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SPECTRAL ASYMPTOTICS OF NONSELFADJOINT DEGENERATE ELLIPTIC OPERATORS WITH SINGULAR MATRIX COEFFICIENTS ON AN INTERVAL

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Abstract. Some spectral asymptotic properties of the nonselfadjoint operator A associated with a noncoercive bilinear form in the space $\mathcal{H}^l = L_2(0,1)^l$ are investigated in the article. Such problems as summability of the Fourier series of elements $f \in \mathcal{H}^l$ with respect to the system of root vector-functions of the operator A by the Abel method with brackets, estimate for the resolvent of the operator A are considered.

Keywords: Elliptic differential operators, resolvent of operator, summability by the Abel method with brackets, system of root vector-functions.

Introduction

The paper is devoted to investigation of some spectral properties of a nonselfadjoint elliptic operator A in the space $\mathcal{H}^l = L_2(0,1)^l$, associated with a noncoercive bilinear form.

Such questions as summability of Fourier series of elements $f \in \mathcal{H}^l$ with respect to the system of root vector functions of the operator A by means of the Abel method with the brackets, resolvent estimate of the operator A are considered.

Spectral asymptotics of degenerate elliptic operators, degenerate elliptic operators, that are far from self-adjoint ones, was investigated in [1–6] in the case when, eigenvalues of the operator are devided into two series, one of which lies outside the angle $|\, arg\, z| \leqslant \varphi$, $\varphi < \pi$, and the other one is localizing to the line $R_+ = (0, \infty)$. This article is adjacent to the works [1, 2, 6], among which [6] contains most general results and the assumption that the highest-order coefficient of the operator A

$$a(t) \in C^m([0,1]; End\mathbf{C}^l) \tag{0.1}$$

has different simple eigenvalues for every $t \in [0,1]$.

Instead of (0.1) we require only that $a(t) \in C([0,1];\ End\,\mathbf{C}^l)$. The results §2–§5 join the work [7], where conditions similar to [6] are imposed on $a(t)$. We generalize results of the work [7] with minimal restrictions on $a(t) \in C([0,1];\ End\,\mathbf{C}^l)$.

Results §2–§5 are qualitatively new even with weaker restrictions on $a(t)$. Spectral problems of a closed expansion, given by boundary-value conditions different from the Dirichlet boundary-value conditions, are investigated here.

The method applied is based on approximating $a(t)$ by smooth matrix functions $a_\delta(t)$. However, resolvent estimates determined in [6] are inapplicable to the corresponding operator $\mathcal{A}_\delta$, because $a_\delta(t)$ may have nonsimple eigenvalues. Therefore, the work is partially devoted to

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the resolvent estimate for the operator $\mathcal{A}_\delta$. This method is also used in investigation of the spectrum asymptotics.

Results of the present paper are partially described in [9–11].

§1. Formulation of basic results

1. An operator A, given in the Hilbert space H, is said to be far from self-adjoint unless it is reduced to the form

$$A = B(E+S), \quad B = B^*, \quad S \in \sigma_\infty(H). \tag{1.1}$$

Here and in what follows the symbol $\sigma_\infty(H)$ denotes the class of linear completely continuous operators in H; B^* is the operator adjoint to B.

Spectral properties of elliptic differential and pseudo-differential operators close to self-adjoint ones, i.e. reducible to the form (1.1), are investigated in sufficient details (see [12, 13]). Spectral properties of elliptic differential operators and pseudo-differential operators, which are far from self-adjoint ones, are investigated also in detail in case if they are given on a compact manifold without a boundary (see [6, 14–16], with the bibliography). In case of domains with boundaries, differential operators and pseudo-differential operators far from self-adjoint are investigated in [2, 3, 17, 18, 19–22]; among which [2, 3, 17] are devoted to degenerate elliptic problems.

2. The present paper investigates spectral properties of a nonselfadjoint operator in $L_2(0,1)^l$, generated by a bilinear from

$$\mathcal{A}[u,v] = \sum_{i,j=0}^{m} \int_0^1 < p_i(t)a_{ij}(t)u^{(i)}(t), p_j(t)v^{(j)}(t) >_{\mathbf{C}^l} dt. \tag{1.2}$$

Here

$$p_i(t) = \{t(1-t)\}^{\theta+i-m} \quad (i=\overline{0,m}),\ \theta < m,\ u^{(i)}(t) = \frac{d^i u(t)}{dt^i},$$

$$a_{ij} \in L_\infty(J;\, End\, \mathbf{C}^l) \quad (i,j=\overline{0,m}),$$

where $J=(0,1)$. The symbol $<\,,\,>_{\mathbf{C}^l}$ stands for the scalar product in $\mathbf{C}^l$.

Let us denote by $\mathcal{H}_+$ the closure of the linear manifold $C_0^\infty(J)$ by the norm

$$|\varphi|_+ = (\int_J p_m^2(t)|\varphi^{(m)}(t)|^2 dt + \int_J |\varphi(t)|^2 dt)^{1/2}.$$

Suppose that

$$\mathcal{H} = L_2(J), \quad \mathcal{H}^l = \mathcal{H} \oplus \cdots \oplus \mathcal{H} \quad (l-\text{times}),$$

$$\mathcal{H}_+^l = \mathcal{H}_+ \oplus \cdots \oplus \mathcal{H}_+ \quad (l-\text{times}).$$

In what follows the scalar product in the spaces $\mathcal{H}, \mathcal{H}^l$ will be designated by one and the same symbol $(\,,\,)$. Likewise, norms in the spaces $\mathcal{H}_+, \mathcal{H}_+^l$ and $\mathcal{H}, \mathcal{H}^l, \mathbf{C}^l$ will be denoted by $|\ |_+, |\ |$, respectively. The symbol $\|T\|$ will denote the norm of the bounded operator T, given in $\mathcal{H}$ or $\mathcal{H}^l$.

Let us take the space $\mathcal{H}_+^l$ as the domain of definition of the bilinear form $\mathcal{A}[u,v]$ (1.2).

Let us assume that the following conditions are satisfied:

$$|a_{ij}(t)| \leqslant Mt^\delta(1-t)^\delta \quad (i+j<2m), \quad \delta>0, \tag{1.3}$$

$$\mu_j(t) \notin S \qquad (j=\overline{1,l},\ t\in \bar{J}), \tag{1.3$'$}$$

where $S \subset \mathbf{C}$ is a certain closed angle with the origin at zero, and $\mu_j(t)$ is the eigenvalue of the matrix $a(t)$.

The following theorem holds if the above conditions are satisfied.

Theorem 1.1. *There exists a single closed operator A in $\mathcal{H}^l$, possessing the following properties:*

(i) $\quad D(A) \subset \mathcal{H}^l_+$, $(Au, v) = \mathcal{A}[u, v] \quad (\forall u \in D(A),\ v \in \mathcal{H}^l_+)$,

(ii) *for a certain $z_0 \in \mathbf{C}$ there is a continuous inverse*

$$(A - z_0 E)^{-1} : \mathcal{H}^l \to \mathcal{H}^l.$$

3. Let us denote by $\mathcal{H}_-$ completion of the space $\mathcal{H}$ by the norm

$$|u|_- = \sup_{0 \neq \varphi \in \mathcal{H}_+} \frac{|(u, \varphi)|}{|\varphi|_+}.$$

Suppose that $\mathcal{H}^l_- = \mathcal{H}_- \oplus \cdots \oplus \mathcal{H}_-$ (l - times). The element $F = (F_1, \ldots, F_l) \in \mathcal{H}^l_-$ generates an anti-linear continuous functional over $\mathcal{H}^l_+$ by the formula

$$< F, v >= \lim_{i \to +\infty} (u_i, v), \qquad v \in \mathcal{H}^l_+,$$

where the sequence of vector-functions $u_1, u_2, \ldots \in \mathcal{H}^l$ is chosen so that $u_i \to F$ $(i \to +\infty)$ in $\mathcal{H}^l_-$.

Note that if $v = (v_1, \ldots, v_l) \in \mathcal{H}^l_+$, then

$$< F, v >= \sum_{i=1}^{l} < F_i, v_i >, \quad |F|_- = (\sum_{i=1}^{l} |F_i|^2_-)^{1/2}.$$

Here and in what follows the same notation is accepted for $l = 1$, as well as for an arbitrary $l \in N$: $|\ |_-$, $<\,,\,>$.

Conversely, for any anti-linear continuous functional $g(v)$ $(v \in \mathcal{H}^l_+)$ there exists a single element $F \in \mathcal{H}^l_-$ such that $g(v) =< F, v >$, $\forall v \in \mathcal{H}^l_+$. Meanwhile, the norm of the functional g equals to $|F|_-$.

In what follows anti-linear continuous functionals over $\mathcal{H}^l_+$ are identified with elements of the space $\mathcal{H}^l_-$.

4. If the condition (1.3) is satisfied, one has

$$|\mathcal{A}[u, v]| \leqslant M|u|_+|v|_+ \quad (\forall u, v \in \mathcal{H}^l_+)$$

according to the Hardy inequality. Therefore, we can introduce the operator $\mathcal{A} : \mathcal{H}^l_+ \to \mathcal{H}^l_-$, acting by the formula

$$< \mathcal{A}u, v >= \mathcal{A}[u, v] \qquad (\forall u, v \in \mathcal{H}^l_+).$$

§2. A lemma on matrix functions

1. Let us formulate and prove an analogue of the Schur lemma for matrix functions in the present section.

Let us consider a matrix function $a(t) \in C^m(\overline{J}; End\mathbf{C}^l)$.

Suppose that the matrix $a(t)$, for every $t \in \overline{J}$, has l different eigenvalues $\mu_1(t), ..., \mu_l(t)$. Then eigenvalues of the matrix $a(t)$ $(t \in \overline{J})$ can be enumerated so that $\mu_j(t) \in C(\overline{J})$, $(j = \overline{1, l})$. The following lemma holds.

Lemma 2.1. *There is a matrix function*

$$U(t) \in C^m(\overline{J}; End\mathbf{C}^l)$$

such that

$$U^{-1}(t) \in C^m(\overline{J}; End\mathbf{C}^l)$$

and

$$a(t) = U(t)\Lambda(t)U^{-1}(t), \tag{2.1}$$

where $\Lambda(t)$ *is a diagonal matrix:*

$$\Lambda(t) = diag\{\mu_1(t), ..., \mu_l(t)\}, \mu_j(t) \in C^m(\overline{J}).$$

The proof is given at items 2 and 3.

2. Let $t_0 \in \overline{J}$. Let $r \in \{1, ..., l\}$ be a fixed index.

Let us introduce the matrix

$$P(t) = \frac{1}{2\pi i}\int\limits_{\gamma_\varepsilon}(a(t) - zI)^{-1}dz, \quad (|t - t_0| < \varepsilon') \qquad (2.1'),$$

where I is a unit matrix, $\gamma_\varepsilon = \{z \in C : |z - \mu_r(t_0)| = \varepsilon\}$ is the outline oriented counterclockwise. Let us introduce the notation $D_\varepsilon = \{z \in C : |z - \mu_r(t_0)| < \varepsilon\}$,

$$\Delta(\varepsilon') = \{t \in \overline{J} :]t - t_0| < \varepsilon'\}, \ \mu_j(\Delta(\varepsilon')) = \{\mu_i(t) : t \in \Delta(\varepsilon')\}.$$

When $\varepsilon, \varepsilon'$ are sufficiently small, one has

$$\mu_i(\Delta(\varepsilon')) \cap \mu_j(\Delta(\varepsilon')) = \emptyset \quad (i \neq j),$$

$$\mu_i(\Delta(\varepsilon')) \cap D_\varepsilon = \emptyset \quad (i \neq r),$$

$$\mu_r(\Delta(\varepsilon')) \subset D_\varepsilon.$$

Invoking that

$$tr\int\limits_{\gamma_\varepsilon} z(a(t) - zI)^{-1}dz = \sum_{i=1}^{l}\int\limits_{\gamma_\varepsilon} z(\mu_j(t) - z)^{-1}dz,$$

where $I \in End\mathbf{C}^l$ is a unit matrix, one obtains

$$\mu_r(t) = \frac{1}{2\pi i}tr\int\limits_{\gamma_\varepsilon} z(a(t) - zI)^{-1}dz. \tag{2.2}$$

Since $a(t) \in C^m(\overline{J}; End\mathbf{C}^l)$ then, $\mu_j(t) \in C^m(\overline{J})(j = \overline{1, l})$.

Let $y_\nu(t) = (y_{1\nu}(t), ..., y_{l\nu}(t))$ be an eigenvector of the matrix $a(t) = (a_{ij}(t))_{i,j=1}^l$, corresponding to eigenvalues $\mu_\nu(t)$, i.e.

$$\sum_{\nu=1}^{l} a_{ij}(t)y_{j\nu}(t) = \mu_\nu(t)y_{i\nu}(t) \quad (i = \overline{1, l}).$$

One can readily verify that the matrix $U(t) = (y_{ij}(t))_{i,j=1}^l$ satisfies the equalities

$$(a(t)U(t))_{pq} = \sum_{\nu=1}^{l} a_{pq}(t)y_{\nu q}(t) = \mu_q(t)y_{pq}(t),$$

$$(U(t)\Lambda(t))_{pq} = \sum_{\nu=1}^{l} y_{pq}(t)\delta_{\nu q}\mu_\nu(t) = \mu_q(t)y_{pq}(t),$$

where $\delta_{\nu q}$ denotes the Kronecker-Capelli symbol. Hence, $a(t)U(t) = U(t)\Lambda(t)$. Since the columns of the matrix $U(t)$ are composed of linearly independent eigenvectors of the matrix $a(t)$, one has

$$\det U(t) \neq 0 \quad (t \in \Delta(\varepsilon')). \tag{2.3}$$

Invoking that

$$(a(t) - zI)^{-1} = U(t)(\Lambda(t) - zI)^{-1}U^{-1}(t), \quad (t \in \Delta(\varepsilon')),$$

one obtains

$$P(t) = \frac{1}{2\pi i}U(t)(\int\limits_{\gamma_\varepsilon} (\Lambda(t) - zI)^{-1}dz)U^{-1}(t) = U(t)T_r U^{-1}(t),$$

$$T_r = diag\{\delta_{1r}, ..., \delta_{lr}\}. \tag{2.4}$$

One can readily deduce from these equalities that the domain range of the operator $P(t) : C^l \to C^l$ is one-dimensional and contains an eigenvector $y_r(t), (t \in \Delta(\varepsilon'))$. Therefore, the matrix $P(t)$ acts by the formula

$$P(t)h =< h, \varphi_r(t) >_{C^l} y_r(t), \quad (\forall h \in C^l, t \in \Delta(\varepsilon')), \tag{2.5}$$

where $\varphi_r(t) \in C^l, \forall t \in \Delta(\varepsilon')$. Assuming that $h = a(t)h_1 \quad (h_1 \in C^l)$ and invoking the equalities

$$P(t)a(t)h_1 = a(t)P(t)h_1 =< a(t)h_1, \varphi_r(t) >_{C^l} y_r(t) =$$

$$=< h_1, a^*(t)\varphi_r(t) >_{C^l} y_r(t) = \mu_r(t) < h_1, \varphi_r(t) >_{C^l} y_r(t),$$

one obtains

$$a^*(t)\varphi_r(t) = \overline{\mu_r(t)}\varphi_r(t)$$

due to arbitrariness of $h_1 \in C^l$.

