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THE CONVERGENCE DOMAIN FOR SERIES OF EXPONENTIAL MONOMIALS

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Abstract. Problems of convergence for exponential series of monomials are studied in the paper. Exponential series, Dirichlet's series and power series are particular cases of these series. The space of coefficients of exponential series of monomials converging in the given convex domain in a complex plane is described. The complete analogue of the Abel theorem for these series is formulated with a natural restriction. In particular, results on continuation of convergence of exponential series follow from this analogue. A complete analogue of the Cauchy-Hadamard theorem is obtained as well. It provides a formula for finding the convergence domain of these series by their coefficients. The obtained results include all earlier known results connected with the Abel and Cauchy-Hadamard theorems for exponential series, Dirichlet's series and power series as particular cases.

Keywords: exponential series, convex domain, analytic function.

1. Introduction

The paper is devoted to investigation of convergence of series of exponential monomials, i.e. series of the form

$$\sum_{k=1,n=0}^{\infty,m_k-1} d_{k,n} z^n \exp(\lambda_k z). \tag{1.1}$$

The problem of describing the space of coefficients of the converging series (1.1), the character of convergence of the series are investigated, the domain of their convergence is described, and the problem of continuation of convergence of the series (1.1) is studied.

Let $\Lambda = \{\lambda_k, m_k\}_{k=1}^{\infty}$ be a multiple sequence, where λ_k are complex numbers numbered with respect to nondecreasing moduli, $|\lambda_k| \to \infty$ when $k \to \infty$, and m_k are natural numbers. Certain known characteristics of the sequence Λ will be of use:

$$m(\Lambda) = \overline{\lim_{k\to\infty}} \frac{m_k}{|\lambda_k|}, \qquad \sigma(\Lambda) = \overline{\lim_{j\to\infty}} \frac{\ln j}{|\xi_j|},$$

where ξ_k is a non-decreasing in modulus sequence, composed of points λ_k, and every λ_k occurs in it m_k times exactly.

The subject related to series of exponential monomials and their particular cases, namely series of exponents (i.e. series of the form (1.1), where $m_k = 1, k = 1, 2, \ldots$), the Dirichlet series (i.e. series of the form (1.1), where $m_k = 1$ and λ_k are positive integers) and the Taylor series has a rich history. Their investigation originates from works of Taylor, Cauchy, Hadamard, Abel, and Dirichlet. The above problems for such series were studied by E. Hille [3], G.L. Lunts [4,5], A.F. Leont'ev [1,2], A.V. Bratishchev [6] and other mathematicians. Series of exponential monomials are a natural generalization of the series of exponents. The theory of the latter is presented completely enough in the monograph [1] by A.F. Leont'ev. The fundamental result of

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the theory of series of exponentials has become classical and belongs to A.F. Leont'ev as well. He managed to prove that any function, analytical in a convex domain $D \subset \mathbb{C}$, can be expanded into a series of exponentials with fixed coefficients $\lambda_1, \lambda_2, \ldots$, provided that certain conditions are imposed on the coefficients. As it is known, exponents (and they only) are eigenfunctions of a differential operator. Therefore, the problem of representation by exponential series can be considered as a problem of expansion in terms of eigenfunctions of an operator.

Denote the space of functions, analytical in the domain D, having the topology of a uniform convergence on compact subsets D, by $H(D)$. Since there is a sufficiently large supply of eigenfunctions of a differential operator in $H(D)$ (all exponents, to be more exact), there is a variety of sets of coefficients $\lambda_1, \lambda_2, \ldots$, that allow one to obtain the representation of all functions from $H(D)$ in terms of exponential series. If we turn from the whole space $H(D)$ to its closed subspace W, which is invariant with respect to the differential operator (e.g. the space of solutions of a homogeneous convolution equation or a system of such equations), it appears that as a rule, eigenfunctions of the operator by themselves (in this case, there is only a countable set of eigenfunctions) are not sufficient to expand all functions from W. However, the situation changes if together with eigenfunctions of the differential operator in W, adjoint functions are considered. The latter functions are exponential monomials

$$z^n \exp(\lambda_k z), \qquad n = 1, 2, \ldots, m_k - 1,$$

where m_k is the multiplicity of the eigenvalue λ_k. The problem of expanding functions from a closed subspace $W \subset H(D)$, invariant with respect to the differential operator, in terms of eigen and adjoint functions of the differential operator is termed as the fundamental principle problem. Such term is due to the fact that in a particular case, when the invariant subspace is a space of solutions to a linear homogeneous differential equation with constant coefficients, the possibility to expand an arbitrary solution in terms of eigen and adjoint functions of the differential operator is called the fundamental L. Euler principle. Thus, issues related to the behaviour of the series of the form (1.1) become of importance. As well as in the theory of exponential series, the priorities for such series, particularly for the power Dirichlet series, are the problems of describing classes of convergence domains, including the problem on continuation of convergence, and the character of the series convergence, as well as recovery of the convergence domain by coefficients of the series. In the theory of power series, the first two problems are solved by means of the Abel theorem (it is more often called the Abel lemma), and the last one is solved by means of the Cauchy-Hadamard theorem. For the Dirichlet series, there is an analogue of the Abel theorem (see, e.g., [2], Chapter 2, Lemma 1.1), that claims that the convergence of the Dirichlet series

$$\sum_{k=1}^{\infty} d_k \exp(\lambda_k z)$$

at one point z_0 entails its convergence in the half-plane $\{z \in \mathbb{C} : Re\ z < Re\ z_0\}$.
If the value $\sigma(\Lambda)$ vanishes, then (see [2], Chapter 2, Theorem 1.1) the convergence is absolute and uniform in any half-plane $\{z \in \mathbb{C} : Re\ z < Re\ z_0 - \varepsilon\}$, where $\varepsilon > 0$.

