ISSN 2074-1863 Ufa mathematical journal. Volume 3. 3 (2011). Pp. 3-14.

UDC 517.946

THE LOWER ESTIMATE OF DECAY RATE OF SOLUTIONS FOR DOUBLY NONLINEAR PARABOLIC EQUATIONS

E.R. ANDRIYANOVA, F.KH. MUKMINOV

Abstract. Existence of a strong solution to a doubly nonlinear parabolic equation in unbounded domains is established by the method of Galerkin's approximations. In earlier publications, existence was proved usually in bounded domains by approximating the evolutionary part of the equation by finite differences. Usage of Galerkin's approximations makes it possible to prove the second integral identity. On the basis of the identity, the lower estimate of the decay rate of the solution norm is proved in bounded domains. Similar estimates for quasilinear parabolic equations were established earlier by Tedeev A.F. and Alikakos N., Rostmanian R.

Keywords: doubly nonlinear parabolic equation, decay rate of solution, lower estimates, existence of strong solution.

1. Introduction

Let Ω be an unbounded domain of the space $\mathbb{R}_n = \{\mathbf{x} = (x_1, x_2, \dots, x_n)\}$, $n \geq 2$. The first mixed problem

$$(|u|^{\alpha-2}u)_t = \sum_{i=1}^{n}(|u_{x_i}|^{p-2}u_{x_i})_{x_i}, \quad \alpha, p > 1, \quad (t, \mathbf{x}) \in D; \tag{1}$$

$$u(t, \mathbf{x})\Big|_S = 0, \quad S = \{t > 0\} \times \partial\Omega; \quad u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad u_0(\mathbf{x}) \in L_\alpha(\Omega) \tag{2}$$

is considered in a cylinder domain $D = \{t > 0\} \times \Omega$ for a doubly nonlinear parabolic equation.

Existence and uniqueness of the problem solution were considered by Raviart P.A. [1], Lions J.L.[2], Bamberger A.[5], Grange O., Mignot F.[6], Alt, H.W., Luckhaus, S.[7] Bernis F.[10] and others. The problems were basically considered in bounded domains. A strong solution of the problem in a bounded domain was established by Raviart P.A. by means of substitution of the evolutionary derivative by a difference relation. Bernis F. proved that a weak solution to the problem exists in an unbounded domain by means of passing to the limit from solutions constructed in bounded domains by Grange O., Mignot F. However, working with a weak solution one comes across difficulties in investigating, e.g., a decrease of solution when $t \to \infty$. Bamberger A. established uniqueness of the strong positive solution to the problem.

We suggest a usual method for constructing the strong solution to the problem in an unbounded domain at once based on Galerkin's approximations. Their constructions is little different from that suggested by Lions J.L. in [4] for the case $\alpha = 2$. The suggested method can be adapted for a significantly wider class of equations.

Let us define the space $W^1_{\alpha,p}(\Omega)$ as supplements $C_0^\infty(\Omega)$ with respect to the norm

$$||v||_{W^1_{\alpha,p}} = ||v||_\alpha + ||\nabla v||_p,$$

The work is supported by the Russian Foundation for Basic Research (grant 10-01-00118-a).
Submitted on 3 June 2011.

where $||v||_\alpha = ||v||_{\alpha,\Omega}$, $||v||_{\alpha,Q} = \left(\int\limits_Q v^\alpha dx\right)^{1/\alpha}$.

Denote by $V(D^T)$ supplements $C_0^\infty(D^T)$ with respect to the norm

$$||v||_V = ||v||_{\alpha,D^T} + ||\nabla v||_{p,D^T}.$$

The function $u \in V(D^T)$ satisfying the identity

$$\int\limits_{D^T} \left(-|u|^{\alpha-2}u\varphi_t + \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\right) dxdt = \int\limits_\Omega |u_0|^{\alpha-2}u_0\varphi(0,x)dx+ \tag{3}$$

$$+(f,\varphi)_{D^T}$$

when $\varphi \in C_0^\infty(D_{-1}^T)$ is said to be the generalized solution to the problem (1),(2). Here and in what follows $(f,\varphi)_Q$ indicates values of the generalized function f at the element $\varphi \in C_0^\infty(Q)$, where Q is a domain in $\mathbb{R}_n$ or in $\mathbb{R} \times \mathbb{R}_n$.

Theorem 1. *Let $f, f_t \in (V(D^T))', u_0 \in W^1_{\alpha,p}(\Omega)$. Then, there is a generalized solution u to the problem (1),(2) satisfying the conditions*

$$u \in L_\infty((0,T); W^1_{\alpha,p}(\Omega)), \tag{4}$$

$$|u|^{\frac{\alpha-2}{2}}u_t \in L_2(D^T), \alpha > 1, \tag{5}$$

$$u_t \in L_\alpha(D^T),\ C([0,T]; L_\alpha(\Omega)) \qquad \textit{when } \alpha \in (1,2) \tag{6}$$

$$|u|^{\alpha-2}u_t \in L_{\alpha'}(D^T) \qquad \textit{when } \alpha \geq 2. \tag{7}$$

Galerkin's approximations are smooth functions. This facilitates the proof of their various estimates, which afterwards extend to solution of the problem (1),(2) by passage to the limit. In particular, the estimates

$$c(1+t)^{-1/(p-\alpha)} \leq \|u(t)\|_{L_\alpha(\Omega)} \leqslant Mt^{-1/(p-\alpha)}, \quad t > 0 \tag{8}$$

are determined in case of a bounded domain when $p > \alpha$. The estimates (8) when $\alpha = 2$ are obtained by A.F. Tedeev [19] and Alikakos N., Rostmanian R. [15] for the Cauchy problem. Exact two-sided estimates of the decay rate for the solution norm of a linear and quasilinear parabolic equation in an unbounded domain are established in works by L.M. Kojevnikova [16] and R. Kh. Karimov, L.M. Kojevnikova. [17].

