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ASYMPTOTIC PRESENTATION OF EIGENFUNCTIONS OF A TWO-DIMENSIONAL HARMONIC OSCILLATOR

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Abstract. The asymptotics of eigenfunctions of a two-dimensional harmonic oscillator has been obtained all over the space. The need of such presentation arises when studying the spectral characteristics of finite perturbation of the two-dimensional harmonic oscillator. The absence of exact asymptotic equalities for fundamental systems of solutions to the differential equation complicates the study, because eigenfunctions of the two-dimensional harmonic oscillator are represented in the form of a product of normalized eigenfunctions of the one-dimensional harmonic oscillator. The usage of standard solutions helps to solve the problem.

Keywords: harmonic oscillator, eigenfunctions, trace formulas, eigenvalue asymptotics.

The formula for the first regularized trace of perturbation of a two-dimensional harmonic oscillator

$$-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + x_1^2 + x_2^2$$

by a multiplication operator V by a real finite function $V(x_1, x_2) \in C_0^4(\mathbb{R}^2)$ was obtained by Fazullin Z.Yu. and Murtazin Kh.Kh. in [1]. The spectrum of the operator $H_0 = -\Delta + x^2$ is well known and consists of eigenvalues $\lambda_n = 2n + 2$, $n \geq 0$. The corresponding projectors on eigen-subspaces (of the dimension $n + 1$) have the form

$$P_n h = \sum_{l=0}^{n} \left(h, \varphi_l^{(n)}\right) \varphi_l^{(n)},$$

where $(\cdot, \cdot)$ is a scalar product in $L^2(\mathbb{R}^2)$,

$$\varphi_k^{(n)}(x) = f_k(x_1) f_{n-k}(x_2), \qquad (1)$$

$f_l(t) = \left(2^l l! \sqrt{\pi}\right)^{-1/2} e^{-t^2/2} H_l(t)$ are normed eigenfunctions of a one-dimensional harmonic oscillator corresponding to eigen numbers $2l + 1$ $(l \geq 0)$, $H_l(t)$ are the Hermite polynomials. The absence of exact asymptotic equalities, homogeneous with respect to t, for fundamental systems of solutions to the differential equation $y'' + (\lambda - t^2)y = 0$, complicates the study of the asymptotics of the projector $P_n h$ and the spectrum of the perturbed operator $H = H_0 + V$. In order to avoid the problem, the authors had to impose rather severe restrictions on the function $V(x_1, x_2)$. Results of the work [2], where standard solutions are used, give a possibility to write out the asymptotics of eigenfunctions of the harmonic oscillator and thus to avoid the finiteness of $V(x_1, x_2)$. The following theorem holds.

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Theorem 1. *If $x_1 \geq 0$ and $x_2 \geq 0$, eigenfunctions of the operator H_0 have the form*

$$\varphi_k^{(n)}(x) = \frac{\gamma_1 \cos\hat{Q}(x_1, 2k+1)\cos\hat{Q}(x_2, 2(n-k)+1)}{\pi\,|2k+1-x_1^2|^{\frac{1}{4}}\,|2(n-k)+1-x_2^2|^{\frac{1}{4}}} e^{-\left[\hat{Q}_1(x_1,2k+1)+\hat{Q}_1(x_2,2(n-k)+1)\right]} \times$$
$$\times \left[1 + O\left(\frac{1}{\tilde{Q}(x_1, 2k+1)}\right)\right]\left[1 + O\left(\frac{1}{\tilde{Q}(x_2, 2(n-k)+1)}\right)\right]$$
$$\text{when} \quad \tilde{Q}\left(x_1, 2k+1\right) \to +\infty, \quad \tilde{Q}\left(x_2, 2(n-k)+1\right) \to +\infty;$$