Moreover, for the Dirichlet series, there is a complete analogue of the Cauchy-Hadamard theorem, where the distance from the origin of coordinates to the boundary line of the half-plane of convergence is calculated provided that $\sigma(\Lambda) = 0$ (see [2], Chapter 2, Theorem 1.2). In case of exponential series, the complete analogue of the Abel theorem does not exist. There is a result (see [3], [2], Chapter 2, Theorem 2.1) on the convex character of the set of points of absolute convergence of an exponential series. The series converges uniformly on compact subsets of the interior of this set (see [2], Chapter 2, Theorem 2.2). If $\sigma(\Lambda) = 0$, then the simple and absolute convergence of the exponential series in a convex domain are equivalent (see [2], Chapter 2, Theorem 2.3). Moreover, there is an analogue of the Cauchy-Hadamard theorem for exponential series (see [3], [4], [5] and [1], Theorem 3.1.3). It describes the domain of convergence of an exponential series that results as an intersection of a family of half-planes. Moreover, a formula for determining the distance from the origin of coordinates to the boundary lines of these half-planes is given. Only results from [6] can be mentioned in case of general series of the form (1.1). They prove that the domain of absolute convergence of the series (1.1) is convex provided that $m(\Lambda) = 0$.

The present paper presents a complete analogue of the Abel theorem for series of exponential monomials, and particularly, for exponential series provided that the conditions $\sigma(\Lambda) = m(\Lambda) = 0$ are satisfied. It is demonstrated that the domain of convergence of the series (1.1) is a convex domain of a special form. It is proved that the point-to-point convergence of the series (1.1) in the domain is equivalent to its absolute convergence, uniform convergence on compacts and even to convergence in a stronger topology. An analogue of the Cauchy-Hadamard theorem theorem is presented and it contains all the previous similar results for the Dirichlet series and series of exponents as particular cases.

2. The space of coefficients of converging series

Let D be a convex domain in $\mathbb{C}$. Let us describe the space of sequences of coefficients $\{d_{k,n}\}_{k=1,n=0}^{\infty,m_k-1}$, that provide convergence of the series (1.1) in the domain. Denote by $K(D) = \{K_p\}_{p=1}^{\infty}$ the sequence of convex compacts in the domain D, exhausting it strictly, i.e. $K_p \subset int K_{p+1}, p = 1, 2, \dots$ and $D = \bigcup_{p=1}^{\infty} K_p$. Here, the symbol *int* indicates the interior of the set. For every $p = 1, 2, \dots,$ introduce a Banach space of sequences of complex numbers

$$Q_p = \{d = \{d_{k,n}\} : ||d||_p = \sup_{k,n} |d_{k,n}| \exp H_{K_p}(\lambda_k) < \infty\},$$

where $K_p \in K(D)$ and

$$H_M(\xi) = \sup_{z\in M} Re(z\xi)$$

is the support function of the set $M \subset \mathbb{C}$ (to be exact, of the set complex conjugate to M). Let $Q(D) = \bigcap_p Q_p$. Determine the metric

$$\rho(d, d') = \sum_{p=1}^{\infty} 2^{-p} \frac{||d - d'||_p}{1 + ||d - d'||_p}$$

in the space $Q(D)$.

Obviously, $Q(D)$ becomes the Frechet space with this metric. Note that due to the injection $K_p \subset int K_{p+1}$ and the definition of the support function, there is a positive number α_p for every $p = 1, 2, \dots$ such that

$$H_{K_p}(\xi) + \alpha_p|\xi| \leqslant H_{K_{p+1}}(\xi), \quad \forall \xi \in \mathbb{C}. \tag{2.1}$$

Hence, the inequalities

$$||d||_1 \leqslant ||d||_2 \leqslant \dots \leqslant ||d||_p \leqslant \dots$$

hold for every element $d \in Q(D)$.

Let us demonstrate that the space $Q(D)$ coincides with the space of coefficients of series of the form (1.1) converging in the domain D. But first, let us prove some auxiliary statements.

Lemma 2.1. *The series*

$$\sum_{k=1}^{\infty} m_k \exp(-\varepsilon|\lambda_k|) \tag{2.2}$$

converges for any $\varepsilon > 0$ *if and only if* $\sigma(\Lambda) = 0$.

Proof. Let us assume that $\sigma(\Lambda) = 0$, and $\{\xi_j\}$ is a nondecreasing in module sequence composed of points λ_k, and every λ_k occurs in it m_k times exactly. Then, for every $\delta > 0$, there is a number $N(\delta)$ such that

$$\ln j < \delta|\xi_j|, \quad j \geq N(\delta).$$

Fix $\varepsilon > 0$ and select $\delta < \varepsilon$. One has

$$\sum_{j=N(\delta)}^{\infty} \exp(-\varepsilon|\xi_j|) < \sum_{m=N(\delta)}^{\infty} \exp(-\frac{\varepsilon lnj}{\delta}) = \sum_{m=N(\delta)}^{\infty} \frac{1}{j^{\frac{\varepsilon}{\delta}}} < \infty.$$

Hence, the series (2.2) converges for any $\varepsilon > 0$. Let us demonstrate the converse. Let the latter statement be true. Since terms of the series (2.2) are positive, their permutation does not affect the convergence of the series. Therefore, one can suppose that λ_k are numbered with respect to increasing moduli, i.e. $|\lambda_1| \leqslant |\lambda_2| \leqslant \ldots$. Moreover, if the sequence $\{\lambda_k\}_{k=1}^{\infty}$ is bounded, then the series (2.2) diverge. Hence, $|\lambda_k| \to \infty$, when $k \to \infty$. Let us prove the remaining part by contradiction. Assume that $\sigma(\lambda) = 4c > 0$. Then, there is a sequence of natural numbers $\{j(l)\}_{j=1}^{\infty}$ such that