2. Proof of the existence theorem

Conditions on f ensure that

$$f \in C([0,T]; (W^1_{\alpha,p}(\Omega))').$$

In particular, $f(0) \in (W^1_{\alpha,p}(\Omega))'$.

First, consider the case $\alpha > 2$. Let us choose the sequence $\omega_k \in C_0^\infty(\Omega)$ of linearly independent functions whose linear envelope is dense in $W^1_{\alpha,p}(\Omega)$. Let us assume that $I_m = \cup_{k=1}^m \mathrm{supp}\omega_k$. Galerkin's approximations to the solution will be sought in the form

$$u_m(t,x) = \sum_{k=1}^m c_{mk}(t)\omega_k(x),$$

where the functions $c_{mk}(t)$ are determined by the equations

$$\int\limits_{\Omega}\left(\omega_j\frac{\partial}{\partial t}\left(\frac{u_m}{b_m}+|u_m|^{\alpha-2}u_m\right)+\sum_{i=1}^{n}|u_{mx_i}|^{p-2}u_{mx_i}(\omega_j)_{x_i}\right)dx=(f,\omega_j)_{\Omega}, \tag{9}$$

$$j=1,2,...,m.$$

The numbers $b_m>0$ will be chosen later. Let us verify that Equations (9) are solvable with respect to the derivatives c'_{mk}. Obviously, Equations (9) have the form

$$A_{jk}(t)c'_{mk}=F_j(c_{m_1},c_{m_2},...,c_{m_m})+f_j(t).$$

The matrix of coefficients

$$A_{jk}(t)=\int\limits_{\Omega}\left(\frac{1}{b_m}+(\alpha-1)|u_m|^{\alpha-2}\right)\omega_j\omega_k dx$$

for every t is the Gram matrix of a system of linearly independent vectors ω_k, $k=1,2,...,m$ and therefore, it has an inverse one. Equations (9) under the initial conditions $c_{mk}(0)$, chosen so that $u_m(0,x)\to u_0(x)$, provide $c_{mk}(t)$. At first, these functions belong to a small interval of time, but since the Galerkin's approximations are limited, they can be defined at an infinite time interval. Let us select the numbers b_m so that $||u_m(0)||_2^2/b_m\to 0$ when $m\to\infty$.

Let us determine the estimates for Galerkin's approximations. Multiplying Equation (9) by $c_{mj}(t)$ and summing over, we obtain

$$\int\limits_{\Omega}\left(u_m\left(\frac{u_m}{b_m}+|u_m|^{\alpha-2}u_m\right)_t+\sum_{i=1}^{n}|u_{mx_i}|^p\right)dx=(f,u_m)_{\Omega}.$$

Upon integration with respect to t, we have

$$\int\limits_{\Omega}\left(\frac{u_m^2(t)}{2b_m}+\frac{\alpha-1}{\alpha}|u_m(t)|^{\alpha}\right)dx+||\nabla u_m||^p_{p,D_0^t}=(f,u_m)_{D_0^t}+ \tag{10}$$

$$+\int\limits_{\Omega}\left(\frac{u_m^2(0)}{2b_m}+\frac{\alpha-1}{\alpha}|u_m(0)|^{\alpha}\right)dx.$$

The latter integral is limited by a constant independent of m in the right-hand side due to convergencies selected above. Furthermore,

$$|(f,u_m)_{D_0^t}|\leqslant\int\limits_0^t||u_m(\tau)||_{W^1_{\alpha,p}}||f(\tau)||_{(W^1_{\alpha,p})'}d\tau\leqslant c\int\limits_0^t(||u_m(\tau)||_{\alpha}+||\nabla u_m(\tau)||_p)d\tau\leqslant$$

$$\leqslant c(\varepsilon)+\varepsilon\int\limits_0^t(||u_m(\tau)||_{\alpha}^{\alpha}+||\nabla u_m(\tau)||_p^p)d\tau.$$

Therefore, it follows from (10) and the Gronwall lemma that the sequence u_m is limited in spaces $C([0,T];L_{\alpha}(\Omega))$ and $V(D^T)$.

Let us multiply equations (9) by $c'_{mj}(t)$ and make the summation:

$$\int\limits_{\Omega}\left(u'_m\left(\frac{u_m}{b_m}+|u_m|^{\alpha-2}u_m\right)_t+\sum_{i=1}^{n}|u_{mx_i}|^{p-2}u_{mx_i}u'_{mx_i}\right)dx=(f,u'_m)_{\Omega}.$$

integrating with respect to t, we obtain

$$\int\limits_{D_0^T}\left(\frac{1}{b_m}+(\alpha-1)|u_m|^{\alpha-2}\right)(u'_m)^2dxdt+\frac{1}{p}||\nabla u_m(T)||_p^p=$$

$$=\frac{1}{p}||\nabla u_m(0)||_p^p+(f,u'_m)_{D^T}. \tag{11}$$

Let us transform the latter addend by integration by parts

$$(f,u'_m)_{D^T}=(f(T),u_m(T))_\Omega-(f(0),u_m(0))_\Omega-(f',u_m)_{D^T}.$$

Note that

$$|(f(T),u_m(T))_\Omega|\leqslant ||f(T)||_{(W^1_{\alpha,p})'}||u_m(T)||_{W^1_{\alpha,p}}\leqslant c(\varepsilon)+\varepsilon(||u_m(T)||_\alpha^\alpha+||\nabla u_m(T)||_p^p).$$

