}^{n}\|u^M_{x_\alpha}(t)\|^{p_\alpha}_{p_\alpha}\right)dt=\infty,$$

which contradicts the inequalities (33). Then, for almost every $t\in[0,T]$, the inequalities

$$\|u^M(t)\|_k^k+\sum_{\alpha=1}^{n}\|u^M_{x_\alpha}(t)\|^{p_\alpha}_{p_\alpha}\leqslant C_1(t),\quad M=\overline{1,\infty}$$

hold. Whence, in view of the condition $k\leqslant p_1$,

$$\|u^M(t)\|_{W^1_k(Q)}\leqslant C_2(t),\quad M=\overline{1,\infty}.$$

for any bounded domain $Q\subset\Omega$ with a smooth boundary. Since the injection $W^1_k(Q)\subset L_k(Q)$ is compact, it follows that for any $t\in[0,T]\setminus E$, the sequence $u^M(t,\mathbf{x})$ selectively strongly converges to $u(t,\mathbf{x})$ when $M\to\infty$ in $L_k(Q)$.

Let the sequence $\{t_j\}_{j=1}^\infty$ be dense in the interval $[0,T]$ and $t_j\notin E$. By means of a diagonal process, one can single out a subsequence u^{M_i} such that $u^{M_i}(t_j)\to u(t_j)$ converges strongly when $i\to\infty$ in $L_k(Q)$ for any j. $\square$

Questions of existence and uniqueness of solutions of the doubly nonlinear isotropic parabolic equation equation were considered in works by P.A. Raviart [22], J.L. Lions [20], A. Bamberger [23], O. Grange, F. Mignot [24], H.W. Alt, S. Luckhaus [25], F. Bernis [26] and others. Mainly, problem in limited areas were considered. Strong solution of the problem in a bounded domain has been established P.A. Raviart by substituting the evolutionary derivative by a difference ratio. A. Bamberger established the uniqueness of a strong positive solution of the problem. F. Bernis proved the existence of a weak solution to the problem in an unbounded domain passing to the limit of solutions built in bounded domains by O. Grange, F. Mignot. A weak solution of the Cauchy problem for the anisotropic anisotropic equation with $k = 2$ was constructed by M. Bendahmane, K.H. Karlsen [27]. However, in order to obtain the lower estimate for decay rate of the solution for $t \to \infty$ its auxiliary smoothness is necessary.

F.H. Mukminov, E.R. Andriyanova[28] suggested an ad hoc method for constructing a strong solution for a model isotropic parabolic doubly nonlinear equation in an unbounded domain at once, based on Galerkin's approximations. Here the method is adapted to a certain class of anisotropic parabolic equations of the form (1).

Theorem 4. *Let $\varphi(\mathbf{x}) \in \overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega)$, $p_1 > 1$, $k \geq 2$ then, there is a generalized solution $u(t, \mathbf{x})$ to the problem* (1)–(3) *for any $T > 0$, satisfying the conditions*

$$u \in L_\infty((0, T), \overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega)); \tag{34}$$

$$|u|^{(k-2)/2} u_t \in L_2(D^T), \quad \|u(t)\|_k \in C([0, T]); \tag{35}$$

$$|u|^{k-2} u_t \in L_{k'}(D^T), \quad k' = \frac{k}{k-1}. \tag{36}$$

Here the inequalities

$$(k-1)\|u(t)\|_k^k + k\overline{a} \sum_{\alpha=1}^n \int_0^t \|u_{x_\alpha}(\tau)\|_{p_\alpha}^{p_\alpha} d\tau \leqslant (k-1)\|\varphi\|_k^k, \quad t \geq 0; \tag{37}$$

$$(k-1)\frac{d}{dt}\|u(t)\|_k^k + k\overline{a} \sum_{\alpha=1}^n \|u_{x_\alpha}(t)\|_{p_\alpha}^{p_\alpha} \leqslant 0, \quad t > 0 \tag{38}$$

hold.

Proof. Let us choose a linearly independent system of functions $\psi_i(\mathbf{x}) \in C_0^\infty(\Omega)$, $i = \overline{1, \infty}$ such that its linear envelope is a set dense everywhere in the space $\overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega)$. Let us consider the system to be orthonormal in $L_2(\Omega)$.

Let us fix an arbitrary $T > 0$. Approximate solutions $u^M(t, \mathbf{x})$ will be sought for in the form $u^M(t, \mathbf{x}) = \sum\limits_{i=1}^M c_i^M(t)\psi_i(\mathbf{x})$, $M = \overline{1, \infty}$. Meanwhile, the functions $c_i^M(t)$, $t \in [0, T]$ are defined from the system of ordinary differential equations

$$\left(\left(\frac{u^M}{b^M} + |u^M|^{k-2}u^M\right)_t, \psi_j\right) + \sum_{\alpha=1}^n \left(a_\alpha((u_{x_\alpha}^M)^2)u_{x_\alpha}^M, (\psi_j)_{x_\alpha}\right) = 0, \quad j = \overline{1, M}, \tag{39}$$

(the numbers $b^M > 0$ will be chosen later) and the initial conditions

$$c_i^M(0) = c_i^M, \quad i = \overline{1, M}, \tag{40}$$

chosen so that

$$u^M(0, \mathbf{x}) = \sum_{i=1}^M c_i^M \psi_i(\mathbf{x}) \to \varphi(\mathbf{x}) \text{ in } \overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega) \text{ for } M \to \infty. \tag{41}$$

Whence,

$$\|u^M(0)\|_{W^1_{k,\mathbf{p}}(\Omega)} \leqslant E_1(\|\varphi\|_{W^1_{k,\mathbf{p}}(\Omega)}), \quad M = \overline{1,\infty}. \tag{42}$$

Let us make sure that Equations (39) are solvable with respect to the variables $\dfrac{d}{dt}c_i^M(t)$. Manifestly, Equations (39) have the form

$$\sum_{i=1}^{M} A_{ji}(c_1^M(t),...,c_M^M(t))\frac{d}{dt}c_i^M(t) = F_j(c_1^M(t),...,c_M^M(t)), \quad j = \overline{1,M}, \tag{43}$$

$$A_{ji}(c_1,\dots,c_M) = \left(\left(\frac{1}{b^M} + (k-1)\left|\sum_{l=1}^{M} c_l\psi_l\right|^{k-2}\right)\psi_i, \psi_j\right) = (\psi_i,\psi_j)_M, \ i,j = \overline{1,M},$$

$$F_j(c_1,\dots,c_M) =$$

$$= -\sum_{\alpha=1}^{n}\sum_{i=1}^{M} c_i\left(a_\alpha\left(\left(\sum_{l=1}^{M} c_l^M(\psi_l)_{x_\alpha}\right)^2\right)(\psi_i)_{x_\alpha},(\psi_j)_{x_\alpha}\right), \quad j = \overline{1,M}.$$

One can readily verify that $(g,h)_M, \ g,h \in C_0^\infty(\Omega)$ is a scalar product. Hence, the matrix of coefficients $A_{ji}(c_1^M(t),...,c_M^M(t))$ for every t is the Gramian matrix of a system of linearly independent vectors $\psi_i, \ i = \overline{1,M}$, and has an inverse one. Therefore, the system (39) can be written in the form

$$\frac{d}{dt}c_i^M(t) = \sum_{j=1}^{M} A_{ij}^{-1}(c_1^M(t),\dots,c_M^M(t))F_j(c_1^M(t),\dots,c_M^M(t)), \quad i = \overline{1,M}. \tag{44}$$

Let us derive estimates for Galerkin's approximations. Let us multiply the j-th equation (39) by $c_j^M(t)$, and then add all equations in j from 1 to M. The resulting equalities

$$\left(\left(\frac{u^M}{b^M} + |u^M|^{k-2}u^M\right)_t, u^M\right) + \sum_{\alpha=1}^{n}\left(a_\alpha((u_{x_\alpha}^M)^2)u_{x_\alpha}^M, u_{x_\alpha}^M\right) = 0, \quad M = \overline{1,\infty}$$

can be written in the form

$$\frac{d}{dt}\left(\frac{k-1}{k}\|u^M(t)\|_k^k + \frac{1}{2b^M}\|u^M(t)\|^2\right) + \sum_{\alpha=1}^{n}\left(a_\alpha((u_{x_\alpha}^M)^2)u_{x_\alpha}^M, u_{x_\alpha}^M\right) = 0, \quad M = \overline{1,\infty}. \tag{45}$$

