ISSN 2074-1863 Ufa Mathematical Journal. Volume 3. 4 (2011). Pp. 56-61.

UDC 519.1+519.8

COMBINATORIAL COMPLEXITY OF A CERTAIN 1-DIMENSIONAL CUTTING STOCK PROBLEM

V.M.KARTAK, V.V.KARTAK

Abstract. The classical Cutting Stock Problem (1dCSP) is considered. It is known that 1CSP is at least NP-hard. In the present paper a combinatorial algorithm for its solution based on the Branch and Bound Method is described. We estimate the complexity of this algorithm presented for a class of problems that is called compact. The most difficult examples to solve by combinatorial algorithms are identified. This result is consistent with experimental data and could be used to generate difficult test problems, as well as for predicting the time of the algorithm.
Keywords: Cutting Stock Problem, Branch and Bound Method, Integer Programming, Combinatorial complexity.

1. Introduction

The classical problem of a *One-Dimensional Cutting Stock Problem (1dCSP)* consists in the following: given a set of nonnegative numbers $l_i \in \mathbb{R}_+$, $i \in I = \{1, \ldots, m\}$ and some positive number $L > 0$. Find a minimum natural number n such that I splits into n nonintersecting subsets $I = \cup_{k=1}^{n} I_k$ and $\sum_{i \in I_k} l_i \leqslant L$. Let us denote this problem as follows: $E = (L, m, (l_1, l_2, \ \ldots \ l_m))$. Without loss of generality, we assume that l_i are arranged in the descending order $l_1 \geq l_2 \geq ... \geq l_m$.

This problem is known in literature as container allocation problem, one-dimensional cutting stock problem. A number of scheduling problems is also reduced to this statement. The given problem is related to the class of NP-hard problems hence, it can not be solved by means of a pseudo-polynomial algorithm (see [1]).

I.V. Romanovskii interpreted the one-dimensional cutting stock problem as a combinatorial optimization problem. He suggested the general idea of the iterative method to solve the problem and specified it in the form of the "Method of Branch and Bound" (MBB) [2], which was realized by S.V. Katsev, see [3]. Later on, S.Martello and P.Toth in [4] and E.A. Mukhacheva with V.M. Kartak in [5] suggested improving a version of MBB by adding additional restrictions. In 1997, D. Schwerin and G. Wascher classified input data for the one-dimensional cutting stock problem, see [6]. In their subsequent work [7] they, as well as E.A. Mukhacheva and V.M. Kartak in [5], singled out most difficult classes for obtaining the optimum (the difficulty of the class is defined by the number of examples with optimum obtained during a given period time). "'Triplets"' appeared to be most difficult examples, where $L/4 < l_i \leqslant L$.

The given article is devoted to investigation of a computational complexity of combinatorial algorithms in case of a dense problem.

2. Scheme of the combinatorial algorithm

Let us correlate every subset of the partition I_k to an m-dimensional binary vector $\mathbf{a}^k = (a_1^k, \ldots, a_m^k)^T$, where $a_i^k = 1$ if $i \in I_k$, or $a_i = 0$. Then, any solution of the problem,

The work is supported by RFBR 10-07-91330-NNIO-a, 09-01-00046-a and NSh-65497.2010.9.
Submitted on 25 June 2011.

consisting of n subsets, can be represented in the form of a binary matrix with the partition $\mathcal{A} = (a^1, a^2, \dots, a^n)$ of the size $m \times n$. Partition when n reaches its minimum is said to be *optimal.*

To obtain the optimal partition, combinatorial algorithms of the branch and bound type are forced to look through all possible partition variants sequentially at worst [2], [4], [5]. To avoid repetition, a *lexicographic ordering of partition matrices* is introduced.

The vector a^j is of a higher *priority* as compared to a^k, provided that $\sum_{i=m}^{1} a_i^j \cdot 2^i > \sum_{i=m}^{1} a_i^k \cdot 2^i$. Let us impose a requirement that columns in the matrix $\mathcal{A}$ should be ordered by non-increasing priorities. A priority matrix between two matrices is the one that contains a vector of a higher priority. Thus, all matrices can be sorted in a lexicographic ordering by their non-increasing priorities $\mathcal{A}_1 \geq \mathcal{A}_2 \geq \dots \geq \mathcal{A}_K$. The general enumeration scheme looks as follows:

Let $\Lambda(\mathcal{A}_i)$ be a number of columns in the matrix $\mathcal{A}_i$.