Since u_m is limited in the space $V(D^T)$, we have

$$|(f',u_m)_{D^T}|\leqslant ||f_t||_{(V(D^T))'}||u_m||_{V(D^T)}\leqslant c.$$

Therefore, from the equalities (11) we establish the boundedness of the sequences $|u_m|^{\frac{\alpha-2}{2}}u'_m$, and ∇u_m in the spaces $L_2(D^T)$, and $C([0,T];L_p(\Omega))$, respectively. The established facts allow us to choose a subsequence u_{m_k} converging weakly in the spaces indicated below. In order to simplify the notation, the subindex k will be omitted.

$$u_m\to u \text{ weakly in } V(D^T).$$

$$A(u_m)=-\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(|u_{mx_i}|^{p-2}u_{mx_i}\right)\to\chi \qquad \text{weakly in } (V(D^T))'.$$

$$\left(|u_m|^{\frac{\alpha-2}{2}}u_m\right)'\to\tilde{u} \text{ weakly in } L_2(D^T).$$

In what follows, we will prove that one can choose the sequence u_{m_k} converging to u almost everywhere in D^T. This will help to determine that $\tilde{u}=\left(|u|^{\frac{\alpha-2}{2}}u\right)'$.

The sequence $u_m\in C([0,T];W^1_{\alpha,p}(\Omega))$ is limited. For every bounded domain $Q\subset\Omega$ with a smooth boundary, one obtains compactness of the injection $L_1(Q)\subset W^1_1(Q)$. Therefore, by means of a diagonal process one can single out a subsequence $u_{m_k}(t_s)\to h_s$ strongly in $L_1(Q)$ on a countable dense subset $t_s\subset[0,T]$. Choosing a subsequence once more, we can also consider (omitting the subindices), that $u_m(t_s,x)\to h_s(x)$ almost everywhere in Q for every t_s. Likewise, when $\alpha\le p$, we can consider that the sequence $u_m(t_s)\to h_s$ strongly in $L_\alpha(Q)$ for every t_s.

Let us us determine now an equicontinuity with respect to t of the sequence $v_m(t)$ in $L_2(\Omega)$, $v_m=|u_m|^{\frac{\alpha-2}{2}}u_m$.

$$||v_m(t_2)-v_m(t_1)||_2\leqslant\int\limits_{t_1}^{t_2}||v'_m(t)||_2dt\leqslant \tag{12}$$

$$\left(|t_2-t_1|\int\limits_{t_1}^{t_2}||v'_m(t)||_2^2dt\right)^{\frac{1}{2}}\leqslant c|t_2-t_1|^{\frac{1}{2}}.$$

Furthermore, the sequence $v_m(t)$ is bounded in the space $C([0,T];L_2(\Omega))$. Then, one can single out the subsequence $v_{m_k}(t)$, converging weakly in $L_2(\Omega)$ with the same t_s as above. Together with the above convergence almost everywhere in $Q\subset\Omega$, this entails a strong convergence in $L_1(Q)$ for every t_s (see J.L. Lions [4]).

One can readily determine the uniform mutual convergence $v_m(t)$ with respect to the norm $L_1(Q)$:

$$||v_n(t) - v_m(t)||_{1,Q} = ||v_n(t) - v_n(t_{s_k}) + v_n(t_{s_k}) - v_m(t_{s_k}) + v_m(t_{s_k}) - v_m(t)||_{1,Q} \leqslant$$

$$\leqslant C_Q|t - t_{s_k}|^{\frac{1}{2}} + ||v_n(t_{s_k}) - v_m(t_{s_k})||_{1,Q}$$

for the bounded domain Q from (12). Selecting a finite set of numbers t_{s_k} with a small step and then increasing n, m, one achieves the smallness of the right-hand side uniform in t.

Thus, the convergence $v_{m_k} \to v$ in $C([0,T]; L_1(Q))$ is determined. Convergence will also occur in $L_1((0,T) \times Q)$ therefore, one can single out a subsequence converging in $(0,T) \times Q$ almost everywhere. Since Q is arbitrary, the diagonal process can single out the subsequence v_{m_k}, converging in D^T almost everywhere. Then, the sequence u_{m_k} will converge as well almost everywhere in D^T to u (Lemma 1.3. J.L. Lions [4]). Thus, it is established that $v_{m_k} \to v = |u|^{\frac{\alpha-2}{2}}u$.

Furthermore, $(v'_m, \varphi)_{D^T} = -(v_m, \varphi')_{D^T}$. Turning to the limit, we obtain

$$(\tilde{u}, \varphi)_{D^T} = -(v, \varphi')_{D^T}.$$

Whence, $\tilde{u} = v' = (|u|^{\frac{\alpha-2}{2}}u)'$.

Let us demonstrate that the sequence $|u_m|^{\alpha-2}u'_m$ is bounded in $L_{\alpha'}(D^T)$. Indeed,

$$|(|u_m|^{\alpha-2}u'_m, \varphi)_{D^T}| = \left|\left(|u_m|^{\frac{\alpha-2}{2}}u'_m, \varphi|u_m|^{\frac{\alpha-2}{2}}\right)_{D^T}\right| \leqslant C||\varphi|u_m|^{\frac{\alpha-2}{2}}||_{2,D^T}$$

$$\leqslant C||\varphi||_{\alpha,D^T}||u_m||^{\frac{\alpha-2}{2}}_{\alpha,D^T} \leqslant C_1||\varphi||_{\alpha,D^T}.$$

Then, we can consider that $|u_m|^{\alpha-2}u'_m \to |u|^{\alpha-2}u'$ weakly in $L_{\alpha'}(D^T)$.