$$lnj(l) \geq 2c|\xi_{j(l)}|, \quad l = 1, 2, \ldots.$$

Turning once more to the subsequence, one can assume that

$$2|\xi_{j(l)}| \leqslant |\xi_{j(l+1)}|, \quad l = 1, 2, \ldots.$$

Let us compose a new subsequence of natural numbers

$$j(l, s), \quad l = 1, 2, \ldots, \quad s = 1, 2, \ldots, l'(l),$$

where $l'(l)$ is an integer part of the number $j(l)/2$. Suppose that

$$j(l, s) = j(l) - l'(l) + s.$$

Since the moduli ξ_j are nondecreasing, one has

$$\frac{lnj(l, s)}{|\xi_{j(l,s)}|} \geq \frac{lnj(l, s)}{|\xi_{j(l)}|} \geq \frac{lnj(l, 1)}{|\xi_{j(l)}|} \geq \frac{lnj(l) - ln2}{|\xi_{j(l)}|} \geq 2c - \frac{ln2}{|\xi_{j(l)}|}.$$

Since $|\xi_j| \to \infty$, there is a number l_0 such that

$$\frac{lnj(l, s)}{|\xi_{j(l,s)}|} \geq c, \quad l \geq l_0, \quad s = 1, 2, \ldots, l'(l).$$

Whence, for all $l \geq l_0$ and $\varepsilon = c$, one obtains

$$\sum_{j=j(l)-l'(l)+1}^{j(l)} \exp(-\varepsilon|\xi_j|) = \sum_{s=1}^{l'(l)} \exp(-\varepsilon|\xi_{j(l,s)}|) \geq \sum_{s=1}^{l'(l)} \exp(\frac{-\varepsilon}{c} \ln j(l, s)) =$$

$$= \sum_{s=1}^{l'(l)} \frac{1}{j(l, s)^{\frac{\varepsilon}{c}}} = \sum_{l=1}^{l'(l)} \frac{1}{j(l, s)} \geq \frac{l'(l)}{j(l)} \geq \frac{2^{-1}j(l) - 1}{j(l)}.$$

Since $j(l) \to \infty$, when $l \to \infty$, this contradicts the convergence of the series (2.2) when $\varepsilon = c$. Thus, $\sigma(\Lambda) = 0$ and the lemma is proved.

Denote by $\mathbb{S}$ a unit circle with the centre at the origin of coordinates. Let E be a set in $\mathbb{C}$, Θ be a closed subset of the circle $\mathbb{S}$. A convex hull Θ of E is the set

$$E(\Theta) = \{z \in \mathbb{C} : Re(z\xi) < H_E(\xi), \xi \in \Theta\}.$$

Note that the interior E lies in $E(\Theta)$. Indeed, if z is an interior point of E then, the definition of the support function provides the inequalities $Re(z\xi) < H_E(\xi)$, $\forall \xi \in \Theta$. It means that $z \in E(\Theta)$. In a particular case, when $\Theta = \mathbb{S}$, the Θ – convex hull of the set coincides with its ordinary convex hull (to be more exact with the interior of the convex hull) and thus, is a convex domain. The latter takes place in the general case as well. This is verified by the following statement.

Lemma 2.2. *Let E be a set in $\mathbb{C}$, Θ be a closed subset of the circle $\mathbb{S}$. Then, the set $E(\Theta)$ is a convex domain.*

Proof. By definition, the set $E(\Theta)$ is an intersection of half-planes and therefore, it is convex. Convexity entails connectivity $E(\Theta)$. It remains to demonstrate that $E(\Theta)$ is an open set. Suppose this is not true. Then, there is a point $z_0 \in E(\Theta)$ and a sequence $\{z_k\}$ such that $z_k \to z_0$ if $k \to \infty$ and $z_k \notin E(\Theta)$ for all $k \geq 1$, i.e. $Re(z_k\xi_k) \geq H_E(\xi_k)$ for some $\xi_k \in \Theta$, $k = 1, 2, \ldots$. Turning to the subsequence, one can believe that $\{\xi_k\}$ converges to the point $\xi_0 \in \Theta$. Then, the latter inequality provides

$$Re(z_0\xi_0) = \lim_{k\to\infty} Re(z_k\xi_k) \geq \underline{\lim}_{k\to\infty} Re(z_k\xi_k) \geq \underline{\lim}_{k\to\infty} H_E(\xi_k) \geq H_E(\xi_0)$$

in view of the lower semicontinuity of the support function (see [7]).
We have arrived to a contradiction with the definition of $E(\Theta)$, because $z_0 \in E(\Theta)$, and $\xi_0 \in \Theta$. The lemma is proved.

Let $\Lambda = \{\lambda_k, m_k\}_{k=1}^{\infty}$. Denote by $\Theta(\Lambda)$ the set of all partial limits of the sequence $\{\lambda_k/|\lambda_k|\}_{k=1}^{\infty}$ (except for the point $\lambda_k = 0$, if it exists). Manifestly, $\Theta(\Lambda)$ is a closed subset of the circle $\mathbb{S}$.

Lemma 2.3. *Let the sequence Λ be such that $m(\Lambda) = 0$. Assume that the common term of the series (1.1) is bounded on the set $E \subset \mathbb{C}$, i.e.*

$$|d_{k,n}z^n \exp(\lambda_k z)| \leqslant A(z), \quad k = 1, 2, \ldots, \quad n = 0, 1, \ldots, m_k - 1, \quad z \in E.$$

Moreover, if $0 \in E$, the sequence $\{d_{k,n}\}_{k=1,n=0}^{\infty,m_k-1}$ is bounded as well.

Then, there is an injection $d = \{d_{k,n}\} \in Q(D)$, where $D = E(\Theta(\Lambda))$.

