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ON DECAY RATE OF SOLUTION TO DEGENERATING LINEAR PARABOLIC EQUATIONS

V.F. GILIMSHINA, F.KH. MUKMINOV

Abstract. Existence and uniqueness of the solution to a linear degenerating parabolic equation is established in unbounded domains by the method of Galerkin's approximations. The first and the third boundary-value conditions are considered. The upper estimate of the solution decay rate is established when $x \to \infty$ in view of the influence of higher-order coefficients of the equation. The upper estimate of the decay rate of the solution $t \to \infty$ depending on the geometry of the unbounded domain is proved as well.

Keywords: degenerating parabolic equation, decay rate of solution, upper estimates, existence of solution.

1. Introduction

Let Ω be an unbounded domain of the space $\mathbb{R}^n$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $n \geq 2$. Let us consider a linear second-order equation

$$u_t = \sum_{i,j=1}^{n} (a_{ij}(t,x)u_{x_i})_{x_j} + \sum_{i=1}^{n} (b_i u_{x_i} + (c_i u)_{x_i}) - d(t,x)u \tag{1}$$

in the cylindric domain $D = \{t > 0\} \times \Omega$. The following condition is imposed on elements of the symmetric matrix $\{a_{ij}\}$: there exist a positive function $s(t,x)$ continuous in D, and a positive number Υ such that the following inequalities hold for any vector $y \in \mathbb{R}^n$ and almost for all $(t,x) \in D$:

$$s(t,x)|y|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(t,x) y_i y_j \leqslant s(t,x)\Upsilon |y|^2. \tag{2}$$

The function $s(t,x)$ can vanish on the boundary of the domain, and the functions $s(t,x)$, $d(t,x)$ and $s^{-1}(t,x)$ are assumed to be integrable with respect to any bounded subset D. The following restrictions are imposed on measurable lower coefficients:

$$\sum_{i=1}^{n} (b_i(t,x) - c_i(t,x))^2 \leqslant \frac{1}{2} s(t,x) d(t,x). \tag{3}$$

We suppose that there are numbers C and $\delta_0 > 0$ such that the inequalities

$$d(\tau,x) \leqslant C d(t,x), \quad s(\tau,x) \leqslant C s(t,x), \quad |\tau - t| \leq \delta_0, x \in \Omega \tag{4}$$

hold for all $t > 0$.

Boundary conditions of the first and the third type are given on the side boundary of the cylinder D:

$$u(t,x)\Big|_{\Gamma_1} = 0; \qquad \Big(\frac{\partial u}{\partial N} + \sum_{i=1}^{n} n_i c_i u\Big)\Big|_{\Gamma_2} = 0. \tag{5}$$

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Here $\Gamma_1 \subset \Gamma = (0,\infty)\times\partial\Omega$ is an arbitrary closed subset of the side boundary of the cylinder Γ and Γ_2 is its complement $\Gamma_2 = \Gamma\backslash\Gamma_1$; $\frac{\partial u}{\partial N} = \sum\limits_{i,j=1}^{n} a_{ij}u_{x_i}n_j$. We will deal with the generalized solution of the problem (1), (5) with the initial condition

$$u(0,x) = \varphi(x) \in L_2(\Omega), \tag{6}$$

defined (see §2 below) without formal participation of the condition (5). Nevertheless, this generalized solution satisfies the conditions (5) under the condition of sufficient "regularity" of the set Γ_1, smoothness of the boundary $\partial\Omega$ and coefficients of Equation (1).

The present paper is devoted to investigation of dependence of the decay rate of solution to the problem (1), (5), (6) when $t\to\infty$ on the geometry of an unbounded domain Ω and behaviour of eigennumbers of the matrix $\{a_{ij}(t,x)\}$ for $x\to\infty$.

A.K. Gushchin obtain the following result for the second mixed problem for a second-order parabolic equation in the works [2, 3]. The estimate

$$|u(t,y)| \leqslant \frac{\|\varphi\|_{L_1(Q)}}{v(\sqrt{t})}, \qquad y\in Q,$$

where $v(r) = \mathrm{mes}\{y\in Q : |y| < r\}$, is established there for a wide class of domains in order to solve the second mixed problem. The estimate is also proved to be exact. A more complete investigation of the dependence of behaviour when the time value is large of solution to the second mixed problem on the domain geometry and on the initial function has been carried out by A.V. Lezhnev in [14]. V.I. Ushakov [21] obtained results close to that of A.K. Gushchin for the third mixed problem in a noncylindric domain. F.Kh. Mukminov proved the decay rate estimate of solution of the first mixed problem in case of a second-order parabolic equation and demonstrated that it is exact in a class of unbounded monotonously expanding domains of rotation in [17]. A series of technical requirements for obtaining the upper estimate as well as for proving that this estimate is exact is imposed in the works [17, 8] about the decay rate of solution of the first mixed problem. In particular, these conditions for the domain Ω are as follows in [17]:

$$\lim_{r\to\infty} r^2\lambda(r) = \infty, \qquad \lim_{r\to\infty}\lambda(r) = 0, \tag{7}$$

where $\lambda(r)$ is the first eigenvalue of the Dirichlet problem for the Laplace operator in the intersection of the domain with a ball of the radius r. The following estimate of solution with a finite nonnegative initial function $\varphi\not\equiv 0$ is established under these conditions:

$$|u(t,x)| \leqslant M\exp(-\chi\frac{r^2(t)}{t})\|\varphi\|_{L_2(\Omega)} \tag{8}$$

with positive constants χ, M. Here $r(t)$ is a function inverse to $F(r) = \frac{r}{\sqrt{\lambda(r)}}$, $r>0$. In [1], exact estimates of solution to a parabolic equation of the fourth and sixth orders with the Rickyies boundary conditions on the side border of an unbounded cylindric domain are obtained. Decay rate estimates for solutions of pseudo-differential and quasi-linear parabolic equations are obtained in the works [12] and [13], respectively.