Let us prove the equality $\chi = A(u)$. To this end, integral correlations are necessary. Let us multiply Equation (9) by a smooth function $d_j(t)$, integrate it with respect to t and proceed to the limit when $m \to \infty$, denoting $d_j(t)\omega_j(x)$ by φ in the final expression:

$$((|u|^{\alpha-2}u)', \varphi)_{D^T} + (\chi, \varphi)_{D^T} = (f, \varphi)_{D^T}. \tag{13}$$

Note, that

$$\left(\frac{u'_m}{b_m}, \varphi\right)_{D^T} = \frac{1}{b_m}\left(-(u_m, \varphi')_{D^T} + (u_m(T), \varphi(T))_\Omega - (u_m(0), \varphi)\right) \to 0,$$

due to boundedness of u_m in $C([0,T]; L_\alpha(\Omega))$, and to $b_m \to \infty$. Thus, u is a generalized solution of the problem (1),(2), if it is determined that $\chi = A(u)$.

Obviously, the function $u \in V(D^T)$ can be approximated by linear combinations

$$\sum_{j=1}^{N} d_j(t)\omega_j(x).$$

Therefore, (13) yields

$$(f - \chi, u)_{D^T} = ((|u|^{\alpha-2}u)', u)_{D^T} = \frac{\alpha-1}{\alpha}\left(||u(T)||^\alpha_\alpha - ||u(0)||^\alpha_\alpha\right). \tag{14}$$

Note that the inclusion $v, v' \in L_2(D^T)$ entails $v \in C([0,T]; L_2(\Omega))$ and $\|u(t)\|_\alpha \in C([0,T]), \alpha > 1$.

In what follows, standard arguments of monotony are given. One can easily verify that

$$X_m = \int_0^T (A(u_m(t)) - A(h(t)), u_m(t) - h(t))_\Omega dt \geq 0 \quad \forall h \in V(D^T). \tag{15}$$

One can readily deduce the relations

$$(A(u_m), u_m)_{D^T} = (f, u_m)_{D^T} + \frac{\alpha-1}{\alpha}(||u_m(0)||_\alpha^\alpha - ||u_m(T)||_\alpha^\alpha) +$$

$$+\frac{1}{2b_m}\left(||u_m(0)||_2^2 - ||u_m(T)||_2^2\right)$$

from Equations (9). Therefore,

$$X_m = (f, u_m)_{D^T} + \frac{\alpha-1}{\alpha}(||u_m(0)||_\alpha^\alpha - ||u_m(T)||_\alpha^\alpha) +$$

$$+\frac{1}{2b_m}\left(||u_m(0)||_2^2 - ||u_m(T)||_2^2\right) - (A(u_m), h)_{D^T} - (A(h), u_m - h)_{D^T}.$$

Whence, (since $\liminf ||u_m(T)||_\alpha \geq ||u(T)||_\alpha$)

$$\limsup X_m \leqslant (f, u)_{D^T} + \frac{\alpha-1}{\alpha}(||u_0||_\alpha^\alpha - ||u(T)||_\alpha^\alpha) - (\chi, h)_{D^T} - (A(h), u-h)_{D^T}.$$

Applying (14), one obtains

$$(\chi - A(h), u - h) \geq 0$$

from (15). Assume that $h = u - \lambda\omega, \quad \lambda > 0, \quad \omega \in V(D^T):$

$$\lambda(\chi - A(u - \lambda\omega), \omega)_{D^T} \geq 0.$$

Turning $\lambda \to 0$, we have $(\chi - A(u), \omega) \geq 0, \quad \forall\omega$. Whence, $\chi = A(u)$.

Let us assume now that $\alpha \leqslant 2$. Galerkin's approximations will be sought for in the same form, but the functions $c_{mk}(t)$ are to be determined from the equations

$$\int_\Omega \left(\omega_j \frac{\partial}{\partial t}\left(v_m^{\frac{\alpha}{2}-1} u_m\right) + \sum_{i=1}^n |u_{mx_i}|^{p-2} u_{mx_i} (\omega_j)_{x_i}\right) dx = \tag{16}$$

$$= (f, \omega_j)_\Omega, \quad j = 1, 2, ..., m.$$

Here, the functions $v_m = u_m^2 + \varepsilon_m$ are introduced for the sake of regularization, the numbers $\varepsilon_m > 0$ are to be selected in what follows. Let us verify that Equations (16) are solvable with respect to the derivatives c'_{mk}. Manifestly, they have the form

$$A_{jk}(t) c'_{mk} = F_j(c_{m_1}, c_{m_2}, ..., c_{m_m}) + f_j(t).$$

The coefficients matrix

$$A_{jk}(t) = \int_\Omega ((\alpha-1)u_m^2 + \varepsilon_m) v_m^{\frac{\alpha}{2}-2} \omega_j \omega_k dx$$

for every t is the Gram matrix of the system of linearly independent vectors ω_k, $k = 1, 2, ...m$ and therefore, has the inverse one. One obtains the functions $c_{mk}(t)$ from Equations (16) with the initial conditions $c_{mk}(0)$, selected so that $u_m(0, x) \to u_0(x)$.