Step 1. Preparation for solving:
- calculate the lower bound L_d;
- construct $\mathcal{A}_1$ by means of the *first fit* algorithm [1]. $L_u = \Lambda(\mathcal{A}_1)$ is the upper bound;
- $\mathcal{A}_{best} = \mathcal{A}_1$ is the best cutting plan.

Step 2. If $L_d = L_u$, proceed to **Step 4.**

Step 3. Look through $\mathcal{A}_i$ sequentially using the lexicographic ordering and reduction ([4], [5]). If $\Lambda(\mathcal{A}_i) < L_u$ then $\mathcal{A}_{best} = \mathcal{A}_i$, $L_u = \Lambda(\mathcal{A}_i)$, proceed to **Step 2**.

Step 4. The optimal solution $\mathcal{A}_{best}$ is obtained.

Let us estimate the computational complexity of the suggested algorithm. Obviously, the maximum number of iterations is made by the algorithm when it is impossible to reach the lower bound, because in this case the algorithm is forced to generate all possible partition matrices.

3. Estimation of the algorithm complexity in a dense case

A vector a is said to be *dense*, if $\sum_{i=1}^{m} a_j l_j < \min_{k \in I/\{t:a_t=1\}} l_k$. A matrix is said to be *dense*, if it consists of dense vectors.

Let n be an optimal value of the problem E. The problem E is said to be *dense* if any matrix of $\mathcal{A}$ partition, corresponding to the optimal solution, is dense.

One can see from the above scheme that in order to prove that the value n is optimal, the algorithm has to construct all admissible matrices $\mathcal{A}$ of $m \times n$ dimensions and demonstrate that an admissible matrix with the number of columns $n-1$ does not exist. Let us estimate the maximum possible number of such matrices.

To this end, let us consider a certain sequence of nonrecurrent numbers $Q = (q_1, q_2, \dots q_m)$, where $q_i \in I$ (there can be various $m!$ such sequences in total). Let us set the matrix $\mathcal{A}$ to one-to one correspondence with every such sequence by the following rule.

- Let k_1 be a number such that $\sum_{i=1}^{k_1} l_{l_{q_i}} \leqslant L$ and $\sum_{i=1}^{k_1+1} l_{l_{q_i}} > L$. Then the vector a^1 is constructed by the rule $a^1_{q_i} = 1, i = \overline{1, k_1}$, the remaining elements are zeroes.
- The number k_2 is such that $\sum_{i=k_1+1}^{k_1+k_2} l_{l_{q_i}} \leqslant L$ and $\sum_{i=k_1+k_2+1}^{k_1+k_2+1} l_{l_{q_i}} > L$. The vector a^2 is constructed by the rule $a^2_{q_i} = 1, i = \overline{k_1+1, k_1+k_2}$, the remaining elements are zeroes, etc.

When the matrix $\mathcal{A}$ has been constructed, its columns are lexicographically ordered.

Note that one and the same matrix $\mathcal{A}$ can be obtained from several various sequences. Let $K(\mathcal{A})$ be a number of sequences generating the matrix $\mathcal{A}$. Manifestly, if $\mathcal{A}$ is a dense matrix, consisting of n columns, then $K(\mathcal{A}) = k_1!k_2!..k_n!n!$.

Lemma 1. *The estimate $k_1!k_2!..k_n! \geq (k!)^{n-r} \cdot ((k+1)!)^r$ holds. Here $k = [m/n]$ is the integer part of dividing m by n, r is the residue of dividing m by n.*

$\triangleleft$ Let us find the minimum of the expression $k_1!k_2!..k_n!$, using the Ferrer graph ([8], p. 21). Let us compose a table of m rows and n columns. In the first column, fill in the k_1 row, counting from below, in the second column, fill in the k_2 row, etc, in the last n-th column fill in k_n rows and assign weights to the filled cells as it is demonstrated on the picture.

		k_3		
k_1		k_3-1		
k_1-1		k_3-2	$\dots$	k_n
k_1-2	k_2	k_3-3	$\dots$	k_n-1
k_1-3	k_2-1	k_3-4	$\dots$	k_n-2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
1	1	1	$\dots$	1

The general number of all filled cells equals to $k_1+k_2+\dots+k_n=m$. We are looking for the minimum of the product $k_1!k_2!k_3!\dots k_n!$, containing exactly n multipliers.