Let us formulate our result. Define the function

$$\lambda(\tau,r) = \inf_{g\in C_0^\infty(\mathbb{R}^n\backslash\Gamma_1^\tau)} \frac{\int\limits_{\Omega[r]} (s(\tau,x)|\nabla g|^2 + d(\tau,x)g^2)dx}{\int\limits_{\Omega[r]} g^2dx}, \tag{9}$$

where $\Omega[r] = \{x\in\Omega \mid |x| < r\}$, $\Gamma_1^\tau = \Gamma_1\cap\{t=\tau\}$.

Theorem 1. *Let us assume that* $u(t,x)$ *is a solution to the problem* (1), (5), (6) *and the scalar product* $(x,\mathbf{c}) \geq 0$. *Then, there are numbers* $\kappa > 0$, C, T, *depending only on* n, Υ, R_0 ($\mathrm{supp}\varphi \subset \Omega[R_0]$), *such that the following inequality holds for all* $t > T$:

$$\int\limits_{\Omega} u^2(t,x)dx \leqslant C\exp\left(-\kappa M_m(t)\right)\|\varphi\|^2_{L_2(\Omega)}, \tag{10}$$

where

$$M_m(t) = \sup_{r\geq R_0} \min\left(\frac{1}{t}(\int\limits_{R_0}^{r}\frac{d\tau}{\sqrt{s_c(\tau)}})^2, \int\limits_{0}^{t}\lambda(\tau,r)d\tau\right),$$

$s_c(\tau) = \sup\{s(t,x) \mid t > 0,\ |x| = \tau\}$.

In case of a uniformly parabolic equation ($s \equiv 1$) and the Dirichlet boundary conditions ($\Gamma_1 = \Omega$), the estimate of the theorem is reduced to the one known from [17].

Note that the function $\lambda(\tau,r)$ may vanish for some values of τ (e.g., if $\Gamma_1^\tau = \emptyset$). In this case, methods based on the notion of a λ–sequence [11] are inapplicable. Therefore, in Proposition 1 decrease of solution is established beforehand for $x \to \infty$, which is similar to the decrease of the fundamental solution of the heat equation, but in view of the behaviour of the function $s(t,x)$ at infinity.

Section §3 provides examples demonstrating the estimate of the theorem for specific domains and equations.

2. Existence and uniqueness of the generalized solution of the mixed problem for a parabolic equation

Let us introduce the following notation: $D_a^b = (a,b)\times\Omega$, $D^T = D_0^T$, $D = D_0^\infty$,

$$(u,w)_{D^T} = \int\limits_{D^T} uwdxdt, \quad (u,w)_{A,D^T} = \sum_{\alpha,\beta=1}^{n}\int\limits_{D^T}(a_{\alpha\beta}(t,x)u_{x_\alpha}w_{x_\beta} + duw)dxdt.$$

The norm in $L_2(D^T)$ is denoted by $\|u\|_{D^T}$. Let us define the norms

$$\|u\|^2_{H^{0,1}(D^T)} = \|u\|^2_{D^T} + (u,u)_{A,D^T}; \|u\|^2_{H^{1,1}(D^T)} = \|u\|^2_{H^{0,1}(D^T)} + \|u_t\|^2_{D^T}$$

on the set of restrictions on D^T for functions from $C_0^\infty(\mathbb{R}^{n+1}\backslash\Gamma_1 \cup \{t = T\})$. The corresponding complements of this linear normalized spaced are denoted by $\overset{\circ}{H}{}^{0,1}_{A}(D^T;\Gamma_1)$ and $\overset{\circ}{H}{}^{1,1}_{A}(D^T;\Gamma_1)$.

A generalized solution of the problem (1), (5), (6) in D^T is a function $u(t,x) \in \overset{\circ}{H}{}^{0,1}_{A}(D^T;\Gamma_1)$, satisfying the integral identity:

$$\int\limits_{D^T}\left(-uv_t + \sum_{i,j=1}^{n}a_{ij}(t,x)u_{x_i}v_{x_j} + \sum_{i=1}^{n}(c_i uv_{x_i} - b_i u_{x_i}v) + duv\right)dxdt =$$

$$= \int\limits_{\Omega}\varphi(x)v(0,x)dx, \tag{11}$$

for any function $v(t,x) \in \overset{\circ}{H}{}^{1,1}_{A}(D^T;\Gamma_1)$.

The function $u(t,x)$ is a solution to the problem (1), (5), (6) in D if it is a solution to the problem (1), (5), (6) for all $T > 0$ in D^T.

The generalized solution to the problem (1), (5), (6) in D^T exists and it is unique. In order to prove this statement we use the method described by V.I. Ushakov in [21] consisting in construction of the functions $u^n(t,x)$, converging weakly in to the solution $u(t,x)$.

Let us choose a set of linearly independent functions $w_i(t,x) \in C_0^\infty(\mathbb{R}^{n+1}\backslash\Gamma_1 \cup \{t=T\})$ so that their linear envelope is dense in $\overset{\circ}{H}{}^{1,1}_A(D^T;\Gamma_1)$.

