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APPROXIMATE SOLUTION OF NONLINEAR EQUATIONS WITH WEIGHTED POTENTIAL TYPE OPERATORS

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Abstract. Global theorems on existence, uniqueness and ways of finding solutions are proved in a real space $L_2(-\infty,\infty)$ for different classes of nonlinear integral equations with weighted potential type operators

$$F(x,u(x))+\int\limits_{-\infty}^{\infty}\frac{[a(x)-a(t)]\,u(t)}{|x-t|^{1-\alpha}}dt=f(x)\,,$$

$$u(x)+\int\limits_{-\infty}^{\infty}\frac{[a(x)-a(t)]\,F(t,u(t))}{|x-t|^{1-\alpha}}dt=f(x)\,,$$

$$u(x)+F\left(x,\int\limits_{-\infty}^{\infty}\frac{[a(x)-a(t)]\,u(t)}{|x-t|^{1-\alpha}}dt\right)=f(x)$$

by means of combining the basic principle of monotone operators theory by Browder-Minty with the Banach contraction mapping principle. It is shown that the solutions can be found by using the Picard successive approximations method and speed estimates of their convergence are proved. The obtained results cover, in particular, the linear integral equations case with potential type kernels of a special form.

Keywords: nonlinear integral equations, potential type operator, monotone operator.

Nonlinear integral equations of the form

$$F(x,u(x))+\int\limits_{-\infty}^{\infty}\frac{[a(x)-a(t)]\,u(t)}{|x-t|^{1-\alpha}}dt=f(x)\,, \tag{1}$$

$$u(x)+\int\limits_{-\infty}^{\infty}\frac{[a(x)-a(t)]\,F(t,u(t))}{|x-t|^{1-\alpha}}dt=f(x)\,, \tag{2}$$

$$u(x)+F\left(x,\int\limits_{-\infty}^{\infty}\frac{[a(x)-a(t)]\,u(t)}{|x-t|^{1-\alpha}}dt\right)=f(x)\,, \tag{3}$$

are considered in a real space $L_2(R^1)=L_2(-\infty,\infty)$. Global theorems on existence, uniqueness and ways of finding solution are proved for them by means of combining the method of monotone (according to Browder-Minty) operators (see, e.g., [1]) with the contraction mapping principle. It is demonstrated that solutions can be found by the Picard successive approximations method and speed estimates of their convergence are obtained.

Interest to nonlinear equations with potential type kernels is aroused by their numerous and various applications (see [1], Chapter 2, and [2], Chapter 7).

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For the sake of simplicity, let us introduce the following notation:

$$L_p(R^1) = L_p \ , \quad \|\cdot\|_{L_p(R^1)} = \|\cdot\|_p \ , \quad \langle u, v\rangle = \int\limits_{-\infty}^{\infty} u(x)\, v(x)\, dx \ ,$$

$$(I^\alpha u)\,(x) = \int\limits_{-\infty}^{\infty} \frac{u(t)\, dt}{|x-t|^{1-\alpha}} \ , \qquad (A^\alpha u)\,(x) = \int\limits_{-\infty}^{\infty} \frac{[a(x)-a(t)]\, u(t)}{|x-t|^{1-\alpha}} dt \ .$$

By virtue of the known Hardy-Littlewood theorem (see, e.g., [1]), potential type operator I^α acts continuously from L_p to $L_{p/(1-\alpha p)}$, if $0<\alpha<1$ and $1<p<1/\alpha$, while

$$\|I^\alpha u\|_{p/(1-\alpha p)} \leqslant \|I^\alpha\|_{p\to p/(1-\alpha p)} \|u\|_p \qquad \forall u \in L_p \ , \tag{4}$$

where $\|I^\alpha\|_{p\to p/(1-\alpha p)}$ is the norm of the operator $I^\alpha : L_p \to L_{p/(1-\alpha p)}$, i.e acting from L_p to $L_{p/(1-\alpha p)}$.

In this connection the following lemma, playing a significant part in investigation of Equations (1)–(3), is of interest.

Lemma 1. *Let $0<\alpha<1/2$ and $a \in L_{1/\alpha}$. Then, the operator A^α acts continuously from L_2 to L_2 and is positive and*

$$\|A^\alpha u\|_2 \leqslant 2\, \|I^\alpha\|_{2\to 2/(1-2\alpha)} \|a\|_{1/\alpha} \|u\|_2, \tag{5}$$

$$(A^\alpha u, u) = 0 \qquad \forall u(x) \in L_2 \ , \tag{6}$$

where $(\cdot,\cdot)$ denotes a scalar product in L_2.