Integration from 0 to $t \in [0,T]$ yields

$$\frac{1}{2b^M}\|u^M(t)\|^2 + \frac{k-1}{k}\|u^M(t)\|_k^k + \sum_{\alpha=1}^{n}\left(a_\alpha((u_{x_\alpha}^M)^2)u_{x_\alpha}^M, u_{x_\alpha}^M\right)_{D^t} =$$

$$= \frac{1}{2b^M}\|u^M(0)\|^2 + \frac{k-1}{k}\|u^M(0)\|_k^k, \quad M = \overline{1,\infty}. \tag{46}$$

Invoking (4), assuming that $b^M = M\|u^M(0)\|^2$, and joining (46) with (42), we deduce the inequalities

$$\frac{1}{b^M}\max_{[0,T]}\|u^M(t)\|^2 + \max_{[0,T]}\|u^M(t)\|_k^k + \sum_{\alpha=1}^{n}\|u_{x_\alpha}^M\|_{p_\alpha,D^T}^{p_\alpha}d\tau \leqslant E_2, \quad M = \overline{1,\infty}. \tag{47}$$

Here and in what follows, the constants E_i depend only on $\widehat{a}, \overline{a}, \widehat{b}, \mathbf{p}, \|\varphi\|_{W^1_{k,\mathbf{p}}(\Omega)}$.

The inequalities (4), (47) allow us to establish the estimates

$$\sum_{\alpha=1}^{n}\|a_\alpha((u_{x_\alpha}^M)^2)u_{x_\alpha}^M\|_{p_\alpha/(p_\alpha-1),D^T} \le \widehat{a}\sum_{\alpha=1}^{n}\|u_{x_\alpha}^M\|_{p_\alpha,D^T}^{p_\alpha-1} \leqslant E_3, \quad M = \overline{1,\infty}. \tag{48}$$

Let us demonstrate that all possible solutions to the problem (40), (44) are uniformly bounded on $[0,T]$. Indeed, using (47), we deduce

$$\max_{[0,T]}|c_i^M(t)|^2 \leq \sum_{j=1}^{M}\max_{[0,T]}|c_j^M(t)|^2 = \max_{[0,T]}\|u^M(t)\|^2 \leq E_2 b^M, \quad i=\overline{1,M}.$$

In view of continuity of the right-hand side of Equations (44), there are absolutely continuous functions $c_i^M(t),\ t\in[0,T],\ i=\overline{1,M}$, that satisfy the system (44) almost everywhere as well as the initial condition (40) (see [29, p. 120]).

Let us multiply the j-the equation (39) by $\dfrac{d}{dt}c_j^M(t)$ and then add all equations with respect to j from 1 to M. The resulting equalities

$$\left(\left(\frac{u^M}{b^M}+|u^M|^{k-2}u^M\right)_t, u_t^M\right)+\sum_{\alpha=1}^{n}\left(a_\alpha((u_{x_\alpha}^M)^2)u_{x_\alpha}^M, u_{tx_\alpha}^M\right)=0, \quad M=\overline{1,\infty},$$

can be written in the form

$$\frac{1}{b^M}\|u_t^M\|^2+(k-1)\||u^M|^{(k-2)/2}u_t^M\|^2+\frac{1}{2}\frac{d}{dt}\sum_{\alpha=1}^{n}\int_{\Omega}A_\alpha((u_{x_\alpha}^M(t))^2)d\mathbf{x}=0, \quad M=\overline{1,\infty}. \tag{49}$$

Upon integrating from 0 to $t\in[0,T]$, using (32), we have

$$\frac{1}{b^M}\|u_t^M\|^2_{D^t}+(k-1)\||u^M|^{(k-2)/2}u_t^M\|^2_{D^t}+\frac{1}{2\widehat{b}}\sum_{\alpha=1}^{n}\left(a_\alpha((u_{x_\alpha}^M(t))^2)u_{x_\alpha}^M(t), u_{x_\alpha}^M(t)\right)\leqslant$$

$$\leqslant\frac{1}{p_1}\sum_{\alpha=1}^{n}\left(a_\alpha((u_{x_\alpha}^M(0))^2)u_{x_\alpha}^M(0), u_{x_\alpha}^M(0)\right), \quad M=\overline{1,\infty}.$$

Applying (4) and using (42), we obtain

$$\||u^M|^{(k-2)/2}u_t^M\|^2_{D^T}+\max_{[0,T]}\sum_{\alpha=1}^{n}\|u_{x_\alpha}^M(t)\|_{p_\alpha}^{p_\alpha}\leqslant E_3, \quad M=\overline{1,\infty}. \tag{50}$$

The inequalities (47), (50) entail the boundedness of the sequence $\{u^M\}_{M=1}^\infty$ in spaces $C([0,T],\overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega))$, $\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$ and $\{|u^M|^{(k-2)/2}u_t^M\}_{M=1}^\infty$ in $L_2(D^T)$. Moreover, it follows from the inequalities (48), that the sequences $a_\alpha((u_{x_\alpha}^M)^2)u_{x_\alpha}^M$ are bounded in spaces $L_{p_\alpha/(p_\alpha-1)}(D^T)$, $\alpha=\overline{1,n}$. The established facts ensure a selective weak convergence of the above sequences when $M\to\infty$ in the corresponding spaces:

$$u^M\rightharpoonup u \quad \text{in} \quad \overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T),$$

$$a_\alpha((u_{x_\alpha}^M)^2)u_{x_\alpha}^M\rightharpoonup b_\alpha \quad \text{in} \quad L_{p_\alpha/(p_\alpha-1)}(D^T), \quad \alpha=\overline{1,n},$$

$$v_t^M=\left(|u^M|^{(k-2)/2}u^M\right)_t=\frac{k}{2}|u^M|^{(k-2)/2}u_t^M\rightharpoonup g \quad \text{in} \quad L_2(D^T).$$

In what follows we prove that u^M selectively converges to u almost everywhere in D^T. This allows us to establish that $g=v_t=(|u|^{(k-2)/2}u)_t$.

The sequence $u^M\in C([0,T],\overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega))$, $M=\overline{1,\infty}$ is bounded in this space. For every bounded domain $Q\subset\Omega$ with a smooth boundary, we have the compactness of the injection $L_1(Q)\subset W_1^1(Q)$. Therefore, one can establish a selective strong convergence $u^M(t_j,\mathbf{x})\to h(t_j,\mathbf{x})$ in $L_1(Q)$ on a countable dense set $\{t_j\}_{j=1}^\infty\subset[0,T]$ by means of the diagonal process. One can also assume that $u^M(t_j,\mathbf{x})\to h(t_j,\mathbf{x})$ selectively almost everywhere in Q for every $t_j,\ j=\overline{1,\infty}$. Likewise, when $k\leq p_1$ one can also assume that the sequence $u^M(t_j,\mathbf{x})\to h(t_j,\mathbf{x}))$ strongly in $L_k(Q)$ for every $t_j,\ j=\overline{1,\infty}$.

Following J.L. Lions [20, Ch. I, §12.2], let us prove the selective strong convergence of the sequence v^M in the space $C([0,T],L_1(Q))$. First, we establish the equipower continuity with respect to t for the sequence $v^M=|u^M|^{(k-2)/2}u^M$ in $L_2(\Omega)$:

$$\|v^M(t_2)-v^M(t_1)\|=\left\|\int\limits_{t_1}^{t_2} v_t^M(t)dt\right\|\leqslant\int\limits_{t_1}^{t_2}\|v_t^M(t)\|dt\leqslant$$

$$\leqslant|t_2-t_1|^{1/2}\left(\int\limits_{t_1}^{t_2}\|v_t^M(t)\|^2dt\right)^{1/2}\leqslant E_4|t_2-t_1|^{1/2},\quad t_1,t_2\in[0,T],\quad M=\overline{1,\infty}. \tag{51}$$