According to (2.4), $trP(t) = trT_r = 1$. Therefore, $< y_r(t), \varphi_r(t) >_{C^l}= 1$.

Applying (2.5), one can readily find elements of the matrix $P(t)$:

$$(P(t))_{ij} = y_{ir}(t)\overline{\varphi_{jr}(t)},$$

where $\varphi_{jr}(t) \quad (j = \overline{1, l})$ are components of the vector $\varphi_r(t)$.

Whence and from (2.1′), it follows that

$$y_{ir}(t)\overline{\varphi_{jr}(t)} \in C^m(\Delta(\varepsilon')) \quad (i, j = \overline{1, l}). \tag{2.6}$$

Substituting the number ε' by a smaller positive number if necessary, one can find the index $w \in \{1, ..., l\}$ such that $\varphi_{wr}(t) \neq 0 \quad (\forall t \in \Delta(\varepsilon'))$. Further substituting if necessary $y_r(t), \varphi_r(t)$ by $\varphi_{wr}^{-1}(t)y_r(t), \varphi_{wr}^{-1}(t)\varphi_r(t)$, respectively one can assume without loss of generality that $\varphi_{wr}(t) \equiv 1 \quad (t \in \Delta(\varepsilon'))$.

Assuming that $j = w$ in (2.6), one obtains $y_{ir}(t) \in C^m(\Delta(\varepsilon')), i = 1, ..., l$.

By virtue of (2.5), one has

$$U^{-1}(t) \in C^m(\Delta(\varepsilon'); End\mathbf{C}^l).$$

3. Let $\Delta_1, \Delta_2 \subset \overline{J}$ be closed intervals, $mes\Delta \neq 0$, $\Delta = \Delta_1 \cap \Delta_2$.

Likewise, let us assume that the matrix functions

$$U_j(t) \in C^m(\Delta_j; End\mathbf{C}^l) \quad (j = 1, 2)$$

are constructed so that

$$U_j^{-1}(t) \in C^m(\Delta_j; End\mathbf{C}^l) \quad (j = 1, 2),$$

$$a(t) = U_j(t)\Lambda(t)U_j^{-1}(t) \quad (t \in \Delta_j), \quad (j = 1, 2).$$

Let us construct a matrix function

$$U(t) \in C^m(\Delta_1 \cup \Delta_2); End\mathbf{C}^l)$$

such that

$$U^{-1}(t) \in C^m(\Delta_1 \cup \Delta_2); End\mathbf{C}^l),$$

$$a(t) = U(t)\Lambda(t)U^{-1}(t) \ \ (t \in \Delta_1 \cup \Delta_2).$$

The columns of the matrices $U_1(t), U_2(t) \quad (t \in \Delta)$ are composed of eigenvectors of the matrix $a(t)$ and therefore they are collinear. Hence,

$$U_1(t) = U_2(t)\Omega(t), \quad \Omega(t) = diag\{w_1(t), ..., w_l(t)\}, \quad t \in \Delta,$$

where $w_j(t), w_j^{-1}(t) \in C^m(\Delta) \quad (j = \overline{1, l})$. Let us extend the functions $w_j(t)$ $(t \in \Delta)$,$j = \overline{1, l}$ up to functions $\tilde{w}_j(t) \in C^m(\Delta_2)$ so that $\tilde{w}_j^{-1}(t) \in C^m(\Delta_2) \quad (j = \overline{1, l})$.

Let us assume that

$$\tilde{\Omega}(t) = diag\{\tilde{w}_1(t), ..., \tilde{w}_l(t)\} \quad (t \in \Delta_2).$$

One can readily verify that the matrix function

$$U(t) = \begin{cases} U_1(t), t \in \Delta_1 \\ U_2(t)\tilde{\Omega}(t), t \in \Delta_2 \end{cases}$$

satisfies the above conditions. The proof of the lemma is completed by applying the factor method.

§ 3. Differential operators with matrix coefficients

1. Let us consider the following bilinear form in the space $\mathcal{H}^r = L_2(J)^r$, $r \in \{1, \dots, l\}$:

$$Q'[u, v] = \int\limits_J \rho^{2\theta}(t) < Q(t)u^{(m)}(t), v^{(m)}(t) >_{\mathbf{C}^r} dt, \; D[Q'] = \mathcal{H}^r_+,$$

where $\rho(t) = t(1-t), \theta < m$. The space $\mathcal{H}^r_+$ is the same as in §1, the matrix function $Q(t)$ has the form

$$Q(t) = \begin{pmatrix} q(t) & 1 & 0 & \dots & 0 \\ 0 & q(t) & 1 & \dots & 0 \\ 0 & 0 & q(t) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & q(t) \end{pmatrix},$$

$Q(t) \in C^m(\bar{J}; End\,\mathbf{C}^r), q(t) \in C^m(\bar{J}), q(t) \bar{\in} S \; (\forall t \in \bar{J})$, where $S \subset \mathbf{C}$ is a certain closed sector with the vertex at zero, located in the left semi-plane.

Let us denote by $\mathcal{H}^r_\nu, \nu > 0$ the space of functions $u \in \mathcal{H}^r_+$ with the norm

$$|u|_\nu = \left(\int\limits_J \rho^{2\theta}(t)|u^{(m)}(t)|^2 dt + \nu \int\limits_J |u(t)|^2 dt \right)^{1/2}.$$

Obviously, $\mathcal{H}_\nu^r = \mathcal{H}_+^r$, $\forall \nu > 0$, and the norms in the spaces are equivalent. Let $\mathcal{H}_{-\nu}^r, \nu > 0$ denotes the space of elements $F \in \mathcal{H}_-^r$ with the norm

$$|F|_{-\nu,r} = \sup_{\substack{v \in \mathcal{H}_+^r \\ |v|_\nu \leqslant 1}} |< F, v >|.$$

Let us introduce the operator $Q_{\nu,r} : \mathcal{H}_\nu^r \to \mathcal{H}_{-\nu}^r, \nu > 0$ according to the formula

$$< Q_{\nu,r} u, v >= Q'[u,v], \quad \forall u, v \in \mathcal{H}_\nu^r.$$

The following lemma holds.

Lemma 3.1. *There exists a sufficiently large number $c' > 0$ such that for $\lambda \in S$, $|\lambda| > c'$ and $\nu \in [1, 2|\lambda|]$ there is a continuous inverse*

$$(Q_{\nu,r} - \lambda E)^{-1} : \mathcal{H}_{-\nu}^r \to \mathcal{H}_\nu^r$$

with the norm not exceeding a certain number M, independent of λ, ν.

Proof. For the sake of simplicity let us consider that the sector S is located in the left semi-plane, is symmetric with respect to R_-, and has an angle of spread less than $\pi/2$.

Since $q(t)$ is a continuous function, there is a sector $\tilde{S}$ which is also located in the left semi-plane and is symmetric with respect to $R_- = (-\infty, 0)$, and has an angle of spread less than $\pi/2$, so that $\bar{S} \subset Int\, \tilde{S}$, and $q(t) \bar{\in} \tilde{S}$.

Let b_+ be a bisectrix of the angle composed by sides of the sectors S and $\tilde{S}$ from the upper semi-plane, and β_+ be the angle from the bisectrix to the imaginary axis.

Obviously, $\beta_+ < \pi/2$ and the inequity $Re\, \lambda e^{-i\beta_+} \leqslant 0$ holds for every $\lambda \in S\, (|\lambda| > 1)$ and $Re\, \lambda e^{-i\beta_+} q(t) > 0$ for all t such that $Im\, q(t) \geq -\frac{\beta_+}{4}$. Likewise, for $Im\, q(t) < \frac{\beta_+}{4}$ we find the number β_- such that the inequalities

$$Re\, \lambda e^{-i\beta_-} \leqslant \frac{\beta_-}{4}, \forall \lambda \in S\, (|\lambda| > 1);\ Re\, e^{-i\beta_-} q(t) > 0$$

hold.

Let us cover now the segment $[0, 1]$ by intervals $I_1, \dots, I_k$ so that the right-hand end I_i intersects the left-hand end I_{i+1}, and

$$mes\, (I_i \cap I_{i+1}) \neq 0, \quad i = \overline{1, k-1},$$

the multiplicity of covering equals to 2, and for every fixed i one has

$$\text{either} \quad Im\, q(t) \geq -\frac{\beta_+}{2}, \forall t \in I_i, \quad \text{or} \quad Im\, q(t) \leqslant \frac{\beta_+}{2}, \forall t \in I_i.$$

Let us construct nonnegative functions $\varphi_1(t), \varphi_2(t), \dots, \varphi_k(t), \psi_1(t)$, $\psi_2(t), \dots, \psi_k(t) \in C^\infty(\bar{J})$ such that

$$\sum_{j=1}^k \varphi_j(t) \equiv 1 \quad (t \in \bar{J})$$

and

$$\psi_j(t) = 1, \quad \forall t \in supp\, \varphi_j, \quad supp\, \psi_j \subset I_j.$$

Therefore,

$$Re\, \lambda e^{i\alpha_j} < 0, \quad Re\, q(t) e^{i\alpha_j} > 0, \quad \forall t \in supp\, \psi_j,$$

where α_j equals to $-\beta_+$ or $-\beta_-$.

Since $q(t)$ is a continuous function on $[0, 1]$, and $supp\, \psi_j$ is a compact, one has

$$Re\, e^{i\alpha_j} q(t) > c_j > 0, \quad \forall t \in supp\, \psi_j,$$

and therefore,

$$Re\, e^{i\alpha_j} q(t) > c > 0, \quad \text{where} \quad c = min\, c_j,\ j = \overline{1, k}.$$

Multiplying the initial bilinear form by the number $\frac{8}{c}$ if necessary, one can obtain the following inequality

$$Re\, e^{i\alpha_j} q(t) \geq 8.$$

Several statements are necessary to complete the proof. Therefore, the remaining part of the proof will be given in item 6 of the present section.

The following lemma holds.

Lemma 3.2. *Let the above conditions hold and $Re\, e^{i\alpha_j} q(t) \geq 8$. Then, for any vector $h \in \mathbf{C}^r$ the inequality*

$$Re < e^{i\alpha_j} Q(t)h, h >_{\mathbf{C}^r} \geq 7|h|^2_{\mathbf{C}^r}$$

holds.

Proof. One has

$$< e^{i\alpha_j} Q(t)h, h >_{\mathbf{C}^r} = \sum_{k=1}^{r-1} \left(\frac{e^{i\alpha_j} q(t)}{2} |h_k|^2 + e^{i\alpha_j} \bar{h}_k h_{k+1} + \frac{e^{i\alpha_j} q(t)}{2} |h_{k+1}|^2\right) +$$

$$+ \frac{e^{i\alpha_j} q(t)}{2} |h_1|^2 + \frac{e^{i\alpha_j} q(t)}{2} |h_r|^2.$$

Whence, in view of the inequality $Re\, e^{i\alpha_j} q(t) \geq 8$, one obtains

$$Re < e^{i\alpha_j} Q(t)h, h >_{\mathbf{C}^r} \geq \sum_{k=1}^{r-1} (4|h_k|^2 - |h_k||h_{k+1}| + 4|h_{k+1}|^2) +$$

$$+4(|h_1|^2 + |h_r|^2) \geq \sum_{k=1}^{r-1} (4|h_k|^2 - \frac{1}{2}|h_k|^2 - \frac{1}{2}|h_{k+1}|^2 + 4|h_{k+1}|^2) +$$

$$+4(|h_1|^2 + |h_r|^2) = \frac{7}{2}[\sum_{k=1}^{r-1} (|h_k|^2 + |h_{k+1}|^2)] +$$

$$+4(|h_1|^2 + |h_r|^2) = 7(\sum_{k=1}^{r} |h_k|^2) + \frac{13}{2}(|h_1|^2 + |h_r|^2) \geq 7|h|^2_{\mathbf{C}^r}.$$

Lemma 3.2 is proved.

2. Let us formulate Theorem 2.0.1 from [23] in the necessary form.

Consider the following bilinear form in the space $\mathcal{H}^r_\nu$:

$$B[u, v] = \sum_{i,j=0}^{m} (a_{ij} p_i u^{(i)}, p_j v^{(j)})_{L_2(J)^r},$$

where a_{ij}, p_i are the same objects as in §1, $r \in \{1, \ldots, l\}$.

Statement 3.1. *(see Proposition 2.0.1 in [23]) Let the bilinear from $B[u, v]$ satisfy the inequalities*

$$|B[u, v]| \leqslant M |u|_{\mathcal{H}^r_+} |v|_{\mathcal{H}^r_+}, \tag{3.1}$$

$$Re\, B[u, u] + \lambda_0(u, v) \geq \delta |u|^2_{\mathcal{H}^r_\nu}. \tag{3.2}$$

Then:

1) there is a linear operator Λ, realizing the homomorphism of the spaces $\mathcal{H}^r_\nu$ and $\mathcal{H}^r_{-\nu}$, such that

$$< \Lambda u, v >= B[u, v] + \lambda_0(u, v), \quad \forall u, v \in \mathcal{H}^r_\nu,$$

where the symbol $< f, v >$ denote the action of the functional f upon the element v;

2) any anti-linear continuous functional $l(v)$ over $\mathcal{H}^r_\nu$ admits a representation

$$l(v) = B[u_0, v] + \lambda_0(u_0, v) =< \Lambda u_0, v >, \quad \forall u \in \mathcal{H}^r_{-\nu},$$

where u_0 is a certain element from $\mathcal{H}_+^r$.

The latter means that the operator Λ^{-1} exists and has a finite form

$$|\Lambda^{-1}|_{\mathcal{H}_{-\nu}^r\to\mathcal{H}_\nu^r} < +\infty.$$

The following lemma complements Statement 3.1.