$$\varphi_k^{(n)}(x) = \frac{\gamma_2 \cos\hat{Q}(x_2, 2(n-k)+1)}{\sqrt{\pi}\,|2(n-k)+1-x_2|^{\frac{1}{4}}} e^{-\hat{Q}_1(x_2,2(n-k)+1)} \times$$
$$\times \left[1 + O\left(\frac{1}{\tilde{Q}(x_2, 2(n-k)+1)}\right)\right] \times \left[\frac{1}{\sqrt[4]{2}\sqrt[3]{9}\Gamma\left(\frac{4}{3}\right) n^{1/12}} - \right.$$
$$\left. - \frac{x_1^2 - 2k - 1}{2^{5/4}3^{4/3}\Gamma\left(\frac{2}{3}\right) n^{5/12}} + O\left(\frac{|2k+1-x_1^2|}{k^{13/12}}\right)\right]$$
$$\text{when} \quad \tilde{Q}\left(x_2, 2(n-k)+1\right) \to +\infty, \quad \left|x_1 - \sqrt{2k+1}\right| \leq C\,k^{-1/6};$$

$$\varphi_k^{(n)}(x) = \frac{\gamma_3 \cos\hat{Q}(x_1, 2k+1)}{\sqrt{\pi}\,|2k+1-x_1|^{\frac{1}{4}}} e^{-\hat{Q}_1(x_1,2k+1)} \left[1 + O\left(\frac{1}{\tilde{Q}(x_1, 2k+1)}\right)\right] \times$$
$$\times \left[\frac{1}{2^{1/4}3^{2/3}\Gamma\left(\frac{4}{3}\right)(n-k)^{1/12}} - \right.$$
$$\left. - \frac{x_2^2 - 2(n-k) - 1}{2^{5/4}3^{4/3}\Gamma\left(\frac{2}{3}\right)(n-k)^{5/12}} + O\left(\frac{|2(n-k)+1-x_2^2|}{(n-k)^{13/12}}\right)\right]$$
$$\text{when} \quad \tilde{Q}\left(x_1, 2k+1\right) \to +\infty, \quad \left|x_2 - \sqrt{2(n-k)+1}\right| \leq C\,(n-k)^{-1/6};$$

$$\varphi_k^{(n)}(x) = \frac{2^{5/6}}{3^{3/4}\Gamma^2\left(\frac{4}{3}\right)(n-k)^{1/12}k^{1/12}} - \frac{x_2^2 - 2(n-k) - 1}{2^{4/3}\sqrt{3}\pi k^{1/12}(n-k)^{5/12}} -$$
$$- \frac{x_1^2 - 2k - 1}{2^{4/3}\sqrt{3}\pi k^{5/12}(n-k)^{1/12}} + \frac{(x_1^2 - 2k - 1)\left(x_2^2 - 2(n-k) - 1\right)}{2^{3/2}3^{8/3}\Gamma^2\left(\frac{2}{3}\right)(n-k)^{5/12}k^{5/12}} +$$
$$+ O\left(\frac{|2(n-k)+1-x_2^2|}{k^{1/12}(n-k)^{13/12}}\right) + O\left(\frac{|2k+1-x_1^2|}{k^{13/12}(n-k)^{1/12}}\right)$$
$$\text{when} \quad \left|x_1 - \sqrt{2k+1}\right| \leq C\,k^{-1/6}, \quad \left|x_2 - \sqrt{2(n-k)+1}\right| \leq C\,(n-k)^{-1/6},$$

where

$$\gamma_1 = \begin{cases} 2, & \text{if} \quad x_1 \le \sqrt{2k+1}, \quad x_2 \le \sqrt{2(n-k)+1}; \\ 1, & \text{if} \quad x_1 \le \sqrt{2k+1}, \quad x_2 > \sqrt{2(n-k)+1}; \\ 1, & \text{if} \quad x_1 > \sqrt{2k+1}, \quad x_2 \le \sqrt{2(n-k)+1}; \\ \frac{1}{2}, & \text{если} \quad x_1 > \sqrt{2k+1}, \quad x_2 > \sqrt{2(n-k)+1}, \end{cases}$$