It is allowed to carry out the following operation on the graph: ”‘shift”’ the upper filled cell from one column into the upper position of another column. For example, let us shift the upper cell from the third column to the second one:

k_1		k_3-1		
k_1-1	k_2+1	k_3-2	$\dots$	k_n
k_1-2	k_2	k_3-3	$\dots$	k_n-1
k_1-3	k_2-1	k_3-4	$\dots$	k_n-2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
1	1	1	$\dots$	1

There are only two possibilities

$$k_1!(k_2+1)!(k_3-1)!\dots k_n! < k_1!k_2!k_3!\dots k_n! \quad \text{if} \quad k_2+1<k_3,$$
$$k_1!(k_2+1)!(k_3-1)!\dots k_n! = k_1!k_2!k_3!\dots k_n! \quad \text{if} \quad k_2+1=k_3.$$

As one can see, ”‘shifting”’ the cell lower does not increase the desired product. It reaches its minimum when the product remains unaltered with any shift of the cell into a lower row. Then, the operation of ”‘shifting”’ is reduced to permutation of columns as such.

Thus, when m and n are fixed, the product $k_1!k_2!k_3!\dots k_n!$ reaches its minimum in the variables k_1, k_2, $\dots$, k_n if:

$$k_1!k_2!k_3!\dots k_n! = (k!)^{n-r}\cdot((k+1)!)^r\cdot n! = (k!)^n\cdot(k+1)^r\cdot n!. \quad \triangleright$$

Let E be a dense problem and $S(m,n)$ is the number of various possible dense matrices $\mathcal{A}$ of dimensions $m\times n$. Then, the following estimate holds:

$$\sum_{i=1}^{S(m,n)} K(\mathcal{A}) \leqslant m!.$$

Lemma 1 entails that

$$\sum_{i=1}^{S(m,n)} K(\mathcal{A}) = \sum_{i=1}^{S(m,n)} k_1^i!k_2^i!..k_n^i!n! \leqslant m! \quad \Rightarrow \quad S(m,n)\cdot(k!)^{n-r}\cdot((k+1)!)^r n! \leqslant m! \quad \Rightarrow$$

$$S(m,n) \leqslant \frac{m!}{(k!)^{n-r}\cdot((k+1)!)^r\cdot n!}.$$

Note that there are problems E such that the equality is reached. For instance: $l_1 = l_2 = \cdots = l_m$ and $m = n\lfloor L/l_1 \rfloor$.

Let us answer the question: **what correlation of m and n allows the function $S(m,n)$ to reach its maximum ?**(m is a fixed number.) To this end, let us neglect the integrality and substitute all factorials by a Gamma function by the rule: $\Gamma(n+1) = n!$

Lemma 2. *The function $S(m,n)$ is majorized by the function $F(m,n)$*

$$S(m,n) \le F(m,n) = \frac{\Gamma(m+1)}{\Gamma^n(\frac{m}{n}+1)\Gamma(n+1)}.$$

$\triangleleft$ Let us prove that the inequality

$$\frac{m!}{(k!)^n \cdot (k+1)^r \cdot n!} \le \frac{\Gamma(m+1)}{\Gamma^n(\frac{m}{n}+1)\Gamma(n+1)}$$

holds. The problem is equivalent to the proving that

$$(k!)^n \cdot (k+1)^r \ge \Gamma^n\left(\frac{m}{n}+1\right) = \Gamma^n\left(\frac{kn+r}{n}+1\right) = \Gamma^n\left(k+\frac{r}{n}+1\right).$$

Let us introduce the notation $x = k+1$, $s = r/n$, and $0 \le s < 1$. Then,

$$\Gamma(x)\cdot(x)^s \ge \Gamma(x+s). \tag{1}$$

The inequality (1) holds by virtue of the known inequality (see [9], [10], [11], [12])

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)}, \qquad 0 < s < 1,$$

since it takes the following form upon transformation:

$$\frac{x}{x^s} < \frac{x\Gamma(x)}{\Gamma(x+s)}, \qquad 0 < s < 1.$$

If $s = 1$, the inequality (1) turns into an identity $\triangleright$.

Let us formulate the following auxiliary lemma .