Galerkin's approximations will be sought for in the form

$$u^l(t,x) = \sum_{i=1}^n C_i^l w_i(t,x). \tag{12}$$

Equations in the unknown coefficients are derived from the requirement

$$\int\limits_{D^T} (u^l(w_s)_t + \sum_{i,j=1}^n a_{ij}(t,x)u^l_{x_i}(w_s)_{x_j} + \sum_{i=1}^n (c_i u^l (w_s)_{x_i} - b_i(t,x)u^l_{x_i} w_s) + \tag{13}$$

$$+ d(t,x)u^l w_s) dxdt = \int\limits_\Omega \varphi(x) w_s(0,x)dx, \qquad s=\overline{1,n}.$$

The conditions (13) lead to the system of linear equations

$$\sum_{k=1}^n A_{ks}C_k^l = b_s, \qquad s=\overline{1,n}. \tag{14}$$

In what follows, a one-valued solvability of the linear system (14) is to be established.

First, let us assume that the system (14) has a solution (e.g., $C_k^l = 0$ when $b_s = 0$). Note that substituting $u = e^t$, one can achieve the inequality

$$\widetilde{d} = d + 1 \geq 1. \tag{15}$$

Let us prove that the set u^l of Galerkin's approximations is bounded in the space $\overset{\circ}{H}{}^{0,1}_A(D^T;\Gamma_1)$. Let us multiply the equality (13) by C_s^l and make the summation. We obtain

$$\int\limits_{D^T} (-u^l_t u^l + \sum_{i,j=1}^n a_{ij}(t,x)u^l_{x_i}u^l_{x_j} + \sum_{i=1}^n (c_i u^l u^l_{x_i} - b_i(t,x)u^l_{x_i}u^l) + d(t,x)u^l u^l) dxdt =$$

$$= \int\limits_\Omega \varphi(x)u^l(0,x)dx. \tag{16}$$

Integrating the first addend in (16) with respect to $t \in (0,T)$ and making use of the condition (3), we have

$$\frac{1}{2}\int\limits_\Omega (u^l(0))^2 dx + \int\limits_{D^T}\left(\sum_{i,j=1}^n a_{ij}(t,x)u^l_{x_i}u^l_{x_j} + d(t,x)u^l u^l\right)dxdt \leqslant$$

$$\leqslant \int\limits_{D^T} |c-b||u^l\nabla u^l| dxdt + \int\limits_\Omega \varphi(x)u^l(0,x)dx \leqslant$$

$$\leqslant \int\limits_{D^T} \sqrt{s(t,x)d(t,x)}|u^l\nabla u^l|dxdt + \int\limits_\Omega \varphi^2(x)dx + \int\limits_\Omega \frac{u^{l2}(0,x)}{4}dx \leqslant$$

$$\leqslant \int\limits_{D^T} (\frac{s(t,x)(\nabla u^l)^2}{2} + \frac{d(u^l)^2}{2})dxdt + \int\limits_\Omega \varphi^2(x)dx + \int\limits_\Omega \frac{u^{l2}(0,x)}{4}dx.$$

Using the condition (2), we establish that

$$\int\limits_\Omega \frac{(u^l(0))^2}{2}dx + \|u^l\|^2_{H^{0,1}_A(D^T;\Gamma_1)} \leqslant 2\int\limits_\Omega \varphi^2(x)dx. \tag{17}$$

It follows from this estimate that if $\varphi = 0$ then $u^l = 0$. Linear independence of the functions $w_i(t,x)$ provides $C_i^l = 0$. It means that the homogeneous system (14) has only a zero solution. Hence, the nonhomogeneous system is solvable uniquely.

Whence, one concludes that the set u^l is bounded in the space $\overset{\circ}{H}{}_A^{0,1}(D^T;\Gamma_1)$. Therefore, one can single out a subsequence converging weakly in this space to a certain function $u \in \overset{\circ}{H}{}_A^{0,1}(D^T,\Gamma_1)$. In order to avoid the pile of indices, we consider that the sequence itself converges weakly.

Obviously, (13) takes the form

$$\int\limits_{D^T} \left(-u(w_s)_t + \sum_{i,j=1}^{n} a_{ij}(t,x) u_{x_i} (w_s)_{x_j} + \right. \tag{18}$$

$$\left. + \sum_{i=1}^{n} (c_i u (w_s)_{x_i} - b_i u_{x_i}(w_s)) + du(w_s) \right) dxdt = \int\limits_{\Omega} \varphi(x) w_s(0,x) dx$$

upon turning to the limit when $l \to \infty$. Note that (18) holds not only for the functions $v = d_s w_s$ with the constants d_s, but for sums of such functions as well. It remains only to mention that one can approximate any function w from $C_0^\infty(\mathbb{R}^{n+1} \backslash \Gamma_1 \cup \{t = T\})$ with respect to the norm of the space $\overset{\circ}{H}{}_A^{1,1}(D^T;\Gamma_1)$ by means of functions of the form $v^m = \sum\limits_{s=1}^{m} d_s w_s$

Now, let us demonstrate that solution to the problem (1), (5), (6) is unique.

Let us denote by $v_h(t,x)$ the Steklov averaging of the function $v(t,x)$:

$$v_h(t,x) = \frac{1}{h} \int_t^{t+h} v(\tau,x) d\tau,$$

having the following properties:

1)$(v, u_{-h}) = (v_h, u)_{L_2(\mathbb{R}^{n+1})}$,

2)if $v \in \overset{\circ}{H}{}_A^{0,1}(D_0^T;\Gamma_1)$ then $(v_h)_{x_i} = (v_{x_i})_h$,

3)if $v, v_t \in L_2(\mathbb{R}^{n+1})$ then $(v_t)_h = (v_h)_t$,

4) if $v \in L_2(D^T)$ then for any $\delta > 0$, the convergence $v_h \to v$ exists in $L_2(D^{T-\delta})$ when $h \to 0$ $(h < \delta)$.