$$\gamma_2 = \begin{cases} 2, & \text{if} \quad x_2 \le \sqrt{2(n-k)+1}; \\ 1, & \text{if} \quad x_2 > \sqrt{2(n-k)+1}, \end{cases} \qquad \gamma_3 = \begin{cases} 2, & \text{if} \quad x_1 \le \sqrt{2k+1}; \\ 1, & \text{if} \quad x_1 > \sqrt{2k+1}, \end{cases}$$

$$\hat{Q}(t,\lambda) = \begin{cases} 0, & \text{when} \quad t \ge \sqrt{\lambda}; \\ Q(t,\lambda), & \text{when} \quad t < \sqrt{\lambda}, \end{cases} \qquad \hat{Q}_1(t,\lambda) = \begin{cases} 0, & \text{when} \quad t \le \sqrt{\lambda}; \\ Q_1(t,\lambda), & \text{when} \quad t > \sqrt{\lambda}, \end{cases}$$

$$\tilde{Q}(t,\lambda) = \begin{cases} Q_1(t,\lambda) = \int\limits_{\sqrt{\lambda}}^{t} \sqrt{z^2-\lambda}\, dz, & \text{when} \quad t \ge \sqrt{\lambda}; \\ Q(t,\lambda) = \int\limits_{t}^{\sqrt{\lambda}} \sqrt{\lambda - z^2}\, dz, & \text{when} \quad t < \sqrt{\lambda}. \end{cases}$$

Proof. Let us write out the asymptotics of eigenfunctions of a one-dimensional harmonic oscillator. To this end, consider the operators

$$L_D^+ u = -u'' + x^2 u, \quad u(0) = 0 \quad \text{и} \quad L_N^+ u = -u'' + x^2 u, \quad u'(0) = 0$$

in $L^2(0,+\infty)$. Ine can readily observe that the spectrum L_D^+ consists of the numbers $\lambda_n = 4n+3$, $n \ge 0$, and the operator L_N^+ has eigenvalues $\lambda_n = 4n+1$, $n \ge 0$. In [2], integral equations for the kernels $B_D^+(x,t,\lambda)$ and $B_N^+(x,t,\lambda)$ of the operators $B_D^+(\lambda)$ and $B_N^+(\lambda)$, respectively, are studied by means of standard solutions. Let us take the following functions as standard solutions:

$$z_1(x,\lambda) = S(x,\lambda) Ai(\xi(x,\lambda)), \quad z_2(x,\lambda) = S(x,\lambda) Bi(\xi(x,\lambda)),$$

where $Ai(\xi)$, $Bi(\xi)$ are the Airy real functions,

$$\xi(x,\lambda) = \left(\frac{3}{2} \int\limits_{\sqrt{\lambda}}^{x} \left| t^2 - \lambda \right|^{1/2} dt \right)^{\frac{2}{3}} \operatorname{sgn}\left(x - \sqrt{\lambda}\right), \quad S(x,\lambda) = \left|\xi'(x,\lambda)\right|^{-\frac{1}{2}}.$$

The asymptotic representation for the Airy functions readily provide the asymptotics of standard solutions $z_k(x,\lambda)$, $k = 1, 2$. One has

$$z_k(x,\lambda) = \frac{e^{(-1)^k Q_1(x,\lambda)}}{2\sqrt{\pi}\,(x^2-\lambda)^{1/4}} \left[1 + O\left(\frac{1}{Q_1(x,\lambda)} \right) \right] \quad \text{when} \quad Q_1(x,\lambda) \to +\infty, \tag{2}$$

$$z_k(x,\lambda) = \frac{\cos\left(Q(x,\lambda) + (-1)^k \frac{\pi}{4}\right)}{\sqrt{\pi}\,(\lambda - x^2)^{1/4}} \left[1 + O\left(\frac{1}{Q(x,\lambda)} \right) \right] \text{ when } Q(x,\lambda) \to +\infty, \tag{3}$$