Proof. Suppose that $d \notin Q(D)$. Then, $d \notin Q_p$ for some number $p = 1, 2, \ldots$. It means that there is a subsequence $\{d_{k_l,n_l}\}$ such that

$$|d_{k_l,n_l}| \exp H_{K_p}(\lambda_{k_l}) \to +\infty, \quad p \to \infty. \tag{2.3}$$

Turning to the subsequence once more, one can consider that $\{\lambda_{k_l}/|\lambda_{k_l}|\}$ converges to a point $x_0 \in \Theta(\Lambda)$. Since K_{p+1} is a compact in the domain $D = E(\Theta(\Lambda))$, the definition of the set $E(\Theta(\Lambda))$ and of the support function entails that the estimate $Re(z_0x_0) > H_{K_{p+1}}(x_0)$ is true for some $z_0 \in E$.
Then, in view of (2.1) and the continuity of the support function of the compact (see [7]), there is $\delta > 0$ such that

$$Re(z_0x_0) > H_{K_{p+1}}(x) \geq H_{K_p}(x) + \alpha_p|x|, \quad x \in B(x_0, \delta). \tag{2.4}$$

Select a number l_0 such that

$$\lambda_{k_l}/|\lambda_{k_l}| \in B(x_0, \delta), \quad l \geq l_0.$$

First, let us assume that $z_0 \neq 0$. By condition, $m(\Lambda) = 0$. Hence, by virtue of the definition of the quantity $m(\Lambda)$, for every $\varepsilon > 0$ there is $l_1 \geq l_0$ such that $m_{k_l} \leqslant \varepsilon|\lambda_{k_l}|$ for all $l \geq l_1$. Let us fix $\varepsilon > 0$ such that $\varepsilon \ln|z_0| > -\alpha_p$. Then, in view of (1.4) and the positive homogeneity of the support function for all $l \geq l_1$, one obtains

$$|z_0^{n_l} \exp(\lambda_{k_l} z_0)| = \exp(n_o l \ln|z_0| + Re(\lambda_{k_l} z_0)) >$$

$$> \exp(-n_l \alpha_p \varepsilon^{-1} + Re(\lambda_{k_l} z_0)) > \exp(-m_{k_l} \alpha_p \varepsilon^{-1} + Re(\lambda_{k_l} z_0)) \geq$$

$$\geq \exp(-\alpha_p|\lambda_{k_l}| + H_{K_p}(\lambda_{k_l}) + \alpha_p|\lambda_{k_l}|) = \exp H_{K_p}(\lambda_{k_l}).$$

Thus, due to (2.3), one has

$$|d_{k_l,n_l} z_0^{n_l} \exp(\lambda_{k_l} z_0)| \to +\infty, \quad p \to \infty.$$

This contradicts the condition of the lemma.

Now, let $z_0 = 0$. Then, invoking (2.3) and (2.4) for all $l \geq l_0$, one obtains

$$|d_{k_l,n_l}| = |d_{k_l,n_l} \exp(\lambda_{k_l} z_0)| \geq |d_{k_l,n_l}| \exp H_{K_p}(\lambda_{k_l}) \to +\infty, \quad p \to \infty.$$

This contradicts the condition of the lemma as well. Thus, $d \in Q(D)$. The lemma is proved.

Remark. If $0 \in E$, an additional condition of boundedness of the sequence of coefficients $\{d_{k,n}\}$ is imposed on the lemma. It is important if 0 is an isolated point of the set E. As an example, consider the series

$$\sum_{k=1}^{\infty} \exp(2k) z \exp(kz).$$

Here $\Theta(\Lambda) = \{1\}$. Let us take the set consisting of the two points $\{-2, 0\}$ as E. Then, $E(\Theta(\Lambda))$ coincides with the half-plane $Re z < 0$, and the common term of the series is bounded on E. However, the series does not converge in the half-plane (it diverges on the circle $\mathbb{S}$). It converges in the half-plane $Re z < -2$, which coincides with the set $E'(\Theta(\Lambda))$, where $E' = \{-2\}$. In this case, the condition of boundedness of the coefficients is violated (the remaining conditions are satisfied), and the lemma becomes untrue. If $0 \in E$ is not an isolated point of E, the lemma holds true even without the condition of boundedness of the coefficients. Indeed, under the above assumptions, the point 0 lies in the closure of the set $E' = E \setminus \{0\}$. It remains to mention that in this case, the domains $E(\Theta(\Lambda))$ and $E'(\Theta(\Lambda))$ coincide (because the support functions of the sets E and E' coincide obviously).

Assume that

$$c_{p,k,n} = \sup_{z \in K_p} |z^n \exp(z\lambda_k)|, \quad p, k = 1, 2, \dots, \quad n = 0, 1, \dots, m_k - 1.$$

Lemma 2.4. *Let the sequence Λ be such that $m(\Lambda) = 0$. Then, for any number p, there is a constant $C > 0$ such that the inequalities $c_{p,k,n} \leqslant C \exp H_{K_{p+1}}(\lambda_k)$, $\forall k = 1, 2, \dots$, $\forall n =$* $0, 1, \dots, m_k - 1$ hold.

Proof. Let us assume that the statement of the lemma is not true. Then, for some number p, there is a subsequence $\{k_l, n_l\}$ such that

$$c_{p,k_l,n_l} > l \exp H_{K_{p+1}}(\lambda_{k_l}), \quad l = 1, 2, \dots \tag{2.5}$$

Denote by z_0, a point of the compact K_p with a maximum modulus. One can assume that $K_p \neq \{0\}$. Then, $z_0 \neq 0$ and one has

$$c_{p,k_l,n_l} \leqslant |z_0|^{n_l} \exp H_{K_p}(\lambda_{k_l}) = \exp(n_l \ln|z_0| + H_{K_p}(\lambda_{k_l})) \leqslant$$

$$\leqslant \exp(m_{k_l}|\ln|z_0|| + H_{K_p}(\lambda_{k_l})).$$

By condition, $m(\Lambda) = 0$. Hence, for every $\varepsilon > 0$, there is a number l_0 such that $m_{k_l} \leqslant \varepsilon|\lambda_{k_l}|$ for all $l \geq l_0$. Let us fix $\varepsilon > 0$ such that $\varepsilon|\ln|z_0|| < \alpha_p$. Then, according to the above, and in view of (2.1), one obtains

$$c_{p,k_l,n_l} \leqslant \exp(\varepsilon|\lambda_{k_l}|\ln|z_0| + H_{K_p}(\lambda_{k_l})) \leqslant$$

$$\leqslant \exp(\alpha_p|\lambda_{k_l}| + H_{K_p}(\lambda_{k_l})) \leqslant \exp H_{K_{p+1}}(\lambda_{k_l}), \quad l \geq l_0.$$

This contradicts (2.5). The lemma is proved.