5) if $v \in \overset{\circ}{H}{}_A^{0,1}(D_0^T;\Gamma_1)$ then for any $\delta \in (0, \delta_0)$ the convergence $v_h \to v$ takes place in $\overset{\circ}{H}{}_A^{0,1}(D_0^{T-\delta};\Gamma_1)$ when $h \to 0$ $(h < \delta)$.

Let us prove the property 5). First, et us establish the continuity of the shift operator $T_z f = f(t+z,x)$, $T_z f \to f$ when $z \to 0$ in the weighted space $L_{2,d}(\mathbb{R}^{n+1})$ with the norm

$$\|f\|_{L_{2,d}}^2 = \int\limits_{\mathbb{R}^{n+1}} d(t,x) f^2(x) dxdt,$$

where $d(t,x)$ is a function integrable with respect to any compact. Let us demonstrate that $T_z f$ is a uniformly bounded operator for $z \in [z - \delta_0, z + \delta_0]$ using the inequality (4),

$$\|T_z f\|_{L_{2,d}}^2 = \int\limits_{\mathbb{R}^{n+1}} d(t,x) f^2(t+z,x) dxdt \leqslant$$

$$\leqslant \int\limits_{\mathbb{R}^{n+1}} C d(t+z,x) f^2(t+z,x) dxdt \leqslant C \|f\|_{L_{2,d}}^2.$$

Then, let $v \in C_0^\infty(\mathbb{R}^{n+1})$ and $\operatorname{supp} v \subset B_R$, B_R be a ball of the radius R. In this case, we have

$$\|T_z v - v\|^2_{L_{2,d}} = \int\limits_{B_{R+1}} d(t,x)(v(t+z,x) - v(t,x))^2 dxdt \leqslant$$

$$\int\limits_{B_{R+1}} d(t,x)\varepsilon^2 dxdt \leqslant C_1\varepsilon$$

due to the uniform continuity of the function v in the ball B_R. Thus, $T_z v \to v$ when $z \to 0$. Since T_z is bounded and $C_0^\infty(\mathbb{R}^{n+1})$ is dense in $L_{2,d}(\mathbb{R}^{n+1})$, then $T_z f \to f$ when $z \to 0$ for an arbitrary function $f \in L_{2,d}(\mathbb{R}^{n+1})$.

Let us prove now that $v_h \to v$ when $h \to 0$ in $L_{2,d}(\mathbb{R}^{n+1})$:

$$(v_h - v)^2 = \left(\frac{1}{h}\int\limits_t^{t+h} v(\tau,x)d\tau - v(t,x)\right)^2 \leqslant$$

$$\leqslant \frac{1}{h^2}\int\limits_t^{t+h} 1^2 d\tau \int\limits_t^{t+h} (v(\tau,x) - v(t,x))^2 d\tau.$$

Upon substituting $\tau = t + z$, we have

$$(v_h - v)^2 \leqslant \frac{1}{h}(\int\limits_0^h \big(v(t+z,x) - v(t,x))^2 dz\big).$$

Let us integrate the latter inequality with respect to t and x :

$$\int\limits_{\mathbb{R}^{n+1}} d(t,x)(v_h - v)^2 dtdx \leqslant \int\limits_{\mathbb{R}^{n+1}} \frac{d(t,x)}{h}(\int\limits_0^h \big(v(t+z,x) - v(t,x))^2 dz\big)\, dtdx =$$

$$= \frac{1}{h}\int\limits_0^h \|T_z v - v\|^2_{L_{2,d}} dz.$$

Whence, due to the convergence $T_z v \to v$ for $z \to 0$, we obtain that $v_h \to v$ when $h \to 0$.

Likewise, $(v_h)_{x_i} = (v_{x_i})_h \to v_{x_i}$ in $L_{2,s}(\mathbb{R}^{n+1})$ when $h \to 0$. In total, one can readily deduce the property 5) from these two congruencies.

Let us substitute the test function v_{-h}, where v is from the space $C_0^\infty(D_0^{T-\delta}\backslash\Gamma_1)$, into the integral identity (11). This is possible because $v_{-h} \in C_0^\infty(D_0^T)$ when $0 < h < \delta$. Using properties of the Steklov averaging, we have

$$\int\limits_{D^T}\left[(u_h)_t v + \sum_{i,j=1}^n (a_{ij}u_{x_i})_h v_{x_j} + \sum_{i=1}^n ((c_i u)_h v_{x_i} - (b_i u_{x_i})_h v) + (du)_h v\right] dxdt = 0. \tag{19}$$

Passing to the limit, one proves that the latter correlation holds not only for the functions $v \in C_0^\infty(D_0^{T-\delta}\backslash\Gamma_1)$, but also for functions $v \in \overset{\circ}{H}{}^{0,1}_A(D_0^{T-\delta};\Gamma_1)$.

Note that the equalities (19) have the form

$$\int\limits_{D^T} (u_h)_t v dxdt = l_h(v), \tag{20}$$

where $l_h(v)$ is a linear functional in the space $\overset{\circ}{H}{}^{0,1}_A(D_0^{T-\delta};\Gamma_1)$.

