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ABOUT FILTERING PROBLEM OF DIFFUSION PROCESSES

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Abstract. The filtering problem of nonlinear one-dimensional diffusion processes is considered. The structures of observable and nonobservable processes are found. It is shown, that solution of the optimal filtering problem can be reduced to solution of the filtering problem for the case when a nonobservable process has a simpler structure and an observable process is the Wiener process with a random smooth trend. The equation connecting a conditional expectation for the initial filtering problem with an unnormalized filtering density for the reduced filtering problem is obtained.

Key words: diffusion processes, optimal filtering problem, filtering density

1. Introduction

Given a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with a flow of σ-algebras $\{F_t\}$, $t \in [0,T]$, and independent Wiener processes $W_1(t)$, $W_2(t)$, $t \in [0,T]$, consistent with the flow $\{F_t\}$. Consider a diffusion process $(x(t),\, y(t))$, satisfying the following system of stochastic differential equations:

$$x(t) = x_0 + \int_0^t b^1(s,x(s),y(s))\; ds + \int_0^t \sigma^1(s,x(s),y(s))\; dW_1(s) + \\ + \int_0^t \sigma^2(s,x(s),y(s))\; dW_2(s), \tag{1}$$

$$y(t) = y_0 + \int_0^t b^2(s,x(s),y(s))\; ds + \int_0^t \sigma^0(s,y(s))dW_2(s), \tag{2}$$

where integrals with respect to the Wiener processes are Ito stochastic integrals. The coefficients of the equations are supposed to be jointly continuous with respect to (t,x,y), locally Lipschitzian and satisfy conditions of a linear growth with respect to (x,y). These conditions guarantee the existence of a unique solution to the system (1)–(2). The function $\sigma^0(s,y)$ is also supposed to be separated from zero.

The process $y(t)$ is observable and the process $x(t)$ is not.

The filtering problem of diffusion processes consists in finding the conditional expectation $m_t = \mathbf{E}[f(x(t))|\,Y_t]$, where $Y_t = \sigma\{y(s), s \le t\}$ is the σ-algebra, generated by values of the process $y(s)$ when $s \in [0,t]$, $f(x)$ is the determinate function such that $\mathbf{E}|f(x(t))| < \infty$.

In works of Liptser R.Sh., Shiryaev A.N. [1], Kallianpur G. [2], Rozovskii B.L. [7] and many other researchers, the given problem (in a multi-dimensional case) has been reduced to the problem of finding an unnormalized filtering density, which is a solution to the stochastic partial differential equation. It is known [7], that the conditional expectation m_t can be calculated by the formula

$$m_t = \mathbf{E}[f(x(t))|\,Y_t] = \int_R f(x)\; \pi(t,x)\; dx, \tag{3}$$

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where $\pi(t,x) = \frac{1}{\int_R V(t,x)dx} V(t,x)$ and $V(t,x)$ are normalized, and unnormalized filtering densities, respectively. Here, the density $V(t,x)$ satisfies the linear Ito stochastic partial differential equation

$$\begin{aligned} V(t,x) - V(0,x) &= \\ &= \int_0^t \left\{ [a(s,x,y(s))\; V(s,x)]''_{xx} - \left[b^1(s,x,y(s))\; V(s,x)\right]'_x \right\} ds + \\ &+ \int_0^t \left\{ h(s,x,y(s))\; V(s,x) - \left[\sigma^2(s,x,y(s))\; V(s,x)\right]'_x \right\} d\widetilde{W}(s), \\ & \qquad V(0,x) = \pi(0,x). \quad (4) \end{aligned}$$

In the latter formula $a = \frac{1}{2}[(\sigma^1)^2 + (\sigma^2)^2]$, $h = \frac{b^2}{\sigma^0}$, $\pi(0,x)$ is a conditional density $x(0)$ with respect to Y_0, and $\widetilde{W}(t)$ is a Wiener process resulting from application of the Girsanov theorem aiming to "remove the trend" in the equation for the observed component (2):

$$y(t) = y_0 + \int_0^t \sigma^0(s,y(s))\; d\widetilde{W}(s). \quad (5)$$

The normalized density $\pi(t,x)$, in its turn, satisfies a stochastic differential equation which is already nonlinear.

