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ON SOLUTION OF A TWO KERNEL EQUATION REPRESENTED BY EXPONENTIALS

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Abstract. The integral equation with two kernels

$$f(x) = g(x) + \int_0^\infty K_1(x-t)f(t)dt + \int_{-\infty}^0 K_2(x-t)f(t)dt, \ \ -\infty < x < +\infty,$$

where the kernel functions $K_{1,\,2}(x) \in L$, is considered on the whole line. The present paper is devoted to solvability of the equation, investigation of properties of solutions and description of their structure. It is assumed that the kernel functions $K_m \geq 0$ are even and represented by exponentials as a mixture of the two-sided Laplace distributions:

$$K_m(x) = \int_a^b e^{-|x|\,s} d\sigma_m(s) \geq 0, \ \ m = 1,\, 2\,.$$

Here $\sigma_{1,2}$ are nondecreasing functions on $(a, b) \subset (0, \infty)$ such that

$$0 < \lambda_1 \leq 1, \quad 0 < \lambda_2 < 1, \quad \text{where} \quad \lambda_i = \int_{-\infty}^\infty K_i(x)dx = 2\int_a^b \frac{1}{s} d\sigma_i(s)\,, \ i = 1,\, 2\,.$$

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1. Introduction, formulation of the problem

A prominent place among convolution equations is occupied by the integral equation on the whole line with two kernels (see [1]-[3]):

$$f(x) = g(x) + \int_0^\infty K_1(x-t)f(t)dt + \int_{-\infty}^0 K_2(x-t)f(t)dt, \ \ -\infty < x < +\infty, \tag{1}$$

where the kernel functions $K_{1,\,2}(x) \in L \equiv L_1\,(-\infty,\, \infty)$.

In a nonsingular (elliptic) case, Equation (1) has a unique solution in L, when $g \in L$ is arbitrary. The necessary and sufficient condition for elipticity of Equation (1) in terms of the symbol properties of the equation (see [1]–[2]) is known. The monographs [1]–[3] contain also results on Equation (1) in some special cases when the symbol index is other than zero. In [1]–[3], methods of harmonic analysis are mainly used. The method used in [3], gives a possibility to construct a solution of the nonsingular equation (1) in a closed form in special classes via several (direct and inverse) Fourier transformations.

In [4]–[5], some singular and nonsingular equations of the form (1) are investigated without application of the harmonic analysis methods. In [4], the problem was reduced to the equation

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on a semi-line with the kernel depending on the sum of the arguments. The work [5] provides the proof (applying the method of [4]) of solvability in classes of locally integrable functions of some homogeneous and nonhomogeneous equations of the form (1) in a twice conservative case (TCC), when $K_{1,2} \geq 0$, $\int\limits_{-\infty}^{\infty} K_{1,2}(x)\, dx = 1$. The TCC is related to the special case of Equation (1) with a degenerate symbol.

The present paper is devoted to questions of solvability of Equation (1), investigation of properties of solutions and description of their structure. It is assumed that the kernel functions $K_m \geq 0$ are even and are represented interms of exponents in the form of a mixture of two-sided Laplace distributions:

$$K_m(x) = \int\limits_a^b e^{-|x|\, s} d\sigma_m(s) \geq 0, \quad m = 1,\, 2\,. \tag{2}$$

Here $\sigma_{1,2}$ are nondecreasing functions on $(a, b) \subset (0, \infty)$ such that

$$0 < \lambda_1 \leq 1, \quad 0 < \lambda_2 < 1, \tag{3}$$

where

$$\lambda_i = \int\limits_{-\infty}^{\infty} K_i(x) dx = 2 \int\limits_a^b \frac{1}{s} d\sigma_i(s)\,, \quad i = 1,\, 2\,.$$

Convolution equations with kernels of the form (2) have a number of applications in mathematical physics. They describe a certain range of problems of random walk in the space consisting of two homogeneous half-spaces.