Let us prove the uniform boundedness of the linear functional $l_h(v)$ when $|h| < \delta_0$ in a unit sphere of the space $\overset{\circ}{H}{}^{0,1}_{A}(D_0^{T-\delta};\Gamma_1)$.

$$l_h(v) = l_h^a(v) + l_h^c(v) + l_h^b(v) + l_h^d(v), \tag{21}$$

where $l_h^a(v) = -\int\limits_{D^{T-\delta}} \sum\limits_{i,j=1}^{n} (a_{ij}u_{x_i})_h v_{x_j} dxdt$, $l_h^c(v) = -\int\limits_{D^{T-\delta}} \sum\limits_{i=1}^{n} (c_i u)_h v_{x_i} dxdt$,
$l_h^b(v) = \int\limits_{D^{T-\delta}} \sum\limits_{i=1}^{n} (b_i u_{x_i})_h v) dxdt$, $l_h^d(v) = -\int\limits_{D^{T-\delta}} (du)_h v dxdt$. Consider $l_h^a(v)$, in view of $s(\tau,x) \leqslant Cs(t,x)$, $\tau \in [t-\delta; t+\delta]$, we have:

$$|l_h^a(v)| \leqslant |\int\limits_{D^{T-\delta}} \sum_{i,j=1}^{n} (a_{ij}u_{x_i})_h v_{x_j} dxdt| \leqslant$$

$$\leqslant \int\limits_{D^{T-\delta}} \left(\frac{\Upsilon}{h} \int\limits_{t}^{t+h} s(\tau,x)|\nabla u(\tau,x)| d\tau \right) |\nabla v(t,x)| dxdt \leqslant$$

$$\leqslant \int\limits_{D^{T-\delta}} \left(\frac{C_1 s(t,x)}{h} \int\limits_{t}^{t+h} |\nabla u(\tau,x)| d\tau \right) |\nabla v(t,x)| dxdt \leqslant$$

$$\leqslant \int\limits_{D^{T-\delta}} C_1 s(t,x) \left(\frac{1}{h^2} \left(\int\limits_{t}^{t+h} 1 \cdot |\nabla u(\tau,x)| d\tau \right)^2 + |\nabla v(t,x)|^2 \right) dxdt.$$

Upon changing the order of integration, we have

$$\frac{C_1}{h} \int\limits_{D^{T-\delta}} \left(\int\limits_{t}^{t+h} s(t,x)|\nabla u(\tau,x)|^2 d\tau \right) dxdt \leqslant$$

$$\leqslant \frac{C_1}{h} \int\limits_{\Omega} \int\limits_{0}^{T} |\nabla u(\tau,x)|^2 \left(\int\limits_{\tau-h}^{\tau} s(t,x) dt \right) dxd\tau \leqslant$$

$$\leqslant C_2 \int\limits_{\Omega} \int\limits_{0}^{T} s(\tau,x)|\nabla u(\tau,x)|^2 dxd\tau = C_3$$

in the first addend.

Thus, $|l_h^a(v)| \leqslant C_4$.

$$|l_h^c(v)| = |\int\limits_{D^{T-\delta}} \sum_{i=1}^{n} (c_i u)_h v_{x_i} dxdt| \leqslant |\int\limits_{D^{T-\delta}} \sum_{i=1}^{n} \left(\frac{1}{h} \int\limits_{t}^{t+h} c_i u d\tau \right) v_{x_i} | dxdt \leqslant$$

$$\leqslant \int\limits_{D^{T-\delta}} \sum_{i=1}^{n} \left(\frac{\left(\frac{1}{h^2} \int\limits_{t}^{t+h} 1 \cdot c_i u d\tau \right)^2}{s(t,x)} + v_{x_i}^2(t,x) s(t,x) \right) dxdt \leqslant$$

$$\leqslant \int\limits_{D^{T-\delta}} \left(\frac{1}{h} \int\limits_{t}^{t+h} \frac{nA^2 s(\tau,x) d(\tau,x) u^2(\tau,x) d\tau}{s(t,x)} + |\nabla v|^2(t,x) s(t,x) \right) dxdt \leqslant$$

$$\leqslant \int\limits_{D^{T-\delta}} \left(\frac{nA^2C}{h} \int\limits_{t}^{t+h} d(\tau,x) u^2(\tau,x) d\tau \right) dxdt + C_5.$$

Upon changing the order of integration, we have

$$|l_h^c(v)| \leqslant \frac{nA^2C}{h} \int\limits_{\Omega} \int\limits_{0}^{T} \int\limits_{\tau-h}^{\tau} d(\tau,x) u^2(\tau,x) dt dx d\tau + C_5 =$$

$$= nA^2C \int\limits_{\Omega} \int\limits_{0}^{T} d(\tau,x) u^2(\tau,x) dx d\tau + C_5 \leqslant C_6$$

in the first expression.

Let us consider $l_h^b(v)$:

$$|l_h^b(v)| = |\int\limits_{D^{T-\delta}} \sum_{i=1}^{n} (b_i u_{x_i})_h v dxdt| \leqslant |\int\limits_{D^{T-\delta}} \sum_{i=1}^{n} \left(\frac{1}{h} \int\limits_{t}^{t+h} b_i u_{x_i} d\tau \right) v dxdt| \leqslant$$

$$\leqslant \int\limits_{D^{T-\delta}} \sum_{i=1}^{n} \left(\frac{\left(\frac{1}{h^2} \int\limits_{t}^{t+h} 1 \cdot b_i u_{x_i} d\tau \right)^2}{d(t,x)} + v^2(t,x) d(t,x) \right) dxdt \leqslant$$

$$\leqslant nA^2C \int\limits_{\Omega} \int\limits_{0}^{T} s(\tau,x) |\nabla u|^2(\tau,x) dt dx d\tau + C_7 \leqslant C_8.$$

Likewise, we obtain that $|l_h^d(v)| \leqslant C_9$. Thus, it is proved that the linear functional $l_h(v)$ is bounded.