Earlier, it has been possible to solve a stochastic differential equation (4) in a linear case only (the Kalman-Bucy filter). In other separate cases, attempts have been made to solve the problem by methods of statistical modelling, which is rather laborious and complicated. In [1], we reduced the problem to solving a pair of non-stochastic partial differential equations. Moreover, construction of the solution to the problem of nonlinear filtering of diffusion processes was exemplified.

The aim of the present work is to simplify the filtering equation (4) itself by substituting the original filtering problem (1)–(2) by a simpler problem, where the stochastic differential equation for a non-normalized filtering density is much simpler, and the renewal process coincides with the new observed process. Thus, the given result allows one to simplify solution of practical problems of noise filtering.

Let us introduce the necessary notation. The sets $R = (-\infty, +\infty)$, $[0,T]$, $T > 0$ are supposed to be provided with σ-algebras of the Borel sets, indicated by $B(R)$ and $B([0,T])$, respectively; the Lebesque measure is considered to be given on these subsets. Let us indicate by $\mathbf{1}(A)$ the indicator of the set A, i.e. the function equal to 1 within A, and 0 outside A.

2. Structure of processes $x(t)$ and $y(t)$

The below considerations are in many respects based on the structure of solutions to the system of equations (1)–(2). Therefore, our immediate aim is to investigate the structure of the processes $x(t)$ and $y(t)$. Let us rewrite Equations (1) and (2) with Stratonovich stochastic integrals

$$\begin{aligned} x(t) = x_0 + \int_0^t \widetilde{b}^1(s,x(s),y(s))\; ds + \int_0^t \sigma^1(s,x(s),y(s)) * dW_1(s) + \\ + \int_0^t \sigma^2(s,x(s),y(s)) * dW_2(s), \end{aligned} \quad (6)$$

$$y(t) = y_0 + \int_0^t \widetilde{b}^2(s,x(s),y(s))\; ds + \int_0^t \sigma^0(s,y(s)) * dW_2(s), \quad (7)$$

where

$$\widetilde{b}^1(s,x(s),y(s)) = b^1(s,x(s),y(s)) - \frac{1}{2}[\sigma^1(\sigma^1)'_x + \sigma_1(\sigma^2)'_x + \sigma_0(\sigma^2)'_y],$$

$$\widetilde{b}^2(s,x(s),y(s)) = b^2(s,x(s),y(s)) - \frac{1}{2}\sigma^0(s,y(s))(\sigma^0)'_y(s,y(s)).$$

It is known [5, 6] that the solution of the given system can be reduced to the solution of a finite chain of systems of differential equations that do not contain stochastic integrals. Therefore, let us find the solution to the system (6)–(7) in the form

$$x(t) = \widetilde{\phi}(t, W_1(t), W_2(t)), \quad y(t) = \widetilde{\psi}(t, W_2(t)),$$

where $\widetilde{\phi}(t,u,v)$, $\widetilde{\psi}(t,u,v)$ are smooth arbitrary functions. Given

$$\widetilde{\phi}'_u(s,u,W_2(s)) = \sigma^1(s,\widetilde{\phi}(s,u,W_2(s)),\widetilde{\psi}(s,W_2(s))), \tag{8}$$

$$\widetilde{\phi}'_v(s,W_1(s),v) = \sigma^2(s,\widetilde{\phi}(s,W_1(s),v),\widetilde{\psi}(s,v)), \tag{9}$$

$$\widetilde{\psi}'_v(s,v) = \sigma^0(s,\widetilde{\psi}(s,v)). \tag{10}$$

It follows from the formulae (10), that $\int \frac{d\widetilde{\psi}}{\sigma^0(s,\widetilde{\psi})} = v + C(s)$. Hence, $\widetilde{\psi} = \psi(s, v + C(s))$, where $\psi(s,v)$ is an already determinate function, defined by the latter relation, and $C(s)$ is an unknown arbitrary smooth function. Hence, the observed process is represented in the form

$$y(s) = \psi(s, W_2(s) + C(s)). \tag{11}$$

Then, upon similar reasoning, the relation (8) provides

$$x(s) = \phi(s, y(s), W_1(s) + \widetilde{D}(s, y(s))), \tag{12}$$

where $\phi(s,y,u)$ is a determinate function defined from the equality $\int \frac{d\phi}{\sigma^1(s,\phi,\psi)} = u + \widetilde{D}(s,\psi)$. Here $\widetilde{D}(s,\psi)$ is an unknown function.