Proof. Let $u \in L_2$. Then, applying the Hölder inequality with the coefficients $1+2\alpha$ and $(1+2\alpha)/(2\alpha)$, one has

$$\|a\cdot u\|_{2/(1+2\alpha)} \leqslant \|a\|_{1/\alpha} \|u\|_2 \ . \tag{7}$$

Thus, $a\cdot u \in L_{2/(1+2\alpha)}$. Since $1 < 2/(1+2\alpha) < 1/\alpha$ (the first inequality is equivalent to the condition that $\alpha<1/2$, and the validity of the second one is obvious) then, according to the Hardy-Littlewood theorem $I^\alpha(a\,u) \in L_2$ because $\dfrac{\frac{2}{1+2\alpha}}{1-\frac{2\alpha}{1+2\alpha}} = 2$ and $\|I^\alpha(a\cdot u)\|_2 \leqslant \|I^\alpha\|_{2/(1+2\alpha)\to 2} \|a\cdot u\|_{2/(1+2\alpha)}$. Making use of the estimate (7), one obtains

$$\|I^\alpha(a\cdot u)\|_2 \leqslant \|I^\alpha\|_{2/(1+2\alpha)\to 2} \|a\|_{1/\alpha} \|u\|_2 \tag{8}$$

from the latter inequality.

Since $u \in L_2$ and $0<\alpha<1/2$, according to the Hardy-Littlewood theorem $I^\alpha u \in L_{2/(1-2\alpha)}$, and by virtue of the inequality (4),

$$\|I^\alpha u\|_{2/(1-2\alpha)} \leqslant \|I^\alpha\|_{2\to 2/(1-2\alpha)} \|u\|_2 \ . \tag{9}$$

Then, applying the Hölder inequality with the coefficients $1/(1-2\alpha)$ and $1/(2\alpha)$, one has $\|a\cdot I^\alpha u\|_2 \leqslant \|a\|_{1/\alpha} \|I^\alpha u\|_{2/(1-2\alpha)}$. Therefore, the latter inequality, in view of the estimate (9), immediately provides

$$\|a\cdot I^\alpha u\|_2 \leqslant \|I^\alpha\|_{2\to 2/(1-2\alpha)} \|a\|_{1/\alpha} \|u\|_2. \tag{10}$$

Since by virtue of the inequalities (8) and (10), $A^\alpha u = a\cdot I^\alpha u - I^\alpha(a\cdot u) \in L_2$ and

$$\|A^\alpha u\|_2 \leqslant \|a\cdot I^\alpha u\|_2 + \|I^\alpha(a\cdot u)\|_2 \leqslant$$
$$\leqslant \left(\|I^\alpha\|_{2\to 2/(1-2\alpha)} + \|I^\alpha\|_{2/(1+2\alpha)\to 2}\right) \|a\|_{1/\alpha} \|u\|_2$$

then, the latter inequality, in view of the obvious (see, e.g., [3], p. 247) equality $\|I^\alpha\|_{2\to 2/(1-2\alpha)} = \|I^\alpha\|_{2/(1+2\alpha)\to 2}$ (since I^α is a self-adjoint operator) readily provides the inequality (5).

It remains to prove the equality (6). Since the operator I^α is symmetric then,

$$(A^\alpha u, u) = (a\, I^\alpha u, u) - (I^\alpha(a\, u), u) = \langle I^\alpha u, a\, u\rangle - \langle a\, u, I^\alpha u\rangle = 0,$$

which was to be proved.

Let us investigate the nonlinear equations (1)–(3), containing a weighted potential type operator A^α. Denote the set of all natural numbers $\mathbf{N}$. In what follows, the function $F(x,t)$, generating the Nemytsky operator $Fu = F[x,u(x)]$, is supposed to be defined when $x,t \in (-\infty,\infty)$ and to satisfy the Karatheodori conditions: it is measurable with respect to x for every fixed t and is continuous with respect to t almost for all x.