One can easily verify that in the dissipative case $\lambda_{1,2} < 1$, Equation (1) is an equation with a contraction operator in L (with the contraction coefficient $q = \max(\lambda_1, \lambda_2)$) and (1) solvable uniquely in L. A half-conservative case when (HCC), when $\lambda_1 = 1$, $\lambda_2 < 1$, relates to special cases of Equation (1) (with a degenerate symbol).

Following [4],[5], let us write Equation (1) in the form of the following system with a sum-and-difference kernel with respect to $f_{1,2}$:

$$\begin{aligned} f_1(x) &= g_1(x) + \int\limits_0^{\infty} K_1(x-t) f_1(t) dt + \int\limits_0^{\infty} K_2(x+t) f_2(t) dt\,, \\ f_2(x) &= g_2(x) + \int\limits_0^{\infty} K_1(x+t) f_1(t) dt + \int\limits_0^{\infty} K_2(x-t) f_2(t) dt\;. \end{aligned} \tag{4}$$

Here $f_{1,2}$ are new unknown functions on a positive semi-axis defined by

$$f_1(x) = f(x), \;\; f_2(x) = f(-x), \quad x > 0, \tag{5}$$

The functions $g_{1,2}$ are defined likewise:

$$g_1(x) = g(x), \;\; g_2(x) = g(-x), \quad x > 0.$$

Without loss of generality one can consider that the functions $g_{1,2}$ ar nonnegative, it is also assumed that $g \in L$ and thus:

$$0 \leq g_m \in L^+ \equiv L_1(0, \infty)\,, \;\; m = 1, 2. \tag{6}$$

In the dissipative case a nonnegative solution of the system (4) in $L^+ \times L^+$ will be constructed under the conditions (2).

In the half-conservative case $\lambda_1 = 1$, an additional condition of finiteness of the first momentum is imposed on the function g_1 :

$$\int_0^\infty g_1(x)\,xdx = \int_0^\infty g(x)\,xdx < +\infty. \tag{7}$$

A solution to the system (4) will be constructed so that $0 \le f_2 \in L^+$, and $f_1 \ge 0$ is a function locally integrable on $[0,\infty)$, having the asymptotics

$$\int_0^x f_1(t)\,dt = o\left(x^2\right), \quad x \to \infty.$$

2. Auxiliary facts

2.1. The Wiener-Hopf equation. Let us consider the following Wiener-Hopf equation:

$$f(x) = g(x) + \int_0^\infty K(x-t)f(t)dt\,, \quad x > 0, \tag{8}$$

where $K(x) \ge 0$, $\lambda = \int\limits_{-\infty}^{\infty} K(x)dx \le 1$.

Let $\hat{K}$ be an integral operator occurring in (8):

$$\left(\hat{K} f\right)(x) = \int_0^\infty K(x-t)\,f(t)\,dt. \tag{9}$$

This operator acts boundedly in the space E^+, which coincides with one of Banach spaces $L_p(0,\infty)$, $1 \le p \le \infty$ and $C_0[0,\infty)$.

In the dissipative case $\lambda < 1$, Equation (8) is an equation with a contracting operator and possessing a unique solution $f \in E^+$ when $g \in E^+$. In the conservative case $\lambda = 1$, with arbitrary $g \in L^+$, there is the so-called basic solution f (BS) of Equation (8). BS is a limit of simple iterations with a zero initial approximation. It has the asymptotics

$$\int_0^x |f(t)|\,dt = o\left(x^2\right), \quad x \to \infty. \tag{10}$$

If a free term has a finite first momentum, i.e.

$$\int_0^\infty |g(t)|\;t\,dt < +\infty, \tag{11}$$

then there is an asymptotics

$$\int_0^x |f(t)|\,dt = o(x)\,, \quad x \to \infty. \tag{12}$$

2.2. The Ambartsumian equation. Let us consider the equation (8) in case, when

$$K(x) = \int_a^b e^{-|x|\,s} d\sigma(s). \tag{13}$$

Here σ is a nondecreasing function on $(a, b) \subset (0, \infty)$, and

$$\lambda = \int_{-\infty}^{\infty} K(x)dx = 2\int_a^b \frac{1}{s} d\sigma(s) \leq 1. \tag{14}$$

Equation (8), (13) will be used in cases when the kernel K coincides with one of the kernels K_m, given by means of (2).