Let us use the relation (9) to find $\widetilde{D}(s,\psi)$. To this end, note that by virtue of (11), one has

$$\widetilde{\phi}(s,W_1(s),v) = \phi(s,\psi(s,v+C(s)),W_1(s)+\widetilde{D}(s,\psi(s,v+C(s)))).$$

Therefore, in view of the formulae (8) and (10), the relation (9) takes the form

$$\begin{aligned}
&\widetilde{\phi}'_v(s,W_1(s),v) = \\
&\quad = \phi'_y(s,\psi(s,v+C(s)),W_1(s)+\widetilde{D}(s,\psi(s,v+C(s))))\sigma^0(s,\psi(s,v+C(s)))+ \\
&\quad + \sigma^1\Big(s,\phi(s,\psi(s,v+C(s)),W_1(s)+\widetilde{D}(s,\psi(s,v+C(s)))),\psi(s,v+C(s))\Big)\times \\
&\qquad \times \widetilde{D}'_\psi\,(s,\psi(s,v+C(s)))\,\sigma^0(s,\psi(s,v+C(s))) = \\
&\quad = \sigma^2\Big(s,\phi(s,\psi(s,v+C(s)),W_1(s)+\widetilde{D}(s,\psi(s,v+C(s)))),\psi(s,v+C(s))\Big).
\end{aligned}$$

Assuming that $\psi = \psi(s,v+C(s))$ in the latter equality, one obtains the equation for an unknown function $\widetilde{D}$:

$$\begin{aligned}
&\widetilde{D}'_\psi(s,\psi) = \\
&\quad = \frac{\sigma^2(s,\phi(s,\psi,W_1(s)+\widetilde{D}(s,\psi)),v) - \phi'_\psi(s,\psi,W_1(s)+\widetilde{D}(s,\psi))\sigma^0(s,\psi)}{\sigma^1(s,\phi(s,\psi,W_1(s)+\widetilde{D}(s,\psi)),\psi)\sigma^0(s,\psi)}.
\end{aligned} \tag{13}$$

Equation (13) allows one to find the unknown function $\widetilde{D}(s,\psi)$ with the accuracy to an unknown arbitrary function $P(s)$:

$$\widetilde{D}(s,\psi) = D(s,P(s),\psi), \tag{14}$$

where $D(s,p,\psi)$ is derived from Equation (13). The unknown functions $C(s)$ and $P(s)$ are in their turn found (see [6]) from the relations

$$\begin{aligned} \widetilde{\phi}'_s(s,u,v)|_{u=W_1(s),v=W_2(s)} &= \widetilde{b}^1(s,x(s),y(s)),\\ \widetilde{\psi}'_s(s,v)|_{v=W_2(s)} &= \widetilde{b}^2(s,x(s),y(s)), \end{aligned} \tag{15}$$

which represent a system of differential equations in $C(s)$ and $P(s)$. Indeed, in view of the formulae (8)–(14), the first relation of (15) takes the form

$$\begin{aligned} &\phi'_s(s,\psi(s,W_2+C),W_1+D(s,P,\psi(s,W_2+C)))+\\ &\quad+\phi'_y(s,\psi(s,W_2+C),W_1+D(s,P,\psi(s,W_2+C)))\times\\ &\quad\times\left[\psi'_s(s,W_2+C)+\sigma^0(s,\psi(s,W_2+C))\,C'\right]+\\ &\quad+\sigma^1(s,\phi(s,\psi(s,W_2+C),W_1+D(s,P,\psi(s,W_2+C))),\psi(s,W_2+C)))\times\\ &\quad\times\left[D'_s(s,P,\psi(s,W_2+C))+D'_p(s,P,\psi(s,W_2+C))P'+\right.\\ &\quad+D'_\psi(s,P,\psi(s,W_2+C))\,(\psi'_s(s,W_2+C)+\\ &\quad\left.+\,\sigma^0(s,\psi(s,W_2+C))\,C')\right]=\\ &\quad=\widetilde{b}^1(s,\phi(s,\psi(s,W_2+C),W_1+D(s,P,\psi(s,W_2+C))),\psi(s,W_2+C)). \end{aligned}$$