For the monotone operators method and the contraction mapping principle to be applicable to Equations (1)-(3) it is necessary to require that the function $F(x,t)$, defining the nonlinearity in Equations (1)-(3), should possess the monotonicity property and satisfy the Lipschitz conditio, respectively. In this connection, it is assumed throughout this work that the nonlinearity $F(x,t)$ satisfies the following conditions almost for every fixed $x \in (-\infty,\infty)$ and for any $t_1,t_2 \in (-\infty,\infty)$:

1) $|F(x,t_1) - F(x,t_2)| \leqslant M \cdot |t_1 - t_2|$, where $M > 0$;

2) $\Big(F(x,t_1) - F(x,t_2)\Big) \cdot (t_1 - t_2) \geq m \cdot |t_1 - t_2|^2$, where $m > 0$.

Condition 1 entails that the Nemytsky operator F acts continuously from L_2 to L_2 and satisfies the Lipschitz condition:

$$\|Fu - Fv\|_2 \leqslant M \cdot \|u - v\|_2 \ , \qquad \forall u,v \in L_2 \ , \tag{11}$$

and Condition 2 entails that it is strongly monotone:

$$(Fu - Fv, u - v) \geq m \cdot \|u - v\|_2^2 \ , \qquad \forall u,v \in L_2 \ . \tag{12}$$

Obviously, any linear function $F(x,t) = a \cdot t + b$, $a > 0$, such that $m = M = a$, satisfies Conditions 1 and 2. A simplest example of nonlinear function, satisfying Conditions 1 and 2, can be the function $F(x,t) = (t + 2t^3)/(1 + t^2)$ such that $m = 1$, $M = 17/8$.

In what follows we will need the following known theorem (see [1], p. 13, with its detailed proof), which is a corollary of more general results by F. Browder and V. Petryshyn.

Theorem 1. *Let H be a real Hilbert space, and the operator A act from H into H. If there are constants $m > 0$ and $M > 0$ ($M > m$) such that for any $u,v \in H$ the equalities*

$$\|Au - Av\|_H \leqslant M \cdot \|u - v\|_H \ , \tag{13}$$

$$(Au - Av, u - v) \geq m \cdot \|u - v\|_H^2 \ , \tag{14}$$

hold, then equation $Au = f$ has a unique solution $u^* \in H$ for any $f \in H$. This solution can be found by the successive approximations method by the formula ($n \in \mathbf{N}$):

$$u_n = u_{n-1} - \frac{m}{M^2}(Au_{n-1} - f) \ , \tag{15}$$

with the error estimate:

$$\|u_n - u^*\|_H \leqslant \frac{m}{M^2} \cdot \frac{\alpha^n}{1 - \alpha}\|Au_0 - f\|_H \ , \tag{16}$$

where $\alpha = \sqrt{1 - m^2 M^{-2}}$, $u_0 \in H$ ia an arbitrary element (initial approximation).

First let us consider the nonlinear equation (1), which is simplest for investigation, by means of the method under consideration.

Theorem 2. *Let $0 < \alpha < 1/2$, $a \in L_{1/\alpha}$ and the nonlinearity $F(x,t)$ satisfy Conditions 1) and 2). Then, Equation(1) has a unique solution $u^* \in L_2$ for any $f \in L_2$. This solution can be obtained by the iterative method according to the scheme:*

$$u_n = u_{n-1} - \mu_1 \cdot (Fu_{n-1} + A^\alpha u_{n-1} - f) \ , \tag{17}$$

with the error estimate

$$\|u_n - u^*\|_2 \leqslant \mu_1 \frac{\alpha_1^n}{1 - \alpha_1}\|Fu_0 + A^\alpha u_0 - f\|_2 \ , \tag{18}$$

where $\mu_1 = m/(M + 2\|I^\alpha\|_{2\to 2/(1-2\alpha)}\|a\|_{1/\alpha})^2$, $\alpha_1 = \sqrt{1 - m\cdot\mu_1}$, $u_0 \in L_2$ *is the initial approximation (arbitrary function).*

Доказательство. Let $u, v \in L_2$ be any functions. Let us write the given equation (1) in the operator form: $Au = f$, where $A = F + A^\alpha$. Using the Minkowski inequality first, and then the inequalities (5) and (11), one has

$$\|Au - Av\|_2 \leqslant \big(M + 2\|I^\alpha\|_{2\to 2/(1-2\alpha)}\|a\|_{1/\alpha}\big)\cdot\|u - v\|_2$$

on the one hand. On the other hand, using the equality (6) and the inequality (12), one obtains

$$(Au - Av, u - v) = (A^\alpha u - A^\alpha v, u - v) + (Fu - Fv, u - v) \geq m\cdot\|u - v\|_2^2 \ .$$