Let us set the estimates for Galerkin's approximations. Multiplying Equations (16) by $c_{mj}(t)$ and summing over, we obtain

$$\int_\Omega \left(u_m\left((\alpha-1)v_m^{\frac{\alpha}{2}-1} - \varepsilon_m(\alpha-2)v_m^{\frac{\alpha}{2}-2}\right) u'_m + \sum_{i=1}^n |u_{mx_i}|^p\right) dx = (f, u_m)_\Omega.$$

Since $u_m u'_m = v'_m/2$, upon integrating with resect to t one has

$$\int_{I_m} \left(\frac{\alpha-1}{\alpha} v_m(t)^{\frac{\alpha}{2}} - \varepsilon_m v_m(t)^{\frac{\alpha}{2}-1}\right) dx + ||\nabla u_m||_{p, D_0^t}^p = (f, u_m)_{D_0^t} \tag{17}$$

$$+\int\limits_{I_m}\left(\frac{\alpha-1}{\alpha}v_m(0)^{\frac{\alpha}{2}}-\varepsilon_m v_m(0)^{\frac{\alpha}{2}-1}\right)dx.$$

The latter integral in the right-hand side is bounded by a constant independent of m due to the convergencies selected above. Similarly to the above,

$$|(f,u_m)_{D_0^t}|\leqslant c(\varepsilon)+\varepsilon\int\limits_0^t(||u_m(\tau)||_\alpha^\alpha+||\nabla u_m(\tau)||_p^p)d\tau.$$

Note that the choice ε_m ensures the validity of the inequalities

$$\int\limits_{I_m}\varepsilon_m v_m(t)^{\frac{\alpha}{2}-1}dx\leq\int\limits_{I_m}\varepsilon_m^{\frac{\alpha}{2}}dx\leq\varepsilon_m^{\frac{\alpha}{4}}. \tag{18}$$

Therefore, (17) and the Gronwall lemma entail the uniform boundedness of the integrals $\int\limits_{I_m} v_m(t)^{\frac{\alpha}{2}}dx$ with respect to t and m and hence, of the sequence u_m in spaces $C([0,T];L_\alpha(\Omega))$ and $V(D^T)$.

Let us multiply Equations (16) by $c'_{mj}(t)$ and make the summation:

$$\int\limits_\Omega\left((u'_m)^2((\alpha-1)u_m^2+\varepsilon_m)v_m^{\frac{\alpha}{2}-2}+\sum_{i=1}^n|u_{mx_i}|^{p-2}u_{mx_i}u'_{mx_i}\right)dx=(f,u'_m)_\Omega.$$

Integration with respect to t provides

$$\int\limits_{D_0^T}(u'_m)^2((\alpha-1)u_m^2+\varepsilon_m)v_m^{\frac{\alpha}{2}-2}dxdt+\frac{1}{p}||\nabla u_m(T)||_p^p= \tag{19}$$

$$=\frac{1}{p}||\nabla u_m(0)||_p^p+(f,u'_m)_{D^T}.$$

Similarly to the above, one has $|(f,u'_m)_{D^T}|\leqslant c$. Moreover, $(\alpha-1)u_m^2+\varepsilon_m\geq(\alpha-1)v_m$. Assuming that $g(u)=\int\limits_0^u(t^2+\varepsilon_m)^{\frac{\alpha-2}{4}}dt$, we determine from the equalities (19) the boundedness of the sequences $v_m^{\frac{\alpha-2}{4}}u'_m=(g(u_m))'$ and ∇u_m in the spaces $L_2(D^T)$ and $C([0,T];L_p(\Omega))$, respectively. The established facts allow us to choose the subsequence u_{m_k}, converging weakly in the below spaces. For the sake of simplicity in notation, the subindex k is omitted.

$$u_m\to u \text{ weakly in } V(D^T).$$

$$A(u_m)\to\chi\qquad\text{weakly in } (V(D^T))'.$$

$$(g(u_m))'\to\tilde{u} \text{ weakly in } L_2(D^T).$$

In what follows, we prove that one can chose a subsequence u_{m_k}, converging to u almost everywhere on D^T. This allows us to determine that $\tilde{u}=|u|^{\frac{\alpha-2}{2}}u'$.

Proceeding as above, one can consider (omitting subindices) that $u_m(t_s,x)\to h_s(x)$ almost everywhere in Q for every t_s, and when $\alpha\leq p$ one can consider that the sequence $u_m(t_s)\to h_s$ strongly in $L_\alpha(Q)$ for every t_s.

Let us determine now the equicontinuity with respect to t of the sequence $g(u_m(t))$ in $L_2(Q)$.

$$||g(u_m(t_2))-g(u_m(t_1))||_{2,Q}\leqslant\int\limits_{t_1}^{t_2}||(g(u_m(t)))'||_{2,\Omega}dt\leqslant$$

$$\leqslant \left(|t_2 - t_1| \int\limits_{t_1}^{t_2} ||(g(u_m(t)))'||_2^2 dt \right)^{\frac{1}{2}} \leqslant c|t_2 - t_1|^{\frac{1}{2}}. \tag{20}$$

The sequence $g(u_m(t))$ is bounded in the space $C([0,T]; L_2(Q))$. Then, one can singe out the subsequence $g(u_{m_k}(t))$, converging weakly in $L_2(\Omega)$ with t_s the same as above. Together with the convergence almost everywhere in $Q \subset \Omega$ determined above, this entails a strong convergence in $L_1(Q)$ for every t_s. For a bounded domain Q from (20) this leads to a uniform mutual convergence $g(u_m(t))$ according to the norm $L_1(Q)$. Thus, the convergence $g(u_{m_k}) \to v$ is determined in $C([0,T]; L_1(Q))$. Convergence will also occur in $L_1((0,T) \times Q)$, therefore, one can single out a subsequence, converging in $(0,T) \times Q$ almost everywhere. Since Q is arbitrary, we can single out the subsequence $g(u_{m_k})$, converging in D^T almost everywhere by means of the diagonal process. Then the sequence u_{m_k} will converge almost everywhere in D^T to u as well. Thus, it is determined that $g(u_{m_k}) \to v = \frac{2}{\alpha}|u|^{\frac{\alpha-2}{2}}u$. Meanwhile, $\tilde{u} = |u|^{\frac{\alpha-2}{2}}u'$.