Here, we write $C=C(s)$, $P=P(s)$, $W_1=W_1(s)$, $W_2=W_2(s)$ for the sake of brevity. Likewise, the second relation of (15) yields the equation

$$\begin{aligned} &\psi'_s(s,W_2+C)+\sigma^0(s,\psi(s,W_2+C))C'=\\ &\quad=\widetilde{b}^2(s,\phi(s,\psi(s,W_2+C),W_1+D(s,P,\psi(s,W_2+C))),\psi(s,W_2+C)). \end{aligned}$$

Making use of the latter relation, one arrives to the Cauchy problem

$$C'=\frac{1}{\sigma^0}[\,\widetilde{b}^2-\psi'_s\,], \tag{16}$$

$$P'=\frac{1}{\sigma^1 D'_p}\left[\widetilde{b}^1-\phi'_s-\phi'_y\widetilde{b}^2-\sigma^1 D'_s-\sigma^1 D'_\psi\widetilde{b}^2\right], \tag{17}$$

$$\begin{aligned} \phi(s,\psi(s,W_2(s)+C(s)),W_1(s)+\widetilde{D}(s,P(s),\psi(s,W_2(s)+C(s))))|_{s=0}&=x_0,\\ \psi(s,W_2(s)+C(s))|_{s=0}&=y_0. \end{aligned}$$

We have previously omitted arguments of functions involved in (16) and (17).

Thus, the system of stochastic equations (1)–(2) can be reduced to the solution of a chain of systems of differential equation that do not include stochastic integrals any more.

3. Reduction of the original problem

Further considerations are connected with the formulae (11)–(12). Let us use the notation

$$\widetilde{y}(s)=W_2(s)+C(s),\quad \widetilde{x}(s)=W_1(s)+D(s,P(s),\psi(s,\widetilde{y}(s))). \tag{18}$$

Then, $y(s)=\psi(s,\widetilde{y}(s))$, $x(s)=\phi(s,\psi(s,\widetilde{y}(s)),\widetilde{x}(s))$. Hence, the original observed process $y(s)$ represents a determinate function of the process $\widetilde{y}(s)$.

Assume that $Y_t=\sigma(y(s),s\le t)$, $\widetilde{Y}_t=\sigma(\widetilde{y}(s),s\le t)$.

Lemma 1. *For any t, the equality $Y_t=\widetilde{Y}_t$ holds.*

Proof. Manifestly $Y_t\subseteq\widetilde{Y}_t$. It remains only to verify the reverse injection. Indeed, by virtue of the relation (10) and assumptions made for the function σ^0, the function $\psi(s,v)$ is strictly monotonous in the variable v with every s. Therefore, the equality $\{\omega:\widetilde{y}(s,\omega)\le x\}=\{\omega: y(s,\omega)=\psi(s,\widetilde{y}(s,\omega))\le\psi(s,x)\}\in Y_t$ holds for preimages.

Consider the filtering problem of the processes $(\widetilde{x}(s), \widetilde{y}(s))$, where the first component is unobservable and is to be estimated, and the second one is observable. By virtue of the formulae (12) and (11), Lemma 1 entails that the conditional expectation m_t for the original problem (1)–(2) equals

$$m_t = \mathbf{E}[f(x(t))|Y_t] = \mathbf{E}[f(x(t))|\widetilde{Y}_t] = \mathbf{E}[f(\phi(s, \psi(s, \widetilde{y}(s)), \widetilde{x}(s)))|\widetilde{Y}_t]. \tag{19}$$