On the basis of the inequalities (47), we make the conclusion that the sequence $v^M(t,\mathbf{x})$ is uniformly bounded with respect to $t\in[0,T]$ in $L_2(\Omega)$:

$$\|v^M(t)\|=\|u^M(t)\|_k^{k/2}\leqslant E_5,\quad M=\overline{1,\infty}.$$

Since the sequence $v^M(t,\mathbf{x})$, $M=\overline{1,\infty}$ is bounded, it selectively weakly converges in $L_2(\Omega)$ for the same t_j as above in the space $C([0,T],L_2(\Omega))$. From the above selective convergence $u^M(t_j,\mathbf{x})\to h(t_j,\mathbf{x})$ almost everywhere in Q for every t_j follows the selective convergence $v^M(t_j,\mathbf{x})\to v(t_j,\mathbf{x})=|h(t_j,\mathbf{x})|^{(k-2)/2}h(t_j,\mathbf{x})$ almost everywhere in Q. Then, on the basis of the Egorov theorem for any $\delta>0$ we establish the uniform convergence $v^M(t_j,\mathbf{x})\rightrightarrows v(t_j,\mathbf{x})$ on Q_δ, $\operatorname{mes}(Q\backslash Q_\delta)<\delta$. Whence, due to validity of the inequalities

$$\|v^M(t_j)-v(t_j)\|_{1,Q}\leqslant\operatorname{mes}Q\max_{\mathbf{x}\in Q_\delta}|v^M(t_j,\mathbf{x})-v(t_j,\mathbf{x})|+\|v^M(t_j)-v(t_j)\|_{1,Q\backslash Q_\delta}\leqslant$$

$$\leqslant\operatorname{mes}Q\max_{\mathbf{x}\in Q_\delta}|v^M(t_j,\mathbf{x})-v(t_j,\mathbf{x})|+\delta^{1/2}\|v^M(t_j)-v(t_j)\|_{2,Q\backslash Q_\delta},$$

follows the strong convergence $v^M(t_j,\mathbf{x})\to v(t_j,\mathbf{x})$ in $L_1(Q)$ for every t_j.

For a bounded domain Q from (51) one can readily establish a uniform mutual convergence of the sequence $v^M(t,\mathbf{x})$ with respect to the norm $L_1(Q)$:

$$\|v^N(t)-v^M(t)\|_{1,Q}=\|v^N(t)-v^N(t_{j_l})+v^N(t_{j_l})-v^M(t_{j_l})+v^M(t_{j_l})-v^M(t)\|_{1,Q}\leqslant$$

$$\leqslant(\operatorname{mes}Q)^{1/2}E_6|t-t_{j_l}|^{1/2}+\|v^N(t_{j_l})-v^M(t_{j_l})\|_{1,Q}.$$

Choosing the finite set of numbers t_{j_l} with a small step and then increasing N,M, we achieve a uniform smallness of the right-hand side with respect to t.

Thus, the selective strong convergence $v^M\to v$ is established in $C([0,T],L_1(Q))$. Convergence will also occur in $L_1((0,T)\times Q)$ therefore, $v^M\to v$ converges selectively almost everywhere in $(0,T)\times Q$. Since Q is arbitrary, the sequence v^M selectively converges to v almost everywhere in D^T. Then the sequence $u^M(t,\mathbf{x})$ selectively converges to $h(t,\mathbf{x})$ almost everywhere in D^T as well. According to Lemma 2 $u^M(t,\mathbf{x})\rightharpoonup h(t,\mathbf{x})$ in $L_k(D^T)$ because the limit $h(t,\mathbf{x})=u(t,\mathbf{x})$ is unique almost everywhere in D^T. Thus, v^M converges selectively to $v=|u|^{(k-2)/2}u$ almost everywhere in D^T.

According to Lemma 2, $v^M\rightharpoonup v$ weakly in $L_2(D^T)$. Furthermore, $(v_t^M,w)_{D^T}=-(v^M,w_t)_{D^T}$ for any function $w\in C_0^\infty(D^T)$, passing to the limit when $M\to\infty$, we obtain

$$(g,w)_{D^T}=-(v,w_t)_{D^T}.$$

Whence, it follows that $g=v_t=(|u|^{(k-2)/2}u)_t$. Note that the membership $v,v_t\in L_2(D^T)$ entails $v\in C([0,T],L_2(\Omega))$.

Let us demonstrate that the sequence $(|u^M|^{k-2}u^M)_t = (k-1)|u^M|^{k-2}u^M_t$, $M = \overline{1,\infty}$, is limited in $L_{k'}(D^T)$. Indeed,

$$\||u^M|^{k-2}u^M_t\|_{k',D^T} = \left(\int\limits_{D^T} |u^M|^{k(k-2)/(2(k-1))}(|u^M|^{(k-2)/2}|u^M_t|)^{k/(k-1)}d\mathbf{x}dt\right)^{(k-1)/k} \leqslant$$

$$\leqslant \frac{2}{k}\|u^M\|^{(k-2)/2}_{k,D^T}\|v^M_t\|_{2,D^T}.$$

The boundedness $\||u^M|^{k-2}u^M_t\|_{k',D^T}$ entails that $(|u^M|^{k-2}u^M)_t \rightharpoonup b$ in $L_{k'}(D^T)$. It follows from Lemma 2 that $|u^M|^{k-2}u^M \rightharpoonup |u|^{k-2}u$ in $L_{k'}(D^T)$. Whence, $((|u^M|^{k-2}u^M)_t, w)_{D^T} = -(|u^M|^{k-2}u^M, w_t)_{D^T}$ for any function $w \in C_0^\infty(D^T)$, passing to the limit when $M \to \infty$, we obtain

$$(b, w)_{D^T} = -(|u|^{k-2}u, w_t)_{D^T}.$$

Hence, $b = (|u|^{k-2}u)_t$. Then, we can consider that $(|u^M|^{k-2}u^M)_t \rightharpoonup (|u|^{k-2}u)_t$ weakly in $L_{k'}(D^T)$.

Let us prove that the function $u(t,\mathbf{x})$ satisfies the integral identity (29). The identities

$$\left(\left(\frac{u^M}{b^M} + |u^M|^{k-2}u^M\right)_t, w\right)_{D^T} + \sum_{\alpha=1}^n \left(a_\alpha((u^M_{x_\alpha})^2)u^M_{x_\alpha}, w_{x_\alpha}\right)_{D^T} = 0, \quad M = \overline{1,\infty}, \tag{52}$$

valid for any function $w(\tau,\mathbf{x}) \in P = \bigcup_{L=1}^\infty P_L$ follow from (39).

Note that

$$\frac{1}{b^M}\left(u^M_t, w\right)_{D^T} = \frac{1}{b^M}\left\{-(u^M, w_t)_{D^T} + (u^M(T), w(T)) - (u^M(0), w(0))\right\} \to 0,$$

because u^M is bounded in $C([0,T], L_k(\Omega))$, and $b^M \to \infty$ when $M \to \infty$.

We can pass to the limit in (52) when $M \to \infty$, to obtain the identity

$$\left((|u|^{k-2}u)_t, w\right)_{D^T} + \sum_{\alpha=1}^n (b_\alpha, w_{x_\alpha})_{D^T} = 0, \tag{53}$$

which holds true for any function $w \in P$. Since P is dense in the space $\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$ (Lemma 3), the identity (53) holds for an arbitrary $w \in \overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$. Here we make use of the fact that $(|u|^{k-2}u)_t \in L_{k'}(D^T)$, $b_\alpha \in L_{p_\alpha/(p_\alpha-1)}(D^T)$, $\alpha = \overline{1,n}$. In particular, for $w = u$ we deduce

$$\begin{gathered}\sum_{\alpha=1}^n (b_\alpha, u_{x_\alpha})_{D^T} + ((|u|^{k-2}u)_t, u)_{D^T} = \\ = \frac{k-1}{k}\left(\|u(T)\|_k^k - \|u(0)\|_k^k\right) + \sum_{\alpha=1}^n (b_\alpha, u_{x_\alpha})_{D^T} = 0.\end{gathered} \tag{54}$$

Let us prove that for any function $v \in \overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$ the equality

$$\sum_{\alpha=1}^n (b_\alpha, v_{x_\alpha})_{D^T} = \sum_{\alpha=1}^n (a_\alpha((u_{x_\alpha})^2)u_{x_\alpha}, v_{x_\alpha})_{D^T} \tag{55}$$

holds. Let us subtract the equalities (52) from (46) when $t = T$ and obtain the following relation for $w \in P$:

$$-\left(\left(\frac{u^M}{b^M} + |u^M|^{k-2}u^M\right)_t, w\right)_{D^T} + \sum_{\alpha=1}^n \left(a_\alpha((u^M_{x_\alpha})^2)u^M_{x_\alpha}, (u^M - w)_{x_\alpha}\right)_{D^T} +$$

$$+\frac{k-1}{k}\|u^M(t)\|_k^k\Big|_{t=0}^{t=T} + \frac{1}{2b^M}\|u^M(t)\|^2\Big|_{t=0}^{t=T} = 0, \quad M = \overline{1,\infty}.$$

Using the condition of monotonous nondecreasing of functions $a_\alpha(z^2)z,\ z\in\mathbb{R},\ \alpha=\overline{1,n}$ (see (31)) from the latter relations, we deduce the inequalities

$$-\left(\left(\frac{u^M}{b^M}+|u^M|^{k-2}u^M\right)_t,w\right)_{D^T}+\sum_{\alpha=1}^n\big(a_\alpha((w_{x_\alpha})^2)w_{x_\alpha},(u^M-w)_{x_\alpha}\big)_{D^T}+$$
$$+\frac{k-1}{k}\|u^M(t)\|_k^k\Big|_{t=0}^{t=T}+\frac{1}{2b^M}\|u^M(t)\|^2\Big|_{t=0}^{t=T}\leqslant 0,\quad M=\overline{1,\infty}.$$

Let us pass to the limit in $M\to\infty$ for a fixed $w\in P$ using the convergence obtained above.