The theory of the Wiener-Hopf equation with kernels, represented in terms of exponentials, is well developed (see, e.g. [7]). Many statements and constructions of this theory can be expanded to the two kernel equation (1) under the condition (2), (3). One step in this direction was done in the work [6] in connection with solving the translation equation in adjacent semispaces.

The Ambartsumian equation (AE) (see [7])

$$\varphi\,(s) = 1 + \varphi\,(s) \int_a^b \frac{1}{s+p} \varphi\,(p)\; d\sigma\,(p) \tag{15}$$

plays an important part in the theory of integral Wiener-Hopf equations (and a number of other convolution equations) with the kernel (13).

Let $L\left(\frac{1}{s} d\sigma\,(s)\right)$ be a space of functions integrable with respect to the measure $\frac{1}{s} d\sigma\,(s)$ with the norm $\|\varphi\| = \int_a^b |\varphi\,(s)|\, \frac{1}{s} d\sigma\,(s) < +\infty$. Similarly to the dissipative case $\lambda < 1$, in the conservative case $\lambda = 1$, there is a basic solution $\varphi \in L\left(\frac{1}{s} d\sigma\,(s)\right)$ to Equation (15), which is a limit of simple iterations with a zero initial approximation in $L\left(\frac{1}{s} d\sigma\,(s)\right)$. The function φ has the properties:

$$\varphi\,(s) \downarrow \text{ by} > \; s,\; \varphi\,(0) = (1-\lambda)^{-1} \;\; (\leq +\infty)\,,\; \varphi\,(+\infty) = 1,$$

$$\int_a^b \frac{\varphi\,(s)}{s} d\sigma\,(s) = 1 \; - \sqrt{1-\lambda}.$$

In DC, one has $\varphi \in C\,[a\,,\,b] \subset L\left(\frac{1}{s} d\sigma\,(s)\right)$. When $b = \infty$ the continuity of the function φ at the point b is understood in the sense of existence of its finite limit in infinity. In the CC the function φ is continuous on $(0\,,\,b]$.

Let us assume that I is an identity operator, and the operator $\hat{K}$ is given by (9). Let us describe the connection of the function φ with factorization of the Wiener-Hopf operator $\mathrm{I} - \hat{K}$. Consider the function

$$V\;(x) = \int_a^b e^{-x\,s} \varphi\,(s)\; d\sigma\,(s) \in L^+. \tag{16}$$

It is nonnegative and completely monotonous on $(0\,,\;\infty)$. One has the identity

$$\int_0^\infty V\,(x)\,dx = 1 - \sqrt{1-\lambda}, \tag{17}$$

the Wiener-Hopf factorization is constructed via the function V (see [7]):

$$\mathrm{I} - \hat{K} = \left(\mathrm{I} - \hat{V}^-\right)\,\left(\mathrm{I} - \hat{V}^+\right), \tag{18}$$

where $\hat{V}^{\pm}$ are the following formally Volterra operators:

$$\left(\hat{V}^{+} f\right)(x)=\int_{0}^{x} V(x-t)f(t)\,dt\,,\quad \left(\hat{V}^{-} f\right)(x)=\int_{x}^{\infty} V(t-x)f(t)\,dt\,. \tag{19}$$

Factorization (18) takes place as identity of operators, acting in E^{+} and in a series of other spaces. Note that in the CC one has the expansion (18), though $\mathrm{I}-\hat{K}$ is irreversible in spaces E^{+}.