When $\alpha \leqslant 2$, the sequence u'_m is bounded in $L_\alpha(D^T)$. Indeed, (19) provides that

$$|(u'_m, \varphi)_{D^T}| = |\left(v_m^{\frac{\alpha-2}{4}} u'_m, \varphi v_m^{\frac{2-\alpha}{4}} \right)_{D^T}| \leqslant C||\varphi v_m^{\frac{2-\alpha}{4}}||_{2,D_m^T}$$

$$\leqslant C||\varphi||_{\alpha',D^T}||v_m||_{\frac{\alpha}{2},D_m^T}^{\frac{2-\alpha}{4}} \leqslant C_1||\varphi||_{\alpha',D^T}; \quad D_m^T = (0,T) \times I_m.$$

Therefore, one can consider that $u'_m \to u'$ weakly in $L_\alpha(D^T)$, and then $u \in C([0,T]; L_\alpha(\Omega))$.

Let us prove the equality $\chi = A(u)$. To this end, integral relations are necessary. Let us multiply Equations (16) by a smooth function $d_j(t)$, integrate with respect to $t \in (0,T)$ and integrate by parts in the first term. Then, denoting $d_j(t)\omega_j(x)$ by φ, we have

$$(v_m^{\frac{\alpha}{2}-1}(T)u_m(T), \varphi(T))_\Omega - (v_m^{\frac{\alpha}{2}-1}u_m, \varphi_t)_{D^T} + (|\nabla u_m|^{p-2}\nabla u_m, \varphi_{x_i})_{D^T}$$

$$= (f, \varphi)_{D^T} + (v_m^{\frac{\alpha}{2}-1}(0)u_m(0), \varphi(0))_\Omega.$$

Note that $|v_m^{\frac{\alpha}{2}-1}u_m| \leqslant v_m^{\frac{\alpha}{2}-1}v_m^{\frac{1}{2}} \in C([0,T], L_{\alpha'}(Q))$, since $\left(v_m^{\frac{\alpha-1}{2}}\right)^{\alpha'} = v_m^{\frac{\alpha}{2}}$ is a bounded sequence in $C([0,T]; L_1(Q))$. Hence, one can single out a subsequence so that the weak convergencies $v_m^{\frac{\alpha}{2}-1}u_m \to |u|^{\alpha-2}u$ in $L_{\alpha'}(D^T)$ and $v_m^{\frac{\alpha}{2}-1}(T)u_m(T) \to |u|^{\alpha-2}u(T)$ in $L_{\alpha'}(\Omega)$ are provided. The fact that the limiting functions will be of this very type, is provided by the above determined convergence of the subsequence u_m almost everywhere in D^T, and almost everywhere in Ω when $t = T$ as well. Then, upon passage to the limit $m \to \infty$, we have

$$(|u|^{\alpha-2}(T)u(T), \varphi(T))_\Omega - (|u|^{\alpha-2}u, \varphi_t)_{D^T} + (\chi, \varphi)_{D^T} \tag{21}$$

$$= (f, \varphi)_{D^T} + (|u|^{\alpha-2}(0)u(0), \varphi(0))_\Omega.$$

Substituting $\varphi = u$ into (21), we obtain

$$(f - \chi, u)_{D^T} = \frac{\alpha - 1}{\alpha}\left(||u(T)||_\alpha^\alpha - ||u(0)||_\alpha^\alpha\right). \tag{22}$$

The correlation (21) indicates that u is a generalized solution to the problem (1),(2), if it is determined that $\chi = A(u)$.

Further, standard arguments of monotony follow. One can readily deduce the relations

$$(A(u_m), u_m)_{D^T} = (f, u_m)_{D^T} + \int\limits_{I_m} \left(\frac{\alpha-1}{\alpha} v_m^{\frac{\alpha}{2}}(0) - \varepsilon_m v_m^{\frac{\alpha}{2}-1}(0) \right) dx$$

$$- \int\limits_{I_m} \left(\frac{\alpha-1}{\alpha} v_m^{\frac{\alpha}{2}}(T) - \varepsilon_m v_m^{\frac{\alpha}{2}-1}(T) \right) dx \tag{23}$$

from Equations (17).

Let us make use of the inequality (18), then (23) entails

$$X_m \leqslant (f, u_m)_{D^T} + \frac{\alpha-1}{\alpha} \int\limits_{I_m} \left(v_m^{\frac{\alpha}{2}}(0) - v_m^{\frac{\alpha}{2}}(T) \right) dx + 2\varepsilon_m^{\frac{\alpha}{4}} -$$

$$-(A(u_m), h)_{D^T} - (A(h), u_m - h)_{D^T}.$$

Whence, (since $\liminf ||v_m^{\frac{\alpha}{4}}(T)||_2 \geq ||u(T)||_\alpha$ and $\varepsilon_m \to 0$)

$$\limsup X_m \leqslant (f, u)_{D^T} + \frac{\alpha-1}{\alpha} (||u_0||_\alpha^\alpha - ||u(T)||_\alpha^\alpha) - (\chi, h)_{D^T} - (A(h), u - h)_{D^T}.$$

Then, similarly to the case $\alpha > 2$, it is proved that $\chi = A(u)$. The theorem is proved.

3. Lower estimate for the solution norm in a bounded domain

Let us assume that $\alpha \leqslant p$ and the domain Ω is bounded. Let us determine lower estimates of decay rate of solutions when $t \to \infty$. Since the uniqueness of the solution is not established so far, in fact, we will determine lower estimates only for the constructed solution in the domain D^T for every sufficiently large T.