Thus, for an arbitrary $w\in P$ we have the inequality

$$-\big((|u|^{k-2}u)_t,w\big)_{D^T}+\sum_{\alpha=1}^n\big(a_\alpha((w_{x_\alpha})^2)w_{x_\alpha},(u-w)_{x_\alpha}\big)_{D^T}+$$
$$+\frac{k-1}{k}\|u(t)\|_k^k\Big|_{t=0}^{t=T}\leqslant 0. \tag{56}$$

According to Lemma 4, the set P is dense in the space $\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$. Then, for an arbitrary function $w\in\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$, there is a sequence $w^l\in P$ such that $\|w^l-w\|_{W^{0,1}_{k,\mathbf{p}}(D^T)}\to 0$ when $l\to\infty$. Let us write (56) for $w=w^l$, then pass to the limit when $l\to\infty$.

Let us justify passing to the limit when $l\to\infty$ in the integrals

$$\big(a_\alpha((w^l_{x_\alpha})^2)w^l_{x_\alpha},(u-w^l)_{x_\alpha}\big)_{D^T}\to\big(a_\alpha((w_{x_\alpha})^2)w_{x_\alpha},(u-w)_{x_\alpha}\big)_{D^T},\quad \alpha=\overline{1,n}. \tag{57}$$

For an arbitrary function $v\in\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$, there are $\theta_l\in[0,1]$ such that

$$|\big(a_\alpha((w^l_{x_\alpha})^2)w^l_{x_\alpha}-a_\alpha((w_{x_\alpha})^2)w_{x_\alpha},v_{x_\alpha}\big)_{D^T}|\leqslant$$
$$\leqslant\big(|a_\alpha((w^l_{x_\alpha})^2)w^l_{x_\alpha}-a_\alpha((w_{x_\alpha})^2)w_{x_\alpha}|,|v_{x_\alpha}|\big)_{D^T}\leqslant$$
$$\leqslant\int\limits_{D^T}|w^l_{x_\alpha}-w_{x_\alpha}||v_{x_\alpha}|(a_\alpha(z^2)z)'\big|_{z=(\theta^lw^l+(1-\theta^l)w)_{x_\alpha}}d\mathbf{x}dt.$$

Using the conditions (31), (4), we deduce

$$|\big(a_\alpha((w^l_{x_\alpha})^2)w^l_{x_\alpha}-a_\alpha((w_{x_\alpha})^2)w_{x_\alpha},v_{x_\alpha}\big)_{D^T}|\leqslant$$
$$\leqslant\widehat{c}\int\limits_{D^T}|w^l_{x_\alpha}-w_{x_\alpha}||v_{x_\alpha}|a_\alpha(z^2)\big|_{z=(\theta^lw^l+(1-\theta^l)w)_{x_\alpha}}d\mathbf{x}dt\leqslant$$
$$\leqslant\widehat{c}\widehat{a}\int\limits_{D^T}|w^l_{x_\alpha}-w_{x_\alpha}||v_{x_\alpha}|(|w^l_{x_\alpha}|+|w_{x_\alpha}|)^{p_\alpha-2}\leqslant$$
$$\leqslant\widehat{c}\widehat{a}\||w^l_{x_\alpha}|+|w_{x_\alpha}|\|^{p_\alpha-2}_{p_\alpha,D^T}\|w^l_{x_\alpha}-w_{x_\alpha}\|_{p_\alpha,D^T}\|v_{x_\alpha}\|_{p_\alpha,D^T}\to 0 \tag{58}$$

when $l\to\infty$ in particular, for $v=u$ and $v=w$. Moreover, using (5), we establish

$$|\big(a_\alpha((w^l_{x_\alpha})^2)w^l_{x_\alpha},w^l_{x_\alpha}-w_{x_\alpha}\big)_{D^T}|\leqslant\big(a_\alpha((w^l_{x_\alpha})^2)|w^l_{x_\alpha}|,|w^l_{x_\alpha}-w_{x_\alpha}|\big)_{D^T}\leqslant$$
$$\leqslant\widehat{a}\int\limits_{D^T}|w^l_{x_\alpha}|^{p_\alpha-1}|w^l_{x_\alpha}-w_{x_\alpha}|d\mathbf{x}dt\leqslant\widehat{a}\|w^l_{x_\alpha}-w_{x_\alpha}\|_{p_\alpha,D^T}\|w^l_{x_\alpha}\|^{p_\alpha-1}_{p_\alpha,D^T}\to 0 \tag{59}$$

when $l\to\infty$.

The inequalities

$$|\big(a_\alpha((w^l_{x_\alpha})^2)w^l_{x_\alpha},w^l_{x_\alpha}\big)_{D^T}-\big(a_\alpha((w_{x_\alpha})^2)w_{x_\alpha},w_{x_\alpha}\big)_{D^T}|\leqslant$$
$$\leqslant|\big(a_\alpha((w^l_{x_\alpha})^2)w^l_{x_\alpha},w^l_{x_\alpha}-w_{x_\alpha}\big)_{D^T}|+$$

$$+\left|\left(a_\alpha((w^l_{x_\alpha})^2)w^l_{x_\alpha}-a_\alpha((w_{x_\alpha})^2)w_{x_\alpha},w_{x_\alpha}\right)_{D^T}\right|,$$

from (58), (59) entail (57). Thus, the identity (56) is established for an arbitrary $w\in\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$.

Let us subtract (54) from (56) and add (53). This provides the inequality

$$\sum_{\alpha=1}^{n}(a_\alpha((w_{x_\alpha})^2)w_{x_\alpha}-b_\alpha,(u-w)_{x_\alpha})_{D^T}\leq 0, \tag{60}$$

which holds for every $w\in\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$. Let us assume that $w=u+\varepsilon v$, $\varepsilon>0$ in (60), where $v\in\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$. Then,

$$\sum_{\alpha=1}^{n}(a_\alpha((u_{x_\alpha}+\varepsilon v_{x_\alpha})^2)(u_{x_\alpha}+\varepsilon v_{x_\alpha})-b_\alpha,v_{x_\alpha})_{D^T}\geq 0.$$

When $\varepsilon\to 0$, the latter inequality yields the relation

$$\sum_{\alpha=1}^{n}(a_\alpha((u_{x_\alpha})^2)u_{x_\alpha}-b_\alpha,v_{x_\alpha})_{D^T}\geq 0,$$

which provides the equality (55) due to arbitrariness of v. On the basis of (53) and (55), we conclude that the identity

$$\left((|u|^{k-2}u)_t,v\right)_{D^T}+\sum_{\alpha=1}^{n}\left(a_\alpha((u_{x_\alpha})^2)u_{x_\alpha},v_{x_\alpha}\right)_{D^T}=0 \tag{61}$$

holds for $v\in\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$. Integrating the first addend by parts, we arrive to the equality

$$-\left(|u|^{k-2}u,v_t\right)_{D^T}+\sum_{\alpha=1}^{n}\left(a_\alpha((u_{x_\alpha})^2)u_{x_\alpha},v_{x_\alpha}\right)_{D^T}+(|u|^{k-2}u,v)\Big|_{t=0}^{t=T}=0,\quad v\in\overset{\circ}{W}{}^{1,1}_{k,\mathbf{p}}(D^T).$$

Thus, we have obtained (29).