2.3. Resolvent function. Inversion of operators occurring in the factorization (18) is connected with construction of the resolvent kernel Φ. It is determined from the following renewal type equation:

$$\Phi\left(x\right)=V\left(x\right)+\int_{0}^{x} V\left(x-t\right)\Phi\left(t\right)\,dt. \tag{20}$$

There is a unique solution $\Phi\in L_{loc}\left[0,\infty\right)$ of Equation (20). The form (16) of the function V entails (see [8]) that the resolvent function admits the representation

$$\Phi(x)=\int_{0}^{b} e^{-x\,p}\,d\omega(p)\geq 0, \tag{21}$$

where ω is a nondecreasing function.

In the dissipative case $\lambda<1$, one has $\Phi\in L_{1}\left(0,\infty\right)$ and

$$\left(\mathrm{I}-\hat{V}^{\pm}\right)^{-1}=\mathrm{I}+\hat{\Phi}^{\pm}, \tag{22}$$

where the operators $\hat{\Phi}^{\pm}$ are determined by means of

$$\left(\hat{\Phi}^{+} f\right)(x)=\int_{0}^{x} \Phi\left(x-t\right)f\left(t\right)\,dt\ ,\quad \left(\hat{\Phi}^{-} f\right)(x)=\int_{x}^{\infty} \Phi\left(t-x\right)f\left(t\right)\,dt\ . \tag{23}$$

In the conservative case, the operators $\mathrm{I}-\hat{V}_{\pm}$ are irreversible. Then $\Phi\notin L^{+}$, the following asymptotics takes place:

$$\int_{0}^{x} \Phi\left(t\right)dt=O\left(x\right),\quad x\to\infty, \tag{24}$$

and the equalities (22) hold true in the sense of equality of the operators translating the space L^{+} into the corresponding space of locally integrable functions. In the conservative case, as well as in the dissipative case, the basic solution f of Equation (8) with $g\in L^{+}$ has the form

$$f=\left(I+\hat{\Phi}^{+}\right)\left(I+\hat{\Phi}^{-}\right)g. \tag{25}$$

i.e. $f\left(x\right)=F\left(x\right)+\int_{0}^{x}\Phi\left(x-t\right)F\left(t\right)dt$, where $F\left(x\right)=g\left(x\right)+\int_{x}^{\infty}\Phi\left(t-x\right)g\left(t\right)dt$.

3. Transformation of Equation (1)

Let us consider the problem of existence and construction of the basic solution of the system (4), which is its minimal positive solution.

The applied approach is connected with using the Laplace transformations $\alpha_{1,2}$ from the desired functions $f_{1,2}$:

$$\alpha_m(s) = \int_0^\infty e^{-t\,s} f_m(t)dt\,, \quad m = 1, 2. \tag{26}$$

The solution (4) will initially be constructed in the class of functions $f_{1,2} \geq 0$ such that the corresponding functions α_m satisfy the conditions

$$\alpha_m(s) = \int_0^\infty e^{-t\,s} f_m(t)dt \in L\left(\frac{1}{s}d\sigma_m(s)\right), \quad m = 1, 2. \tag{27}$$

In what follows properties of $f_{1,2}$ will be specified. Note that if $f_m \in L^+$, then

$$\alpha_m \in C[a, b] \subset L\left(\frac{1}{s}d\sigma_m(s)\right). \tag{28}$$

Integrating both parts of (27) with respect to the measure $\frac{1}{s}d\sigma_m(s)$, one arrives to the equalities

$$\int_a^b \alpha_m(s)\frac{1}{s}d\sigma_m(s) = \int_0^\infty f_m(t)\,dt \int_a^b e^{-ts}\frac{1}{s}d\sigma_m(s) = \int_0^\infty f_m(t)\,\rho_m(t)\,dt,$$

where $\rho_m(x) = \int_a^b e^{-xs}\frac{1}{s}d\sigma_m(s) = \int_x^\infty K_m(t)\,dt > 0$. Therefore, the condition (28) is equivalent to integrability of functions f_1 and f_2 with the weights ρ_1 and ρ_2, respectively.