where

$$Q_1(x,\lambda) = \int\limits_{\sqrt{\lambda}}^{x} \sqrt{t^2-\lambda}\, dt, \quad Q(x,\lambda) = \int\limits_{x}^{\sqrt{\lambda}} \sqrt{\lambda - t^2}\, dt,$$

$$z_k(x,\lambda)=\frac{1}{\lambda^{1/12}6^{1/6}\sqrt{3}\Gamma\left(\frac{4}{3}\right)}+$$
$$+(-1)^k\frac{\operatorname{sgn}\left(x-\sqrt{\lambda}\right)|x^2-\lambda|}{\lambda^{5/12}6^{5/6}\sqrt{3}\Gamma\left(\frac{2}{3}\right)}+O\left(\frac{|x^2-\lambda|}{\lambda^{13/12}}\right) \tag{4}$$
$$\text{when}\quad \left|x-\sqrt{\lambda}\right|\le C\lambda^{-1/6},\quad C>0,\quad \text{is independent of } \lambda.$$

The asymptotics of derivatives of functions $z_k(x,\lambda)$, $k=1,2$ with respect to λ are also easily written out

$$\frac{\partial z_k(x,\lambda)}{\partial\lambda}=\frac{1}{4\sqrt{\pi}}\ln\left(\frac{x+\sqrt{x^2-\lambda}}{\sqrt{\lambda}}\right)\frac{e^{(-1)^kQ_1(x,\lambda)}}{(x^2-\lambda)^{1/4}}+$$
$$+O\left(\frac{e^{(-1)^kQ_1(x,\lambda)}}{\lambda(x^2-\lambda)^{1/4}}\right)\quad\text{when}\quad Q_1(x,\lambda)\to+\infty, \tag{5}$$

$$\frac{\partial z_k(x,\lambda)}{\partial\lambda}=(-1)^{k+1}\frac{\cos\left(Q(x,\lambda)+(-1)^{k+1}\frac{\pi}{4}\right)}{2\sqrt{\pi}(\lambda-x^2)^{1/4}}\arccos\frac{x}{\sqrt{\lambda}}+$$
$$+O\left(\frac{1}{\lambda(\lambda-x^2)^{1/4}}\right),\quad\text{when}\quad Q(x,\lambda)\to+\infty, \tag{6}$$

and when $|x-\sqrt{\lambda}|\leqslant C\lambda^{-1/6}$

$$\frac{\partial z_k(x,\lambda)}{\partial\lambda}=O\left(\frac{1}{\lambda^{13/12}}\right). \tag{7}$$

Asymptotic representations of derivatives of functions $z_k(x,\lambda)$ with respect to the variable x when $Q(x,\lambda)\to+\infty$

$$z_k'(x,\lambda)=\frac{x}{2}\left(\lambda-x^2\right)^{-\frac{5}{4}}\left[\frac{1}{\sqrt{\pi}}\cos\left(Q(x,\lambda)+(-1)^k\frac{\pi}{4}\right)+O\left(\frac{1}{Q(x,\lambda)}\right)\right]-$$
$$-\left(\lambda-x^2\right)^{\frac{1}{4}}\left[\frac{(-1)^{k+1}}{\sqrt{\pi}}\cos\left(Q(x,\lambda)+(-1)^{k+1}\frac{\pi}{4}\right)+O\left(\frac{1}{Q(x,\lambda)}\right)\right], \tag{8}$$

as well as asymptotics of the derivatives $\partial z_k'(0,\lambda)/\partial\lambda$ when $\lambda\to+\infty$

$$\frac{\partial z_k'(0,\lambda)}{\partial\lambda}=\frac{\sqrt{\pi}}{4}\lambda^{1/4}\cos\left(\frac{\pi}{4}\left[\lambda+(-1)^k\right]\right)\left[1+O\left(\frac{1}{\lambda}\right)\right]+$$
$$+\frac{1}{4\sqrt{\pi}}\lambda^{-3/4}\sin\left(\frac{\pi}{4}\left[\lambda+(-1)^k\right]\right)\left[1+O\left(\frac{1}{\lambda}\right)\right] \tag{9}$$

will also be necessary.