Here $f(x)$ and $\phi(s, y, u)$ are determinate functions. On the other hand, the unnormalized conditional density $\widetilde{V}(t, x)$ for the $(\widetilde{x}(s), \widetilde{y}(s))$ filtering problem exists and the conditional expectation $\widetilde{m}_t$ is determined by the formula

$$\widetilde{m}_t = \mathbf{E}[f(\widetilde{x}(t))|Y_t] = \frac{1}{\int_R \widetilde{V}(t, u)\, du} \int_R f(u)\, \widetilde{V}(t, u)\, du. \tag{20}$$

In what follows we will need a property of conditional expectations

Lemma 2. *The conditional expectation m_t from the formula* (19) *equals*

$$m_t = \frac{1}{\int_R \widetilde{V}(t, u)\, du} \int_R f(\phi(t, \psi(t, \widetilde{y}(t)), u))\, \widetilde{V}(t, u)\, du. \tag{21}$$

Proof. Note that in order to prove (21), it suffices to verify that for any determinate function $g(t, v, u)$, for which $\mathbf{E}|g(t, \widetilde{y}(t), \widetilde{x}(t))| < \infty$ is true, the following equality holds:

$$\mathbf{E}[g(t, \widetilde{y}(t), \widetilde{x}(t))|Y_t] = \frac{1}{\int_R \widetilde{V}(t, u)\, du} \int_R g(t, \widetilde{y}(t), u)\, \widetilde{V}(t, u)\, du. \tag{22}$$

Indeed, taking $g(t, v, u) = f(\phi(t, \psi(t, v), u))$ in the latter formula, one obtains (21).

Let $S \in B(R^+), U, V \in B(R)$. Then, assuming that $f(u) = \mathbf{1}(u \in U)$ in the formula (20) and multiplying both sides of the formula by $\mathbf{1}((t, \widetilde{y}(t)) \in S \times V)$, due to properties of conditional expectations and additivity of the integral, one obtains the formula (22) with the function $g(t, v, u) = \mathbf{1}((t, v, u) \in S \times V \times U)$. Manifestly, the formula (22) remains valid for linear combinations of such functions $g(t, y, u)$. Then, using standard passages to the limit, one concludes that our formula (22) holds for an arbitrary bounded function or a function of fixed signs $g(t, v, u)$ such that $\mathbf{E}|g(t, \widetilde{y}(t), \widetilde{x}(t))| < \infty$.

Lemma 3. *Let $\widetilde{\pi}(t, x)$ be a normalized filtering density for the filtering problem of $(\widetilde{x}(s), \widetilde{y}(s))$. Then, the equality*

$$\pi(t, x) = \widetilde{\pi}(t, \phi^{-1}(t, y(t), x))\, \left(\sigma^1(t, x, Y(t))\right)^{-1} \tag{23}$$

holds for any t and x.

Proof. Let us substitute the variables $x = \phi(t, y(t), u)$ in the integral from the right-hand side of the relation (21). Then, by virtue of (8), one has

$$dx = \sigma^1(t, \phi(t, y(t), u), y(t))du, \quad u = \phi^{-1}(t, y(t), x),$$

where $\phi^{-1}(t, y(t), x) = \int \frac{dx}{\sigma^1(t, x, y(t))} - D(t, P(t), y(t))$ is a function inverse to the function $\phi(t, y(t), x)$ for every t.

Hence, the right-hand side of the formula (22) equals

$$m_t = \frac{1}{\int_R \widetilde{V}(t, u)\, du} \int_R f(x)\, \widetilde{V}(t, \phi^{-1}(t, y(t), x))\, \left(\sigma^1(t, x, y(t))\right)^{-1} dx.$$

On the other hand, according to (3), one has $m_t = \int_R f(x)\pi(t, x)dx$. Since the function $f(x)$ is arbitrary, one arrives to the formula (23).