Lemma 3. *If $z \ge 3$ the digamma function $\Psi(z) = \Gamma'(z)/\Gamma(z)$ is estimated as follows:*

$$\ln z - C(z) \le \Psi(z) \le \ln z + C(z), \quad \textit{where } C(z) \textit{ is a finite expression.}$$

$\triangleleft$ Let us write the expression

$$\Psi(z+1) = \Psi(z) + \frac{1}{z} = \Psi(z-1) + \frac{1}{z-1} + \frac{1}{z} = \cdots =$$

in more detail. Let us represent the variable z in the form $z = [z] + \{z\} = n + \alpha$, $0 \le \alpha < 1$ then, the previous equality continues:

$$= \Psi(\alpha+2) + \frac{1}{\alpha+2} + \frac{1}{\alpha+3} + \cdots + \frac{1}{z}.$$

The estimate:

$$\int_{\alpha+k}^{\alpha+k+1} \frac{dt}{t} \le \frac{1}{\alpha+k} \le \int_{\alpha+k-1}^{\alpha+k} \frac{dt}{t}$$

holds. Summation over all inequalities provides

$$\int_{\alpha+2}^{\alpha+n+1} \frac{dt}{t} \le \sum_{k=2}^{n} \frac{1}{\alpha+k} \le \int_{\alpha+1}^{\alpha+n} \frac{dt}{t}.$$

The left- and the right-hand sides of the inequality are integrable:

$$\ln(z+1) - \ln(\alpha+2) \le \sum_{k=2}^{n} \frac{1}{\alpha+k} \le \ln z - \ln(\alpha+1).$$

Adding $\Psi(\alpha+2)$ to all parts of the inequality, we obtain:

$$\ln(z+1) + \Psi(\alpha+2) - \ln(\alpha+2) \le \Psi(z+1) \le \ln z + \Psi(\alpha+2) - \ln(\alpha+1).$$

Thus, when $z \ge 3$, we have the estimate

$$\ln z - C(z) \le \Psi(z) \le \ln z + C(z),$$

where $C(z)$ is a finite expression. $\triangleright$

Let us introduce a function, having the meaning of the number of workpieces in the cutting card

$$\Phi(m) = \left\{ \frac{m}{n} \ : \ F(m,n) \to max \right\}.$$

Theorem 1. *If $m \to \infty$, the value $\Phi(m) \simeq \beta \cdot \ln m$, where β is a constant.*

$\triangleleft$ When m is fixed, find the denominator minimum:

$$f(t) = \left(\Gamma\left(t+1\right)\right)^{\frac{m}{t}} \Gamma\left(\frac{m}{t}+1\right)$$

For this purpose, differentiate it with respect to t:

$$\left(\Gamma\left(t+1\right)\right)^{\frac{m}{t}} m\Gamma\left(\frac{m+t}{t}\right)\left(-\ln\left(\Gamma\left(t+1\right)\right) + \Psi\left(t+1\right)t - \Psi\left(\frac{m}{t}+1\right)\right)t^{-2}$$

Consider the equation

$$\ln\Gamma(t+1) - t\Psi(t+1) + \Psi\left(\frac{m}{t}+1\right) = 0. \tag{2}$$

Let us introduce the notation $A(t) = \ln\Gamma(t+1) - t\Psi(t+1)$ and use the known equalities $\Gamma(z+1) = z\Gamma(z)$ and $\Psi(z+1) = \Psi(z) + 1/z$ for calculating $A(t+1)$:

$$A(t+1) = \ln\Gamma(t+2) - (t+1)\Psi(t+2) = \ln((t+1)\Gamma(t+1)) - \left(\Psi(t+1) + \frac{1}{t+1}\right)(t+1) =$$

$$= \ln(t+1) + \ln\Gamma(t+1) - t\Psi(t+1) - \Psi(t+1) - 1 = A(t) + \ln(t+1) - \Psi(t+1) - 1.$$

Thus,

$$A(t+1) - A(t) = \ln(t+1) - \Psi(t+1) - 1. \tag{3}$$

The right-hand side (3) is finite according to Lemma 3. Therefore, we can estimate the rate of growth $A(t) : A(t) \simeq \alpha t$, $\alpha = const$. Let us assume that a certain t_m is a solution to Equation (2). Then, substituting the estimate for $A(t)$ into it, we obtain $\Psi\left(\frac{m}{t_m}+1\right) \simeq \alpha t_m$.

Let us use Lemma 3 once more: $\ln\left(m/t_m\right) \simeq \alpha t_m$, whence $t_m \simeq \beta \ln m$, $\beta = const$. $\triangleright$

4. Conclusion

Graph of the function $\Phi(m)$ was constructed pointwise for integer values $m \in \{1, \ldots, 1000\}$, see Fig.1. The value of the constant $\beta \approx 0.98$ is defined by the graph.