Linearly independent solutions $y_k(x,\lambda)$, $k=1,2$, to the equation

$$-y''+x^2y=\lambda y \tag{10}$$

have the form

$$y_1(x,\lambda)=z_1(x,\lambda)+\int_x^{\infty}H(x,t,\lambda)y_1(t,\lambda)\,dt, \tag{11}$$

$$y_2(x,\lambda)=z_2(x,\lambda)-\int_0^{x}H(x,t,\lambda)y_2(t,\lambda)\,dt, \tag{12}$$

where

$$H(x,t,\lambda)=\{z_1(x,\lambda)z_2(t,\lambda)-z_1(t,\lambda)z_2(x,\lambda)\}S''(t,\lambda)S^{-1}(t,\lambda),$$

$y_1(x,\lambda) \in L^2(0,\infty)$. The formulae (11), (12) provide that the following representations hold:

$$y_k(x,\lambda) = z_k(x,\lambda)\left(1 + z_k^{(1)}(x,\lambda)\right), \tag{13}$$

$$\frac{\partial y_1(x,\lambda)}{\partial\lambda} = \frac{\partial z_1(x,\lambda)}{\partial\lambda}\left(1 + z_1^{(2)}(x,\lambda)\right), \tag{14}$$

$$y_1'(x,\lambda) = z_1'(x,\lambda)\left(1 + \hat{z}_1^{(1)}(x,\lambda)\right) + z_2'(x,\lambda)\hat{z}_2^{(1)}(x,\lambda)e^{-2\hat{Q}_1(x,\lambda)}, \tag{15}$$

where $\sup\limits_{x\ge 0,\lambda>1}\left|z_k^{(1)}(x,\lambda)\right| \le C\lambda^{-1}$, $\sup\limits_{x\ge 0,\lambda>1}\left|z_1^{(2)}(x,\lambda)\right| \le C\lambda^{-1}$, $\sup\limits_{x\ge 0,\lambda>1}\left|\hat{z}_k^{(1)}(x,\lambda)\right| \le C\lambda^{-1}$, $k = 1,2$,

$$\hat{Q}(x,\lambda) = \begin{cases} Q_1(x,\lambda) = \int\limits_{\sqrt{\lambda}}^{x}\sqrt{z^2-\lambda}\,dz, & \text{when} \quad x > \sqrt{\lambda}; \\ 0, & \text{when} \quad x \le \sqrt{\lambda}. \end{cases}$$

Assuming that $u(x,\lambda) = [B_D^+(\lambda)h](x)$, where $\lambda \ne 4n+3$, $n \ge 0$, $h(x) \in L^2(0,\infty)$, one concludes that $u(x,\lambda)$ satisfies the nonhomogeneous equation

$$-u''(x) + x^2u(x) - \lambda u(x) = h(x), \tag{16}$$

and its conditions

$$u(0,\lambda) = 0, \quad u(x,\lambda) \in L^2(0,\infty).$$

Let us introduce the kernel

$$G(x,t,\lambda) = \frac{1}{W(\lambda)}\begin{cases} y_1(x,\lambda)y_2(t,\lambda), & \text{если } 0 \le t \le x < +\infty; \\ y_1(t,\lambda)y_2(x,\lambda), & \text{если } 0 \le x \le t < +\infty, \end{cases}$$

where

$$W(\lambda) = y_1(x,\lambda)y_2'(x,\lambda) - y_1'(x,\lambda)y_2(x,\lambda). \tag{17}$$