Lemma 3.3. *Let Conditions 1) and 2) be satisfied. Then, the following inequality holds:*

$$|\Lambda^{-1}|_{\mathcal{H}_{-\nu}\to\mathcal{H}_{+\nu}} \leqslant \frac{1}{\delta}, \quad \delta > 0. \tag{3.3}$$

Proof. The latter inequality means that

$$\delta|\Lambda^{-1}v|_{\mathcal{H}_\nu} \leqslant |v|_{\mathcal{H}_{-\nu}} \quad \forall v \in \mathcal{H}_{-\nu}. \tag{3.4}$$

Since $< \Lambda u, v >= B[u,v] + \lambda_0(u,v)_{\mathcal{H}}$ then,

$$Re\ < \Lambda u, u > \geq \delta|u|^2_{\mathcal{H}_{+\nu}}, \quad \forall u \in \mathcal{H}_+.$$

However, according to definition of the functional norm

$$|Re\ < \Lambda u, u > | \leqslant | < \Lambda u, u > | \leqslant |\Lambda u|_{\mathcal{H}_{-\nu}}|u|_{\mathcal{H}_{+\nu}}.$$

Therefore,

$$|\Lambda u|_{\mathcal{H}_{-\nu}}|u|_{\mathcal{H}_{+\nu}} \geq \delta|u|^2_{\mathcal{H}_{+\nu}}$$

and

$$|\Lambda u|_{\mathcal{H}_{-\nu}} \geq \delta|u|_{\mathcal{H}_{+\nu}}, \quad \forall u \in \mathcal{H}_+.$$

Substituting $v = \Lambda u$, one obtains (3.4). Lemma 3.3 is proved.

Let us introduce a function $\tilde{q}_j(t)$ coinciding with $q(t)$ on the interval I_j, and varying smoothly outside $I_j, j = \overline{1,k}$.

Obviously, the following inequality holds (see Lemma 3.2)

$$Re\ < e^{i\alpha_j}\tilde{Q}_j(t)h, h >_{\mathbf{C}^r} \geq c_0|h|^2_{\mathbf{C}^r}, \quad \forall t \in [0,1], \tag{3.5}$$

where $0 < c_0 < 7$, and the matrix function $\tilde{Q}_j(t)$ is derived from $Q(t)$ via substituting $q(t)$ by $\tilde{q}_j(t)$.

Let us introduce the bilinear form

$$Q_j^0[u,v] = \int\limits_J \rho^{2\theta}(t) < \tilde{Q}_j(t)u^{(m)}(t), v^{(m)}(t) >_{\mathbf{C}^r} dt,\ u,v \in \mathcal{H}_\nu^r.$$

The following lemma holds true.

Lemma 3.4. *For any $u \in \mathcal{H}_\nu^r$ the inequality*

$$Re\, e^{i\alpha_j}Q_j^0[u,u] \geq c_0|u^{(m)}|^2_{L_2(J)^r},\ 0 < c_0 < 7 \tag{3.6}$$

holds.

Proof. Substituting $h = \rho^\theta(t)u^{(m)}(t)$ to inequality (3.5), one obtains

$$Re\ < e^{i\alpha_j}Q_j(t)\rho^\theta u^{(m)}(t), \rho^\theta u^{(m)}(t) >_{\mathbf{C}^r} \geq$$

$$\geq c_0|\rho^\theta u^{(m)}(t)|^2_{\mathbf{C}^r}, \quad 0 \leqslant t \leqslant 1.$$

Integrating with respect to t from zero to 1, one obtains (3.6), which proves the lemma.

Since $Re\,\lambda e^{i\alpha_j} \leqslant -c|\lambda|$, Lemma 3.4 entails that

$$Re\,[e^{i\alpha_j}Q_j^0[u,u] - \lambda e^{i\alpha_j}(u,u)] \geq$$

$$\geq c'|u|^2_{\mathcal{H}_\nu^r} = c'|u|^2_{\mathcal{H}_+^r} + \nu c'|u|^2_{L_2(J)^r},\ (1 \leqslant \nu < 2|\lambda|,\ |\lambda| \geq 1).$$

Let us introduce the operator $L_{j,\nu} : \mathcal{H}_\nu^r \to \mathcal{H}_{-\nu}^r$ according to the formula

$$< L_{j,\nu} u, v >= Q_j^0[u,v], \quad \forall u, v \in \mathcal{H}_\nu^r.$$

Applying the Cauchy-Bunyakovsky inequality, one can readily demonstrate that the bilinear from $Q_j^0[u,v]$ satisfies the inequality

$$|Q_j^0[u,v]| \leqslant M|u|_{\mathcal{H}_\nu^r}|v|_{\mathcal{H}_\nu^r}.$$

Then, on the basis of Lemma 3.4 we conclude that all the conditions of Statement 3.1 are satisfied. Hence, there exists a continuous inverse

$$\mathcal{R}_{j,\nu}(\lambda) = (e^{i\alpha_j} L_{j,\nu} - \lambda e^{i\alpha_j} E)^{-1} : \mathcal{H}_{-\nu}^r \to \mathcal{H}_\nu^r,$$

$$(\lambda \in S, \quad |\lambda| \geq 1, \quad \nu \in [1; 2|\lambda|),$$

and

$$|\mathcal{R}_{j,\nu}(\lambda)| \leqslant \delta^{-1},$$

$0 < \delta$ is a certain number independent of λ, ν.

For $F \in \mathcal{H}_{-\nu}^r, v \in \mathcal{H}_\nu^r$ one has

$$< \psi_j F, \varphi_j v >= (\rho^\theta(t) Q(t) e^{i\alpha_j} \partial_t^m \mathcal{R}_{j,\nu}(\lambda)\psi_j F, \rho^\theta \partial_t^m (v\varphi_j))-$$

$$-\lambda e^{i\alpha_j}(\mathcal{R}_{j,\nu}(\lambda)\psi_j F, \varphi_j v), \tag{3.7}$$

aaa $\partial_t = \frac{d}{dt}$. Here and in what follows in the present item (,) indicates the scalar product in $L_2(J)^r$.

Let us introduce the operator

$$\mathcal{R}_\nu(\lambda) = \sum_{j=1}^k \varphi_j e^{i\alpha_j} \mathcal{R}_{j,\nu}\psi_j : \mathcal{H}_{-\nu}^r \to \mathcal{H}_\nu^r, (\lambda \in S, |\lambda| \geq 1, \nu \in [1, 2|\lambda|]). \tag{3.7'}$$

Obviously,

$$< (Q_\nu - \lambda E)\mathcal{R}_\nu(\lambda)F, v >= \sum_{j=1}^k e^{i\alpha_j}(Q(t)\rho^\theta(t)\partial_t^m \varphi_j \mathcal{R}_{j,\nu}(\lambda)\psi_j F, \rho^\theta(t) v^{(m)}(t))-$$

$$-\lambda \sum_{j=1}^k e^{i\alpha_j}(\varphi_j \mathcal{R}_{j,\nu}(\lambda)\psi_j F, v).$$

In view of (3.7) and the equality

$$\sum_{j=1}^k < \psi_j F, \varphi_j v >=< F, v >,$$

one obtains

$$< (Q_\nu - \lambda E)\mathcal{R}_\nu(\lambda)F, v >=< F, v > + X_\lambda(F,v) + Y_\lambda(F,v), \tag{3.8}$$

where

$$X_\lambda(F,v) = \sum_{j=1}^k e^{i\alpha_j}\{\sum_{\substack{m_1+m_2=m \\ m_2\neq 0}} C_{m_1,m_2}(\rho^\theta Q \partial_t^m \mathcal{R}_{j,\nu}(\lambda)\psi_j F, \rho^\theta v^{(m_1)}\varphi_j^{(m_2)})\}, \tag{3.8'}$$

$$Y_\lambda(F,v) = \sum_{j=1}^k e^{i\alpha_j}\{\sum_{\substack{m_1+m_2=m \\ m_2\neq 0}} C'_{m_1,m_2}(\rho^\theta Q \varphi_j^{(m_2)} \partial_t^m \mathcal{R}_{j,\nu}(\lambda)\psi_j F, \rho^\theta v^{(m)})\}. \tag{3.9}$$

Here $C_{m_1,m_2}, C'_{m_1,m_2}$ are some constant numbers depending only on m_1, m_2. Integrating by parts once, one obtains

$$X_\lambda(F,v) = -\sum_{j=1}^{k} e^{i\alpha_j}\{\sum_{\substack{m_1+m_2=m \\ m_2\neq 0}} C_{m_1,m_2}(\rho^\theta \partial_t^{m-1}(\mathcal{R}_{j,\nu}(\lambda)\psi_j F),$$

$$\rho^{-\theta}\partial_t(Q^*(t)\rho^{2\theta}v^{(m_1)}\varphi_j^{(m_2)}))\}. \tag{3.10}$$

3. Let P be a selfadjoint operator in $\mathcal{H}$ associated with the bilinear from

$$P'[u,v] = (\rho^\theta u^{(m)}, \rho^\theta v^{(m)}), \quad D[P'] = \mathcal{H}_+.$$

In what follows the following lemma will be of use.

Lemma 3.5. *There exists a continuous inverse operator $T_\omega : \mathcal{H}_- \to \mathcal{H}, \omega \geq 1$ such that $T_\omega u = (P+\omega E)^{-1/2}u, \forall u \in \mathcal{H}$, and*

$$|T_\omega F| \leqslant M|F|_{-\nu} \quad (\forall \omega \geq 1, \nu \in [1,2\omega), \forall F \in \mathcal{H}_{-\nu}),$$

where the number $M > 0$ is independent of ω, ν.

Proof. Let $F \in \mathcal{H}_{-\nu}$. By virtue of density of the space $\mathcal{H}$ in $\mathcal{H}_-$, there are elements $u_1, u_2, \ldots \in \mathcal{H}$ such that $u_j \to_{\mathcal{H}_{-\nu}} F\ (j \to +\infty)$. Then,

$$|((P+\omega E)^{-1/2}(u_j - u_k), v)| = |(u_j - u_k, (P+\omega E)^{-1/2}v)| \leqslant$$

$$\leqslant |u_j - u_k|_{-\nu}|(P+\omega E)^{-1/2}v|_\nu \leqslant M|v||u_j - u_k|_{-\nu}$$

for all $v \in \mathcal{H}_+$. Since $\mathcal{H}_+$ is dense in $\mathcal{H}$, then

$$|(P+\omega E)^{-1/2}(u_j - u_k)| \leqslant M|u_j - u_k|_{-\nu}.$$

Therefore, the sequence $(P+\omega E)^{-1/2}u_j, j = 1, 2, \ldots$ is fundamental in $\mathcal{H}$ and one has

$$(P+\omega E)^{-1/2}u_j \to_{\mathcal{H}} g \quad (j \to +\infty),$$

where g is a certain element from $\mathcal{H}$. Let us assume that $T_\omega F = g, F \in \mathcal{H}$. Obviously, $|g| \leqslant M \lim\limits_{j\to+\infty} |u_j|_{-\nu} = M|F|_{-\nu}, \forall F \in \mathcal{H}_{-\nu}$ and $T_\omega F = (P+\omega E)^{-1/2}F, \forall F \in \mathcal{H}$. The lemma is proved.

4. Our nearest goal is the proof of the inequality (see $(3.8')$ – (3.10))

$$|X_\lambda(F,v)| + |Y_\lambda(F,v)| \leqslant M|\lambda|^{-\varepsilon'}|F|_{-\nu}|v|_\nu \tag{3.11}$$

$$(\forall F \in \mathcal{H}^r_-, v \in \mathcal{H}^r_+, \lambda \in S, |\lambda| \geq 1, \nu \in [1, 2|\lambda|)),$$

with some $\varepsilon' > 0$. Note that the bilinear form

$$P'_{j,\lambda}[u,v] = e^{i\alpha_j}(\rho^\theta Q u^{(m)}, \rho^\theta Q^{(m)}) - \lambda e^{i\alpha_j}(u,v), \ (u, v \in \mathcal{H}^r_+),$$

where $\lambda \in S, |\lambda| \geq 1$, is defined densely in $\mathcal{H}^r$, closed, and sectorial. According to the known theorem (see Theorem 2.1 from [24, Ch.VI, §2]), there is an m-sectorial operator $P_{j,\lambda}$ in $\mathcal{H}^r$ such that $D(P_{j,\lambda}) \subset \mathcal{H}^r_+$, and

$$(P_{j,\lambda}u, v) = P'_{j,\lambda}[u,v] \quad (\forall u \in D(P_{j,\lambda}),\ v \in \mathcal{H}^r_+).$$

Likewise, there is a positive selfadjoint operator $P^0_{j,\lambda}$ in $\mathcal{H}$ such that $D(P^0_{j,\lambda}) = \mathcal{H}^r_+$ and

$$(P^0_{j,\lambda}u, v) = \frac{1}{2}(P'_{j,\lambda}[u,v] + \overline{P'_{j,\lambda}[v,u]}),\ \forall u, v \in \mathcal{H}_+.$$

Applying Theorem 3.2 from [24, Ch.VI, §3], one obtains

$$P^{-1}_{j,\lambda} = (P^0_{j,\lambda})^{-1/2}\mathcal{F}_{j,\lambda}(P^0_{j,\lambda})^{-1/2},$$

where

$$\|\mathcal{F}'_{j,\lambda}\|_{\mathcal{H}^r \to \mathcal{H}^r} \leqslant M;$$

the number M is independent of $\lambda \in S\,(|\lambda| \geq 1)$. Whence, one readily concludes that

$$P_{j,\lambda}^{-1} = (P + |\lambda|E)^{-1/2}\mathcal{F}_{j,\lambda}(P + |\lambda|E)^{-1/2},$$

where P is the same operator as in Lemma 3.5, and

$$\|\mathcal{F}_{j,\lambda}\|_{\mathcal{H}^r \to \mathcal{H}^r} \leqslant M', \quad (\lambda \in S, |\lambda| \geq 1).$$

Let us prove that

$$\mathcal{R}_{j,\nu}(\lambda) = (P + |\lambda|E)^{-1/2}\mathcal{F}_{j,\lambda}T_{|\lambda|},\ (\lambda \in S, |\lambda| \geq 1, \nu \in [1, 2|\lambda|)). \tag{3.12}$$

The definition of operators $\mathcal{R}_{j,\nu}(\lambda), P_{j,\lambda}$ entails that

$$P_{j,\lambda}^{-1}u = \mathcal{R}_{j,\nu}(\lambda)u \quad (\forall u \in \mathcal{H}_+^r, \lambda \in S, |\lambda| \geq 1, \nu \in [1, 2|\lambda|)).$$

Therefore,

$$\mathcal{R}_{j,\nu}(\lambda)u = (P + |\lambda|E)^{-1/2}\mathcal{F}_{j,\lambda}T_{|\lambda|}u \quad (\forall u \in \mathcal{H}^r).$$

Since the operators $\mathcal{R}_{j,\nu}(\lambda), (P + |\lambda|E)^{-1/2}\mathcal{F}_{j,\lambda}T_{|\lambda|}$ are continuous from $\mathcal{H}_\nu^r$ to $\mathcal{H}_{-\nu}^r$, and the space $\mathcal{H}^r$ is dense in $\mathcal{H}_{-\nu}^r$ then,

$$\mathcal{R}_{j,\nu}(\lambda)F = (P + |\lambda|E)^{-1/2}\mathcal{F}_{j,\lambda}T_{|\lambda|}F,$$
$$(\forall F \in \mathcal{H}_{-\nu}, \lambda \in S, |\lambda| \geq 1, \nu \in [1, 2|\lambda|)),$$

which proves (3.12).