First, consider the case $\alpha \geq 2$. Let us introduce the notation

$$E(t) = \int\limits_\Omega \left(\frac{u_m^2(t)}{2b_m} + \frac{\alpha-1}{\alpha} |u_m(t)|^\alpha \right) dx,$$

$$H(t) = ||\nabla u_m(t)||_p^p.$$

Upon differentiation with respect to t, Formula (10) for $f = 0$ takes the form

$$E' + H = 0. \tag{24}$$

Formula (11) is written after differentiation as follows:

$$\int\limits_\Omega \left(\frac{1}{b_m} + (\alpha-1)|u_m(t)|^{\alpha-2} \right) u_m'^2(t) dx + \frac{1}{p} H'(t) = 0. \tag{25}$$

The estimates

$$(E')^2 = \left(\int\limits_\Omega \left(\frac{u_m u_m'(t)}{b_m} + (\alpha-1)|u_m(t)|^{\alpha-2} u_m'(t) \right) dx \right)^2 \leqslant$$

$$\left(\left(\int\limits_\Omega \frac{u_m^2(t)}{b_m} dx \int\limits_\Omega \frac{u_m'^2(t)}{b_m} dx \right)^{\frac{1}{2}} + (\alpha-1) \left(\int\limits_\Omega |u_m|^\alpha dx \int\limits_\Omega |u_m|^{\alpha-2} u_m'^2(t) dx \right)^{\frac{1}{2}} \right)^2 \leqslant$$

hold. Applying the Cauchy-Bunyakowsky inequality for the scalar product in $\mathbb{R}_2$, we deduce

$$\leqslant \left(\int\limits_\Omega \frac{u_m'^2(t)}{b_m} dx + (\alpha-1) \int\limits_\Omega |u_m|^{\alpha-2} u_m'^2(t) dx \right) \left(\int\limits_\Omega \frac{u_m^2(t)}{b_m} dx + (\alpha-1) \int\limits_\Omega |u_m|^\alpha dx \right),$$

whence,

$$(E')^2 \leqslant -\frac{\alpha}{p} H'(t) E(t). \tag{26}$$

By means of (24), the latter is rewritten in the from

$$-HE' \leqslant -\frac{1}{\gamma}EH', \quad \gamma = \frac{p}{\alpha}.$$

Whence, $\gamma\frac{E'}{E} \geq \frac{H'}{H}$ or, upon integration,

$$H(t) \leqslant \frac{H(0)E^{\gamma}(t)}{E^{\gamma}(0)}.$$

Then,

$$E'(t) = -H(t) \geq -\frac{H(0)E^{\gamma}(t)}{E^{\gamma}(0)},$$

or

$$\frac{E'}{E^{\gamma}} \geq -\frac{H(0)}{E^{\gamma}(0)}.$$

Whence,

$$E^{1-\gamma}(t) - E^{1-\gamma}(0) \leqslant (\gamma-1)\frac{H(0)t}{E^{\gamma}(0)}.$$

Thus,

$$E(t) \geq E(0)\left(1 + (\gamma-1)\frac{H(0)t}{E(0)}\right)^{\frac{1}{1-\gamma}}. \tag{27}$$

For a fixed $t \in [0,T]$ when $\alpha \leq p$ in case of a bounded domain we can single out the sequence $u_{m_k}(t)$, converging strongly in the space $L_\alpha(\Omega)$. Therefore,

$$E_m(t) \to \frac{\alpha-1}{\alpha}||u(t)||_\alpha^\alpha.$$

The functions

$$u_m(t) = \sum_{k=1}^{n} c_{mk}(t)\omega_k$$

belong to the linear envelope of the functions $\omega_1, \omega_2, ..., \omega_m$. All norms are equivalent in a finite-dimensional space therefore,

$$\int\limits_\Omega u_m^2(t)dx \leqslant c_m\|u_m(t)\|_\alpha^2 \leqslant \widetilde{c_m}.$$

Let us choose the numbers b_m so that $\widetilde{c_m} \leqslant b_m/m$. Upon passing to the limit in (27) when $m \to \infty$, we obtain

$$||u(t)||_\alpha^\alpha \geq ||u(0)||_\alpha^\alpha(1 + C(u_0)t)^{-\frac{\alpha}{p-\alpha}}. \tag{28}$$

Let $\alpha \leqslant 2$. In this case, let us use the notation

$$E(t) = \int\limits_{I_m}\left(\frac{\alpha-1}{\alpha}v_m^{\frac{\alpha}{2}} - \varepsilon_m v_m^{\frac{\alpha-2}{2}}\right)dx + 2\varepsilon_m^{\frac{\alpha}{4}}.$$

Note, that (18) entails the inequality $E(t) \geq \varepsilon_m^{\frac{\alpha}{4}}$.