Then, the function $w(x,\lambda) = \int\limits_0^\infty G(x,t,\lambda)h(t)dt$ satisfies Equations (16) and the condition $w(x,\lambda) \in L^2(0,\infty)$ (see [2]). Therefore, the function $f(x,\lambda) = u(x,\lambda) - w(x,\lambda)$ satisfies the homogeneous equation (10) and belongs to $L^2(0,\infty)$. Hence, $f(x,\lambda) = Ay_1(x,\lambda)$. The constant A is provided by the condition $u(0,\lambda) = 0$. This yields the representation for the kernel $B_D^+(x,t,\lambda)$

$$B_D^+(x,t,\lambda) = G(x,t,\lambda) - \frac{y_2(0,\lambda)y_1(x,\lambda)y_1(t,\lambda)}{W(\lambda)y_1(0,\lambda)}. \tag{18}$$

Similar reasoning is given for the Neumann problem

$$B_N^+(x,t,\lambda) = G(x,t,\lambda) - \frac{y_2'(0,\lambda)y_1(x,\lambda)y_1(t,\lambda)}{W(\lambda)y_1'(0,\lambda)}. \tag{19}$$

Let us obtain formulae for the eigenfunctions $f_l(x)$. By definition

$$B_D^+(x,t,\lambda) = 2\sum_{l=0}^{\infty}\frac{f_{2l+1}(x)f_{2l+1}(t)}{4l+3-\lambda}, \quad B_N^+(x,t,\lambda) = 2\sum_{l=0}^{\infty}\frac{f_{2l}(x)f_{2l}(t)}{4l+1-\lambda}.$$

Whence,

$$2f_{2l+1}(x)f_{2l+1}(t) = \lim_{\lambda\to 4l+3}(4l+3-\lambda)B_D^+(x,t,\lambda),$$
$$2f_{2l}(x)f_{2l}(t) = \lim_{\lambda\to 4l+1}(4l+1-\lambda)B_N^+(x,t,\lambda).$$

Since the function $G(x,t,\lambda)$ does not have singularities in the neighborhood of eigen-numbers $\lambda_l = 2l+1$, the formulae (18), (19) provide

$$f_{2l+1}^2(x) = \frac{y_2(0,4l+3)y_1^2(x,4l+3)}{2W(4l+3)(y_1)'_\lambda(0,4l+3)}, \tag{20}$$

$$f^2_{2l}(x) = \frac{y_2'(0,4l+1)y_1^2(x,4l+1)}{2W(4l+1)(y_1)''_{x\lambda}(0,4l+1)}. \tag{21}$$

Let us investigate the behaviour of functions involved in the right-hand side of the formulae (20) and (21). One obtains directly from the formula (12) that $y_2(0,\lambda) = z_2(0,\lambda)$, $y_2'(0,\lambda) = z_2'(0,\lambda)$. Then, the formulae (3), (8) yield

$$y_2(0,\lambda) = \frac{\cos\left(\frac{\pi}{4}(\lambda+1)\right)}{\sqrt{\pi}\lambda^{1/4}}\left[1+O\left(\frac{1}{\lambda}\right)\right], \tag{22}$$

$$y_2'(0,\lambda) = \frac{\lambda^{1/4}\cos\left(\frac{\pi}{4}(\lambda-1)\right)}{\sqrt{\pi}}\left[1+O\left(\frac{1}{\lambda}\right)\right]. \tag{23}$$

According to (13) and (3)

$$y_1(0,\lambda) = \frac{\cos\left(\frac{\pi}{4}(\lambda-1)\right)}{\sqrt{\pi}\lambda^{1/4}}\left[1+O\left(\frac{1}{\lambda}\right)\right], \tag{24}$$

and the formulae (15), (8) provide

$$y_1'(0,\lambda) = \frac{\lambda^{1/4}\cos\left(\frac{\pi}{4}(\lambda+1)\right)}{\sqrt{\pi}}\left[1+O\left(\frac{1}{\lambda}\right)\right]. \tag{25}$$