In what follows the important property of the system (4) that variables separate under off-diagonal integrals will be used. Using the representations (2), from (4) one obtains

$$\begin{aligned} \left(I - \hat{K}_1\right) f_1(x) &= g_1(x) + \int_a^b e^{-x\,s}\alpha_2(s)\,d\sigma_2(s)\;, \\ \left(I - \hat{K}_2\right) f_2(x) &= g_2(x) + \int_a^b e^{-x\,s}\alpha_1(s)\,d\sigma_1(s)\;, \end{aligned} \tag{29}$$

where $\hat{K}_{1,2}$ are the following Wiener-Hopf operators:

$$\left(\hat{K}_m f\right)(x) = \int_0^\infty K_m(x-t)\,f(t)\,dt, \ \ m = 1,\,2\,, \tag{30}$$

and the functions $\alpha_{1,2}$ are given by the formulae (26).

The functions $\varphi,\ V,\ \Phi$ introduced in Section 2 will be used. These functions, corresponding to the kernel $K = K_m,\quad m = 1, 2$, will be supplied by the index m (and denoted by $\varphi_m,\ V_m,\ \Phi_m$).

Let the functions $P_m(x, s)$, $m = 1, 2$ be basic solutions of the following Wiener-Hopf equations ($s > 0$ -parameter)

$$P_m(x, s) = e^{-xs} + \int_0^\infty K_m(x-t)P_m(t, s)dt\,, \qquad m = 1, 2\,. \tag{31}$$

The following formulae hold (see [7]):

$$P_m(x,s) = \varphi_m\,(s)\; e^{-x\,s} \left(1 + \int_0^x \Phi_m\,(t)\, e^{t\,s} dt \right), \quad m = 1, 2\,. \tag{32}$$

The Ambartsumian functions φ_m are determined from (15) when $\sigma = \sigma_m$:

$$\varphi_m\,(s) = 1 + \varphi_m\;(s) \int_a^b \frac{1}{s+p} \varphi_m\,(p)\; d\sigma_m\,(p)\,, \quad m = 1,\,2\,, \tag{33}$$

and the functions Φ_m are determined from Equation (20) when $V = V_m$.

Equations (29) can be considered as the Wiener-Hopf equations with respect to f_1, f_2. Meanwhile, the right-hand sides of these equations play the part of free terms. Comparing the free terms of Equations (29) and (31) (with the use of superposition property for basic solutions), one arrives at the following relations:

$$\begin{aligned} f_1(x) &= \tilde{g}_1(x) + \int_a^b P_1\,(x,s)\,\alpha_2\,(s)\,d\sigma_2\,(s)\;\;, \\ f_2(x) &= \tilde{g}_2(x) + \int_a^b P_2\,(x,s)\,\alpha_1\,(s)\,d\sigma_1\,(s)\;\;. \end{aligned} \tag{34}$$

The functions $\tilde{g}_m$ represent basic solutions of the following Wiener-Hopf equations:

$$\tilde{g}_m(x) = g_m\,(x) + \int_0^\infty K_m(x-t)\tilde{g}_m(t)dt\,, \qquad m = 1, 2\,. \tag{35}$$

According the formula (25), one has:

$$\tilde{g}_m = \left(I + \hat{\Phi}_m^+\right)\left(I + \hat{\Phi}_m^-\right) g_m. \tag{36}$$

One can obtain from (34) a system of integral equations with respect to the functions $\alpha_{1,2}$. To begin with consider the functions

$$\tilde{\alpha}_m\,(s) = \int_0^\infty e^{-x\,s} \tilde{g}_m(t)dt\,, \quad m = 1, 2. \tag{37}$$

Using the formula (36) and the Laplace transformation formula for convolution, one obtains

$$\tilde{\alpha}_m\,(s) = \left(1 + \int_0^\infty \Phi_m\,(x)\, e^{-sx} dx \right) \beta_m\,(s)\,, \quad m = 1, 2, \tag{38}$$

from (37). Here

$$\beta_m\,(s) = \int_0^\infty e^{-xs} \left[g_m\,(x) + \int_0^\infty \Phi_m\,(t)\, g_m\,(x+t)\, dt \right] dx\,, \quad m = 1, 2. \tag{39}$$