The obtained result is consistent with the data of Schwerin P., Wascher G., who singled out hard classes for solving by sequencial algorithms for $m \in [40..200]$ experimentally, see [7]. The result is singled out by a rectangle on Fig.1.

Results of Theorem 1 can also be used for formulating more laborious test problems with a maximum number of admissible solutions.

The authors are sincerely grateful Professor R.S. Yulmukhametov for valuable remarks and proof of Theorem 1.

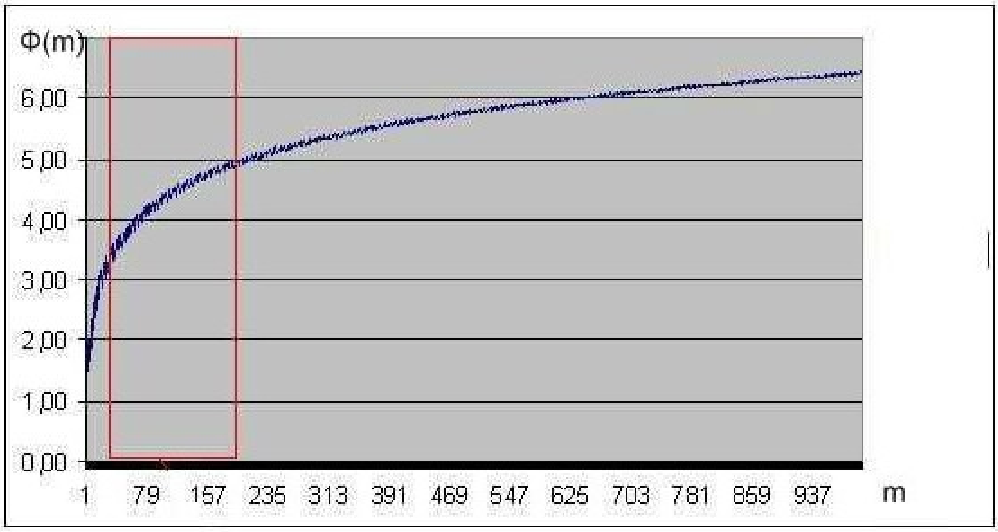

FIGURE 1. Graph $\Phi(m)$, obtained numerically

BIBLIOGRAPHY

1. M.P. Gary, D.S. Johnson. *Calculating machines and intractable problems.* Moscow. Mir. 1982.
2. I.V. Romanovskii *Algorithms for soling extreme problems.* Moscow. Nauka. 1977. 88 p.
3. S.V. Katsev. *On a class of discrete minimax problems* // Cybernetics. 1979. No. 5. P. 139–141.
4. S. Martello and P.Toth. *Lower Bounds and Reduction procedures for the Bin Packing Problem.* Discrete Applied Mathematics, 1990.
5. E.A. Mukhacheva, V.M. Kartak. *Modified branch and bound method: algorithm and numerical experiment for a one-dimensional cutting stock problem* // Information technologies. 2000. No. 9. P. 15–21.
6. P. Schwerin, G. Wašcher *The Bin-Packing Problem: A problem Generator and Some Numerical Experiments with FFD Packing and MTP* // International Transactions in Operational Research. 1997. V. 4. ?5/6. Pp. 377–389.
7. P. Schwerin, G. Wašcher *A new lower bound for the Bin-Packing Problem and its integration into MTP* // Betreils-wirtschaftliche Diskussionsbeitrage. Beitrag 1998. No. 26 Martin-Luher Universitat Hall-Wittenberg. 23 p.
8. G. Andrews *Splitting theory.* Moscow. Nauka. 1982.
9. Andrea Laforgia *Further Inequalities for the Gamma Function* // Mathematics of Computation. 1984. Vol. 42. No. 166. Pp. 597–600.
10. W. Gautschi *A harmonic mean inequality for the gamma function* // SIAM J. Math. Anal. 1974. V. 5. Pp. 278–281.
11. W. Gautschi *Some mean value inequalities for the gamma function* // SIAM J. Math. Anal. 1974. V. 5. Pp. 282–292.
12. Digital Library of Mathematical Functions. http://dlmf.nist.gov/5.6.E4.

Kartak Vadim Mikhailovich,
Ufa State Aviation Technical University,
450000, K.Marx Str. 12, Ufa, Russia
E-mail: kvmail@mail.ru

Kartak Vera Valer'evna,
Bashkir State University,
450074, Z.Validi Str., 32, Ufa, Russia
E-mail: kvera@mail.ru

Translated from Russian by E.D. Avdonina.