Lemma 2.5. *Let us assume that D is a convex domain in $\mathbb{C}$, and the sequence Λ is such that $\sigma(\Lambda) = m(\Lambda) = 0$ and $d = \{d_{k,n}\} \in Q(D)$. Then, for every $p = 1, 2, ...$, there is a number $C_p > 0$ (independent of the sequence d) such that*

$$\sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}|c_{p,k,n} \leqslant C_p||d||_{p+2}.$$

Proof. Let $d = \{d_{k,n}\} \in Q(D)$. By virtue of Lemma 2.4, in view of (2.1) and the definition of a norm in the space Q_{p+2}, one obtains

$$\sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}|c_{p,k,n} \leqslant C \sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}| \exp H_{K_{p+1}}(\lambda_k) \leqslant$$

$$\leqslant C \sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}| \exp(H_{K_{p+2}}(\lambda_k) + H_{K_{p+1}}(\lambda_k) - H_{K_{p+2}}(\lambda_k)) \leqslant$$

$$C||d||_{p+2} \sum_{k=1}^{\infty} m_k \exp(-\alpha_{p+1}|\lambda_k|).$$

By condition, $\sigma(\Lambda) = 0$. Hence, the latter series converges according to Lemma 2.1. This provides the necessary inequality with a constant $C_p > 0$, independent of $d = \{d_{k,n}\}$. The lemma is proved.

The following theorem describes the space of sequences of coefficients of series of exponential monomials converging in the convex domain $D \subset \mathbb{C}$.

Theorem 2.6. *Let D be a convex domain in $\mathbb{C}$ and the sequence Λ be such that $\sigma(\Lambda) = m(\Lambda) = 0$. Then, the following statements are equivalent.*
1) The series (1.1) converges in the domain D.
2) There exists an injection $d = \{d_{k,n}\} \in Q(D)$.

Proof. Let us assume that 1) holds. Then, the common term of the series (1.1) is bounded at every point of the domain D. Hence, according to Lemma 2.3 and in view of the remark to it, the sequence $d = \{d_{k,n}\}$ belongs to the space $Q(D(\Theta(\Lambda)))$. Since the domain D lies in $D(\Theta(\Lambda))$, the definition of the space $Q(D)$ readily provides the injection $Q(D(\Theta(\Lambda))) \subset Q(D)$. Thus, $d \in Q(D)$.

Assume now that 2) holds. Then, it follows from the inequality in Lemma 2.5 that the series (1.1) converges on any compact of the domain D and hence, in the domain D itself. The theorem is proved.

Remark. Lemma 2.5 has been used while proving the implication 2)→1). According to it, the injection $d = \{d_{k,n}\} \in Q(D)$ entails not only point-to-point convergence of the series (1.1), but also its absolute and uniform convergence on compacts in the domain D (and even convergence in a stronger topology). Keeping in mind the implication 1)→2), one can claim that the simple convergence of the series (1.1) in the domain D is equivalent to its absolute and uniform convergence on compacts of the domain.

3. An analogue of the Abel theorem for series of exponential monomials

The following result is an analogue of the Abel theorem for the series (1.1).

Theorem 3.1. *Let the sequence Λ be such that $\sigma(\Lambda) = m(\Lambda) = 0$. Suppose that the common term of the series (1.1) is bounded on the set $E \subset \mathbb{C}$. Moreover, if the origin of coordinates is an isolated point of the set E, the sequence $\{d_{k,n}\}_{k=1,n=0}^{\infty,m_k-1}$ is bounded as well. Then, for every number $p = 1, 2, \ldots$, there is a number $C_p > 0$ (independent of the sequence d) such that*

$$\sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}| c_{p,k,n} \leqslant C_p ||d||_{p+2},$$

where the numbers $c_{p,k,n}$ and the norms $||d||_p$ are constructed according to the sequence of compacts $K(D)$ and $D = E(\Theta(\Lambda))$. In particular, the series (1.1) converges absolutely and uniformly on any compact from the domain D.

Proof. Let us assume that the conditions of the theorem hold. Then, according to lemma 2.3 and the remark to it, there is an injection $d = \{d_{k,n}\} \in Q(D)$. Whence, according to Lemma 2.5, there is a number $C_p > 0$ (independent of the sequence d) for every $p = 1, 2, \ldots$ such that

$$\sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}| c_{p,k,n} \leqslant C_p ||d||_{p+2}.$$

In particular, it means that the series (1.1) converges absolutely and uniformly on any compact from the domain D. The theorem is proved.

Remark. 1. Theorem 3.1 entails that under the condition $\sigma(\Lambda) = m(\Lambda) = 0$, the interior of the set of convergence of the series (1.1) is always a convex and even a Θ — convex domain (i.e. a domain which is an intersection of half-planes $\{z : Re(z\xi) < h(\xi), \xi \in \Theta\}$), h is a lower semicontinuous function.

2. If we exclude the condition $\sigma(\Lambda) = 0$ from Theorem 3.1, the latter becomes untrue. The book [2, Chapter 2] contains an example of the Dirichlet series for which $\sigma(\Lambda) > 0$. The series converges in the half-plane (and hence, its common term is bounded in the half-plane), but diverges absolutely at every point of the plane.