Now the proof of the inequality (3.11) can be easily completed. Let us substitute $\mathcal{R}_{k,\nu}(\lambda)$ by the right-hand side (3.12) in the formulae (3.9), (3.10). Then, it remains only to demonstrate that the inequalities

$$\|\rho^\theta \partial_t^{m_1}(P + |\lambda|E)^{-1/2}\|_{\mathcal{H}\to\mathcal{H}} \leqslant M|\lambda|^{-\varepsilon'}\,(m_1 < m),$$
$$\|\rho^{\theta-\delta}\partial_t^{m_1}(P + |\lambda|E)^{-1}\|_{\mathcal{H}\to\mathcal{H}} \leqslant M|\lambda|^{-\varepsilon'},$$
$$|\rho^{\delta-\theta}\partial_t(\bar{Q}\rho^{2\theta}v^{(m_1)}\psi_j^{(m_2)})|_{\mathcal{H}^r} \leqslant M|v|_+,\ (m_1 + m_2 = m, m_2 \neq 0)$$

hold for $\lambda \in S, |\lambda| \geq 1$. Here $\varepsilon', \delta > 0$ are sufficiently small numbers. The first two inequalities above are deduced from the known multiplicative inequalities (see, e.g., [25]). While the number δ can be arbitrary from the interval $(0, 1)$. The latter estimate follows from the Hardy inequality:

$$\sum_{m_1+m_2=m} |\rho^{\theta-m_2+\delta}v^{(m_1)}|_{\mathcal{H}^r} \leqslant M_\delta|v|_+ \quad (\forall v \in \mathcal{H}_+^r).$$

If $\theta \neq \frac{1}{2}, \ldots, m - \frac{1}{2}$, the inequality holds for $\delta = 0$ as well.

6. According to (3.8), the inequality (3.11) entails that

$$(Q_\nu - \lambda E)\mathcal{R}_\nu(\lambda) = E + G_\nu(\lambda),\ (\lambda \in S, |\lambda| \geq 1, \nu \in [1, 2|\lambda|)),$$

where $G_\nu(\lambda) : \mathcal{H}_{-\nu}^r \to \mathcal{H}_{-\nu}^r$ is a continuous operator,

$$\|G_\nu(\lambda)\|_{\mathcal{H}_{-\nu}^r\to\mathcal{H}_{-\nu}^r} \leqslant M|\lambda|^{-\varepsilon'},\ \varepsilon' > 0.$$

Let us select the number $\sigma_0 > 0$ such that $M|\lambda|^{-\varepsilon'} \leqslant \frac{1}{2}$ for all $|\lambda| \geq \sigma_0$. Then,

$$(Q_\nu - \lambda E)\mathcal{R}_\nu(\lambda)G'_\nu(\lambda) = E,\ G'_\nu(\lambda) = (E + G_\nu(\lambda))^{-1},$$
$$\|(E - G'_\nu(\lambda)\|_{\mathcal{H}_{-\nu}^r\to\mathcal{H}_{-\nu}^r} < 1.$$

Let us demonstrate that $ker\,(Q_\nu - \lambda E) = 0, \forall\lambda \in S, |\lambda| \geq \sigma_1$, where σ_1 is a sufficiently large number, $\nu \in [1, 2|\lambda|)$. Then, for $\lambda \in S, |\lambda| \geq \sigma' = max\{\sigma_0, \sigma_1\}, \nu \in [1, 2|\lambda|)$ one has the equality

$$(Q_\nu - \lambda E)^{-1} = \mathcal{R}_\nu(\lambda)G'_\nu(\lambda). \tag{3.13}$$

Let us consider the operator $Q_{*,\nu} : \mathcal{H}_\nu^r \to \mathcal{H}_{-\nu}^r, \nu > 0$, acting by the formula

$$< Q_{*,\nu}u, v >= (\rho^\theta\bar{Q}u^{(m)}, \rho^\theta v^{(m)}),\ (\forall u, v \in \mathcal{H}_\nu^r).$$

Similarly to the above, one constructs the operators

$$\mathcal{R}_{*,\nu}(\bar\lambda)\,:\,\mathcal{H}^r_{-\nu}\to\mathcal{H}^r_{\nu},\; G_{*,\nu}(\bar\lambda)\,:\,\mathcal{H}^r_{-\nu}\to\mathcal{H}^r_{-\nu},$$

such that

$$(Q_{*,\nu}-\bar\lambda E)\mathcal{R}_{*,\nu}(\bar\lambda)=E+G_{*,\nu}(\bar\lambda), \tag{3.14}$$

$$\|G_{*,\nu}(\bar\lambda)\|_{\mathcal{H}_{-\nu}\to\mathcal{H}_{-\nu}}\leqslant\frac{1}{2}\quad(\lambda\in S,|\lambda|\geq\sigma_1,\nu\in[1,2|\lambda|)). \tag{3.14$'$}$$

Let $u\in\mathcal{H}^r_{-\nu}$ be such an element that $(Q_\nu-\lambda E)u=0$. Moreover, let $|\lambda|\geq\sigma'=max\{\sigma_0,\sigma_1\}$. Then,

$$<(Q_\nu-\lambda E)u,v>=0,\quad\forall v\in\mathcal{H}^r_\nu,$$

i.e.

$$(\rho^\theta Qu^{(m)},\rho^\theta v^{(m)})-\lambda(u,v)=0\quad(\forall v\in\mathcal{H}^r_\nu).$$

Hence,

$$<(Q_{*,\nu}-\bar\lambda E)v,u>=(\bar Q(t)\rho^\theta(t)v^{(m)}(t),u^{(m)}(t))-\bar\lambda(v,u)=0.$$

Suppose that $v=\mathcal{R}_{*,\nu}(\bar\lambda)F, F\in\mathcal{H}^r_{-\nu}$. Then, due to (3.9),

$$<(E+G_{*,\nu}(\bar\lambda))F,u>=0,\quad\forall F\in\mathcal{H}^r_{-\nu}.$$

Since the operator

$$(E+G_{*,\nu}(\bar\lambda))\,:\,\mathcal{H}^r_{-\nu}\to\mathcal{H}^r_\nu$$

has a continuous inverse one when $\lambda\in S,|\lambda|\geq\sigma',\nu\in[1,2|\lambda|)$ then, $<F_1,u>=0,\forall F_1\in\mathcal{H}^r_{-\nu}$. Assuming that $F_1=0$, one obtains $u=0$. Thus, the equality (3.13) is proved, which entails the estimate

$$\|(Q_\nu-\lambda E)^{-1}\|_{\mathcal{H}^r_{-\nu}\to\mathcal{H}^r_\nu}\leqslant2\|\mathcal{R}_\nu(\lambda)\|_{\mathcal{H}^r_{-\nu}\to\mathcal{H}^r_\nu}.$$

Applying now (3.7$'$), (3.12), and Lemma 3.5 we complete the proof of Lemma 3.1.

Remark. Results of the present section are also valid when the matrix $Q(t)$ is diagonal and they are deduced easier due to absence of the unit.

§ 4. Proof of Theorem 1.1.

1. Let us consider a bilinear form $\mathcal{A}[u,v]$ (1.2). Let us assume that all the conditions of Theorem 1.1 are satisfied. Here we assume that the matrix $a(t)$ has the form

$$a(t)=U(t)\Lambda(t)U^{-1}(t), \tag{4.1}$$

where the matrix functions are $U(t),U^{-1}(t)\in C^m(\bar J;End\,\mathbf{C}^l)$, and $\Lambda(t)$ is the Jordan matrix of the following structure. There are numbers $r_1,\dots,r_p$ such that $\sum\limits_{i=1}^p r_i=l$, and if the complex l-dimensional space $\mathbf{C}^l$ is represented in the from $\mathbf{C}^{r_1}\times\cdots\times\mathbf{C}^{r_p}$ then,

$$\Lambda(t)=diag\,\{Q_1(t),\dots,Q_p(t)\},$$

where $Q_i(t)$ is an $r_i\times r_i$-matrix of the form

$$\begin{pmatrix} q_i(t) & 1 & 0 & \dots & 0\\ 0 & q_i(t) & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & q_i(t)\end{pmatrix}$$

or

$$Q_i(t)=diag\,\{q_i(t),\dots,q_i(t)\}\quad(r_i\quad\text{- times}). \tag{4.2}$$

Increasing the number of blocks if necessary, one can achieve that $r_i=1$ in the case (4.2) and then one deals with a scalar case (i.e. a one-dimensional matrix).

It is clear that $\{q_i(t)\}$ are eigenvalues of the matrix $a(t)$.

According to Lemma 2.1, the same representations take place, in particular, when eigenvalues of the matrix $q(t)$ in the coordinate plane are all different at the ends of the interval. Because in this case they are different by continuity in some neighborhood of the interval ends.

Let us introduce the operator $Q_{\nu,i} : \mathcal{H}^{r_i}_\nu \to \mathcal{H}^{r_i}_{-\nu}\,(\nu > 0, i = \overline{1,p})$ by the formula

$$< Q_{\nu,i}u, v >= Q^0_{\nu,i}[u, v] = (\rho^\theta Q_i(t)u^{(m)}, \rho^\theta v^{(m)})_{L_2(J)^l}\,(u, v \in \mathcal{H}^{r_i}_\nu).$$

In the direct sum $\mathcal{H}^l_\nu = \mathcal{H}^{r_1}_\nu \oplus \mathcal{H}^{r_2}_\nu \oplus \cdots \oplus \mathcal{H}^{r_p}_\nu$, let us introduce the operator

$$\mathcal{B}_\nu = diag\,\{Q_{\nu,1}, \ldots, Q_{\nu,p}\} \,:\, \mathcal{H}^l_\nu \to \mathcal{H}^l_{-\nu},$$

where $\mathcal{H}^l_{-\nu} = \mathcal{H}^{r_1}_{-\nu} \oplus \mathcal{H}^{r_2}_{-\nu} \oplus \cdots \oplus \mathcal{H}^{r_p}_{-\nu}$. The norm $|F|_{-\nu}$ of the element $F \in \mathcal{H}^l_{-\nu}$ is equal to the upper bound of numbers $| < F, v > |$ with respect to $v \in \mathcal{H}^l_\nu$ such that $|v|_\nu = 1$.

If $r \in \{1, \ldots, p\}, \lambda \in S, |\lambda| \geq c', \nu \in [1, 2|\lambda|)$, where $c' > 0$ is a sufficiently large number then according to Lemma 3.1, there are continuous inverse

$$(Q_{\nu,r} - \lambda E)^{-1} \,:\, \mathcal{H}^l_{-\nu} \to \mathcal{H}^l_\nu, \quad r = \overline{1,p}.$$

Obviously,

$$(\mathcal{B}_\nu - \lambda E)^{-1} = diag\,\{(Q_{\nu,1} - \lambda E)^{-1}, \ldots, (Q_{\nu,p} - \lambda E)^{-1}\} \,:\, \mathcal{H}^l_{-\nu} \to \mathcal{H}^l_\nu$$

is a continuous operator. Let us assume that

$$X_\nu(\lambda) = U(\mathcal{B}_\nu - \lambda E)^{-1}U^{-1}, \tag{4.3}$$

where U denotes the operator acting in $\mathcal{H}^l_{-\nu}$ by the formula

$$< UF, v >=< F, U^*(t)v(t) > \quad (\forall F \in \mathcal{H}^l_{-\nu},\, v \in \mathcal{H}^l_{-\nu}).$$

Obviously, $U \,:\, \mathcal{H}^l \to \mathcal{H}^l$, $U \,:\, \mathcal{H}^l_\nu \to \mathcal{H}^l_\nu$.

Note that if

$$F = (F_1, \ldots, F_l) \in \mathcal{H}^l_{-\nu},\, v = (v_1, \ldots, v_l) \in \mathcal{H}^l_\nu,\, \nu > 0,$$

then

$$|F|_{-\nu} = (\sum_{i=1}^{l} |F_i|^2_{-\nu})^{1/2}, \quad |v|_\nu = (\sum_{i=1}^{l} |v_i|^2_\nu)^{1/2},$$

$$< F, v >= \sum_{i=1}^{l} < F_i, v_i > .$$

Likewise, if $F = (F_1, \ldots, F_p)$, aaa $F_i \in \mathcal{H}^l_{-\nu}$, $v = (v_1, \ldots, v_p) \in \mathcal{H}^l_\nu,\ \nu > 0$ then,

$$|F|_{-\nu} = (\sum_{i=1}^{p} |F_i|^2_{-\nu})^{1/2}, \quad |v|_\nu = (\sum_{i=1}^{p} |v_i|^2_\nu)^{1/2},$$

$$< F, v >= \sum_{i=1}^{p} < F_i, v_i > .$$

For $F \in \mathcal{H}^l$, one has $< F, v >= (F, v)$, $\forall v \in \mathcal{H}^l_\nu$.