Let us substitute the function $v = (u_{h_1} - u_{h_2})\chi(t_1,t_2) \in \overset{\circ}{H}{}^{0,1}_{A}(D_0^{T-\delta};\Gamma_1)$, where $\chi(t_1,t_2)$ is a characteristic function of the interval (t_1,t_2), into the equality $(20)_{h_1}$–$(20)_{h_2}$. We have

$$|\int\limits_{t_1}^{t_2} \int\limits_{\Omega} ((u_{h_1})_t - (u_{h_2})_t)(u_{h_1} - u_{h_2}) dxdt| =$$

$$= |(l_{h_1} - l_{h_2})(\chi(u_{h_1} - u_{h_2}))| \leqslant C \|(u_{h_1} - u_{h_2})\|_{H_A^{0,1}(D^T;\Gamma_1)} \leqslant \varepsilon.$$

The latter inequality follows from the convergence $u_h \to u$ in the space $\overset{\circ}{H}{}^{0,1}_{A}(D_0^{T-\delta};\Gamma_1)$ when h_1, h_2 are sufficiently small. Upon integrating with respect to t, we have

$$\int\limits_{\Omega} (u_{h_1} - u_{h_2})^2(t_1,x) dx \leqslant \int\limits_{\Omega} (u_{h_1} - u_{h_2})^2(t_2,x) dx + 2\varepsilon.$$

Let us integrate the latter inequality with respect to $t_2 \in [t_1, T-\delta]$

$$(T - \delta - t_1) \int\limits_{\Omega} (u_{h_1} - u_{h_2})^2(t_1,x) dx \leqslant \|(u_{h_1} - u_{h_2})\|^2_{L_2(D^{T-\delta})} + 2\varepsilon(T - \delta - t_1).$$

Since $u_h \to u$ in $L_2(D^{T-\delta})$ then we have the inequality

$$\int\limits_{\Omega} (u_{h_1} - u_{h_2})^2(t_1, x)dx \leqslant \frac{\varepsilon_1}{\delta} + 2\varepsilon$$

when $t_1 < T - 2\delta$. Whence follows the uniform mutual convergence of the family of functions $u_h(t_1, x)$ in $L_2(\Omega)$ with respect to t_1. Therefore, $u_h(t, x) \rightrightarrows u(t, x)$ in $L_2(\Omega)$ when $h \to 0$ uniformly with respect to $t \in [0, T - 2\delta]$, and the limiting function is continuous with respect to t in the norm $L_2(\Omega)$. Let us substitute the function $v = u_h\chi(0, t)$ into (19)

$$\int\limits_{D_0^t} ((u_h)_t u_h + \sum_{i,j=1}^{n} (a_{ij}u_{x_i})_h (u_h)_{x_j} + \sum_{i=1}^{n} ((c_i u)_h (u_h)_{x_i} -$$

$$-(b_i u_{x_i})_h u_h) + (du)_h u_h) dxdt = 0.$$

Upon integration of the first addend with respect to t and passing to the limit $h \to 0$, we have

$$\frac{1}{2}\int\limits_{\Omega} u^2(t, x)dx + \int\limits_{D_0^t} [\sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j} + \sum_{i=1}^{n} ((c_i - b_i)uu_{x_i} + du^2]dxdt = \tag{22}$$

$$= \frac{1}{2}\int\limits_{\Omega} u^2(0, x)dx.$$

If we prove that $u(0, x) = \varphi(x)$, the latter correlation corresponds to (11). To this end, let us substitute into the identity (11) the test function $v(t, x) = \eta(\frac{t}{\varepsilon})\psi(x)$, where $\eta(t) = 1 - t$ for $t \in [0, 1]$ and $\eta(t)$ is constant in the remaining intervals $(-\infty, 0]$, $[1, \infty)$.

Since $v_t = -\frac{1}{\varepsilon}\psi(x)$, the identity (11) takes the form

$$\int\limits_0^{\varepsilon}\int\limits_{\Omega} \frac{1}{\varepsilon}\psi(x)u(t, x)dtdx + l^{\varepsilon}(\psi) = \int\limits_{\Omega} \varphi(x)\psi(x)dx,$$

where the linear functional $l^{\varepsilon}(\psi)$ is tending to zero when $\varepsilon \to 0$. Upon passing to the limit with $\varepsilon \to 0$, we have

$$\int\limits_{\Omega} \psi(x)u(0, x)dx = \int\limits_{\Omega} \varphi(x)\psi(x)dx$$

for any $\psi \in C_0^{\infty}(\Omega)$. This proves the validity of the initial condition $u(0, x) = \varphi(x)$.

3. Upper estimate for solution of a parabolic equation

In what follows, we deduce Theorem 1 from the following statement in case of the function $s(t, x)$ bounded in D.

Proposition 1. *Let us assume that $u(t, x)$ is a solution to the problem (1)-(3) with the initial function φ, equal to zero outside the sphere of the radius R_0, and the scalar product $(x, \mathbf{c}) \geq 0$ in D^T. Then, for all $t > 0$, $r \geq R_0$ the following inequality holds:*

$$\int\limits_{\Omega\setminus\Omega[r]} u^2(t, x)dx \leqslant A_1 \exp\left(-\widetilde{C}t^{-1}(\int\limits_{R_0}^{r} \frac{d\tau}{\sqrt{s_c(\tau)}})^2\right), \tag{23}$$

where A_1, $\widetilde{C}$ are constants depending on Υ.

Proof.