3. The condition $m(\Lambda) = 0$ is also essential. Indeed, let the sequence $\Lambda = \{k, m_k\}$ be such that $m(\Lambda) = \tau > 0$, and $\sigma(\Lambda) = 0$ (for example, $m_k = 2k$). Consider the series

$$\sum_{k=1}^{\infty} \exp(2k) z^{m_k-1} \exp(kz).$$

One can readily demonstrate that the series converges in a domain lying in the half-plane $Rez < -a$, where $a > 1$ is selected from the condition $a > 2(\tau \ln a + 1)$, and in the circle $B(0, r)$, where $r \in (0, 1)$ is such that $-2^{-1}\tau \ln r > 3$. Meanwhile, it obviously diverges on the circle $\mathbb{S}$. Thus, the interior of the set of convergence of the given series is not a convex domain and even not a domain at all (it is not connected).

As it was mentioned, Theorem 3.1 is an analogue of the Abel theorem for power series. Indeed, Theorem 3.1, as well as the latter theorem, proves that the boundedness of the common term of the series at some boundary points of the domain entails its absolute and uniform convergence inside the domain. A power series is a particular case of an exponential series: by means of a simple transformation of the variable, a power series turn into a series of the form $\sum d_k \exp(kz)$. However, reformulating Theorem 3.1 for the particular case, one obtains a weaker statement than the Abel theorem. This is explained by the fact that the circles, where the power series should converge absolutely and uniformly, turn into unbounded sets under the above transformation. However, the uniform convergence in Theorem 3.1 is guaranteed only on compact subsets. Complicating the proof of this theorem significantly, one can demonstrate that the series (1.1) converges all the same uniformly in some cases on unbounded sets as well. However, this sets will not always contain images of circles during transformation of the variable that turns a power series into an exponential series. The following example clarifies the above. Consider the series

$$\sum_{k=1}^{\infty}(\exp(kz) + z\exp(kz)). \tag{3.1}$$

The set $\Theta(\Lambda)$ in this case is a singleton: $\Theta(\Lambda) = \{1\}$. Coefficients of the series are equal to one and are bounded therefore. Hence, according to Theorem 3.1, the series (3.1) converges in the domain $E(\Theta(\Lambda))$, where $E = \{0\}$, which coincides with the left half-plane, and is uniform on its compacts. One can demonstrate that the series (3.1) converges uniformly on some unbounded sets as well (for example, on angles strictly smaller than π and with the vertexes belonging to a negative real semiaxis). However, it does not converge uniformly in any half-plane of the form $\Pi(a) = \{z : Re z < -a\}, \quad a > 0$.
Consider the series

$$\sum_{k=1}^{\infty}\exp(kz). \tag{3.2}$$

It is derived from the power series w^k by means of the transformation $w = \exp z$. The latter series converges in the circle $B(0, 1)$ and according to the Abel theorem, it converges uniformly in any circle of a smaller radius. The circles transfer into half-planes $\Pi(a)$ under the above transformation. Hence, the series (3.2) converges uniformly at every half-plane. Such difference in sets of uniform convergence of the series (3.1) and (3.2) is connected with the presence of the multipliers z in the series (3.1). As one can see from this example, it is impossible to prove a theorem of the type 3.1 preserving such multiplyers so that its particular case was the Abel theorem for power series. However, the situation can be adjusted by rejecting the cofactors z^n in the series (1.1), i.e. considering only "pure"exponential series, which is verified by the following result.

Together with $E(\Theta)$, define a set

$$E(\Theta, \varepsilon) = \{z \in \mathbb{C} : Re(z\xi) < H_E(\xi) - \varepsilon, \forall\xi \in \Theta\}$$

for every $\varepsilon > 0$.
Note that if Θ lies at a corner with the vertex at zero and of an angle not greater than π, the set $E(\Theta)$, and $E(\Theta, \varepsilon)$ as well, is unbounded for a sufficiently small $\varepsilon > 0$.

Theorem 3.2. *Let us assume that terms of the series*

$$\sum_{k=1}^{\infty} d_k \exp(\lambda_k z) \tag{3.3}$$

are uniformly bounded on the set E, that is

$$|d_k \exp(\lambda_k z)| \leqslant A, \quad k = 1, 2, \dots, \quad z \in E.$$

Furthermore, let $\sigma(\Lambda) = 0$ and the closed set $\Theta \subset \mathbb{S}$ be such that the injection

$$\lambda_k/|\lambda_k| \in \Theta, \quad k \geq k_0$$

holds true for some number k_0. *Then, there is* $c(\varepsilon, \Lambda) > 0$ *for every* ε *such that the inequality*

$$\sum_{k=k_0}^{\infty} |d_k \exp(\lambda_k z)| \leqslant Ac(\varepsilon, \Lambda), \quad z \in E(\Theta, \varepsilon)$$

holds.

In particular, the series (3.3) converges absolutely and uniformly on $E(\Theta(\Lambda), \varepsilon)$.

Proof. Let $\varepsilon > 0$ and $z \in E(\Theta, \varepsilon)$. Since $\xi_k = \lambda_k/|\lambda_k| \in \Theta$ for all $k \geq k_0$ then, by definition of $E(\Theta, \varepsilon)$, one has the estimate

$$\sum_{k=k_0}^{\infty} |d_k \exp(\lambda_k z)| = \sum_{k=k_0}^{\infty} |d_k \exp(|\lambda_k|(\xi_k z))| =$$

$$= \sum_{k=k_0}^{\infty} |d_k| \exp(|\lambda_k| Re(\xi_k z)) \leqslant \sum_{k=k_0}^{\infty} |d_k| \exp(|\lambda_k|(H_E(\xi_k) - \varepsilon)).$$