Since $T>0$ is arbitrary, the equality (61) is written in the form

$$\left((|u|^{k-2}u)_\tau,v\right)_{D^t}+\sum_{\alpha=1}^{n}\left(a_\alpha((u_{x_\alpha})^2)u_{x_\alpha},v_{x_\alpha}\right)_{D^t}=0,\quad t>0. \tag{62}$$

Assuming that $v=u$ and using the equality

$$\int_0^t((|u|^{k-2}u)_\tau,u)d\tau=\frac{k-1}{k}\left(\|u(t)\|_k^k-\|\varphi\|_k^k\right),$$

we obtain the equality

$$\frac{k-1}{k}\|u(t)\|_k^k+\sum_{\alpha=1}^{n}\int_0^t(a_\alpha(u^2_{x_\alpha})u_{x_\alpha},u_{x_\alpha})d\tau=\frac{k-1}{k}\|\varphi\|_k^k,\quad t\geq 0. \tag{63}$$

Differentiating the latter with respect to t, we obtain

$$\frac{k-1}{k}\frac{d}{dt}\|u(t)\|_k^k+\sum_{\alpha=1}^{n}(a_\alpha(u^2_{x_\alpha})u_{x_\alpha},u_{x_\alpha})=0,\quad t>0. \tag{64}$$

Then, applying (4) from (63), (64) we deduce (37), (38). □

3. Admissible decay rate of solution

Since the uniqueness of solution to the problem (1)–(3) is not proved, we will in fact obtain the lower estimate only for the constructed solution.

Proof of Theorem 2. First, et us assume that the domain Ω is bounded and prove the estimate (10) for Galerkin's approximations.

Let us introduce the notation

$$G^M(t)=\sum_{\alpha=1}^{n}\int\limits_{\Omega} a_\alpha((u^M_{x_\alpha})^2)(u^M_{x_\alpha})^2d\mathbf{x},\quad H^M(t)=\sum_{\alpha=1}^{n}\int\limits_{\Omega} A_\alpha((u^M_{x_\alpha})^2)d\mathbf{x},$$

$$E^M(t)=\frac{k-1}{k}\|u^M(t)\|_k^k+\frac{1}{2b^M}\|u^M(t)\|^2,$$

using (32), we obtain the inequalities

$$\frac{p_1}{2}H^M(t)\leqslant G^M(t)\leqslant \widehat{b}H^M(t),\quad t\geq 0. \tag{65}$$

Let us rewrite the inequalities (45), (49) in the form

$$\frac{dE^M(t)}{dt}+G^M(t)=0,\quad t>0, \tag{66}$$

$$(k-1)\||u^M|^{(k-2)/2}u^M_t(t)\|^2+\frac{1}{b^M}\|u^M_t(t)\|^2+\frac{1}{2}\frac{dH^M(t)}{dt}=0,\quad t>0. \tag{67}$$

Applying the integral Cauchy-Bunyakovsky inequality, we obtain the relations

$$\left(\frac{dE^M(t)}{dt}\right)^2=\left(\int\limits_{\Omega}\left(\frac{1}{b^M}+(k-1)|u^M|^{k-2}\right)u^Mu^M_t d\mathbf{x}\right)^2\leqslant$$

$$\leqslant\left(\frac{1}{b^M}\|u^M(t)\|\|u^M_t(t)\|+(k-1)\|u^M(t)\|_k^{k/2}\||u^M|^{(k-2)/2}u^M_t(t)\|\right)^2.$$

Using the Cauchy-Bunyakovsky inequality for the sum, according to (67), we deduce

$$\begin{gathered}\left(\frac{dE^M(t)}{dt}\right)^2\leqslant\\ \leqslant\left(\frac{1}{b^M}\|u^M(t)\|^2+(k-1)\|u^M(t)\|_k^k\right)\left(\frac{1}{b^M}\|u^M_t(t)\|^2+(k-1)\||u^M|^{(k-2)/2}u^M_t(t)\|^2\right)\leqslant\\ \leqslant-\frac{k}{2}\frac{dH^M(t)}{dt}\left(\frac{1}{2b^M}\|u^M(t)\|^2+\frac{k-1}{k}\|u^M(t)\|_k^k\right)=-\frac{k}{2}\frac{dH^M(t)}{dt}E^M(t).\end{gathered} \tag{68}$$

The formulae (68), (66), (65) provide the inequalities

$$\frac{k}{2}E^M(t)\frac{dH^M(t)}{dt}\leqslant\frac{dE^M(t)}{dt}G^M(t)\leqslant\frac{p_1}{2}\frac{dE^M(t)}{dt}H^M(t),$$

that can be rewritten in the form

$$\frac{dH^M(t)}{dt}/H^M(t)\leqslant\frac{p_1}{k}\frac{dE^M(t)}{dt}/E^M(t).$$

Solving the differential inequality, applying (65), we obtain the estimates

$$\frac{1}{\widehat{b}}G^M(t)\leqslant H^M(t)\leqslant H^M(0)(E^M(t))^{p_1/k}/(E^M(0))^{p_1/k},\quad t>0. \tag{69}$$

Then, joining (66), (69), (65), we deduce the relation

$$\frac{dE^M(t)}{dt}\geq-\widehat{b}H^M(0)(E^M(t))^{p_1/k}/(E^M(0))^{p_1/k}\geq$$

$$\geq -\frac{2\widehat{b}}{p_1}G^M(0)(E^M(t))^{p_1/k}/(E^M(0))^{p_1/k},$$

and rewrite it in the form

$$\frac{dE^M(t)}{dt}/(E^M(t))^{p_1/k} \geq -\frac{2\widehat{b}}{p_1}G^M(0)/(E^M(0))^{p_1/k}.$$

Solving the differential inequality, we obtain the estimate

$$E^M(t) \geq E^M(0)\left(t\frac{2(p_1-k)\widehat{b}}{kp_1}G^M(0)/E^M(0)+1\right)^{-k/(p_1-k)}, \quad t>0. \tag{70}$$

When $t \in [0,T]$ is fixed and $k \leq p_1$ in case of a bounded domain Ω, the sequence $u^M(t)$ selectively strongly converges for $M \to \infty$ to $u(t)$ in the space $L_k(\Omega)$. Suppose that $b^M = M\max\{\|u^M(0)\|^2, \mathrm{mes}^{(k-2)/2}\Omega\}$then,

$$E^M(t) \leqslant \frac{k-1}{k}\|u^M(t)\|_k^k + \frac{\mathrm{mes}^{(k-2)/2}\Omega}{2b^M}\|u^M(t)\|_k^2 \leqslant$$

$$\leqslant \frac{k-1}{k}\|u^M(t)\|_k^k + \frac{1}{2M}\|u^M(t)\|_k^2, \quad M=\overline{1,\infty},$$

and

$$\lim_{M\to\infty} E^M(t) \leqslant \frac{k-1}{k}\|u(t)\|_k^k.$$

Moreover, the inequalities

$$\lim_{M\to\infty} E^M(0) \geq \frac{k-1}{k}\lim_{M\to\infty}\|u^M(0)\|_k^k = \frac{k-1}{k}\|\varphi\|_k^k,$$

$$\lim_{M\to\infty} G^M(0) \leqslant \lim_{M\to\infty}\widehat{a}\sum_{\alpha=1}^{n}\|u^M_{x_\alpha}\|_{p_\alpha}^{p_\alpha} = \widehat{a}\sum_{\alpha=1}^{n}\|\varphi_{x_\alpha}\|_{p_\alpha}^{p_\alpha}$$

hold.