According to Lemma 3.1, the representation (4.3) entails that when $\lambda \in S$ are sufficiently large in module, the following inequality, where M is independent of $\lambda, \nu \in [1, 2|\lambda|)$, holds:

$$\|X_\nu(\lambda)\|_{\mathcal{H}^l_{-\nu} \to \mathcal{H}^l_\nu} \leqslant M. \tag{4.3'}$$

By virtue of (1.2) the following equality holds for $F \in \mathcal{H}^l_{-\nu}, v \in \mathcal{H}^l_\nu$:

$$\mathcal{A}[X_\nu(\lambda)F, v] = x_\lambda(F, v) + y_\lambda(F, v),$$

where

$$x_\lambda(F, v) = \sum_{j=0}^{m} \sum_{i=0}^{r_j} (p_i a_{ij} \partial^i_t X_\nu(\lambda)F, p_j v^{(j)}),\, r_j = min\{m, 2m - j - 1\},$$

$$y_\lambda(F,v) = (\rho^\theta a(t)\partial_t^m X_\nu(\lambda)F, \rho^\theta v^{(m)}).$$

Note that $y_\lambda(F,v) = y_\lambda^{(1)}(F,v) + y_\lambda^{(2)}(F,v)$, where

$$y_\lambda^{(1)}(F,v) = (\rho^\theta U\Lambda\partial_t^m(\mathcal{B}_\nu - \lambda E)^{-1}U^{-1}F, \rho^\theta v^{(m)}),$$

$$y_\lambda^{(2)}(F,v) = \sum_{j=0}^{m-1} c_j(\rho^\theta a(t)(\partial_t^{m-j}U)\partial_t^j(\mathcal{B}_\nu - \lambda E)^{-1}U^{-1}F, \rho^\theta v^{(m)});$$

c_j are constant numbers depending only on m, j.

Then, taking into account that

$$y_\lambda^{(1)}(F,v) - \lambda(U^{-1}F, U^*(t)v(t)) =< U^{-1}F, U^*(t)v(t) >=< F, v >,$$

one obtains

$$\mathcal{A}[X_\nu(\lambda)F, v] - \lambda(F,v) =< F, v > +K_\lambda(F,v) + T_\lambda(F,v),$$

where

$$K_\lambda(F,v) = \sum_{j=0}^{m}\sum_{i=0}^{r_j}(p_i a_{ij}\partial_t^i X_\nu(\lambda)F, p_j v^{(j)}), \tag{4.4}$$

$$T_\lambda(F,v) = \sum_{j=0}^{m-1} c_j(\rho^\theta a(t)U^{(m-j)}\partial_t^j(\mathcal{B}_\nu - \lambda E)^{-1}F, \rho^\theta v^{(m)}). \tag{4.5}$$

Here $U^{(m-j)} = \partial_t^{m-j}U$. Using the formulae (3.7$'$), (3.12), (3.13), (4.3) and following the same reasoning as in §3, one establishes that when $\lambda \in S, |\lambda| \geq c', \nu \in [1, 2|\lambda|)$, where $c' > 0$ is a sufficiently large number, there is a continuous inverse operator

$$(\mathcal{A}_\nu - \lambda E)^{-1} = X_\nu(\lambda)(E + \Gamma_\nu(\lambda)), \tag{4.6}$$

$$\|\Gamma_\nu(\lambda)\|_{\mathcal{H}^l_{-\nu}\to\mathcal{H}^l_{-\nu}} \leqslant M|\lambda|^{-\varepsilon'}, \quad \varepsilon' > 0. \tag{4.7}$$

Here the operator $\mathcal{A}_\nu : \mathcal{H}^l_\nu \to \mathcal{H}^l_{-\nu}$ is determined by the formula

$$< \mathcal{A}_\nu u, v >= \mathcal{A}[u,v], \quad (\forall u, v \in \mathcal{H}^l_\nu).$$

2. Let $a_{mm}(t) \in C^m(\bar{J};\, End\,\mathbf{C}^l)$, and the inequality (1.3) hold. Let us assume that eigenvalues of the matrix $a(t) = a_{mm}(t)(t \in \bar{J})$ are situated outside a certain closed sector $S \subset \mathbf{C}$ with the vertex at zero. Let us assume that there is a number $\varepsilon \in (0, 1/2)$ such that the representation

$$a(t) = U_\pm(t)\Lambda_\pm(t)U_\pm^{-1}(t),\ det\,U_\pm(t) \neq 0\,(t \in \Delta_\pm), \tag{4.8}$$

where

$$U_\pm(t), U_\pm^{-1}(t) \in C^m(\Delta_\pm; End\,\mathbf{C}^l),\ \Delta_+ = [0,\varepsilon),\ \Delta_- = (1-\varepsilon, 1] \tag{4.8$'$}$$

exists and $\Lambda_\pm(t)$ is the Jordan matrix of the following structure for every fixed $t \in \Delta_\pm$. There are numbers $r_1^+, \ldots, r_p^+, r_1^-, \ldots, r_p^-$ such that $\sum\limits_{i=1}^{p} r_i^+ = \sum\limits_{j=1}^{p} r_j^- = l$, and if the complex l-dimensional space $\mathbf{C}^l$ is represented in the form $\mathbf{C}^{r_1^+} \times \cdots \times \mathbf{C}^{r_p^+}$ and in the form $\mathbf{C}^{r_1^-} \times \cdots \times \mathbf{C}^{r_p^-}$ then

$$\Lambda_+(t) = diag\,\{Q_1^+(t), \ldots, Q_p^+(t)\},$$

$$\Lambda_-(t) = diag\,\{Q_1^-(t), \ldots, Q_p^-(t)\},$$

where $Q_i^\pm(t)$ is an $r_i^\pm \times r_i^\pm$-matrix of the form

$$\begin{pmatrix} q_i^\pm(t) & 1 & 0 & \ldots & 0 \\ 0 & q_i^\pm(t) & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & q_i^\pm(t) \end{pmatrix},$$

$$q_i^\pm(t) \in C^m(\Delta_\pm; End\,\mathbf{C}^{r_i^\pm}),$$

or

$$Q_i^{\pm}(t) = diag\,\{q_i^{\pm}(t), \dots, q_i^{\pm}(t)\} \quad (r_i^{\pm} \quad \text{- times}). \tag{4.9}$$

Let us extend the functions $q_i^+(t)$ from the interval $(0, \varepsilon)$ to all the segment $[0, 1]$ so that $\tilde{q}_i^+(t) \in C^m(\bar{J};\, End\,\mathbf{C}^l), \tilde{q}_i^+(t) \notin S$. In accordance with this extension, extend the matrix $\Lambda_+(t)$ to the matrix $\tilde{\Lambda}_+(t)$ so that the structure of the Jordan cells remains unaltered.

Likewise, extend the matrix $U_+(t)$ so that $\tilde{U}_+(t) \in C^m(\bar{J};\, End\,\mathbf{C}^l)$ and

$$det\,\tilde{U}_+(t) \neq 0, \qquad t \in \bar{J}.$$

Accordingly, one obtains extension of the matrix $a(t)$ from the interval $(0, \varepsilon)$ to the whole $\bar{J}$:

$$\tilde{a}^+(t) = \tilde{U}_+(t)\tilde{\Lambda}_+(t)\tilde{U}_+^{-1}(t). \tag{$*$}$$

Similarly the matrix $\tilde{a}^-(t)$, to which the matrix functions $\tilde{q}_i^-(t), \tilde{U}_-(t)$ and $\tilde{\Lambda}_-(t)$ correspond, is constructed:

$$\tilde{a}^-(t) = \tilde{U}_-(t)\tilde{\Lambda}_-(t)\tilde{U}_-^{-1}(t), \quad t \in \bar{J}. \tag{$**$}$$

Let us introduce the operator $Q_{\nu,i}^{\pm} : \mathcal{H}_{\nu}^{r_i^{\pm}} \to \mathcal{H}_{-\nu}^{r_i^{\pm}}\,(\nu > 0, i = \overline{1,p})$ by the formula

$$< Q_{\nu,i}^{\pm}u, v >= Q_{\nu,i}^0[u, v] = (\rho^{\theta} Q_i^{\pm}(t)u^{(m)}, \rho^{\theta} v^{(m)})_{L_2(J)^l}\,(u, v \in \mathcal{H}_{\nu}^{r_i^{\pm}}).$$

In the direct sum $\mathcal{H}_{\nu}^l = \mathcal{H}_{\nu}^{r_1^{\pm}} \oplus \mathcal{H}_{\nu}^{r_2^{\pm}} \oplus \dots \oplus \mathcal{H}_{\nu}^{r_p^{\pm}}$ introduce the operator

$$\mathcal{B}_{\nu}^{\pm} = diag\,\{Q_{\nu,1}^{\pm}, \dots, Q_{\nu,p}^{\pm}\} \,:\, \mathcal{H}_{\nu}^l \to \mathcal{H}_{-\nu}^l,$$

where

$$\mathcal{H}_{-\nu}^l = \mathcal{H}_{-\nu}^{r_1^{\pm}} \oplus \mathcal{H}_{-\nu}^{r_2^{\pm}} \oplus \dots \oplus \mathcal{H}_{-\nu}^{r_p^{\pm}}.$$

According to Lemma 3.1, when $r \in \{1, \dots, p\}, \lambda \in S, |\lambda| \geq c', \nu \in [1, 2|\lambda|)$, where $c' > 0$ is a sufficiently large number, there are continuous inverse

$$(Q_{\nu,r}^{\pm} - \lambda E)^{-1} \,:\, \mathcal{H}_{-\nu}^{r_i^{\pm}} \to \mathcal{H}_{\nu}^{r_i^{\pm}} \quad r = \overline{1,p}.$$

Obviously,

$$(\mathcal{B}_{\nu}^{\pm} - \lambda E)^{-1} = diag\,\{(Q_{\nu,1}^{\pm} - \lambda E)^{-1}, \dots (Q_{\nu,p}^{\pm} - \lambda E)^{-1}\} \,:\, \mathcal{H}_{-\nu}^l \to \mathcal{H}_{\nu}^l$$

is a continuous operator.

Let us break the unit of the interval $[0, 1]$.

There are nonnegative functions $\psi_j(t) \in C_0^{\infty}(\bar{J}), i = 1, 2, \dots$ with the following properties:

1) $\sum\limits_{j=-\infty}^{\infty} \psi_j^2(t) \equiv 1\,(t \in R)$.

2) all functions $\psi_j(t)$ are derived by a "shift"from one function.

3) multiplicity of the covering $[0, 1] = \bigcup\limits_{j=1}^{\infty} supp\,\psi_j$ equals to 2.

The inequality

$$|t - \tau| \leqslant c|\lambda|^{-\varepsilon}, \quad c > 0 \tag{4.10}$$

holds for any $t, \tau \in supp\,\psi_j(\cdot, \delta)$, where $\delta > 0$ is a certain number. It follows from 2) that $\sum\limits_{j=-\infty}^{\infty} \psi_j^2(t|\lambda|^{-\varepsilon}) \equiv 1$.

Let us assume that

$$R_0(\lambda, t, \tau) = \sum_{j=-\infty}^{\infty} \psi_j(\cdot, \delta)R_j(\lambda)\psi_j(\cdot, \delta),$$

where $R_j(\lambda)$ is a pseudo-differential operator with the symbol

$$R_j(s,\lambda)=(\rho^{2\theta}(\tau_j)a(\tau_j)s^{2m}-\lambda I)^{-1}, \tau_j\in supp\,\psi_j(\cdot,\delta).$$

Since the norm of the pseudo-differential operator is estimated via the norms of its symbol, one can demonstrate that

$$|R_j(s,\lambda)|\leqslant M(\rho^{2\theta}(t_i)s^{2m}+|\lambda|)^{-1},$$

where $M>0, t_i\in supp\,\psi_j(\cdot,\delta), \lambda\in S$.

Let us construct an operator function $X_\nu(\lambda)$, satisfying relations of the form (4.6), (4.7). Let us fix the nonnegative functions $\psi_+(t),\psi_-(t),\psi(t)\in C^\infty[0,1]$ with the following property:

$$\psi_+^2(t)+\psi_-^2(t)+\psi^2(t)\equiv 1,\quad \psi_-(t)=\psi_+(1-t)\quad (t\in\bar{J}),$$

$$\psi_+(\tau)=0\quad (\frac{3}{4}\varepsilon<\tau<1),\quad \psi_+(\tau)=1\quad (0\leqslant\tau<\varepsilon/2).$$

Let us introduce the operator $X_\nu(\lambda)$ by the formula

$$X_\nu(\lambda)=\psi_+\tilde{U}_+(\mathcal{B}_\nu^+-\lambda E))^{-1}\tilde{U}_+^{-1}\psi_++R_0(\lambda,t,\tau)+\psi_-\tilde{U}_-(\mathcal{B}_\nu^--\lambda E))^{-1}\tilde{U}_-^{-1}\psi_-,$$

where $\psi_\pm(t)$ denotes the operator of multiplication by the function $\psi_\pm(t)$, and $U_\pm:\mathcal{H}_-^l\to\mathcal{H}_-^l$ is a continuous operator such that $(U_\pm u)(t)=\tilde{U}_\pm(t)u(t),\forall u\in\mathcal{H}^l,(\lambda\in S,|\lambda|>c',c'>0$ is a sufficiently large number).

Let us represent $X_\nu(\lambda)$ in the form

$$X_\nu(\lambda)=\sum_{k=1}^{3}X_{\nu,k}(\lambda), \tag{4.11}$$

where

$$X_{\nu,1}(\lambda)=\psi_+\tilde{U}_+(\mathcal{B}_\nu^+-\lambda E)^{-1}\tilde{U}_+^{-1}\psi_+,$$
$$X_{\nu,2}(\lambda)=\psi R_0\psi,$$
$$X_{\nu,3}(\lambda)=\psi_-\tilde{U}_-(\mathcal{B}_\nu^--\lambda E)^{-1}\tilde{U}_-^{-1}\psi_-.$$

In the following items a), b), c) representations for $\mathcal{A}_\nu[X_{\nu,i}F,v]$ are obtained when $i=2,1,3$ respectively.

a) For $F\in\mathcal{H}_{-\nu}^l, v\in\mathcal{H}_\nu^l$ one has

$$\mathcal{A}_\nu[X_{\nu,2}F,v]=\sum_{i,j=0}^{m}(p_ia_{ij}(t)\partial_t^i(X_{\nu,2}F),p_jv^{(j)}(t))_{L_2}=$$
$$=x_\lambda(F,v)+y_\lambda(F,v), \tag{4.12}$$

where

$$x_\lambda(F,v)=\sum_{j=0}^{m}\sum_{i=0}^{r_j}(p_ia_{ij}\partial_t^i(X_{\nu,2}(\lambda)F),p_jv^{(j)}),\ r_j=min\{m,2m-j-1\},$$
$$y_\lambda(F,v)=(\rho^{2\theta}a(t)\partial_t^m(X_{\nu,2}(\lambda)F),v^{(m)}).$$

Here and in what follows the index $L_2(\varepsilon,1-\varepsilon)$ is omitted.