Further, by virtue of the definition of the support function, find the point $z_k \in E$, $k \geq k_0$ such that

$$Re(z_k \xi_k) \geq H_E(\xi_k) - \varepsilon/2.$$

The above and the condition of the theorem provide

$$\sum_{k=k_0}^{\infty} |d_k \exp(\lambda_k z)| \leqslant \sum_{k=k_0}^{\infty} |d_k| \exp(|\lambda_k|(H_E(\xi_k) - \varepsilon)) \leqslant$$

$$\leqslant \sum_{k=k_0}^{\infty} |d_k| \exp(|\lambda_k|(Re(z_k \xi_k) - \varepsilon/2)) = \sum_{k=k_0}^{\infty} |d_k| \exp((Re(z_k \lambda_k) - \varepsilon|\lambda_k|/2)) =$$

$$= \sum_{k=k_0}^{\infty} |d_k \exp(|\lambda_k z_k)| \exp(-\varepsilon|\lambda_k|/2) \leqslant A \sum_{k=k_0}^{\infty} \exp(-\varepsilon|\lambda_k|/2).$$

Since $\sigma(\Lambda) = 0$, than the latter series converges according to Lemma 2.1 and one obtains the required inequality. The theorem is proved.

Remark. Consider an exponential series

$$\sum_{k=1}^{\infty} d_k \exp(kz) \tag{3.4}$$

into which the power series $\sum d_k w^k$ transfers under the transformation $w = \exp z$. In this case $\sigma(\Lambda) = 0$, and the injection $\lambda_k/|\lambda_k| \in \Theta = \{1\}$ is true for every number $k = 1, 2, \ldots$.

Let us assume that the common term of the series (3.4) is bounded at the point z_0 and that $E = \{z_0\}$. Then, according to Theorem 3.2, the series (3.4) converges absolutely ad uniformly on every set $E(\Theta, \varepsilon)$, $\varepsilon > 0$, coinciding with the half-plane $\{z : Rez < Rez_0 - \varepsilon\}$. This yields the Abel theorem for power series.

4. An analogue of the Cauchy-Hadamard theorem for series of exponential monomials

Let us represent a result which is an analogue of the Cauchy-Hadamard theorem for power series. This theorem contains a formula for calculating the radius of convergence of a power series. An analogue of the circle for exponential series is a half-plane, and an analogue of the radius of the circle is the distance from the origin of coordinates to the half-plane. If $\Theta(\Lambda)$ consists of two points, then the corresponding $\Theta(\Lambda)$-convex domain of convergence of the series (1.1) is an intersection of two half-planes. This domain has two "radii of convergence" — the distances from the origin of coordinates to two straight lines, which are the boundaries of these half-planes. If $\Theta(\Lambda)$ is an infinite set, then there are infinitely many corresponding "radii of convergence" of the series (1.1). Note that some distances should be taken with the minus sign. Such a situation occurs when the domain of convergence does not include the origin of coordinates. Consider, for example, the series $\sum\limits_{k=1}^{\infty} 2^k \exp(kz)$.
Applying the Abel theorem to the power series corresponding to the latter series, one readily verifies that the domain of its convergence is the half-plane $\{z : Rez < \ln(1/2)\}$. For the sake of clarity, the "radius of convergence" here is considered to be the quantity $\ln(1/2)$, equal to the distance from the origin of coordinates to the straight line bounding the half-plane, taken with the minus sign, and not the distance itself. Let us illustrate this. Consider another series $\sum\limits_{k=1}^{\infty} 2^{-k} \exp(kz)$.
Similarly to the first case, one finds out that the domain of convergence of the series is a half-plane $\{z : Rez < \ln 2\}$. Here the "radius of convergence" is already equal to $\ln 2$, i.e. to the distance from the origin of coordinates to the line bounding the half-plane.

Before formulating the above stated result, let us introduce some additional notation. Let $\xi \in \Theta(\Lambda)$. Assume that

$$h(d,\xi) = \inf \underline{\lim}_{j\to\infty} \min_{0\leqslant n\leqslant m_{k(j)}-1} \frac{\ln(1/|d_{k(j),n}|)}{|\lambda_{k(j)}|}$$

for a sequence of coefficients $d = \{d_{k,n}\}$ of the series (1.1). Here, the infimum is taken with respect to all subsequences $\{\lambda_{k(j)}\}$ of the sequence $\{\lambda_k\}$ such that $\lambda_{k(j)}/|\lambda_{k(j)}|$ converges to ξ, when $j \to \infty$. Thus, we have obtained the function $h(d,\xi)$, $\xi \in \Theta(\Lambda)$. One can readily deduce from its definition that it is lower semi-continuous. Then, similarly to Lemma 2.2, it is demonstrated that the set

$$D = D(d,\Lambda) = \{z : Re(z\xi) < h(d,\xi), \xi \in \Theta(\Lambda)\}$$

is a $\Theta(\Lambda)$-convex domain .

Theorem 4.1. *Let the sequence Λ be such that $\sigma(\Lambda) = m(\Lambda) = 0$. Then the series (1.1) converges at every point of the domain D and diverges at every point of its exterior $\mathbb{C} \setminus \overline{D}$ except for the origin of coordinates if the series $\sum d_{k,0}$ converges.*

Proof. Let $z \in D$. Choose a number p such that $z \in K_p$, where K_p is an element of the set $K(D)$. Then, according to Lemma 2.4, in view of (2.1), one obtains the estimate

$$\sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}||z|^n \exp(Re(z\lambda_k)) \leqslant C \sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}| \exp H_{K_{p+1}}(\lambda_k) =$$

$$= C \sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}| \exp(H_{K_{p+2}}(\lambda_k) + H_{K_{p+1}}(\lambda_k) - H_{K_{p+2}}(\lambda_k)) \leqslant$$

$$\leqslant \sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}| \exp(H_{K_{p+2}}(\lambda_k) - \alpha_{p+1}|\lambda_k|). \tag{4.1}$$