Let $\xi(\tau,r,\rho)$ be a continuous nonnegative function equal to zero when $\tau\leqslant r$ and to one when $\tau\geq r+\rho$. In the remaining interval it satisfies the condition $\frac{\partial\xi}{\partial\tau}=\frac{1}{z\sqrt{s_c(\tau)}}$, where the parameter z is derived from the condition $\xi(r+\rho,r,\rho)=1$ and $s_c(\tau)=\sup\limits_{t>0,|x|=\tau}s(t,x)$. Substituting the test function $v(t,x)=\eta(x;r,\rho)u_h$ $\eta(x)=\xi^2(|x|,r,\rho)$ into the identity (19), we obtain

$$\int\limits_{D^T}\Bigg[\frac{1}{2}(u_h^2\eta)_t+\sum_{i,j=1}^n(a_{ij}u_{x_i})_h(\eta u_h)_{x_j}+$$

$$+\sum_{i=1}^n((c_iu)_h(\eta u_h)_{x_i}-(b_iu_{x_i})_h(\eta u_h))+(du)_h(\eta u_h)\Bigg]dxdt=0. \tag{24}$$

Upon passing to the limit in the equality (24) when $h\to0$, we have:

$$\int\limits_{\Omega}(u^2(T,x)-\varphi^2(x))\eta dx+$$

$$+2\int\limits_{D^T}\left[\sum_{i,j=1}^n a_{ij}u_{x_i}(\eta u)_{x_j}+\sum_{i=1}^n c_iu(\eta u)_{x_i}-b_iu_{x_i}\eta u)+du^2\eta\right]dxdt=0.$$

Whence, by virtue of the condition supp$\varphi\in\Omega[R_0]$, one can readily obtain the following inequality for any $r\geq R_0$ and $\rho>0$:

$$\int\limits_{\Omega}\eta u^2(T,x)dx+2\int\limits_{D^T}\left[\sum_{i,j=1}^n\eta a_{ij}u_{x_i}u_{x_j}+du^2\eta\right]dxdt\leqslant$$

$$\leqslant 2\int\limits_{D^T}\sum_{i,j=1}^n a_{ij}u_{x_i}u\frac{\partial\eta}{\partial x_j}dxdt+2\int\limits_{D^T}\sum_{i=1}^n\eta|c_i-b_i||uu_{x_i}|dxdt-2\int\limits_{D^T}u^2\frac{\partial\eta}{\partial\mathbf{c}}dxdt\leqslant$$

$$\leqslant 2\int\limits_{D^T}(s(t,x)\Upsilon|u\nabla u||\nabla\eta|+\sqrt{sd}|u\nabla u|\eta)dxdt. \tag{25}$$

Transformation of the latter provides

$$\int\limits_{\Omega}\eta u^2(T,x)dx+\int\limits_{D^T}(s\eta|\nabla u|^2+du^2\eta)dxdt\leqslant 2\int\limits_{D^T}s\Upsilon|\nabla u||u||\nabla\eta|dxdt.$$

Making use of the form of the function η, one can readily obtain the inequality

$$\int\limits_{\Omega\backslash\Omega[r+\rho]}u^2(t,x)dx+\int\limits_0^t\int\limits_{\Omega\backslash\Omega[r+\rho]}(s|\nabla u|^2+du^2)dxdt\leqslant$$

$$\leqslant\frac{C}{z^2}\int\limits_0^t\int\limits_{\Omega[r+\rho]\backslash\Omega[r]}u^2dxdt.$$

Introducing the notation

$$H_r(t)=\int\limits_{\Omega\backslash\Omega[r]}u^2(x,t)dx+\int\limits_0^t\int\limits_{\Omega\backslash\Omega[r]}(s|\nabla u|^2+du^2)dxdt,$$

we establish that

$$H_{r+\rho}(t) \leqslant \frac{C}{z^2}\int\limits_0^t H_r(\tau)d\tau. \tag{26}$$

The inequality (26) will be applied inductively for the sequence r_i, $i = 0,1,2,...k$, $r_{i+1} = r_i + \rho_i$, numbers ρ_i are selected so that $z = \int\limits_{r^i}^{r_{i+1}} \frac{d\tau}{\sqrt{s_c(\tau)}}$. Invoking that $H_r(t) \leqslant A$, we have

$$H_{r_0+\rho_0}(t) = \frac{ACt}{z^2}. \tag{27}$$

Further, by means of induction with respect to k we establish the inequality

$$H_{r_k}(t) \leqslant \frac{AC^k t^k}{z^{2k}k!}. \tag{28}$$

Using the Stirling inequality, we arrive to the correlation, from (28) one can readily obtain

$$H_{r_k}(t) \leqslant \frac{AC^k e^k t^k}{\sqrt{2\pi k} z^{2k} k^k} \leqslant Ae^{-k\ln\frac{z^2 k}{Cet}}. \tag{29}$$

We will choose the number k so that $Ce^2t \leqslant \; z^2k \leqslant 2Ce^2t$. Then, (29) provides $H_{r_k}(t) \leqslant Ae^{-k}$. Let us form up a sequence r_i for the pair of numbers r, R_0 so that

$$zk = \sum_{i=0}^{k-1}\int\limits_{r_i}^{r_{i+1}} \frac{d\tau}{\sqrt{s_c(\tau)}} = \int\limits_{R_0}^{r} \frac{d\tau}{\sqrt{s_c(\tau)}} = I.$$

Then, $I^2 = z^2k^2 \leqslant 2Ce^2tk$. Let us select the leas integer satisfying this inequality as k. Then, $k \geq \frac{I^2}{2Ce^2t}$. Thus, the inequality (23) is determined.

Proof of the theorem.