Several auxiliary lemmas are necessary for further investigation.

Let $\hat{\mathcal{H}}_\nu^l$ be a closure of $C_0^\infty(R^n)^l$ by the norm

$$|u|_\nu=(|u^{(m)}(t)|_{L_2}^2+\nu|u|_{L_2}^2)^{1/2}.$$

Let us denote by $\hat{\mathcal{H}}_{-\nu}$ the completion of the space $L_2(J)$ by the norm

$$|F|_{-\nu}=\sup_{v\neq 0}\frac{|(F,v)|}{|v|_{\hat{\mathcal{H}}_\nu^l}}.$$

Elements from $\hat{\mathcal{H}}^l_{-\nu}$ are identified with the corresponding anti-linear continuous functionals over $\hat{\mathcal{H}}^l_{\nu}$.

This yields a triple of densely embedded spaces

$$\hat{\mathcal{H}}^l_{\nu} \subset L^l_2 \subset \hat{\mathcal{H}}^l_{-\nu}.$$

In this embedding $\hat{\mathcal{H}}^l_{\nu}$ is a positive space, and $\hat{\mathcal{H}}^l_{-\nu}$ is a negative space (see [23, §2.0]).

Obviously, there are embeddings

$$\hat{\mathcal{H}}^l_{\nu} \subset \mathcal{H}^l_{\nu} \subset L^l_2 \subset \hat{\mathcal{H}}^l_{-\nu} \subset \mathcal{H}^l_{-\nu}.$$

Lemma 4.1. *The inequality*

$$|F|_{\hat{\mathcal{H}}^l_{-\nu}} \leqslant |F|_{\mathcal{H}^l_{-\nu}}$$

holds.

Proof. The following inequality is obvious

$$|v|_{\hat{\mathcal{H}}^l_{\nu}} \geq |v|_{\mathcal{H}^l_{\nu}}.$$

Using this inequality, one has

$$|F|_{\hat{\mathcal{H}}^l_{-\nu}} = \sup_{v\neq 0} \frac{|(F,v)|}{|v|_{\hat{\mathcal{H}}^l_{\nu}}} \leqslant \sup_{v\neq 0} \frac{|(F,v)|}{|v|_{\mathcal{H}^l_{\nu}}} = |F|_{\mathcal{H}^l_{-\nu}}.$$

The lemma is proved.

Let us introduce the operator $\overset{\circ}{P}{}'_{\nu}$, acting as follows:

$$(\overset{\circ}{P}{}'_{\nu}\, u, v) = (u^{(m)}, v^{(m)}) + \nu(u,v), \quad (u, v \in \hat{\mathcal{H}}^l_{\nu}),$$

with the domain of definition

$$D(\overset{\circ}{P}{}'_{\nu}) = \hat{\mathcal{H}}^l_{\nu}.$$

Let us determine the operator $\overset{\circ}{P}_{\nu} : \hat{\mathcal{H}}_{\nu} \to L_2$ by the formula

$$\overset{\circ}{P}_{\nu} = (\overset{\circ}{P}{}'_{\nu})^{1/2}, \quad D(\overset{\circ}{P}_{\nu}) = D((\overset{\circ}{P}{}'_{\nu})^{1/2}) = \hat{\mathcal{H}}^l_{\nu}.$$

Lemma 4.2. *Let $\alpha(t) \in C^m(J; End\,\mathbf{C}^l)$, $\quad T : L_2(J) \to L_2(J)$ be a bounded operator. Then for any $F \in \mathcal{H}^l_{-\nu}, v \in \mathcal{H}_{\nu}$ the inequality*

$$| < \alpha(t)TF, v^{(j)} > | \leqslant \sup_{t\in J} |\alpha(t)| \| \overset{\circ}{P}_{\nu}\, T^* \|_{L_2\to L_2} |F|_{-\nu} |v|_{\nu}$$

holds

Proof. Since $L_2(J)^l$ is dense in $\mathcal{H}^l_{-\nu}$ then, without loss of generality one can assume that $F \in L_2(J)^l$. Then,

$$\begin{gathered} | < \alpha(t)TF, v^{(j)} > | = |(\alpha(t)TF, v^{(j)})| = \\ = |(\alpha(t)T\, \overset{\circ}{P}_{\nu}\, (\overset{\circ}{P}_{\nu})^{-1}F, v^{(j)})| \leqslant \\ \leqslant (\sup_{t\in J} |\alpha(t)|) \| \overset{\circ}{P}_{\nu}\, T^* \|_{L_2} \| (\overset{\circ}{P}_{\nu})^{-1} F \|_{L_2} |v|_{\nu}. \end{gathered}$$

Whence the statement of the lemma follows provided that it is taken into account that

$$\|(\overset{\circ}{P}_{\nu})^{-1}F\|_{L_2} = |F|_{\hat{\mathcal{H}}^l_{-\nu}} \leqslant |F|_{\mathcal{H}^l_{-\nu}}.$$

Now let us prove the following inequality (see (4.12))

$$|x_\lambda(F,v)| \leqslant M(|\lambda|^{-\varepsilon'} + \nu^{-\varepsilon''})|F|_{-\nu}|v|_{\nu}, \tag{4.13}$$

where $M > 0, \varepsilon' > 0, \varepsilon'' > 0, \nu \in [1, 2|\lambda|], \lambda \in S$.

Applying the Leibnitz formula, one has

$$x_\lambda(F,v)=\sum_{j=0}^{m}\sum_{i=0}^{r_j}(p_i a_{ij}\partial_t^i(X_{\nu,2}(\lambda)F),p_j v^{(j)})=$$

$$=\sum_{i,j=0}^{m-1}(p_ip_ja_{ij}\partial_t^i(X_{\nu,2}(\lambda)F),v^{(j)})+\sum_{i=0}^{m-1}(p_ip_ma_{im}\partial_t^i(X_{\nu,2}(\lambda)F),v^{(m)})+$$

$$+\sum_{j=0}^{m-1}(p_jp_ma_{mj}\partial_t^m(X_{\nu,2}(\lambda)F),v^{(j)})=x_1(F,v)+x_2(F,v)+x_3(F,v).$$

Further

$$x_1(F,v)=\sum_{i,j=0}^{m-1}(p_ip_ja_{ij}\partial_t^i(\sum_{k=1}^{+\infty}\psi\psi_k(\cdot,\delta)R_k(\lambda)\psi\psi_k(\cdot,\delta)F),v^{(j)})=$$

$$=\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}(p_ip_ja_{ij}\partial_t^i(\psi\psi_k(\cdot,\delta)R_k(\lambda)\psi\psi_k(\cdot,\delta)F),v^{(j)})=$$

$$=\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}(p_ip_ja_{ij}\psi\psi_k(\cdot,\delta)\partial_t^i(R_k(\lambda)\psi\psi_k(\cdot,\delta)F),v^{(j)})+$$

$$+\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}(p_ip_ja_{ij}(\sum_{l=0}^{i-1}C_i^l\partial^{i-l}(\psi\psi_k(\cdot,\delta))\partial_t^l(R_k(\lambda)\psi\psi_k(\cdot,\delta)F),v^{(j)})=$$

$$=\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}(p_ip_ja_{ij}\psi\psi_k(\cdot,\delta)\partial_t^i(R_k(\lambda)\psi\psi_k(\cdot,\delta)F),v^{(j)})+$$

$$+\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}\sum_{l=0}^{i-1}(p_ip_ja_{ij}C_i^l\partial^{i-l}(\psi\psi_k(\cdot,\delta))\partial_t^l(R_k(\lambda)\psi\psi_k(\cdot,\delta)F),v^{(j)})=$$

$$=\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}(p_ip_ja_{ij}\psi\psi_k(\cdot,\delta)\partial_t^i(R_k(\lambda)F)\psi\psi_k(\cdot,\delta),v^{(j)})+$$

$$+\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}\sum_{n=0}^{i-1}(p_ip_ja_{ij}\psi\psi_k(\cdot,\delta)C_i^n\partial^{i-n}(\psi\psi_k(\cdot,\delta))\partial_t^n(R_k(\lambda)F),v^{(j)})+$$

$$+\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}\sum_{l=0}^{i-1}(p_ip_ja_{ij}C_i^l\partial^{i-l}(\psi\psi_k(\cdot,\delta))\partial_t^l(R_k(\lambda)\psi\psi_k(\cdot,\delta)F),v^{(j)})=$$

$$=\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}(p_ip_ja_{ij}\psi\psi_k(\cdot,\delta)R_k^i(\lambda)F\psi\psi_k(\cdot,\delta),v^{(j)})+$$

$$+\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}\sum_{n=0}^{i-1}(p_ip_ja_{ij}\psi\psi_k(\cdot,\delta)C_i^n\partial^{i-n}(\psi\psi_k(\cdot,\delta))(R_k^n(\lambda)F),v^{(j)})+$$

$$+\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}\sum_{l=0}^{i-1}(p_ip_ja_{ij}C_i^l\partial^{i-l}(\psi\psi_k(\cdot,\delta))R_k^l(\lambda)F\psi\psi_k(\cdot,\delta),v^{(j)})+$$

$$+\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}\sum_{l=0}^{i-1}\sum_{\eta=0}^{l-1}(p_ip_ja_{ij}C_i^l\partial^{i-l}(\psi\psi_k(\cdot,\delta))C_l^\eta(\psi\psi_k(\cdot,\delta))\partial^{l-\eta}(\psi\psi_k(\cdot,\delta))R_k^\eta(\lambda)F,v^{(j)})=$$

$$= \sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}(p_ip_ja_{ij}\psi\psi_k(\cdot,\delta)R_k^i(\lambda)\overset{\circ}{P}_\nu(\overset{\circ}{P}_\nu^{-1}F)\psi\psi_k(\cdot,\delta),v^{(j)})+$$

$$+\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}\sum_{n=0}^{i-1}(p_ip_ja_{ij}\psi\psi_k(\cdot,\delta)C_i^n\partial^{i-n}(\psi\psi_k(\cdot,\delta))(R_k^n(\lambda)\overset{\circ}{P}_\nu(\overset{\circ}{P}_\nu^{-1}F),v^{(j)})+$$

$$+\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}\sum_{l=0}^{i-1}(p_ip_ja_{ij}C_i^l\partial^{i-l}(\psi\psi_k(\cdot,\delta))R_k^l(\lambda)\overset{\circ}{P}_\nu(\overset{\circ}{P}_\nu^{-1}F)\psi\psi_k(\cdot,\delta),v^{(j)})+$$

$$+\sum_{k=1}^{+\infty}\sum_{i,j=0}^{m-1}\sum_{l=0}^{i-1}\sum_{\eta=0}^{l-1}(p_ip_ja_{ij}C_i^l\partial^{i-l}(\psi\psi_k(\cdot,\delta))\cdot$$

$$\cdot C_l^\eta(\psi\psi_k(\cdot,\delta))\partial^{l-\eta}(\psi\psi_k(\cdot,\delta))R_k^\eta(\lambda)\overset{\circ}{P}_\nu(\overset{\circ}{P}_\nu^{-1}F),v^{(j)})\equiv$$

$$\equiv I_1+I_2+I_3+I_4,$$

where R_k^i indicates $\partial_t^i(R_k(\lambda))$.

Then, assuming in I_j $(j=\overline{1,4})$ that

$$T_1^+=\psi\psi_kR_k^i(\lambda),\quad T_2^+=\psi\psi_kR_k^n(\lambda),$$

$$T_3^+=\psi\psi_k(\cdot,\delta)R_k^l(\lambda),\quad T_4^+=\psi\psi_kR_k^\eta(\lambda)$$

respectively, and then using Lemma 4.2, and taking into account that

$$|v^{(j)}|_{L_2}\leqslant M\nu^{\frac{j-m}{2m}}|v|_\nu,\quad \delta=|\lambda|^{-\varepsilon'},$$

one arrives to the inequality

$$|x_1(F,v)|\leqslant M\sup_{t\in J}|p_ip_ja_{ij}||F|_{\mathcal{H}^l_{-\nu}}\nu^{\frac{j-m}{2m}}|v|_\nu(\sum_{\mu=1}^4\|\overset{\circ}{P}_\nu T_\mu^*\|).$$

Let us turn to the estimate $\|\overset{\circ}{P}_\nu T_\mu^*\|,\mu=\overline{1,4}$. One has

$$\|\overset{\circ}{P}_\nu T_\mu^*\|_{L_2}\leqslant\|\partial^mT_\mu^*\|_{L_2}+\nu^{1/2}\|T_\mu^*\|_{L_2},\ \mu=\overline{1,4}.$$

Using the inequality

$$\frac{s^{i+m}}{s^{2m}+|\lambda|}\leqslant M|\lambda|^{-\varepsilon},\quad \text{where}\quad \varepsilon=\frac{m-i}{2m}>0,$$

which can be easily verified, one obtains (when $\mu=1$)

$$\|\overset{\circ}{P}_\nu T_1^*\|_{L_2}\leqslant\|\partial^mT_1^*\|_{L_2}+\nu^{1/2}\|T_1^*\|_{L_2}\leqslant$$

$$\leqslant M\sup_s(\frac{s^{i+m}}{s^{2m}+|\lambda|}+\nu^{1/2}\frac{s^i}{s^{2m}+|\lambda|})\leqslant M|\lambda|^{-\varepsilon'},\quad \text{where}\quad \varepsilon'=\frac{m-i}{2m}>0.$$

Similar estimates exist for $\|\overset{\circ}{P}_\nu T_\mu^*\|,\mu=\overline{2,4}$:

$$\|\overset{\circ}{P}_\nu T_2^*\|\leqslant M|\lambda|^{-\varepsilon''},\quad \text{where}\quad \varepsilon''=\frac{n-m}{2m}>0,$$

$$\|\overset{\circ}{P}_\nu T_3^*\|\leqslant M|\lambda|^{-\varepsilon'''},\quad \text{where}\quad \varepsilon'''=\frac{l-m}{2m}>0,$$

$$\|\overset{\circ}{P}_\nu T_4^*\|\leqslant M|\lambda|^{-\varepsilon^{IV}},\quad \text{where}\quad \varepsilon^{IV}=\frac{\eta-m}{2m}>0.$$

Thus, the estimate

$$\sum_{\mu=1}^{4} \| \overset{\circ}{P}_{\nu}\, T^{*}_{\mu} \| \leqslant M|\lambda|^{-\varepsilon_1}, \quad \text{where} \quad \varepsilon_1 = min(\varepsilon', \varepsilon'', \varepsilon''', \varepsilon^{IV})$$

holds. Hence,

$$A) \quad |x_1(F,v)| \leqslant M|\lambda|^{-\varepsilon_2}|F|_{-\nu}|v|_{\nu}, \quad \text{where} \quad \varepsilon_2 > 0.$$

Acting similarly, one obtains estimates for $x_2(F,v), x_3(F,v)$:

$$B) \quad |x_2(F,v)| \leqslant M|\lambda|^{-\varepsilon_3}|F|_{-\nu}|v|_{\nu}, \quad \varepsilon_3 > 0,$$
$$C) \quad |x_3(F,v)| \leqslant M\nu^{\frac{j-m}{2m}}|F|_{-\nu}|v|_{\nu}.$$

Now $A), B), C)$ readily provide (4.13).