Hence, according to Theorem 1, the equation $Au = f$, i.e. the given equation (1) has a unique solution $u^* \in L_2$, and this solution can be found by the scheme (17), resulting from the formula (15), with the error estimate (18), following from the inequality (16). □

Let us consider the nonlinear equations (2) and (3), which are more difficult to investigate by the considered method. The general Theorem 1 cannot be applied directly to such classes of equations, because a product of nonlinear monotone operators is not a monotone operator, generally speaking.

Theorem 3. *Let* $0 < \alpha < 1/2$, $a \in L_{1/\alpha}$ *and the nonlinearity* $F(x,t)$ *satisfy Conditions 1) and 2). Then, for any* $f \in L_2$, *the nonlinear equation (2) has a unique solution* $u^* \in L_2$. *This solution can be derived by the iterative method by the scheme:*

$$u_n = F^{-1}v_n, \quad v_n = v_{n-1} - \mu_2\cdot\left(F^{-1}v_{n-1} + A^\alpha v_{n-1} - f\right) \ , \tag{19}$$

with the error estimate

$$\|u_n - u^*\|_2 \leqslant \frac{\mu_2}{m}\cdot\frac{\alpha_2^n}{1-\alpha_2}\|u_0 + A^\alpha F u_0 - f\|_2 \ , \tag{20}$$

where $n \in \mathbf{N}$, $\mu_2 = m/[M(m^{-1} + 2\|I^\alpha\|_{2\to 2/(1-2\alpha)}\|a\|_{1/\alpha})]^2$, $\alpha_2 = \sqrt{1 - m\cdot M^{-2}\cdot\mu_2}$, F^{-1} *is the operator inverse to* F, $v_0 = Fu_0$, $u_0 \in L_2$ *is the initial approximation (arbitrary function).*

Доказательство. Let $u, v \in L_2$ be arbitrary functions. Since the Nemytsky operator F satisfies the inequalities (11) and (12) then, according to Theorem 1.3 from [1], there is an inverse operator F^{-1} such that

$$\|F^{-1}u - F^{-1}v\|_2 \leqslant \frac{1}{m}\|u - v\|_2 \ , \tag{21}$$

$$(F^{-1}u - F^{-1}v, u - v) \geq \frac{m}{M^2}\|u - v\|_2^2 \ . \tag{22}$$

Let us write Equation (2) in the operator form:

$$u + A^\alpha F u = f \ . \tag{23}$$

Direct verification shows that if v^* is a solution to the equation

$$Bv \equiv F^{-1}v + A^\alpha v = f \tag{24}$$

then, $u^* = F^{-1}v^*$ is a solution to Equation (23), and these solutions are unique in L_2, because the operators F and F^{-1} are strictly monotone, and the operator A^α is positive.

Let us prove that Equation (24) has a unique solution $v^* \in L_2$. Since Equation (24) has the same form as Equation (1), and the properties (21) and (22) of the operator F^{-1} are similar to the properties (11) and (12) of the operator F then, using the equality (6), the inequalities (21) and (22), similarly to the proof of Theorem 2, one obtains that

$$\|Bu - Bv\|_2 \leqslant \left(\frac{1}{m} + 2\|I^\alpha\|_{2\to 2/(1+2\alpha)}\|a\|_{1/\alpha}\right)\|u - v\|_2 \ ,$$

$$(Bu - Bv, u - v) = (F^{-1}u - F^{-1}v, u - v) + (A^\alpha u - A^\alpha v, u - v) \geq \frac{m}{M^2}\|u - v\|_2^2 .$$