Passing to the limit in (70) when $M \to \infty$, we obtain

$$\|u(t)\|_k^k \geq \|\varphi\|_k^k(1+C(\|\varphi\|_{W^1_{k,\mathbf{p}}(\Omega)})t)^{-k/(p_1-k)}. \tag{71}$$

Now let us obtain the estimate (71) for solution to the problem (1)–(3) in an unbounded domain Ω. Let $\Omega^{(l)} \subset \Omega$ be bounded subdomains such that $\Omega^{(l)} \subset \Omega^{(l+1)}$, $l=\overline{1,\infty}$, $\bigcup\limits_{l=1}^{\infty}\Omega^{(l)} = \Omega$. Let us denote by $u^{(l)}$ solutions in $\Omega^{(l)}$ with a finite initial function (supp $\varphi \subset \Omega^{(1)}$), these solutions can be considered to be an extended zero outside $\Omega^{(l)}$. The inequality (37) provides that the sequence is bounded in the space $\overset{\circ}{W}{}^{0,1}_{k,\mathbf{p}}(D^T)$ for any $T>0$. Then, according to Lemma 4, there is a countable dense set $\{t_j\}_{j=1}^{\infty} \subset [0,T]$ such that $u^{(l)}$ selectively strongly converges in $L_k(\Omega^r)$ for every t_j and $r>0$. Due to the estimate (9) for any ε there is r such that the inequality

$$\|u^{(l)}(t)\|_{k,\Omega_r}^k \leqslant \varepsilon$$

holds for all $t \geq 0$. For $u^{(l)}$, the estimate (71) holds. Then,

$$\|u^{(l)}(t_j)\|_{k,\Omega^r}^k \geq \|\varphi\|_k^k(1+C(\|\varphi\|_{W^1_{k,\mathbf{p}}(\Omega)})t)^{-k/(p_1-k)} - \varepsilon$$

when $t_j \in (t-\delta,t)$. Using the strong convergence in $L_k(\Omega^r)$, we pass to the limit when $l \to \infty$ then, according to $r \to \infty$ ($\varepsilon \to 0$). Using the continuity of the function $\|u(t)\|_k$, we pass to the limit when $t_j \to t$. Thus, the estimate (10) is established in an unbounded domain Ω. $\square$

4. Upper estimates

Theorems 1,3 of Introduction are proved in this section.

Lemma 5. *If the condition $x_\alpha \neq y_\alpha$ is met for $\mathbf{x} \in \Omega$ when a certain $\alpha \in \overline{1,n}$ is fixed then, we have the inequality*

$$\left\| \frac{u(\mathbf{x})}{|x_\alpha - y_\alpha|} \right\|_{p_\alpha} \leqslant \frac{p_\alpha}{p_\alpha - 1} \|u_{x_\alpha}(\mathbf{x})\|_{p_\alpha} \tag{72}$$

for the function $u(\mathbf{x}) \in C_0^\infty(\Omega)$

Proof. Using the equality

$$\left(\frac{x_\alpha - y_\alpha}{|x_\alpha - y_\alpha|^{p_\alpha}} \right)'_{x_\alpha} = -\frac{p_\alpha - 1}{|x_\alpha - y_\alpha|^{p_\alpha}},$$

integrating by parts, we obtain

$$\int\limits_\Omega \frac{|u(\mathbf{x})|^{p_\alpha}}{|x_\alpha - y_\alpha|^{p_\alpha}} d\mathbf{x} = -\frac{1}{p_\alpha - 1} \int\limits_\Omega |u(\mathbf{x})|^{p_\alpha} \frac{d}{dx_\alpha} \frac{x_\alpha - y_\alpha}{|x_\alpha - y_\alpha|^{p_\alpha}} d\mathbf{x} =$$

$$= \frac{p_\alpha}{p_\alpha - 1} \int\limits_\Omega |u|^{p_\alpha - 2} u u_{x_\alpha} \frac{x_\alpha - y_\alpha}{|x_\alpha - y_\alpha|^{p_\alpha}} d\mathbf{x} \leqslant \frac{p_\alpha}{p_\alpha - 1} \int\limits_\Omega \frac{|u|^{p_\alpha - 1}}{|x_\alpha - y_\alpha|^{p_\alpha - 1}} |u_{x_\alpha}| d\mathbf{x}.$$

Applying the Holder inequality, we obtain

$$\int\limits_\Omega \frac{|u(\mathbf{x})|^{p_\alpha}}{|x_\alpha - y_\alpha|^{p_\alpha}} d\mathbf{x} \leqslant \frac{p_\alpha}{p_\alpha - 1} \left(\int\limits_\Omega \frac{|u(\mathbf{x})|^{p_\alpha}}{|x_\alpha - y_\alpha|^{p_\alpha}} d\mathbf{x} \right)^{(p_\alpha - 1)/p_\alpha} \left(\int\limits_\Omega |u_{x_\alpha}(\mathbf{x})|^{p_\alpha} d\mathbf{x} \right)^{1/p_\alpha}.$$

Whence, (72) follows. □

Corollary 1. *When $0 < a < b$, the inequality*

$$\frac{1}{b} \|u\|_{p_s, \Omega_a^b} \leqslant \frac{p_s}{p_s - 1} \|u_{x_s}\|_{p_s} \tag{73}$$

holds for the function $u(\mathbf{x}) \in \overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega)$, ($\Omega$ is the domain located along the axis Ox_s).

Proof. Let us consider $y_s = 0$ and deduce

$$\left(\int\limits_{\Omega_a^b} |u(\mathbf{x})|^{p_s} d\mathbf{x} \right)^{1/p_s} \leqslant b \left(\int\limits_{\Omega_a^b} \frac{|u(\mathbf{x})|^{p_s}}{|x_s|^{p_s}} d\mathbf{x} \right)^{1/p_s} \leqslant b \frac{p_s}{p_s - 1} \left(\int\limits_\Omega |u_{x_s}(\mathbf{x})|^{p_s} d\mathbf{x} \right)^{1/p_s}$$

from the inequality (72) for $u(\mathbf{x}) \in C_0^\infty(\Omega)$. Whence, it follows that if the sequence $u^k(\mathbf{x}) \in C_0^\infty(\Omega)$ converges in norm of the space $\overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega)$, it converges in $L_{p_s}(\Omega_a^b)$ as well. Passing to the limit, we obtain the inequality (73) for $u \in \overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega)$. □

Proof of Theorem 1. Let $\theta(x)$, $x > 0$, be an absolutely continuous function which is equal to one when $x \geq r$, zero when $x \leq R_0$, linear when $x \in [R_0, 2R_0]$, and satisfying the equation

$$\theta'(x) = \delta \nu(x) \theta(x), \quad x \in (2R_0, r), \tag{74}$$

(the constant δ will be defined later). Solving this equation we obtain, in particular, that

$$\theta'(x) = \frac{\theta(2R_0)}{R_0} = \frac{1}{R_0} \exp\left(-\delta \int\limits_{2R_0}^{r} \nu(\rho) d\rho \right), \quad x \in (R_0, 2R_0). \tag{75}$$

For any function $v(\mathbf{x}) \in C_0^\infty(\Omega)$, the definition of the function $\nu(\rho)$ entail the inequalities

$$\nu(\rho)\|v\|_{p_\alpha,\gamma_\rho} \le \|v_{x_\alpha}\|_{p_\alpha,\gamma_\rho}, \quad \rho > 0, \quad \alpha = 1, n,$$

whence we obtain the relations

$$\int\limits_{2R_0}^{r} \theta^{p_s}(\rho)\nu^{p_\alpha}(\rho)\|v\|^{p_\alpha}_{p_\alpha,\gamma_\rho} d\rho \le \int\limits_{2R_0}^{r} \theta^{p_s}(\rho)\|v_{x_\alpha}\|^{p_\alpha}_{p_\alpha,\gamma_\rho} d\rho, \quad \alpha = 1, n. \tag{76}$$

Applying (76) for any function $v \in C_0^\infty(\Omega)$ when $s \ne 1, n$ we deduce

$$\int\limits_{2R_0}^{r} \nu^{p_s}(\rho)\theta^{p_s}(\rho)\|v\|^{p_s}_{p_s,\gamma_\rho} d\rho \le \int\limits_{2R_0}^{r} \nu^{p_1}(\rho)\theta^{p_s}(\rho)\|v\|^{p_1}_{p_1,\gamma_\rho} d\rho + \int\limits_{2R_0}^{r} \nu^{p_n}(\rho)\theta^{p_s}(\rho)\|v\|^{p_n}_{p_n,\gamma_\rho} d\rho \leqslant$$

$$\leqslant \int\limits_{2R_0}^{r} \theta^{p_s}(\rho)\|v_{x_1}\|^{p_1}_{p_1,\gamma_\rho} d\rho + \int\limits_{2R_0}^{r} \theta^{p_s}(\rho)\|v_{x_n}\|^{p_n}_{p_n,\gamma_\rho} d\rho. \tag{77}$$

Note that the inequalities (1) hold true for any function $v \in \overset{\circ}{W}{}^1_{k,\mathbf{p}}(\Omega)$ (see Corollary 1).