Let us represent the matrix $\rho^{2\theta}(t)a(t)$ in the form

$$\rho^{2\theta}(t)a(t) = \rho^{2\theta}(t_j)a(t_j) + \rho^{2\theta}(t)a(t) - \rho^{2\theta}(t_j)a(t_j),$$

where $t_j \in supp\, \psi_j(\cdot,\delta)$.

Then (see (4.12))

$$\begin{aligned} y_\lambda(F,v) &= (\rho^{2\theta}(t)a(t)\partial_t^m(\psi R_0(\lambda)\psi F), v^{(m)}) = \\ &= (\rho^{2\theta}(t_j)a(t_j)\partial_t^m(\psi R_0(\lambda)\psi F), v^{(m)}) + \\ &+([\rho^{2\theta}(t)a(t) - \rho^{2\theta}(t_j)a(t_j)]\partial_t^m(\psi R_0(\lambda)\psi F), v^{(m)}) = \\ &= y_\lambda^{(1)}(F,v) + y_\lambda^{(2)}(F,v), \end{aligned}$$

where

$$y_\lambda^{(1)}(F,v) = (\rho^{2\theta}(t_j)a(t_j)\partial_t^m(\psi R_0(\lambda)\psi F), v^{(m)}), \tag{4.14}$$

$$y_\lambda^{(2)}(F,v) = ([\rho^{2\theta}(t)a(t) - \rho^{2\theta}(t_j)a(t_j)]\partial_t^m(\psi R_0(\lambda)\psi F), v^{(m)}). \tag{4.15}$$

Applying the Leibnitz formula, one obtains

$$\begin{aligned} y_\lambda^{(1)}(F,v) &= (-1)^m \sum_{j=1}^{+\infty} (\rho^{2\theta}(t_j)a(t_j)\partial_t^{2m}[\psi\psi_j(\cdot,\delta)R_j(\lambda)\psi_j(\cdot,\delta)\psi F], v) = \\ &= (-1)^m \sum_{j=1}^{+\infty} (\rho^{2\theta}(t_j)a(t_j)\psi\psi_j(\cdot,\delta)(\partial_t^{2m}R_j(\lambda))\psi\psi_j(\cdot,\delta)F, v) + \\ &+(-1)^m \sum_{j=1}^{+\infty} (\rho^{2\theta}(t_j)a(t_j) \sum_{k=1}^{2m-1} C_{2m}^k(\partial^{2m-k}(\psi\psi_k(\cdot,\delta))(\partial^k R_j(\lambda))\psi\psi_j(\cdot,\delta)F, v) = \\ &= T_1(F,v) + T_2(F,v), \end{aligned}$$

where

$$T_1(F,v) = (-1)^m \sum_{j=1}^{+\infty} (\rho^{2\theta}(t_j)a(t_j)\psi\psi_j(\cdot,\delta)(\partial_t^{2m}R_j(\lambda))\psi\psi_j(\cdot,\delta)F, v),$$

$$T_2(F,v) = (-1)^m \sum_{j=1}^{+\infty} (\rho^{2\theta}(t_j)a(t_j) \sum_{k=1}^{2m-1} C_{2m}^k(\partial^{2m-k}(\psi\psi_k(\cdot,\delta))\cdot$$
$$\cdot(\partial^k(R_j(\lambda))\psi\psi_j(\cdot,\delta)F, v).$$

Obviously,

$$T_1(F,v) - \lambda(\psi\psi_j(\cdot,\delta)R_j(\lambda)\psi\psi_j(\cdot,\delta)F, v) =< \psi^2 F, v > . \tag{4.16}$$

Let us make the estimate $|T_2(F,v)|$. Acting as in proving the inequality (4.13), using the obvious inequalities

$$\|\partial^{2m-n}(\psi\psi_j(\cdot,\delta)\|_{L_2} \leqslant M|\lambda|^{\varepsilon'(n-2m)},$$

$$\|\partial^n R_j(\lambda)F\|_{L_2} \leqslant \sup_s \frac{|s^n||F|_{-\nu}}{s^{2m}+|\lambda|} \leqslant M|\lambda|^{-\frac{1}{2m}}|F|_{-\nu},$$

$$|v|_{L_2} \leqslant \nu^{-1/2}(|\partial^m v|_{L_2} + \nu^{1/2}|v|_{L_2}) \leqslant M\nu^{-1/2}|v|_\nu,$$

one obtains

$$\|T_2(F,v)\|_{L_2} \leqslant M_1\nu^{-1/2}|\lambda|^{\varepsilon'(n-2m)-\frac{1}{2m}}|F|_{-\nu}|v|_\nu,$$

or

$$\|T_2(F,v)\|_{L_2} \leqslant M|\lambda|^{-\varepsilon''}|F|_{-\nu}|v|_\nu, \tag{4.17}$$

where

$$\varepsilon'' = \frac{m-2m\varepsilon(n-2m)+1}{2m} > 0.$$

Let us prove the estimate (see (4.15))

$$|y_\lambda^{(2)}(F,v)| \leqslant M|\lambda|^{-\varepsilon'}|F|_{-\nu}|v|_\nu,\ M>0,\ \nu\in[1:2|\lambda|],\ \lambda\in S. \tag{4.18}$$

According to the Lagrange theorem

$$|\rho^{2\theta}(t)a(t)-\rho^{2\theta}(t_j)a(t_j)| \leqslant M_1|t-t_j|.$$

If $t\in supp\,\psi_j$, oi $|t-t_j|\leqslant c|\lambda|^{-\varepsilon'}$ then,

$$|\rho^{2\theta}(t)a(t)-\rho^{2\theta}(t_j)a(t_j)| \leqslant M_2|\lambda|^{-\varepsilon'}. \tag{4.19}$$

Using (4.19) and repeating the above reasoning one establishes the inequality (4.18).

Now (4.13), (4.16), (4.17), and (4.8) provide

$$\mathcal{A}_\nu[X_{\nu,2}F,v]-\lambda(\psi\psi_jR_j\psi\psi_jF,v) = <\psi^2F,v> + T(F,v), \tag{4.20}$$

where the operator function $T(F,v)$ satisfies the estimate

$$\|T(F,v)\|_{L_2} \leqslant M|\lambda|^{-\varepsilon'}|F|_{-\nu}|v|_\nu,\quad M>0,\ \nu\in[1,2|\lambda|],\ \lambda\in S.$$

b) For $F\in\mathcal{H}^l_{-\nu}, v\in\mathcal{H}^l_\nu$, one has

$$\mathcal{A}_\nu[X_{\nu,1}(\lambda)F,v] = \sum_{i,j=0}^m (p_ia_{ij}(t)\partial_t^i(X_{\nu,1}F),p_jv^{(j)})_{L_2} =$$

$$= x_\lambda(F,v)+y_\lambda(F,v),$$

where

$$x_\lambda(F,v) = \sum_{j=0}^m\sum_{i=0}^{r_j}(p_ia_{ij}(t)\partial_t^i(X_{\nu,1}(\lambda)F),p_jv^{(j)}),$$

$r_j = min(m,2m-j-1)$,

$$y_\lambda(F,v) = (\rho^{2\theta}(t)a(t)\partial_t^m(X_{\nu,1}(\lambda)F),v^{(m)}) =$$

$$= (\rho^{2\theta}(t)a(t)\partial_t^m(\psi_+\tilde U_+(\mathcal{B}_\nu^+-\lambda E)^{-1}\tilde U_+^{-1}\psi_+F),v^{(m)}).$$

Extending the matrix $a(t)$ with respect to continuity up to $\tilde a^+(t)$ (see the formula $(*)$), one obtains

$$\tilde y_\lambda(F,v) \equiv (\rho^{2\theta}(t)\tilde a(t)\partial_t^m(\psi_+\tilde U_+(\mathcal{B}_\nu^+-\lambda E)^{-1}\tilde U_+^{-1}\psi_+F),v^{(m)}).$$

Obviously,

$$\tilde y_\lambda(F,v)-\lambda(\psi_+\tilde U_+(\mathcal{B}_\nu^+-\lambda E)^{-1}\tilde U_+^{-1}\psi_+F),v) = <\psi_+^2F,v>. \tag{4.21}$$

Considerations given in §3, entail the following estimate for $x_\lambda(F,v)$:

$$|x_\lambda(F,v)| \leqslant M|\lambda|^{-\varepsilon}|F|_{-\nu}|v|_\nu. \tag{4.22}$$

Thus,

$$\mathcal{A}_\nu[X_{\nu,1}F,v]-\lambda(\psi_+\tilde U_+(\mathcal{B}_\nu^+-\lambda E)^{-1}\tilde U_+^{-1}\psi_+F),v) =$$

$$=< \psi_+^2 F, v > + x_\lambda(F,v). \tag{4.23}$$

c) Similarly to the previous item, for $F \in \mathcal{H}^l_{-\nu}, v \in \mathcal{H}^l_\nu$ one has

$$\mathcal{A}_\nu[X_{\nu,3}(\lambda)F, v] = \sum_{i,j=0}^{m} (p_i a_{ij}(t)\partial_t^i (X_{\nu,3}(\lambda)F), v^{(j)})_{L_2(1-\varepsilon,1)} =$$

$$= x_\lambda(F,v) + y_\lambda(F,v),$$

where

$$x_\lambda(F,v) = \sum_{j=0}^{m}\sum_{i=0}^{r_j} (p_i p_j a_{ij}(t)\partial_t^i (X_{\nu,3}(\lambda)F), v^{(j)}),$$

$r_j = min(m, 2m - j - 1)$,

$$y_\lambda(F,v) = (\rho^{2\theta}(t)a(t)\partial_t^m (X_{\nu,3}(\lambda)F), v^{(m)}) =$$

$$= (\rho^{2\theta}(t)a(t)\partial_t^m (\psi_- \tilde{U}_- (\mathcal{B}_\nu^- - \lambda E)^{-1}\tilde{U}_-^{-1}\psi_- F), v^{(m)}).$$

Substituting the matrix $a(t)$ by $\tilde{a}^-(t)$ (see the formula $(**)$), one obtains

$$\tilde{y}_\lambda(F,v) \equiv (\rho^{2\theta}(t)\tilde{a}(t)\partial_t^m (\psi_- \tilde{U}_- (\mathcal{B}_\nu^- - \lambda E)^{-1}\tilde{U}_-^{-1}\psi_- F), v^{(m)}).$$

Clearly,

$$\tilde{y}_\lambda(F,v) - \lambda(\psi_- \tilde{U}_- (\mathcal{B}_\nu^- - \lambda E)^{-1}\tilde{U}_-^{-1}\psi_- F), v) =< \psi_-^2 F, v >,$$

and $x_\lambda(F,v)$ satisfies the inequality of the type (4.22). Thus,

$$\mathcal{A}_\nu[X_{\nu,3}F, v] - \lambda(\psi_- \tilde{U}_- (\mathcal{B}_\nu^- - \lambda E)^{-1}\tilde{U}_-^{-1}\psi_- F), v) =$$

$$=< \psi_-^2 F, v > + x_\lambda(F,v). \tag{4.24}$$

d) representations (4.11), (4.20), (4.23), and (4.24) provide

$$\mathcal{A}_\nu[X_\nu(\lambda)F, v] - \lambda(X_\nu(\lambda)F, v) =< (\psi_+^2 + \psi^2 + \psi_-^2)F, v > + \tilde{T}(F,v) =$$

$$=< F, v > + \tilde{T}(F,v), \tag{4.25}$$

where the operator function $\tilde{T}(F,v)$ satisfies the estimate

$$\|\tilde{T}(F,v)\| \leqslant M|\lambda|^{-\varepsilon'}|F|_{-\nu}|v|_\nu. \tag{4.26}$$

3. According to (4.25), the inequality (4.26) entails that

$$(\mathcal{A}_\nu - \lambda E)X_\nu(\lambda) = E + \Gamma_\nu(\lambda), \quad (\lambda \in S, |\lambda| \geq 1, \nu \in [1, 2|\lambda|),$$

where $\Gamma_\nu(\lambda) : \mathcal{H}_{-\nu} \to \mathcal{H}_{-\nu}$ is a continuous operator,

$$\|\Gamma_\nu(\lambda)\|_{\mathcal{H}_{-\nu}\to\mathcal{H}_{-\nu}} \leqslant M(|\lambda|^{-\varepsilon'} + \nu^{-\varepsilon''}), \quad \varepsilon' > 0,\ \varepsilon'' > 0.$$

Let us select a number $\sigma_0 > 0$ such that $M(|\lambda|^{-\varepsilon'} + \nu^{-\varepsilon''}) \leqslant \frac{1}{2}$ for all $|\lambda| \geq \sigma_0$. Then,

$$(\mathcal{A}_\nu - \lambda E)X_\nu(\lambda)\Gamma'_\nu(\lambda) = E, \quad \Gamma'_\nu(\lambda) = (E + \Gamma_\nu(\lambda))^{-1},$$

$$\|E - \Gamma'_\nu(\lambda)\|_{\mathcal{H}_{-\nu}\to\mathcal{H}_{-\nu}} < 1.$$

Let us demonstrate that $ker(\mathcal{A}_\nu - \lambda E) = 0, \forall \lambda \in S, |\lambda| \geq \sigma_1$, where σ_1 is a sufficiently large number $\nu \in [1, 2|\lambda|)$. Then, when $\lambda \in S, |\lambda| \geq \sigma_1 = max\{\sigma_0, \sigma_1\}, \nu \in [1, 2|\lambda|)$ one has the equality

$$(\mathcal{A}_\nu - \lambda E)^{-1} = X_\nu(\lambda)\Gamma'_\nu(\lambda). \tag{4.27}$$

Consider the operator $\mathcal{A}'_\nu : \mathcal{H}_\nu \to \mathcal{H}_{-\nu}, \nu > 0$, acting by the formula

$$< \mathcal{A}'_\nu u, v >= \sum_{i,j=0}^{m} (a^*_{ij} p_i u^{(i)}, p_j v^{(j)}), \quad \forall u, v \in \mathcal{H}_\nu.$$