Let us differentiate the formula (17) with respect to t and rewrite it for $f = 0$:

$$E' + H = 0. \tag{29}$$

Formula (19) entails that

$$\int_{I_m} ((\alpha-1)u_m^2+\varepsilon)v_m^{\frac{\alpha-4}{2}} u_m'^2 dx = -\frac{1}{p}H'(t). \tag{30}$$

For every $\nu > 0$, the following inequalities are evident:

$$(E'(t))^2 = \left(\int_{I_m}\left(\frac{\alpha-1}{2}v_m^{\frac{\alpha-2}{2}}(2u_m u_m') - \varepsilon_m\frac{\alpha-2}{2}v_m^{\frac{\alpha-4}{2}}(2u_m u_m')\right)dx\right)^2$$

$$\leqslant \frac{1}{4}\left((\alpha-1)\int_{I_m} v_m^{\frac{\alpha-2}{2}}(\nu u_m^2 + u_m'^2/\nu)dx + \varepsilon_m(2-\alpha)\int_{I_m} v_m^{\frac{\alpha-4}{2}}(\nu u_m^2 + u_m'^2/\nu)dx\right)^2$$

$$\leqslant \frac{1}{4}\left(-\frac{1}{p\nu}H' + \alpha E(t)\nu + \varepsilon_m(3-\alpha)\nu\int_{I_m} v_m^{\frac{\alpha-2}{2}}dx - 2\alpha\nu\varepsilon_m^{\frac{\alpha}{4}}\right)^2.$$

Invoking (18), we obtain

$$(E'(t))^2 \leqslant \frac{1}{4}\left(-\frac{1}{p\nu}H' + \alpha E(t)\nu\right)^2.$$

Minimizing the right-hand side with respect to ν, we establish (26).

Further reasoning is similar to the case $\alpha > 2$.

Let us demonstrate that the estimate (28), determined for a bounded domain, is exact. Let us use an inequality of the Steklov-Friedrichs type

$$\|\varphi\|_p \le C\|\nabla\varphi\|_p, \quad \forall\ \varphi \in C_0^\infty(\Omega).$$

If $p > \alpha$, the inequalities

$$\|\varphi\|_\alpha \le C_1\|\varphi\|_p \le C_2\|\nabla\varphi\|_p \tag{31}$$

hold. Differentiating (14) or (22) with respect to T, by means of (31), written for u, we find that

$$\frac{\alpha-1}{\alpha}\frac{d}{dt}\|u(t)\|_\alpha^\alpha = -(A(u),u(t))_\Omega = -\|\nabla u(t)\|_p^p \le -C_2^{-1}\|u(t)\|_\alpha^p.$$

Solving the differential inequality, we obtain the estimate

$$||u(t)||_\alpha^\alpha \leqslant ||u(0)||_\alpha^\alpha(1+c(u_0)t)^{-\frac{\alpha}{p-\alpha}},$$

proving the precision of the inequality (28).

REFERENCES

1. P.A. Raviart *Sur la resolution de certaines equations paraboliques nonlineaires* // J. Funct. Anal. 5 (1970). P. 209–328.
2. J.L. Lions *Quelques methodes de resolution de problemes aux limites non lineaires.* Dunod, Paris, 1969.
3. J.L. Lions, E. Magenes *Problemes aux limites non homogenes et applications. Vol. I.* Paris: Dunod 1968.
4. J.L. Lions *Some methods of solution of nonlinear boundary-value problems.* Izdatel'stvo "Mir". Moskva. 1972. 587 p. In Russian.
5. A. Bamberger *Etude d'une equation doublement non lineaire* // J. Funct. Anal. 24 (1977). P. 148–155.

6. O. Grange, F. Mignot *Sur la resolution d'une equation et d'une inequation paraboliques non lineaires* // J. Funct. Anal. 11 (1972). P. 77–92.
7. H.W. Alt, S. Luckhaus*Quasilinear elliptic-parabolic differential equations* // Math. Z. 1983. 183. P. 311–341.
8. A. Bamberger *Etude d'une equation doublement non lineaire.* Rapport Interne No. 4 du Centre de Mathematiques Appliquees de l'Ecole Polytechnique, Palaiseau 1-34 (1975)
9. F. Bernis *Qualitative properties for some nonlinear higher order degenerate parabolic equations.* IMA Preprint 184, University of Minnesota, 1985.
10. F. Bernis *Existence results for doubly nonlinear higher order parabolic equations on unbounded domains* // Math. Ann. 1988. 279, P. 373-394.
11. H. Brezis *On some degenerate nonlinear parabolic equations* // Proc. Symp. Pure Math. 1970. 18, P. 28–38.
12. H. Brezis *Problemes unilateraux* // J. Math. Pures Appl. 1972. 51, P. 1-168.
13. H. Brezis *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* Math. Studies 5. Amsterdam: North-Holland. 1973.
14. H. Brezis *Analyse fonetionnelle.* Paris: Masson. 1983.
15. N. Alikakos, R. Rostamian *Gradient estimates for degenerate diffusion equation.* // Proc. Roy. Soc. Edinburgh. 1982. V. 91. No. 3–4. P. 335–346.
16. Kozhevnikova L.M. *Stabilization of solutions of pseudo-differential parabolic equations in unbounded domains* // Izv. Math. Ser. math. 2010. V. 74, No. 2. P. 109-–130.
17. Karimov R.Kh., Kozhevnikova L.M. *Stabilization of solutions of quasilinear second order parabolic equations in domains with non-compact boundaries* // Sb. Math. 2010. V. 201, No. 9. P. 1249—1271 .
18. Tedeev A.F. *Estimates for the rate of stabilization $t \to \infty$ of the solution of the second mixed problem for a second-order quasilinear parabolic equation* // Differentsial'nye Uravneniya. 1991. V. 27, No 10. Pp. 1795–1806. In Russian.
19. Tedeev A.F. *Stabilization of solutions of initial boundary-value problems for quasilinear parabolic equations* // Ukr. mat. jurn. 1992 V. 44 No. 10. Pp. 1441–1450. In Russian.

Elina Radikovna Andriyanova,
Ufa State Aviation Technical University,
K. Marx Str., 12,
450000, Ufa, Russia
E-mail: Elina.Andriyanov@mail.ru

Mukminov Farit Khamzaevich
Ufa State Aviation Technical University,
K. Marx Str., 12,
450000, Ufa, Russia
E-mail: mfkh@rambler.ru

Translated from Russian by E.D. Avdonina.