It is known that (see [7])

$$1 + \int_0^\infty \Phi_m\,(x)\, e^{-sx} dx = \varphi_m\,(s)\,, \quad m = 1,\,2\,. \tag{40}$$

Therefore, there is the formula

$$\tilde{\alpha}_m(s) = \varphi_m(s)\,\beta_m(s), \quad m = 1,2. \tag{41}$$

Lemma 1. *If Conditions* (6) *are satisfied, and in the semiconservative case the auxiliary condition* (7) *is satisfied as well, then*

$$\tilde{\alpha}_m \in L\left(\frac{1}{s}d\sigma_m(s)\right), \;\; m = 1,2. \tag{42}$$

Доказательство. In the dissipative case, one has $\tilde{g}_{1,2} \in L^+$. Therefore, according to the formula (37), one obtains $\tilde{\alpha}_{1,2} \in C[a,b] \subset L\left(\frac{1}{s}d\sigma_m(s)\right)$. By virtue of $\tilde{g}_2 \in L^+$ one has $\tilde{\alpha}_2 \in C[a,b]$ in the semi-conservative case. It remains to consider the function $\tilde{\alpha}_1$ in the semi-conservative case. Let us use the formula (41). It follows from (7) that the function $g_1(x) + \int\limits_0^\infty \Phi_1(t)\,g_1(x+t)\,dt \;\in L^+$. The formula (39) entails that $\beta_1 \in C[a,\,b]$. The function $\varphi_1 \in L\left(\frac{1}{s}d\sigma_1(s)\right)$ (see Section 2) therefore, the product $\tilde{\alpha}_1 = \varphi_1\beta_1 \in L\left(\frac{1}{s}d\sigma_1(s)\right)$. The lemma is proved. □

Let us consider the functions

$$U_m(p,s) = \int\limits_0^\infty P_m(x,s)\,e^{-x\,p}dx, \quad m = 1,2. \tag{43}$$

Application of the Laplace transform to to the formula (32) in view of (40) leads to the following known expressions for the functions U_m:

$$U_m(p,s) = \int\limits_0^\infty P_m(x,s)\,e^{-x\,p}dx = \frac{\varphi_m(s)\,\varphi_m(p)}{s+p}, \qquad m = 1,2. \tag{44}$$

Taking into account (37) and (43), applying the Laplace transform to the equalities (34), one obtains the following system of integral equations with respect to $\alpha_{1,2}$:

$$\begin{aligned} \alpha_1(p) &= \tilde{\alpha}_1(p) + \int\limits_a^b U_1(p,s)\,\alpha_2(s)\,d\sigma_2(s)\;\;, \\ \alpha_2(p) &= \tilde{\alpha}_2(p) + \int\limits_a^b U_2(p,s)\,\alpha_1(s)\,d\sigma_1(s)\;\;. \end{aligned} \tag{45}$$

We have proved the following.

Lemma 2. *If the system* (4) *has a solution* $f_1,\ f_2 \geq 0$*, satisfying Condition* (27) *(or the equivalent condition* $\int\limits_0^\infty f_m(t)\,\rho_m(t)\,dt < \infty$*) then, the functions* $\alpha_m \in C[a,b] \subset L\left(\frac{1}{s}d\sigma_m(s)\right)$*,* $m = 1,\,2$ *satisfy the system of integral equations* (45)*.*

4. On solvability of the system (45)

Investigating the system (45), let us use the following estimates for solutions of Equations уравнений (31):

$$\int\limits_a^b P_m(x,s)\,\frac{1}{s}d\sigma_m(s) \leq \lambda_m, \quad m = 1,2. \tag{46}$$

These inequalities have a simple physical (and probability) meaning and are connected with a complete probability of outcome of the wandering particle from a half-space. When $\lambda_1 = 1$, the first inequality (46) turns to an equality.

Note that the estimate (46) was used in [7] however, there is a misprint there (which was mentioned in [9]).

