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THE MATRIX ANALOGS OF THE FIRST PAINLEVÉ EQUATION.

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Abstract. Earlier, Balandin and Sokolov obtained matrix analogs of the first and the second transcendent Painlevé equations and studied them for possession of the Painlevé property. In the present paper the integrability of the generalizations of the first Painlevé equation are studied using Painlevé–Kowalevskaya test. The main result obtained is integrability sufficient conditions for the generalized matrix analogs of the first Painlevé equation. An important role in finding these criteria is played by decomposition of the matrix into blocks. The obtained results are in agreement with the earlier investigations of special cases of our equations.

Keywords: integrability, Painlevé test, matrix equations.

1. Introduction

At the end of the XIX century, the Painlevé school investigated second-order ordinary differential equation for the absence of movable critical points in their general solutions. A long list of such equations was compiled and included six essentially new ones, for example,

$$u'' = 6u^2 + z,$$
$$u'' = 2u^3 + zu + \alpha,$$

where the primes denote differentiation with respect to the variable z, and α is an arbitrary parameter. They are usually termed as the first and the second transcendental Painlevé equations. These equations are a matter of intensive investigations, which is facilitated by their applications in mathematical physics to a great extend. In physical applications, matrix analogs of the Painlevé equations naturally arise as well [1].

In what follows, we say that a differential equation is integrable in the Painlevé (Painlevé-Kovalevskaya) sense if it has a general solution in the form of a formal Laurent series:

$$u = \sum_{k=0}^{\infty} u_k \, (z - z_0)^{k-n}, \tag{1}$$

where n is a certain natural number and z_0 is arbitrary.

The work [2] gives matrix analogs of the above equations and investigates them for integrability by the Painlevé-Kovalevskaya test:

$$U'' = 6U^2 + zB + A, \tag{2}$$
$$U'' = 2U^3 + zU + C$$

in the algebra $n \times n$ of matrices, where $A,\ B,\ C$ are constant matrices. It appears that the integrability test is the condition $B = E,\ C = \alpha E$, i.e. two matrices out of three differ from the unit matrix E only by a scalar coefficient. Note that, unlike the scalar case, the matrix equation (2) contains an arbitrary parameter A and is not reduced to the equation $U'' = 6U^2 + zE$

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mentioned, e.g., in the paper [3]. Later on, Equation (2) was generalized by means of the commutator of the matrices

$$V'' = 6V^2 + [G, V] + zB + A$$

and investigated by the authors [4] in algebras of quadratic matrices. In the present paper, new integrable generalizations of the matrix equation (2) are considered in the same algebra $n \times n$ of matrices of the form

$$u'' = 6u^2 + 60\sum_{k=0}^{l}(\alpha_k u + \beta_k)z^k, \tag{3}$$

where the Greek symbols indicate arbitrary constants $n \times n$ of the matrix, and equations of a more general form

$$u'' = 6u^2 + 60(\alpha(z)u + \beta(z), \tag{4}$$

where $\alpha(z)$, $\beta(z)$ are arbitrary matrix functions analytical on the whole complex plane.

Following the Painlevé-Kovalevskaya test, we arrive at the sequence of relations for matrix coefficients of the Laurent series, into which the desired solutions of the equations (3),(4) are expanded. Meanwhile, for an equation to be integrable in the Painlevé sense it is necessary that the resulting Laurent series should depend on $2n^2$ arbitrary constants.

2. Restrictions on coefficients α_k and β_k

The matrix integrable analog of the first scalar Painlevé equation considered in [2] has the form (2) provided that B=E. Let us investigate a generalization of the form (3) for integrability in the Painlevé sense. A more general equation containing the addends $u\gamma_k z^k$ is reduced to it by a transformation of the form $u \to u + \delta(Z)$.