Likewise, one constructs the operators

$$X'_\nu(\bar\lambda) : \mathcal{H}_{-\nu} \to \mathcal{H}_\nu, \quad G^*_\nu(\bar\lambda) : \mathcal{H}_{-\nu} \to \mathcal{H}_{-\nu},$$

such that

$$(\mathcal{A}'_\nu - \bar\lambda E)X'_\nu(\bar\lambda) = E + G^*_\nu(\bar\lambda), \tag{4.28}$$

$$\|G^*_\nu(\bar\lambda)\|_{\mathcal{H}_{-\nu}\to\mathcal{H}_{-\nu}} \leqslant \frac{1}{2}, \quad (\lambda \in S, |\lambda| \geq \sigma_1, \nu \in [1, 2|\lambda|). \tag{4.29}$$

Let $u \in \mathcal{H}_\nu$ be an element such that $(\mathcal{A}_\nu - \lambda E)u = 0$. Moreover, let $|\lambda| \geq \sigma_1 = max\{\sigma_0, \sigma_1\}$. Then,

$$< (\mathcal{A}_\nu - \lambda E)u, v >= 0, \quad \forall v \in \mathcal{H}_\nu,$$

i.e.

$$\sum_{i,j=1}^m (a_{ij} p_i p_j u^{(i)}, v^{(j)}) - \lambda(u, v) = 0, \quad \forall v \in \mathcal{H}_\nu.$$

Hence,

$$< (\mathcal{A}'_\nu - \bar\lambda E)v, u >= (\sum_{i,j=0}^m (a^*_{ji} p_i p_j v^{(j)}, u^{(i)}) - \bar\lambda(v, u) = 0.$$

Let us assume that $v = X'_\nu(\bar\lambda)F, f \in \mathcal{H}_{-\nu}$. Then,

$$< (E + G^*_\nu(\bar\lambda))F, u >= 0, \quad \forall f \in \mathcal{H}_{-\nu}.$$

Since the operator

$$(E + G^*_\nu(\bar\lambda)) : \mathcal{H}_{-\nu} \to \mathcal{H}_{-\nu}$$

has a continuous inverse one when $\lambda \in S, |\lambda| \geq \sigma', \nu \in [1, 2|\lambda|)$, then $< F, u >= 0, \forall f \in \mathcal{H}_{-\nu}$. Assuming that $F = 0$, one obtains $u = 0$. Whence,

$$ker(\mathcal{A}_\nu - \lambda E) = 0.$$

The estimate

$$\|(\mathcal{A}_\nu - \lambda E)^{-1}\|_{\mathcal{H}_{-\nu}\to\mathcal{H}_\nu} \leqslant 2\|X_\nu(\lambda)\|_{\mathcal{H}_{-\nu}\to\mathcal{H}_\nu}$$

follows from (4.27).

4. To prove Theorem 1.1 assume that $\nu = 1, \mathcal{A} = \mathcal{A}_1$. It is clear that the operator

$$A = \mathcal{A}u, \quad D(A) = \{u \in \mathcal{H}^l_+; \mathcal{A}u \in \mathcal{H}^l\},$$

satisfies the condition (i) of Theorem 1.1. Since the operator $(\mathcal{A} - \lambda E)^{-1}$ performs a one-to-one mapping of $\mathcal{H}^l$ to $D(A)$, there is an inverse

$$(A - \lambda E)^{-1}u = (\mathcal{A} - \lambda E)^{-1}u \quad (\forall u \in \mathcal{H}^l, \lambda \in S, |\lambda| \geq c).$$

The formulae (4.3′), (4.6), (4.7) provide

$$|(A - \lambda E)^{-1}u| \leqslant |(A - \lambda E)^{-1}u|_+ = |(\mathcal{A} - \lambda E)^{-1}u|_+ \leqslant M|u|_- \leqslant M|u|$$

when $\nu = 1$. Whence, it follows that $(A - \lambda E)^{-1} : \mathcal{H}^l \to \mathcal{H}^l(\lambda \in S, |\lambda| \geq c)$ is a continuous operator.

Let us prove the uniqueness of the operator A, possessing the properties $(i), (ii)$ of Theorem 1.1. Let $A^{(1)}, A^{(2)}$ be two operators possessing the mentioned properties. Since $\theta < m$, the injection $\mathcal{H}^l_+ \subset \mathcal{H}^l$ is compact. Hence, the operators $A^{(1)}, A^{(2)}$ have discrete spectrums. Therefore, there is a sufficiently large in module $\lambda \in S$ such that there are continuous inverse $(A^{(1)} - \lambda E)^{-1}, (A^{(2)} - \lambda E)^{-1}$. For

$$u \in \mathcal{H}^l, F = (A^{(1)} - \lambda E)^{-1}u - (A^{(2)} - \lambda E)^{-1}u,$$

one has $(\mathcal{A} - \lambda E)F = 0$. Whence, and from (4.6) it follows that $F = 0$. Thus,

$$(A^{(1)} - \lambda E)^{-1} = (A^{(2)} - \lambda E)^{-1}, \quad \text{i.e.} \quad A_1 = A_2.$$

5. The formula (4.6) in the case $\nu = |\lambda|$ will be necessary further in §5.

§5. Summability of the system of root vector functions of the operator A in the Abel-Lidskii sense

1. In the present section the following results are obtained with the assumption that $a(t) \in C(\bar{J};\, End\,\mathbf{C}^l)$:
a) resolvent estimate for the operator A;
b) estimate of the generalized resolvent (i.e. the case when the operator A acts from $\mathcal{H}_-$ to $\mathcal{H}_+$);
c) summability in the Abel-Lidskii sense for the system of root vector functions of the operator A.

In a special case when the matrix $a(t) \in C^m(\bar{J};\, End\,\mathbf{C}^l)$, and the representation

$$a(t) = U_+(t)\Lambda(t)U_-(t)$$

hold at the ends of the interval, one obtains an integral representation for the resolvent and the generalized resolvent.

These conditions hold, e.g., when $a(0)$, $a(1)$ have simple eigenvalues.

Let the matrix $a(t) \in C(\bar{J};\, End\,\mathbf{C}^l)$, its eigenvalues be located outside the sector S; conditions on $a_{ij}(t)$, $i + j < 2m$ be previous (see §1). For any $\delta > 0$ one can construct a matrix $a_\delta(t)$ such that:

1) $$a_\delta(t) \in C^{4m}(\bar{J};\, End\,\mathbf{C}^l);$$

2) $$|a_\delta(t) - a(t)| < \delta, \quad 0 \leqslant t \leqslant 1;$$

3) $$a_\delta(t) = U_\pm^{(\delta)}(t)\Lambda_\pm^{(\delta)}(t)U_\pm^{(\delta)}(t)^{-1}, \quad t \in \Delta_\pm,$$

the length $\Delta_\pm$ depends on δ;

4) $$a_\delta(t) = a(0), \quad t \in \Delta_+,$$
$$a_\delta(t) = a(1), \quad t \in \Delta_-;$$

5) Eigenvalues $a_\delta(t)$ are located outside the sector S.

In view of 4) in 3) matrices $U_\pm^{(\delta)}(t)$, $\Lambda_\pm^{(\delta)}(t)$ are independent of δ:

$$U_+^{(\delta)}(t) = U_+(0),\, t \in \Delta_+, \quad U_-^{(\delta)}(t) = U_-(1),\, t \in \Delta_-,$$
$$\Lambda_+^{(\delta)}(t) \equiv \Lambda_+(0), \quad t \in \Delta_+$$

and

$$\Lambda_-^{(\delta)}(t) \equiv \Lambda_-(1), \quad t \in \Delta_-.$$

Let $\mathcal{A}^{(\delta)} : \mathcal{H}_\nu^l \to \mathcal{H}_{-\nu}^l$ be an operator generated by the from

$$< \mathcal{A}^{(\delta)}u, v >= \sum_{i+j<2m} \int_0^1 < p_j(t)a_{ij}(t)u^{(j)}(t), p_j(t)v^{(j)}(t) >_{\mathbf{C}^l} dt +$$

$$+ \int_0^1 < \rho^\theta(t)a_\delta(t)u^{(m)}(t), \rho^\theta(t)v^{(m)}(t) >_{\mathbf{C}^l} dt.$$

Let us take $1 \leqslant \nu \leqslant |\lambda|$. Similarly to the item 3 §4, one can demonstrate that when $|\lambda| > c_\delta$ is sufficiently large, the representation

$$(\mathcal{A}^{(\delta)} - \lambda E)^{-1} = R^{(\delta)}(\lambda)(E + \mathcal{J}^{(\delta)}(\lambda))$$

exists and

$$\|\mathcal{J}^{(\delta)}(\lambda)\|_{\mathcal{H}_{-\nu}\to\mathcal{H}_{-\nu}} \to 0, \quad \text{when} \quad \lambda \to +\infty \quad \text{in the sector} \quad S.$$

Here the regularizer $R^{(\delta)}(\lambda)$ consists of three addends

$$R^{(\delta)}(\lambda) = \psi_+(\mathcal{B}^{+,\delta} - \lambda E)^{-1}\psi_+ + \psi R_0^{(\delta)}(\lambda)\psi + \psi_-(\mathcal{B}^{-,\delta} - \lambda E)^{-1}\psi_-.$$

When $\lambda' = \lambda'(\delta)$ is sufficiently large for $|\lambda| > \lambda'$, $\lambda \in S$, one has

$$\|\mathcal{J}^{(\delta)}(\lambda)\|_{\mathcal{H}_{-\nu}\to\mathcal{H}_{-\nu}} < 1/2, \quad \nu = |\lambda|.$$

Likewise,

$$\|R^{(\delta)}(\lambda)\|_{\mathcal{H}^l\to\mathcal{H}^l} \leqslant M(1+|\lambda|)^{-1}, \quad \lambda \in S,\, \lambda \geq M',$$

M' is independent of δ.

Note that in view of 4), $\mathcal{B}^{\pm,\delta}$ is independent of δ. Therefore,

$$\|(\mathcal{B}^{\pm,\delta} - \lambda E)^{-1}\|_{\mathcal{H}^l\to\mathcal{H}^l} \leqslant M''(1+|\lambda|)^{-1}, \quad |\lambda| > M'',\, \lambda \in S,$$

where M'' is independent of δ.

One can readily verify by the explicit form of the operator functions $R^{(\delta)}(\lambda)$ that

$$\|\psi R^{(\delta)}(\lambda)\psi\|_{\mathcal{H}^l_{-\nu}\to\mathcal{H}^l_{-\nu}} \leqslant M(1+|\lambda|)^{-1}, \quad |\lambda| > M''',\, \lambda \in S.$$

Therefore,

$$\|R^{(\delta)}(\lambda)\|_{\mathcal{H}^l_{-\nu}\to\mathcal{H}^l_{-\nu}} \leqslant M(1+|\lambda|)^{-1}, \quad \lambda \in S,\, |\lambda| > M_1,$$

where M_1, M''' are independent of δ.

Now let us prove that

$$(\mathcal{A} - \lambda E)(\mathcal{A}^{(\delta)} - \lambda E)^{-1} = E + \Gamma^\delta(\lambda),$$

where

$$\|\Gamma^\delta(\lambda)\|_{\mathcal{H}^l_{-\nu}\to\mathcal{H}^l_{-\nu}} \leqslant M\delta, \quad \lambda > \lambda'(\delta),\, \lambda \in S. \tag{5.1}$$

For $u, v \in \mathcal{H}_{-\nu}$, one has

$$< (\mathcal{A} - \lambda E)(\mathcal{A}^{(\delta)} - \lambda E)^{-1}u, v >=< u, v > +$$
$$+ < (a(t) - a_\delta(t))\rho^\theta(t)\partial_t^m(\mathcal{A}_\delta - \lambda E)^{-1}u, \rho^\theta(t)\partial_t^m v > .$$

The second addend does not exceed

$$\delta\|(\mathcal{A}^{(\delta)} - \lambda E)^{-1}u\|_{\mathcal{H}^l_\nu}|v|_{\mathcal{H}^l_\nu} \leqslant M\delta|u|_{\mathcal{H}^l_{-\nu}}|v|_{\mathcal{H}^l_\nu},\ \lambda > \lambda'(\delta),\ \lambda \in S$$

by module, which proves (5.1). Similar statement is valid for the selfadjoint bilinear form itself as well. Whence,

$$ker\,(\mathcal{A} - \lambda E)^{-1} = 0, \quad |\lambda| > M,\, \lambda \in S.$$

Thus, for $\lambda \in S$, $|\lambda| > M$:

$$(\mathcal{A} - \lambda E)^{-1} = (\mathcal{A}^{(\delta)} - \lambda E)^{-1}(E + \Gamma^\delta(\lambda))^{-1} =$$
$$= (\mathcal{A}^{(\delta)} - \lambda E)^{-1}(E + \Gamma_0^\delta(\lambda)),$$

where

$$\|\Gamma_0^\delta(\lambda)\|_{\mathcal{H}^l_{-\nu}\to\mathcal{H}^l_{-\nu}} \leqslant M'\delta,$$
$$4 < \nu < 2|\lambda|, \quad \lambda \in S, \quad |\lambda| > \lambda'(\delta).$$

On the basis of this equality and according to the previous scheme, let us prove that contraction of A on $\mathcal{H}^l$ of the operator $\mathcal{A}$ has the following properties:

(i) A is a unique closed operator such that

$$D(A) \subset \mathcal{H}^l_+, \quad (Au, v) = \mathcal{A}[u, v], \quad \forall u \in D(A),\, v \in \mathcal{H}^l_+,$$

(ii)

$$\|(A - \lambda E)^{-1}\|_{\mathcal{H}^l\to\mathcal{H}^l} \leqslant M(1+|\lambda|)^{-1}, \quad \lambda \in S,\, |\lambda| > M.$$

Upon application of Theorem 6.4.2 from [23], on the basis of the estimate (ii), the following result is obtained:

Theorem 5.1. *The Fourier series of any vector function $f \in \mathcal{H}^l$ is summed over the system of root vector functions of the operator A to f by the Abel method with the brackets of the order $\gamma = \frac{1}{2m} + \varepsilon$ with sufficiently small $\varepsilon > 0$.*

The concept of summability by the Abel method is described in [14,26]. The method was introduced by Lidskii [26]. The works [21, 22, 27-30] should also be mentioned (see also [31,Ch.2.§1.3, 32]).

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