Thus, to solve the filtering problem (1)–(2), it is sufficient to find the unnormalized filtering density $\widetilde{V}(t, u)$ for the filtering problem of $(\widetilde{x}(s), \widetilde{y}(s))$. In order to construct a stochastic differential equation for the density $\widetilde{V}(t,u)$, one has to know equations satisfied by the processes $\widetilde{x}(s)$ and $\widetilde{y}(s)$. In view of the formulae (18) and (16), and due to the Ito formula, one has

$$d\widetilde{y}(s) = B^2(s,\widetilde{x}(s),\widetilde{y}(s))ds + dW_2(s), \tag{24}$$

where

$$B^2(s,u,v) = \frac{1}{\sigma^0(s,\psi(s,v))}[\ b^2(s,\phi(s,\psi(s,v),u)),\psi(s,v))- \\ -\frac{1}{2}\sigma^0(s,\psi(s,v))(\sigma^0)'_\psi(s,\psi(s,v)) - \psi'_s(s,v)\ ].$$

In order to introduce an equation for the unobserved component of $\widetilde{x}(s)$, let us find the stochastic Ito differential for the function $\widetilde{x}(s) = W_1(s) + D(s,P(s),\psi(s,W_2(s)+C(s)))$ and transform the resulting expression by means of the formulae (8)–(17). We obtain

$$d\widetilde{x} = dW_1 + D'_\psi\psi'_v dW_2 + \\ + \left\{D'_s + D'_p P'_s + D'_\psi\psi'_v C' + D'_\psi\psi'_s + \frac{1}{2}[D'_\psi\psi'_v]'_v\right\} ds = dW_1 + \widetilde{\sigma}^2 dW_2 + B^1 ds, \tag{25}$$

where

$$\widetilde{\sigma}^2(s,u,v) = \frac{\sigma^2(s,\phi(s,\psi(s,v),u),\psi(s,v)) - \phi'_\psi(s,\psi(s,v),u)\sigma^0(s,\psi(s,v))}{\sigma^1(s,\phi(s,\psi(s,v),u),\psi(s,v))};$$

$$B^1(s,u,v) = \frac{\widetilde{b}^1(s,\phi(s,\psi(s,v),u),\psi(s,v))}{\sigma^1(s,\phi(s,\psi(s,v),u),\psi(s,v))} - \\ - \frac{\phi'_s(s,\psi(s,v),u) + \phi'_\psi(s,\psi(s,v),u)\widetilde{b}^2(s,\phi(s,\psi(s,v),u),\psi(s,v))}{\sigma^1(s,\phi(s,\psi(s,v),u),\psi(s,v))} + \\ + \frac{1}{2}\left[\frac{\sigma^2(s,\phi(s,\psi(s,v),u),\psi(s,v)) - \phi'_\psi(s,\psi(s,v),u)\sigma^0(s,\psi(s,v))}{\sigma^1(s,\phi(s,\psi(s,v),u),\psi(s,v))}\right]'_v.$$

Thus, the formulae (24) and (25) entail that the processes $(\widetilde{x}(s),\widetilde{y}(s))$ are solutions of the stochastic differential equations

$$\widetilde{x}(t) = \widetilde{x}_0 + \int_0^t B^1(s,\widetilde{x}(s),\widetilde{y}(s))ds + W_1(t) + \int_0^t \widetilde{\sigma}^2(s,\widetilde{x}(s),\widetilde{y}(s))dW_2(s), \tag{26}$$

$$\widetilde{y} = \widetilde{y}_0 + \int_0^t B^2(s,\widetilde{x}(s),\widetilde{y}(s))ds + W_2(t), \tag{27}$$

where $\widetilde{x}_0 = \phi^{-1}(0,y_0,x_0)$, $\widetilde{y}_0 = \psi^{-1}(0,y_0)$. Therefore, for the filtering problem of the processes $(\widetilde{x}(s),\widetilde{y}(s))$, one can construct an equation of the form (4) for an unnormalized filtering density. Meanwhile, it follows from the formula (5) that the observed process $\widetilde{y}(s)$ coincides with $\widetilde{W}(t)$.

Theorem 1. *In the above assumptions, the conditional expectation m_t for the filtering problem* (1)*–*(2) *can be derived from the correlation* (21)*, where $\widetilde{V}(t,x)$ is an unnormalized filtering density for the problem* (26)*–*(27)*. Moreover, normalized filtering densities for the above filtering problems are connected by the equality* (23)*.*

In conclusion, we would like to mention that solution of the filtering problem for $(\widetilde{x}(s),\widetilde{y}(s))$ allows one to solve a similar problem not only for original processes $(x(s),y(s))$, but also for a class of diffusion processes, that can be represented in the form of the relations (11) and (12).

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