Let us introduce the notation

$$\varepsilon = A_1 \exp(-I^2/(2Ce^2t)).$$

When $r \geq 2R_0$ and every $t \in (0,T)$, the inequality

$$\int\limits_\Omega u^2(t,x)dx \leqslant \varepsilon + \int\limits_{\Omega[r]} u^2(t,x)dx \tag{30}$$

holds. Since the function $u(t,x)$ is an element of the space $\overset{\circ}{H}{}^1_A(\Omega,\Gamma_1^\tau)$ for almost every $t \in (0,T)$, (9) provides

$$\int\limits_\Omega u^2(t,x)dx \leqslant \varepsilon + \lambda^{-1}(t,r)\int\limits_\Omega (s(t,x)|\nabla u|^2 + du^2)dx. \tag{31}$$

Then, we deduce he inequality

$$(E(t)-\varepsilon)\lambda(t,r) \leqslant -\frac{d}{dt}E(t)$$

for the function $E(t) = \int\limits_\Omega u^2(t,x)dx$ by means of the correlation

$$\frac{d}{dt}\int\limits_\Omega u^2(t,x)dx \leqslant -\int\limits_\Omega (s(\tau,x)|\nabla u|^2 + d(\tau,x)u^2)dx. \tag{32}$$

Solving this inequality we obtain

$$E(t) - \varepsilon \leqslant e^{-\int_0^t \lambda(\tau,r)d\tau}E(0).$$

Substituting the expression for ε, we obtain

$$E(t) \leqslant E(0)(A_1 e^{-I^2/(2Ce^2 t)} + e^{-\int_0^t \lambda(\tau,r)d\tau}). \tag{33}$$

The latter inequality holds for all $r \geq R_0$. Let us choose the number $r = r(t)$ so that $\min\left(\frac{1}{t}(\int\limits_{R_0}^{r} \frac{d\tau}{\sqrt{s_c(\tau)}})^2, \int\limits_0^t \lambda(\tau,r)d\tau\right) \geq M_m(t)$. Then,

$$E(t) \leqslant E(0)\exp\left(-\frac{1}{C_1 t}M_m(t)\right),$$

the inequality (10) of the theorem is proved.

Let us consider the example demonstrating the estimate of Theorem 1 in case of the domain of rotation Ω_f. Obviously, to calculate the function $M_m(t)$ approximately, one can substitute every function of the pair determining it by a smaller one. The inequalities for the function $\lambda(\tau,r)$ are known when $\Gamma_1^\tau = \partial\Omega$

$$\frac{c_1}{\rho^2(r)} \leqslant \lambda(\tau,r) \leqslant \frac{c_2}{\rho^2(r)}, r \geq R_1, \tag{34}$$

where $\rho(r)$ is the radius of the largest sphere inscribed in Ω^r.

Let $P \subset (0,\infty)$ be an arbitrary measurable subset. Suppose that $\Gamma_1 = P \times \partial\Omega$. Using (34), we obtain that

$$\int\limits_0^t \frac{c_1\chi_p(\tau)}{\rho^2(r)}d\tau \leqslant \int\limits_0^t \lambda(\tau,r)d\tau,$$

where $\chi_p(\tau)$ is a characteristic function of the set P. Introducing the notation $q(t) = \int\limits_0^t \chi_p(\tau)d\tau$, we have the inequality

$$\frac{c_1}{\rho^2(r)}q(t) \leqslant \int\limits_0^t \lambda(\tau,r)d\tau. \tag{35}$$

If $s(t,x) \equiv 1$, we define $r(t)$ be the equality

$$\frac{r^2}{t} = \frac{q(t)}{\rho^2(r)}.$$

Let us use the estimate from the proof of Theorem 4:

$$E(t) \leqslant E(0)(A_1 e^{-\frac{r^2}{\widetilde{C}t}} + e^{-\frac{c_1}{\rho^2(r)}q(t)}).$$

Upon selecting $r = r(t)$, we have

$$E(t) \leqslant E(0)(A_1+1)e^{-C_m\frac{r^2(t)}{t}}, \ \ C_m = \min(c_1, 1/\widetilde{C}).$$

In particular, if $f(r) = r^\alpha$ and $q(t) = t/2$, then we obtain the estimate

$$\int\limits_\Omega u^2(t,x)dx \leqslant M\exp\left(-\kappa t^{\frac{1-\alpha}{1+\alpha}}\right), \tag{36}$$

corresponding in form to the one obtained by F.Kh. Mukminov [17] in case when $\Gamma_2 = \emptyset$. But if the density of distribution of the set P is more sparse, e.g., $q(t) = \sqrt{t}$ then,

$$\int\limits_\Omega u^2(t,x)dx \leqslant M\exp\left(-\kappa t^{\frac{1-2\alpha}{2+2\alpha}}\right), \ \ \alpha \in (0, \frac{1}{2}).$$

Let us consider the equation such that $s(t,x) = s(|x|) = |x|^\beta$, $\beta < 2$. Then,

$$\int_{R_0}^{r} \frac{d\tau}{\sqrt{s_c(\tau)}} = \int_{R_0}^{r} \frac{d\tau}{\tau^{\beta/2}} = \frac{\tau^{1-\frac{\beta}{2}}}{1-\frac{\beta}{2}} \,|_{R_0}^{r} \geq r^{1-\frac{\beta}{2}}$$

for sufficiently large r. When $f(r) = r^\alpha$, (35) provides $\int\limits_0^t \lambda(\tau, r)d\tau \geq \frac{c_1 t}{r^{2\alpha}}$. Therefore,

$$M_m(t) = \sup_r \min(t^{-1}r^{2-\beta}, tr^{-2\alpha}) = t^{\frac{2-2\alpha-\beta}{2+2\alpha-\beta}}.$$

When $\beta \geq 2$, the estimate of the Theorem does not provide a qualified decrease of solution.

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Venera Fidarisovna Gilimshina
M. Akmullah Bashkir State Pedagogical University,
Kommunsticheskaya Str. 22, 609
450076, Ufa, Russia
E-mail: gilvenera@mail.ru

Farit Khamzaevich Mukminov
Ufa State Aviation Technical University,
Ukhtomskii Str. 17/2, apt. 177,
450105, Ufa, Russia
E-mail: mfkh@rambler.ru

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