Hence, according to Theorem 1, the equation $Bv = f$, i.e. Equation (24), has a unique solution $v^* \in L_2$, and this solution can be derived by the scheme

$$v_n = v_{n-1} - \mu_2 \cdot (Bv_{n-1} - f) \ , \tag{25}$$

with the error estimate

$$\|v_n - v^*\|_2 \leqslant \mu_2 \cdot \frac{\alpha_2^n}{1 - \alpha_2}\|Bv_0 - f\|_2 \ , \tag{26}$$

where $\mu_2 = m/[M(m^{-1} + 2\|I^\alpha\|_{2\to 2/(1-2\alpha)}\|a\|_{1/\alpha})]^2$, $\alpha_2 = \sqrt{1 - m \cdot M^{-2} \cdot \mu_2}$. However, in this case Equation (23), i.e. the given equation (2), has a unique solution $u^* = F^{-1}v^* \in L_2$ and this solution can be obtained by the scheme (19), resulting from (25), with the error estimate (20), resulting from (26), in view of $Bv = F^{-1}v + A^\alpha v$ and by virtue of the estimate (21), the inequality

$$\|u_n - u^*\|_2 = \|F^{-1}v_n - F^{-1}v^*\|_2 \leqslant \frac{1}{m}\|v_n - v^*\|_2$$

holds. Theorem 3 is proved completely. □

Finally, let us prove the following theorem.

Theorem 4. *Let $0 < \alpha < 1/2$, $a \in L_{1/\alpha}$, and the nonlinearity $F(x,t)$ satisfy Conditions 1) and 2). Then, the nonlinear equation (3) has a unique solution $u^* \in L_2$ for any $f \in L_2$. This solution can be derived by the iterative method by the scheme:*

$$u_n = u_{n-1} + \mu_2 \cdot \left(F^{-1}(f - u_{n-1}) - A^\alpha u_{n-1}\right) \ , \tag{27}$$

with the error estimate

$$\|u_n - u^*\|_2 \leqslant \frac{\mu_2}{m} \cdot \frac{\alpha_2^n}{1 - \alpha_2}\|F^{-1}(f - u_0) - A^\alpha u_0\|_2 \ , \tag{28}$$

where $n \in \mathbf{N}$, $\mu_2 = m/[M(m^{-1} + 2\|I^\alpha\|_{2\to 2/(1-2\alpha)}\|a\|_{1/\alpha})]^2$, $\alpha_2 = \sqrt{1 - m \cdot M^{-2} \cdot \mu_2}$, F^{-1} is the operator inverse to F, $u_0 \in L_2$ is the initial approximation (arbitrary function).

Доказательство. Let $u \in L_2$ be any function. Let us write Equation (3) in the operator form:

$$u + FA^\alpha u = f \ . \tag{29}$$

Assume that $f - u = \varphi$. Then Equation (24) takes the form: $FA^\alpha(f - \varphi) = \varphi$. Applying the operator F^{-1}, whose existence is proved in Theorem 3, to both sides of the latter equation, one arrives to the equation:

$$B\varphi \equiv F^{-1}\varphi + A^\alpha\varphi = A^\alpha f \ . \tag{30}$$

It is verified directly that if φ^* is a solution to Equation(30) then, $u^* = f - \varphi^*$ is a solution to Equation (29), and these solutions are unique in L_2 because the operators F and F^{-1} are strictly monotone and the operator A^α is positive.

Let us prove that Equation (30) has a unique solution $\varphi^* \in L_2$. Since Equation (30) has the same form as Equation (24) then, repeating the reasoning from Theorem 3, we verify that Equation(30) has a unique solution $\varphi^* \in L_2$, which can be derived by a scheme of the form (25):

$$\varphi_n = \varphi_{n-1} - \mu_2(B\varphi_{n-1} - A^\alpha f) \ , \tag{31}$$

with the error estimate of the form (26):

$$\|\varphi_n - \varphi^*\|_2 \leqslant \mu_2 \cdot \frac{\alpha_2^n}{1 - \alpha_2}\|B\varphi_0 - A^\alpha f\|_2 \ . \tag{32}$$

Invoking that $\varphi = f - u$, one obtains the iterative scheme (27) and the error estimate (28) directly from (31) and (32), respectively, which was to be proved.

□

Finally, note that Theorems 2–4 embrace, in particular, the case of the corresponding linear integral equations with potential type kernels in a special form.

BIBLIOGRAPHY

1. Askhabov S.N. *Nonlinear convolution type equations.* Moscow: Fizmatlit, 2009. 304 p. In Russian.
2. R. Gorenflo, S. Vesella *Abel integral equations. Analysis and applications.* Berlin: Springer-Verlag, 1991. 215 p.
3. Kolmogorov A.N., Fomin S.V. *Elements of the theory of functions and functional analysis.* Moscow: Fizmatlit, 2004. 570 p. In Russian.

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