One can readily observe that the formal solution in the form of the Laurent series (1) for Equation (3) has the form:

$$u = u_0(z - z_0)^{-2} + u_1(z - z_0)^{-1} + u_2 + \dots \tag{5}$$

First, let us consider the case $z_0 = 0$, i.e., the series

$$u = u_0 z^{-2} + u_1 z^{-1} + u_2 + \dots + u_k z^{k-2} + \dots \tag{6}$$

Substituting Equation (3) to this series, we obtain

$$\sum_{k=0}^{\infty}(k-2)(k-3)u_k z^{k-4} = 6\sum_{k=0}^{\infty}(\sum_{i=0}^{k} u_i u_{k-i})z^{k-4} + \tag{7}$$

$$+60\sum_{k=2}^{\infty}(\sum_{i=0}^{k-2}\alpha_i u_{k-2-i})z^{k-4} + 60\sum_{k=4}^{l+4}\beta_{k-4}z^{k-4},$$

which leads to the chain of relations on matrix coefficients u_k. In particular, comparing the coefficients of z^{-4}, we obtain

$$u_0^2 = u_0, \tag{8}$$

i.e. u_0 is an idempotent (projector). As it is suggested in [2], it is convenient to write the remaining relations by means of the linear operator

$$L(X) = u_0 X + X u_0 :$$

$$(3L - E)u_1 = 0 \tag{9}$$

$$L(u_2) = -u_1^2 - 10\alpha_0 u_0 \tag{10}$$

$$L(u_3) = -(u_1 u_2 + u_2 u_1) - 10\alpha_0 u_1 - 10\alpha_1 u_0 \tag{11}$$

$$(6L - \lambda_j E)u_j = f_j[u_0, \dots .u_{j-1}], j > 3, \tag{12}$$

where
$$\lambda_j = (j-2)(j-3).$$
Since
$$L^2(X) = u_0^2 X + 2u_0 X u_0 + X u_0^2$$
and
$$L^3(X) = u_0^3 X + 3u_0^2 X u_0 + 3u_0 X u_0^2 + X u_0^3,$$
simple calculations, in view of the relation (8), demonstrate that the operator L satisfies the equation $L^3 - 3L^2 + 2L = 0$. Hence, its spectrum consists of the eigen-numbers 0, 1 and 2, and the coefficients $u_0,\ u_2,\ u_3,\ u_5,\ u_6$ are defined with the accuracy to arbitrary constants, whose number is to be equal to $2n^2-1$, because for Equation (3) to possess the Painlevé property it is necessary that its solution (5) in the form of the Laurent series should depend on $2n^2$ arbitrary constants, and z_0 should be another arbitrary constant of the solution (5). Let us explain in more details.

It follows from the condition (8) (see, e.g., [5, § 25, Theorem 1]), that the lower coefficient is reducible the Jordan form
$$u_0 = T \begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix} T^{-1}, \tag{13}$$
where T is a nondegenerate matrix and E_k is a unit matrix of the order k. Let us consider temporally that u_0 has a block form:
$$u_0 = \begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix}. \tag{14}$$
As a matter of fact, we lose the arbitrariness of arbitrary constants in $2k(n-k)$ (see (22) in what follows). However, this arbitrary constants can be chosen so that u_0 has the form (13) and then for Equation (3) to possess the Painlevé property it is necessary that the series (6) should have other $2n^2 - 2k(n-k) - 1$ arbitrary constants.

For further consideration it is convenient to represent all matrices in the same block form. Then, any matrix X splits into blocks of a similar size:
$$X = \begin{pmatrix} x^s & x^l \\ x^r & x^m \end{pmatrix},$$
where x^s, x^l, x^r and x^m are blocks of the size $k \times k$, $k \times (n-k)$, $(n-k)\times k$ and $(n-k)\times(n-k)$, respectively. In this notation, the operator L acts by the formula
$$L(X) = \begin{pmatrix} 2x^s & x^l \\ x^r & 0 \end{pmatrix}.$$

Thus, the relations (9)–(12) take the form:
$$\begin{pmatrix} (j+1)(j-6)u_j^s & j(j-5)u_j^l \\ j(j-5)u_j^r & (j-2)(j-3)u_j^m \end{pmatrix} = F_j[u_0,\dots .u_{j-1}], \qquad j>0. \tag{15}$$

Here
$$F_1(u_0) = 0,\ F_2(u_0,u_1) = 6(u_1^2 + 10\alpha_0 u_0),\ F_3(u_0,u_1,u_2) = 6((u_1u_2 + u_2u_1) + 10\alpha_0 u_1 + 10\alpha_1 u_0).$$
One can readily see from (15) that when j takes the values 2, 3, 5 and 6, zeroes appear in the corresponding blocks of the left-hand side of this equation and hence, the corresponding block of the matrix u_j becomes arbitrary. Zeroes should also appear in the right-hand side in the same blocks. This imposes restrictions on matrix coefficients of the initial equation (3). The matrices u_j are defined uniquely with respect to the right-hand side when j takes other values.