Multiplying both parts (43) by $\frac{1}{s}$, and integrating with respect to the measure $\sigma_k(s)$ on (a, b) one obtains the inequality (see the inequality (9.11) from [7])

$$\int_a^b U_m(p, s)\, \frac{1}{s} d\sigma_m(s) \leq \frac{\lambda_m}{p}, \quad m = 1,\, 2. \tag{47}$$

Using the symmetry of the kernels U_k (see (44)), one obtains the following formula from (47):

$$\int_a^b U_m(p, s)\, \frac{1}{p} d\sigma_m(p) \leq \frac{\lambda_m}{s}, \quad m = 1,\, 2. \tag{48}$$

Let us rewrite the system (45) in the operator form:

$$\begin{aligned} \alpha_1 &= \tilde{\alpha}_1 + \hat{U}_1 \alpha_2\,, \\ \alpha_2 &= \tilde{\alpha}_2 + \hat{U}_2 \alpha_1\,, \end{aligned} \tag{49}$$

where $\hat{U}_{1,2}$ are the following integral operators:

$$\begin{aligned} \left(\hat{U}_1 \alpha\right)(p) &= \int_a^b U_1(p, s)\, \alpha(s)\, d\sigma_2(s)\;\;, \\ \left(\hat{U}_2 \alpha\right)(p) &= \int_a^b U_2(p, s)\, \alpha(s)\, d\sigma_1(s)\;\;. \end{aligned} \tag{50}$$

It follows from the estimates (48) that the operator $\hat{U}_1$ transforms the space $L\left(\frac{1}{s} d\sigma_2(s)\right)$ into $L\left(\frac{1}{s} d\sigma_1(s)\right)$, and the operator $\hat{U}_2$ transforms $L\left(\frac{1}{s} d\sigma_1(s)\right)$ into $L\left(\frac{1}{s} d\sigma_2(s)\right)$, and the estimates $\left\|\hat{U}_k\right\| \leq \lambda_k$ hold. Whence, it follows that the operator $\hat{U}_1 \hat{U}_2$ acts in $L\left(\frac{1}{s} d\sigma_1(s)\right)$, and the operator $\hat{U}_2 \hat{U}_1$ in $L\left(\frac{1}{s} d\sigma_2(s)\right)$ and the following estimates exist:

$$\left\|\hat{U}_1 \hat{U}_2\right\| \leq q, \quad \left\|\hat{U}_2 \hat{U}_1\right\| \leq q; \quad q = \lambda_1 \lambda_2 < 1. \tag{51}$$

These estimates readily provide solvability of the system (49). For example, one can eliminate α_2 from the system (49), which leads to the following equation in α_1 with the operator, contracting in $L\left(\frac{1}{s} d\sigma_1(s)\right)$:

$$\alpha_1 = \left(\tilde{\alpha}_1 + \hat{U}_1 \tilde{\alpha}_2\right) + \left(\hat{U}_1 \hat{U}_2\right) \alpha_1.$$

It is possible to consider the following successive approximations:

$$\begin{aligned} \alpha_1^{(n)} &= \tilde{\alpha}_1 + \hat{U}_1 \alpha_2^{(n)} \\ \alpha_2^{(n+1)} &= \tilde{\alpha}_2 + \hat{U}_2 \alpha_1^{(n)}, \quad n = 0, 1.2, \cdots \end{aligned}$$

$$\alpha_2^{(0)} = 0,$$

that converge at a geometric rate with the denominator

$$q = \lambda_1 \lambda_2 < 1.$$

This leads to the following theorem.

Theorem 1. *Let us assume that the condition* (6) *holds in the dissipative case, and the auxiliary condition* (7) *is satisfied in the conservative case (conditions of Lemma 1). Then, the system* (49) *has a unique solution in* $L\left(\frac{1}{s}d\sigma_1\left(s\right)\right)\times L\left(\frac{1}{s}d\sigma_2\left(s\right)\right)$.