Let $\xi(\mathbf{x})$ be a Lipschitzian nonnegative patch function. Assuming that $v = u\xi$ in (62), we obtain

$$\frac{k-1}{k}\int\limits_{\Omega} |u|^k \xi \Big|_{\tau=0}^{\tau=t} d\mathbf{x} + \sum_{\alpha=1}^{n} \int\limits_{D^t} a_\alpha(u^2_{x_\alpha}) u_{x_\alpha}(u\xi)_{x_\alpha} d\mathbf{x}d\tau = 0.$$

Suppose that $\xi(\mathbf{x}) = \theta^{p_s}(x_s)$. Then, applying (4), (taking into account that $\theta\varphi = 0$) we arrive at

$$\frac{k-1}{k}\int\limits_{\Omega} |u(t,\mathbf{x})|^k \theta^{p_s}(x_s) d\mathbf{x} + \overline{a}\sum_{\alpha=1}^{n} \int\limits_{D^t} \theta^{p_s}|u_{x_\alpha}|^{p_\alpha} d\mathbf{x}d\tau \leqslant \tag{78}$$

$$\leqslant \widehat{a}\int\limits_{D^t} |u||u_{x_s}|^{p_s-1}(\theta^{p_s}(x_s))' d\mathbf{x}d\tau \equiv \widehat{a} I^t.$$

Let us estimate the integral

$$I^t = \int\limits_0^t \int\limits_{\Omega} |u||u_{x_s}|^{p_s-1} p_s \theta'(x_s)\theta^{p_s-1}(x_s) d\mathbf{x}d\tau.$$

Using the Young inequality, we obtain

$$I^t \leqslant \varepsilon(p_s-1)\int\limits_0^t \int\limits_{\Omega} |u_{x_s}|^{p_s}\theta^{p_s} d\mathbf{x}d\tau + \frac{1}{\varepsilon^{p_s-1}}\int\limits_0^t \int\limits_{\Omega} |u|^{p_s}(\theta'(x_s))^{p_s} d\mathbf{x}d\tau. \tag{79}$$

Let us choose $\varepsilon = \dfrac{\overline{a}}{\widehat{a}}\dfrac{1}{p_s-1}$, joining (78), (79), we deduce the inequality

$$\frac{k-1}{k}\int\limits_{\Omega} |u(t,\mathbf{x})|^k \theta^{p_s}(x_s) d\mathbf{x} + \overline{a}\sum_{\alpha=1,\alpha\ne s}^{n} \int\limits_{D^t} \theta^{p_s}|u_{x_\alpha}|^{p_\alpha} d\mathbf{x}d\tau \leqslant C_1\int\limits_{D^t} |u|^{p_s}(\theta'(x_s))^{p_s} d\mathbf{x}d\tau. \tag{80}$$

Using (74), (75), one can readily reduce (80) to the form

$$\frac{k-1}{k}\int\limits_{\Omega} |u(t,\mathbf{x})|^k \theta^{p_s}(x_s) d\mathbf{x} + \overline{a}\sum_{\alpha=1,\alpha\ne s}^{n} \int\limits_{D^t} \theta^{p_s}|u_{x_\alpha}|^{p_\alpha} d\mathbf{x}d\tau \leqslant \tag{81}$$

$$\leqslant C_1\frac{1}{R_0^{p_s}}\exp\left(-\delta p_s\int\limits_{2R_0}^{r}\nu(\rho)d\rho\right)\int\limits_0^t\int\limits_{\Omega_{R_0}^{2R_0}}|u|^{p_s}d\mathbf{x}d\tau+$$

$$+C_1\delta^{p_s}\int\limits_0^t\int\limits_{\Omega_{2R_0}^{r}}|u|^{p_s}\nu^{p_s}(x_s)\theta^{p_s}(x_s)d\mathbf{x}d\tau=I_1^t+I_2^t.$$

Using the inequalities (73) and (37), we obtain

$$I_1^t\leqslant C_2\exp\left(-\delta p_s\int\limits_{2R_0}^{r}\nu(\rho)d\rho\right)\int\limits_0^t\|u_{x_s}\|_{p_s}^{p_s}d\tau\leqslant C_3\exp\left(-\delta p_s\int\limits_{2R_0}^{r}\nu(\rho)d\rho\right)\|\varphi\|_k^k. \tag{82}$$

Application of (1) provides

$$I_2^t\leqslant C_1\delta^{p_s}\int\limits_0^t\int\limits_{\Omega_{2R_0}^{r}}\left(|u_{x_1}|^{p_1}\theta^{p_s}+|u_{x_n}|^{p_n}\theta^{p_s}\right)d\mathbf{x}d\tau. \tag{83}$$

Choosing $\delta=\left(\dfrac{\overline{a}}{C_1}\right)^{1/p_s}$, joining (81) – (83), we deduce

$$\frac{k-1}{k}\|u(t)\|_{k,\Omega_r}^k+\overline{a}\sum_{\alpha=2,\alpha\neq s}^{n-1}\int\limits_0^t\|u_{x_\alpha}(t)\|_{\Omega_r}^{p_\alpha}d\tau\leqslant C_3\exp\left(-\delta p_s\int\limits_1^r\nu(\rho)d\rho\right)\|\varphi\|_k^k.$$

The inequality (9) is proved. □

Corollary 2. *if the condition*

$$\mu_1=\inf\left\{\|g_{x_1}\|_{p_1}\;\middle|\;g(\mathbf{x})\in C_0^\infty(\Omega),\|g\|=1\right\}>0$$

is met, the estimate (13) *holds for solution* $u(t,\mathbf{x})$ *of the problem* (1)–(3).

Proof. It follows from (38) that

$$\frac{d}{dt}\|u(t)\|_k^k\leqslant-\frac{\overline{a}k}{k-1}\sum_{\alpha=1}^n\|u_{x_\alpha}\|_{p_\alpha}^{p_\alpha}\leqslant-\frac{\overline{a}k}{k-1}\|u_{x_1}\|_{p_1}^{p_1}\leqslant-\frac{\overline{a}k}{k-1}\mu_1^{p_1}\|u\|_k^{p_1}.$$

Solving this differential inequality, we obtain the estimate

$$\|u(t)\|_k\leqslant t^{-1/(p_1-1)}\left(\frac{(p_1-k)\overline{a}}{k-1}\mu_1^{p_1}\right)^{-1/(p_1-k)},$$

whence the inequality (13) follows. □

Proof of Theorem 3. Let us choose a positive number $r\geq 2R_0$. According to Theorem 1, introducing the notation $\varepsilon(r)=\mathcal{M}^k\exp\left(-k\kappa\int\limits_1^r\nu(\rho)d\rho\right)\|\varphi\|_k^k$, we have the relation

$$\|u(t)\|_k^k\leqslant\|u(t)\|_{k,\Omega^r}^k+\varepsilon(r),\quad t\geq 0.$$

The definition (11) entails that

$$\|u(t)\|_k^k\leqslant\left(\mu_1^{-p_1}(r)\|u_{x_1}(t)\|_{p_1,\,\Omega^r}^{p_1}\right)^{k/p_1}+\varepsilon(r),\quad t\geq 0. \tag{84}$$

Denote by t_r a point of the interval $[0,\infty)$ such that $E(t)=\|u(t)\|_k^k=\varepsilon(r)$. If $E(t)>\varepsilon(r)$ for any $t\geq 0$, then $t_r=\infty$. Since the function $E(t)$ is monotonously nonincreasing, the inequality $E(t)>\varepsilon(r)$ holds for $t\in[0,t_r)$. Then, the relation (84) can be written in the form

$$(E(t)-\varepsilon(r))^{p_1/k}\leqslant \mu_1^{-p_1}(r)\sum_{\alpha=1}^{n}\| \, u_{x_\alpha}(t)\|_{p_\alpha,\,\Omega^r}^{p_\alpha},\quad t\in[0,t_r). \tag{85}$$

Joining (38) with (85), we deduce the correlation

$$\frac{dE(t)}{dt}\leqslant -\frac{k\overline{a}}{k-1}\mu_1^{p_1}(r)\,(E(t)-\varepsilon(r))^{p_1/k}\,,\quad t\in(0,t_r). \tag{86}$$

Solving the differential inequality, we obtain

$$(E(t)-\varepsilon(r))^{(p_1-k)/k}\leqslant\frac{k-1}{t\mu_1^{p_1}(r)(p_1-k)},\quad t\in(0,t_r).$$

Substituting the value $\varepsilon(r)$ into the latter inequality, we obtain

$$E(t)\leqslant C_1\left(t\mu_1^{p_1}(r)\right)^{-k/(p_1-k)}+\mathcal{M}^k\exp\left(-k\kappa\int\limits_1^r\nu(\rho)d\rho\right)\|\varphi\|_k^k \tag{87}$$

for $t\in(0,t_r)$. Note that for $t\in[t_r,\infty)$ the inequality $E(t)\leqslant\varepsilon(r)$, as well as the estimate (87), hold.