From the formulae (14), (6), one can readily obtain the representation for $\partial y_1(0,\lambda)/\partial\lambda$

$$\frac{\partial y_1(0,\lambda)}{\partial\lambda} = \frac{\sqrt{\pi}}{4\lambda^{1/4}}\cos\left(\frac{\pi}{4}(\lambda+1)\right)\left[1+O\left(\frac{1}{\lambda}\right)\right]. \tag{26}$$

Then, the relations (17), (22) – (25) provide that

$$W(\lambda) = \frac{1}{\pi} + O\left(\frac{1}{\lambda}\right). \tag{27}$$

Thus, the formulae (20), (22), (26), (27) and (13) yield

$$f_{2l+1}(x) = \sqrt{2}z_1(x,4l+3)\left(1+\hat{f}_{2l+1}(x)\right), \tag{28}$$

where $\sup\limits_{x\geq 0,\, l>1}\left|\hat{f}_{2l+1}(x)\right| \leq Cl^{-1}$.

It remains to investigate the behaviour of $(y_1)''_{x\lambda}(0,\lambda)$. For this purpose, it is more convenient to differentiate the formula (11) with respect to the variable x, and then to substitute the variable $t = \sqrt{\lambda}\tau$ of the integrand and calculate the derivative of the function $y_1(0,\lambda)$ with respect to λ. Using the estimates (2) – (7), (9), one can obtain that

$$\frac{\partial y_1'(0,\lambda)}{\partial\lambda} = \frac{\partial z_1'(0,\lambda)}{\partial\lambda}\left(1+\frac{\tilde{y}^{(1)}(\lambda)}{\lambda}\right) + \frac{\partial z_2'(0,\lambda)}{\partial\lambda}\frac{\tilde{y}^{(2)}(\lambda)}{\lambda},$$

where $\sup\limits_{\lambda>1}|\tilde{y}^{(k)}(\lambda)| \geq C$, $k+1,2$. The latter expression and the formula (9) when $\lambda = 4n+1$ provide

$$\frac{\partial y_1'(0,4n+1)}{\partial\lambda} = \frac{\sqrt{\pi}}{4}(4n+1)^{1/4}\left[1+O\left(\frac{1}{n}\right)\right] + O\left(\frac{1}{n^{7/4}}\right). \tag{29}$$

Thus, the relations (21), (23), (27), (29), (13) entail that the following formula, similar to the formula (28), holds:

$$f_{2l}(x) = \sqrt{2}z_1(x,4l+1)\left(1+\hat{f}_{2l}(x)\right), \tag{30}$$

where $\sup\limits_{x\geq 0,\, l>1}\left|\hat{f}_{2l}(x)\right| \leq Cl^{-1}$.

Thus, (28) and (30) lead to the conclusion that for eigenfunctions of a one-dimensional harmonic oscillator the representation

$$f_k(x) = \sqrt{2} z_1(x, 2k+1) \left(1 + \hat{f}_k(x)\right) \tag{31}$$

holds, and $\sup\limits_{x \geq 0, k > 0} \left|\hat{f}_k(x)\right| \leq Ck^{-1}$. Then, the formulae (31), (1) - (4) provide the asymptotics of eigenfunctions $\varphi_k^{(n)}(x_1, x_2)$ при $x_1 \geq 0$, $x_2 \geq 0$.

Remark 1. *Asymptotics of eigenfunctions* $\varphi_k^{(n)}(x_1, x_2)$ *when* $\forall x_1$, x_2 *can be readily obtained using the equality*

$$f_l(-t) = (-1)^l f_l(t).$$

Indeed, the equality (1) *entails that*

$$\varphi_k^{(n)}(-x_1, x_2) = (-1)^k \varphi_k^{(n)}(x_1, x_2),$$
$$\varphi_k^{(n)}(-x_1, -x_2) = (-1)^n \varphi_k^{(n)}(x_1, x_2),$$
$$\varphi_k^{(n)}(x_1, -x_2) = (-1)^{n-k} \varphi_k^{(n)}(x_1, x_2),$$

where $x_1 \geq 0$, $x_2 \geq 0$.

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