The condition (9) provides vanishing of all elements u_1, because $\frac{1}{3}$ does not belong to the spectrum of the operator $L(X)$.

We obtain the same block structure for the matrices u_2, u_3 :

$$u_2^s = -5\alpha_0^s, u_2^r = -10\alpha_0^r, u_2^l = 0, u_2^m = p, \quad (16)$$

$$u_3^s = -5\alpha_1^s, u_3^r = -10\alpha_1^r, u_3^l = 0, u_3^m = \tilde{p}, \quad (17)$$

where p and $\tilde{p}$ are arbitrary blocks. Thus, u_2, u_3 contain arbitrary matrices p and $\tilde{p}$ in the left lower block, respectively.

Equation (15) is solved uniquely for u_4 because when $j = 4$, all coefficients of the left-hand side (15) are other than zero and the equation itself takes the form:

$$\begin{pmatrix} -10u_4^s & -4u_4^l \\ -4u_4^r & 2u_4^m \end{pmatrix} = 6(u_1u_3 + u_2^2 + u_3u_1) + 60(\alpha_0u_2 + \alpha_1u_1 + \alpha_2u_0) + 60\beta_0,$$

Invoking that $u_1 = 0$, we obtain:

$$u_4^s = -6\alpha_2^s + 15(\alpha_0^s)^2 + 60\alpha_0^l\alpha_0^r - 6\beta_0^s$$
$$u_4^l = -15\alpha_0^l p - 15\beta_0^l$$
$$u_4^r = -15\alpha_2^r + 150\alpha_0^m\alpha_0^r + 15p\alpha_0^r - 15\beta_0^r$$
$$u_4^m = 30\alpha_0^m p + 3p^2 + 30\beta_0^m,$$

where $p = u_2^m$. Turning to the next relation for the matrix u_5 in Equation (15), taking into account that $u_1 = 0$, we have

$$\begin{pmatrix} -6u_5^s & 0 \\ 0 & 6u_5^m \end{pmatrix} = 6(+u_2u_3 + u_3u_2) + 60(\alpha_0u_3 + \alpha_1u_2 + \alpha_3u_0) + 60\beta_1. \quad (18)$$

Thus, we conclude that the blocks u_5^r and u_5^l are arbitrary. The agreement of the left-and the right–hand sides of (18) in these blocks leads to the following restrictions on matrix coefficients of Equation (3):

$$\alpha_1^l p + \alpha_0^l\tilde{p} + \beta_1^l = 0,$$
$$-\alpha_3^r + 10\alpha_1^m\alpha_0^r + 10\alpha_0^m\alpha_1^r + \tilde{p}\alpha_0^r + p\alpha_1^r - \beta_1^r = 0.$$

Since the blocks p and $\tilde{p}$ are arbitrary, it follows from these restrictions that

$$\alpha_0^l = \alpha_1^l = \beta_1^l = 0, \ \alpha_0^r = \alpha_1^r = 0, \ \alpha_3^r = -\beta_1^r. \quad (19)$$

In what follows the blocks u_5^s, u_5^m are not necessary therefore, we do not calculate them. The last arbitrary block appears in the matrix u_6, as one can see from (15) namely, the block u_6^s. Then, taking into account that $u_1 = 0$, we can write Equation (15) in the form

$$\begin{pmatrix} 0 & 6u_6^l \\ 6u_6^r & 6u_6^m \end{pmatrix} = 6(u_3^2 + u_2u_4 + u_4u_2) + 60(\alpha_0u_4 + \alpha_1u_3 + \alpha_2u_2 + \alpha_4u_0) + 60\beta_2. \quad (20)$$

In view of the equalities (19), the agreement of the left and right-hand sides of the relation in the block s leads to the following restriction on matrix coefficients of the equation (3):

$$-2\alpha_4^s + 4\alpha_2^s\alpha_0^s + 5(\alpha_1^s)^2 + 6\alpha_0^s\alpha_2^s + 6\alpha_0^s\beta_0^s - 6\beta_0^s\alpha_0^s - 2\beta_2^s = 0. \quad (21)$$

When $j > 0$, one can see from (15) that the remaining matrices u_j of the solution (6) are defined recurrently via the previous ones.