Let us demonstrate that

$$\overline{\lim}_{k\to\infty} \max_{0\leqslant n\leqslant m_k-1} |d_{k,n}| \exp(H_{K_{m+2}}(\lambda_k)) < +\infty. \tag{4.2}$$

Assume that this is not true. Then, for some subsequence $\{k(j), n(j)\}$, one has

$$\overline{\lim_{j\to\infty}} |d_{k(j),n(j)}| \exp H_{K_{p+2}}(\lambda_{k(j)}) = +\infty,$$

or

$$\overline{\lim_{j\to\infty}} (\ln |d_{k(j),n(j)}| + H_{K_{p+2}}(\lambda_{k(j)})) = +\infty,$$

which is equivalent.
Whence,

$$\overline{\lim_{j\to\infty}} |\lambda_{k(j)}|^{-1} (\ln |d_{k(j),n(j)}| + H_{K_{p+2}}(\lambda_{k(j)})) \geq 0. \tag{4.3}$$

Turning to the subsequence once more, one can consider that $\lambda_{k(j)}/|\lambda_{k(j)}|$ converges to a point $\xi \in \Theta(\Lambda)$. Then, in view of the continuity, positive homogeneity of the support function of the compact, and the definition of the quantity $h(d,\xi)$, one obtains

$$\overline{\lim_{j\to\infty}} |\lambda_{k(j)}|^{-1} (\ln |d_{k(j),n(j)}| + H_{K_{p+2}}(\lambda_{k(j)})) \leqslant$$

$$\leqslant \overline{\lim_{j\to\infty}} |\lambda_{k(j)}|^{-1} \ln |d_{k(j),n(j)}| + \overline{\lim_{j\to\infty}} |\lambda_{k(j)}|^{-1} |H_{K_{p+2}}(\lambda_{k(j)}) \leqslant$$

$$\leqslant \overline{\lim_{j\to\infty}} |\lambda_{k(j)}|^{-1} \ln |d_{k(j),n(j)}| + H_{K_{p+2}}(\xi) \leqslant -h(d,\xi) + H_{K_{p+2}}(\xi) < 0.$$

The latter estimate here follows from the fact that

$$H_{K_{p+2}}(\xi) < H_D(\xi)$$

(because K_{p+2} is a compact in the domain D) and

$$H_D(\xi) \leqslant h(d,\xi)$$

(by virtue of the definition of the domain $D = D(d,\Lambda)$ and the support function H_D). Thus, we have arrived to a contradiction with (4.3). Hence, (4.2) holds true. Therefore, according to (4.1), one has

$$\sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}||z|^n \exp Re(z\lambda_k) \leqslant C' \sum_{k=1,n=0}^{\infty,m_k-1} m_k \exp(-\alpha_{p+1}|\lambda_k|).$$

By condition, $\sigma(\Lambda) = 0$. Then, by virtue of Lemma 2.1, the latter series converges. It means that the series (1.1) converges at the point z.

Now, let $z \in \mathbb{C} \setminus \overline{D}$. If $z = 0$ and the series $\sum d_{k,0}$ converges, then the series (1.1) converges at the point $z = 0$.

Let $z \neq 0$. By definition of the domain D, there is $\xi \in \Theta(\Lambda)$ such that

$$Re(z\xi) > h(d,\xi). \tag{4.4}$$

According to the definition of the quantity $h(d,\xi)$, find the subsequence $\{k(j),n(j)\}$ such that $\lambda_{k(j)}/|\lambda_{k(j)}|$ converges to the point ξ and

$$h(d,\xi)=\lim_{j\to\infty}\frac{\ln(1/|d_{k(j),n(j)}|)}{|\lambda_{k(j)}|}\,. \tag{4.5}$$

Let us assume that the series (1.1) still converges at the point z. Then, the common term of the series (1.1) is bounded on the set $E=\{z\}\cup D$, and according to Lemma 2.3, the sequence of its coefficients d belongs to the space $Q(D')$, where D' is a $\Theta(\Lambda)$ — convex hull of the set E. While z is a boundary point in the domain D' (because it lies outside D). Hence, in view of (4.4), there is a point z' in the domain D' such that

$$Re(z'\xi)>h(d,\xi). \tag{4.6}$$

Let us choose a number p such that the compact $K_p'\in K(D')$ for it contains z'. As it was mentioned, $d\in Q(D')$. Therefore, according to definition of the space $Q(D')$, one has

$$|d_{k,n}|\leqslant B\exp(-H_{K_p'}(\lambda_k)),\quad k=1,2,\ldots,\quad n=0,1,\ldots,m_k-1,$$

where B is a positive constant. Since $z'\in K_p'$, then

$$Re(z'\lambda_k)\leqslant H_{K_p'}(\lambda_k),\quad k=1,2,\ldots.$$

In view of the latter and according to the above, one obtains

$$|d_{k,n}|\leqslant B\exp(-Re(z'\lambda_k)).$$

Whence and from (4.6), it follows that

$$\lim_{j\to\infty}\frac{\ln(1/|d_{k(j),n(j)}|)}{|\lambda_{k(j)}|}\geq\frac{-\ln B+Re(z'\lambda_{k(j)})}{|\lambda_{k(j)}|}=Re(z'\xi)>h(d,\xi).$$

This contradicts (4.5). The theorem is proved.

Remark. In a particular case, one has the formula

$$h(d,1)=\underline{\lim}_{k\to\infty}\frac{\ln(1/|d_k|)}{k}=\underline{\lim}_{k\to\infty}(-\ln\sqrt[k]{|d_k|})$$

for the series (3.4).

Carrying out the transformation $w=\exp z$, that turns the series (3.4) into a power series, one obtains the following formula for the radius of convergence of the latter

$$R=\exp h(d,1)=\underline{\lim}_{k\to\infty}\frac{1}{\sqrt[k]{|d_k|}}.$$

Thus, we have obtained the Cauchy-Hadamard formula for power series.

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