5. On solvability of the system (4) under the conditions (2), (3)

Let us assume that the pair $(\alpha_1,\alpha_2)\in L\left(\frac{1}{s}d\sigma_1\left(s\right)\right)\times L\left(\frac{1}{s}d\sigma_2\left(s\right)\right)$ is a solution of the system (45). Let us demonstrate that one can determine the solution to the system (4) by the formulae (34). Using the representations (37) and (43) for the functions $\tilde{\alpha}_m,\ U_m$, one obtains from (45) the correlations (27), where the functions $f_{1,2}$ are determined by the formulae (34). It follows directly from (27) and (34) that $f_{1,2}$ satisfy the initial requirements on solution of the system (4): nonnegativeness and integrability with the weights $\rho_{1,2}$. Eliminating the functions $\alpha_{1,2}$ from (34),(27) one arrives to the equations (4) with respect to $f_{1,2}$ and thus makes the considered problem solvable.

Let us clarify properties of the functions $f_{1,2}$. Assuming that $s=0$ in the relations (27), one obtains

$$\int\limits_0^\infty f_m\left(x\right)dx=\alpha_m(0)=\tilde{\alpha}_m(0)+\int\limits_a^b U_m\left(0,s\right)\alpha_{3-m}\left(s\right)d\sigma_{3-m}\left(s\right),\ \ m=1,2. \tag{52}$$

Using the formula (52), in view of the equality (44) and monotony of the Ambartsumian function one has

$$\int\limits_0^\infty f_m\left(x\right)dx=\alpha_m(0)=\tilde{\alpha}_m(0)+\int\limits_a^b\frac{\varphi_m\left(s\right)\varphi_m\left(0\right)}{s}\alpha_{3-m}\left(s\right)d\sigma_{3-m}\left(s\right)\leq \tag{53}$$

$$\leq\tilde{\alpha}_m(0)+\varphi_m^2\left(0\right)\left\|\alpha_{3-m}\right\|\ \ (\leq\infty)\,.$$

In the dissipative case $\lambda_1<1$ the numbers $\varphi_m\left(0\right)$ and $\tilde{\alpha}_m\left(0\right)$ are finite. Therefore, inequalities (53) entail integrability of functions $f_{1,2}$. Let us consider a semiconservative case now $\lambda_1=1$. Then, (53) provides integrability of the function f_2 only. It follows from (4) that the function f_1 satisfies the equation

$$f_1(x)=q_1(x)+\int\limits_0^\infty K_1(x-t)f_1(t)dt\,, \tag{54}$$

where $q_1(x)=g_1(x)+\int\limits_0^\infty K_2(x+t)f_2(t)dt\,.$

It follows from $f_2\in L^+$ that $q_1\in L^+$. According to (10), the basic solution of Equation (54) has the asymptotics

$$\int\limits_0^x f_1\left(t\right)dt=\int\limits_0^x f\left(t\right)dt=o\left(x^2\right),\quad x\to\infty. \tag{55}$$

The following theorem is proved.

Theorem 2.
a) In a semi-conservative case, $\lambda_1=1$ *there is a basic solution* $f\in L_{loc}\left[-\infty,\infty\right),\ f\geq 0$ *to Equation* (1) *under the conditions* (2),(3),(6),(7). *It possesses the asymptotics*

$$\int\limits_{-\infty}^x f\left(t\right)dt=o\left(x^2\right),\quad x\to\infty.$$

b) For the basic solution f in the semi-conservative case, as well as for a solution, unique in L, to the dissipative equation (1)*, the formulae* (34) *hold, where* (α_1, α_2) *is a solution to the system* (45)*, that is unique in* $L\left(\frac{1}{s}d\sigma_1(s)\right) \times L\left(\frac{1}{s}d\sigma_2(s)\right)$*, and the functions* $\tilde{g}_{1,2}$ *are defined according to* (36)*.*

Results of the present paper can be used for approximate numerical analytical solution of the considered equation, with the error estimate. A separate work is to be devoted to this question.

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