Now let us demonstrate that the size k of the block s equals to one. To this end, note that the general amount of arbitrary constants in the formal solution (6) (with the arbitrary number k and u_0 of the form (14)) equals to $2(n-k)^2 + 2k(n-k) + k^2$, since it has arbitrary blocks u_2^m, u_3^m, u_5^l, u_5^r, and u_6^s. Meanwhile, as it has been mentioned (see (13)), the matrix u_0 has the following form in the general case:

$$u_0 = T\begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix}T^{-1},$$

where T is an arbitrary non-degenerate matrix, i.e. $T \in GL(n)$. Since the matrix

$$P = \begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix}.$$

is invariant with respect to conjugation by the block-diagonal matrices

$$S = \begin{pmatrix} S_k & 0 \\ 0 & S_{n-k} \end{pmatrix},$$

generating a closed subgroup H of the dimension $k^2 + (n-k)^2$ in the group $GL(n)$ then, this subgroup becomes the stabilizer of the matrix P [6, § 2.5]. Hence, u_0 belongs to the orbit of the matrix P with respect to its conjugation by elements from $GL(n)$, which is diffeomorphic to the set of left cosets $GL(n)/H$. Hence, (see, e.g., [6, Theorem 2.1]), we obtain that u_0 depends on

$$n^2 - (k^2 + (n-k)^2) = 2k(n-k) \tag{22}$$

arbitrary constants. Adding them, we obtain that our solution has

$$2(n-k)^2 + 4k(n-k) + k^2 = 2n^2 - k^2$$

arbitrary constants. Since, solution in the form of the Laurent series should also involve z_0, i.e. the series has the form (5), we obtain that $k = 1$. Thus, we have proved that the block s should be of the size 1×1. Note that in this case, all the blocks s commute and hence, (21) reduces and takes the from

$$\beta_2^s = -\alpha_4^s + 5\alpha_0^s\alpha_2^s + \frac{5}{2}(\alpha_1^s)^2. \tag{23}$$

Let us consider solution in the form of the Laurent series in powers $z - z_0$. To this end, we expand the matrix polynomials once more in the right-hand side in powers $z - z_0$:

$$A(z) = \sum_{k=0}^{l} \alpha_k z^k = A(z_0) + A'(z_0)(z-z_0) + \frac{1}{2}A''(z_0)(z-z_0)^2 + \cdots + \frac{1}{l!}A^{(l)}(z_0)(z-z_0)^l$$

and likewise for $B(z) = \sum_{k=0}^{l} \beta_k z^k$. Then, (3) is written in the form

$$u'' = 6u^2 + 60\sum_{k=0}^{l} (a_k u + b_k)(z-z_0)^k, \tag{24}$$

where $a_k = \frac{1}{k!}A^{(k)}(z_0)$, $b_k = \frac{1}{k!}B^{(k)}(z_0)$. Applying the above analysis, we obtain the same restrictions for the new matrices a_k, b_k. In particular,

$$a_0^l = b_1^l = 0,\ a_0^r = 0\ a_3^r = -b_1^r = 0.$$

However, since

$$a_0 = A(z_0) = \sum_{k=0}^{l} \alpha_k (z-z_0)^k,\ b_1 = B(z_0)' = (\sum_{k=0}^{l} \beta_k (z-z_0)^k)'$$

then, due to arbitrariness of z_0, we obtain that all matrices $\alpha_0, \ldots, \alpha_l, \beta_1, \ldots, \beta_l$ should have zero elements in blocks marked by the indices l and r, i.e. (recall that the block s is of the size 1) the first row and the first column consist of zeroes, except for their common element. The structure of coefficients of the series is preserved as well. The matrix β_0 is still arbitrary. Thus, for the time being we have obtained that all the above matrix coefficients, except for β_0, have a block-diagonal form and satisfy the relation