Let us assume that $r=r(t)$ in (87) and use the definition (14) of the function $r(t)$. This results in

$$E(t)\leqslant C_1\left(t\mu_1^{p_1}(r(t))\right)^{-k/(p_1-k)}+\mathcal{M}^k\exp\left(-k\kappa\int\limits_1^{r(t)}\nu(\rho)d\rho\right)\|\varphi\|_k^k\leqslant$$

$$\leqslant C_2\left(t\mu_1^{p_1}(r(t)\right)^{-k/(p_1-k)}\,,\quad t>0.$$

Thus, the inequality (15) is proved. □

Let us define the function

$$\lambda_1(r)=\inf\left\{\|g_{x_1}\|_{p_1,\Omega^r}\;\middle|\; g(\mathbf{x})\in C_0^\infty(\Omega),\|g\|_{p_1,\Omega^r}=1\right\},\quad r>0.$$

Then the definition of the function $\lambda_1(r)$ entails the inequality

$$\lambda_1^{p_1}(r)\|g\|_{p_1,\Omega^r}^{p_1}\leqslant\|g_{x_1}\|_{p_1,\Omega^r}^{p_1},\quad g(\mathbf{x})\in C_0^\infty(\Omega),\quad r>0.$$

Applying the Holder inequality for $g(\mathbf{x})\in C_0^\infty(\Omega),\ r>0$, we obtain the relations

$$\|g\|_{k,\Omega^r}^{p_1}\leqslant\|g\|_{p_1,\Omega^r}^{p_1}\left(\operatorname{mes}\Omega^r\right)^{(p_1-k)/k}\leqslant\lambda_1^{-p_1}(r)\left(\operatorname{mes}\Omega^r\right)^{(p_1-k)/k}\|g_{x_1}\|_{p_1,\Omega^r}^{p_1},$$

that entail the inequality

$$\lambda_1(r)\leqslant\mu_1(r)\left(\operatorname{mes}\Omega^r\right)^{(p_1-k)/(kp_1)}\,,\quad g(\mathbf{x})\in C_0^\infty(\Omega),\quad r>0. \tag{88}$$

5. Upper estimate for rotation domains

Let $\mathcal{P}(\rho,z)=\{(x,y)\in\mathbb{R}_2\mid z<x<z+\rho,\ 0<y<\rho\}$ is a square with the side ρ and the left lower corner at z of the abscissa. For the positive function $f(x),\ x>0$, the symbol $\Gamma^r(f)$ denotes a curvilinear trapezoid

$$\Gamma^r(f)=\{(x,y)\in\mathbb{R}_2\mid 0<x<r,\ 0<y<f(x)\}.$$

Let us denote the side of the largest square $\mathcal{P}(\rho_*,\,z_*)$, contained in $\Gamma^r(f)$, by $\rho_*(r)$. The estimate

$$\frac{c_1}{\rho_*(r)}\leqslant\lambda_1(r)\leqslant\frac{c_2}{\rho_*(r)},\quad r>0 \tag{89}$$

holds (the proof of a similar statement is available in [30]).

Let us define monotonously increasing functions

$$g(r) = (\operatorname{mes}\Omega^r)^{1/k}\rho_*^{p_1/(p_1-k)}(r), \quad r > 0; \tag{90}$$

$$\widetilde{r}(t): \quad \int_1^{\widetilde{r}} \frac{dx}{f(x)} = \frac{\ln t}{\widetilde{\kappa}(p_1-k)}, \quad t \geq 1. \tag{91}$$

Statement 1. *Let us assume that the condition* (8) *is met then, there is a positive number* $\widetilde{M}$ *such that the estimate* (26) *holds for solution* $u(t, \mathbf{x})$ *of the problem* (1)–(3) *in a cylindric domain* $D(f) = (0, \infty) \times \Omega(f)[s]$ *with the function* $f(x)$*, satisfying the condition* (24).

Proof. Joining (88) and (89), we deduce

$$\mu_1(r) \geq c_1 (\operatorname{mes}\Omega^r)^{-(p_1-k)/(kp_1)} \rho_*{}^{-1}(r) = c_1 g^{-(p_1-k)/p_1}(r), \quad r > 0. \tag{92}$$

Let us fix $t \geq 1$ and assume $r = \widetilde{r}(t)$. Substituting (92), (23) into (87), using the definition (91) of the function $\widetilde{r}(t)$, we obtain the relations

$$\|u(t)\|_{k,\Omega(f)}^k \leqslant C_2 \exp\left(-k\widetilde{\kappa}\int_1^{\widetilde{r}(t)} \frac{dx}{f(x)}\right) + C_2 g^k(\widetilde{r}(t))t^{-k/(p_1-k)} \leqslant C_3 g^k(\widetilde{r}(t))t^{-k/(p_1-k)},$$

whence follows the inequality (26) with the function $\widetilde{g}(t) = g(\widetilde{r}(t))$.

Let us demonstrate that the function $\widetilde{g}(t)$ grows slower than a power function. It follows from the condition (24) that for any $\varepsilon \in (0,1)$, there is r_0 such that

$$\varepsilon\widetilde{\kappa}(p_1-k)\int_1^r \frac{dx}{f(x)} > \ln r, \quad r \geq r_0.$$

Then, it follows from definition (91) of the function $\widetilde{r}(t)$ that the inequality

$$\widetilde{r}(t) < t^{\varepsilon}, \quad t \geq t_0 \tag{93}$$

holds.

Then, definition (90) and the formula

$$\operatorname{mes}\Omega^r(f) = c_n \int_0^r f^{n-1}(x)dx, \tag{94}$$

provide the inequality

$$g(r) \leqslant C_4\rho_*^{p_1/(p_1-k)}(r)\left(\int_0^r f^{n-1}(x)dx\right)^{1/k} \leqslant C_4 r^{p_1/(p_1-k)+1/k} \max_{[0,r]} f^{(n-1)/k}(x)$$

for $r > 0$. Applying the corollary of the inequality (24):

$$f(x) \leqslant Cx, \quad x \geq r_0,$$

we obtain

$$g(r) \leqslant C_5(r_0)r^{p_1/(p_1-k)+n/k}, \quad r \geq r_0. \tag{95}$$

Joining (93), (95) we establish that the function $\widetilde{g}(t)$ grows slower than any power of t. □

Let us assume that there is a constant $\omega \geq 1$ such that

$$\sup\left\{f(z) \mid z \in [x - f(x), x + f(x)]\right\} \leqslant \omega f(x), \quad x \geq 1. \tag{96}$$

For a monotonously nondecreasing function f, satisfying the condition (96), we have the inequality

$$\omega^{-1}f(r) \leqslant \rho_*(r) \leqslant f(r), \quad r \geq r_0 = 1 + f(1). \tag{97}$$

Indeed, due to monotonous nondecreasing of the function $f(x)$, we have the equality $f(z_*) = r - z_* = \rho_*(r)$. According to (96), we have

$$f(r) = \max_{[z_* - f\,(z_*), z_* + f(z_*)]} f(z) \leqslant \omega f(z_*),$$

whence the eft inequality (97) follows.

According to (94), (97), the function $g(r)$ can be defined

$$g(r) = f^{p_1/(p_1-k)}(r) \left(\int\limits_0^r f^{n-1}(x)dx \right)^{1/k}, \quad r \geq r_0.$$

Example 1. For the function $f(x) = x^a,\ 0 \leqslant a < 1,\ x > 0$, one can define the functions

$$\widetilde{r}(t) = (\ln t)^{1/(1-a)}, \quad t \geq t_0, \quad g(r) = r^{\chi}, \quad r \geq r_0.$$

Then the estimate (26) takes the form (27).

Example 2. For the function $f(x) = e,\ 0 < x < e,\ f(x) = x/\ln x,\ x \geq e$, we obtain

$$\widetilde{r}(t) = \exp(\varsigma(\ln t)^{1/2}), \quad t \geq t_0, \quad \varsigma > 0, \quad g(r) = r^{\sigma+1/k}(\ln r)^{-\sigma}, \quad r \geq r_0.$$

Meanwhile the estimate (26) takes the form (28).

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