$$b_2^s = -a_4^s + 5a_0^s a_2^s + \frac{5}{2}(a_1^s)^2. \tag{25}$$

Thus, these conditions are necessary and sufficient for existence of solution in the form of the formal Laurent series (5), with $u_0 = P$. The solution has $2n^2 - 2(n-1)$ arbitrary constants, since the size k of the block s equals to one. The missing $2(n-1)$ arbitrary constants for the Painlevé property are gained by considering solutions in the form (5) with the arbitrary matrix u_0, satisfying the equation $u_0^2 = u_0$ and of the rank 1, because the block s has the size 1.

Indeed, as it follows from (22) when $k = 1$, the set of such matrices depends on $2(n-1)$ arbitrary parameters. If instead of the solution u, beginning from $u_0 = P$, we consider the solution $\tilde{u}$, where $\tilde{u}_0$ is an arbitrary matrix satisfying the equality $\tilde{u}_0^2 = \tilde{u}_0$ then, there is a nondegenerate matrix T, whose conjugation reduces the matrix $\tilde{u_0}$ to the form P. However, in this case, the same transformation turns the equation

$$\tilde{u}'' = 6\tilde{u}^2 + 60\sum_{k=0}^{l}(\alpha_k\tilde{u} + \beta_k)z^k$$

into the equation

$$u'' = 6u^2 + 60\sum_{k=0}^{l}(T\alpha_kT^{-1}u + T\beta_kT^{-1})z^k,$$

where the solution u begins from $u_0 = P$. Then, we can repeat all the above reasoning and then, zero blocks remain in the same places. Hence, the matrices $T\alpha_kT^{-1}$, when $k \geq 0$ and $T\beta_kT^{-1}$, for $k \geq 1$ should be block-diagonal. Since T is arbitrary, this is possible only if the above matrices are scalar, i.e. they differ from the unit matrix only by a scalar factor.

Since we have just established that $a_k = \frac{1}{k!}A^{(k)}(z_0),\ b_k = \frac{1}{k!}B^{(k)}(z_0)$ are scalar matrices, but for b_0, then (25) takes the form

$$\frac{1}{2}B'' = -\frac{1}{24}A^{(4)} + \frac{5}{2}AA'' + \frac{5}{2}(A')^2. \tag{26}$$

One can readily see that Equation (3) integrable in the Painlevé sense takes the following form upon substituting the variables

$$u \to u - 5(\sum_{k=0}^{l}\alpha_kz^k)E:$$

$$u'' = 6u^2 + \beta_0 + \beta_1zE,$$

where β_0 is an arbitrary matrix and β_1 is an arbitrary constant. Note that similar reasoning are also suitable in the case when instead of matrix polynomials $A(z)$ and $B(z)$, we consider matrix functions $\alpha(z)$ and $\beta(z)$ analytical on the whole plane, i.e. Equation (4).

3. Conclusion

We have investigated generalizations of matrix analogues of the first Painlevé equation for integrability and established the corresponding necessary and sufficient integrability conditions. The following theorem is proved.

Theorem 1. *Equation*

$$u'' = 6u^2 + 60(\alpha(z)u + \beta(z),$$

where $\alpha(z)$ and $\beta(z)$ are matrix functions analytical on the whole plane, has a solution in the form of the formal Laurent series

$$u = u_0(z-z_0)^{-2} + u_1(z-z_0)^{-1} + u_2 + \dots$$

depending on $2n^2$ arbitrary constants if and only if $\alpha(z)$ and $\beta(z) - \beta(0)$ are scalar matrices, connected by the relation

$$\beta''(z) = -\frac{1}{12}\alpha^{(4)}(z) + \frac{1}{5}(\alpha(z)^2)'',$$

and $\beta(0)$ *is an arbitrary matrix. The substitution*

$$u \to u - 5\alpha(z)E$$

reduces the equation to the form

$$u'' = 6u^2 + \beta_0 + \beta_1 zE,$$

where β_0 *is an arbitrary matrix, and* β_1 *is an arbitrary constant.*

The results are in complete agreement with the previous ones given in [2] and [4] investigating particular cases of the considered equations.

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