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INEQUALITIES FOR MEROMORPHIC FUNCTIONS WITH PRESCRIBED POLES

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Abstract. The extremal problems for functions of complex variables, as well as approaches for obtaining classical inequalities on the base of various methods of the geometric function theory, are known for various norms and for many classes of functions such as rational functions with various constraints and for various domains in the complex plane. It is important to mention that different types of Bernstein-type inequalities appeared in the literature in more generalized forms in which the underlying polynomial was replaced by a more general class of functions. One such generalization is the passage from polynomials to rational functions. In this paper, we prove some inequalities for meromorphic functions with prescribed poles and restricted zeros. These results not only generalize some Bernstein-type inequalities for rational functions, but also improve and generalize some known polynomial inequalities. These inequalities have their own importance in the approximation theory.

Keywords: polynomials, Blaschke product, inequalities, rational functions.

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1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials $P(z) := \sum_{j=0}^{n} c_j z^j$ of degree at most n. Assume that D_k^- represents the set of all points which lie inside $T_k := \{z : |z| = k\}$ and D_k^+ is the set of the points which lie outside T_k. Concerning the estimate of $|P'(z)|$ in terms of $|P(z)|$ for $z \in T_1$, Bernstein [1] proved the following that if $P \in \mathcal{P}_n$, then

$$\max_{z\in T_1} |P'(z)| \leqslant n \max_{z\in T_1} |P(z)|.$$

The result is sharp and the equality holds for the polynomials of the form $P(z) = az^n, a \neq 0$. This inequality can be sharpened under additional conditions on the zeros of $P(z)$. In fact, if $P(z) \neq 0$ in D_1^-, then

$$\max_{z\in T_1} |P'(z)| \leqslant \frac{n}{2} \max_{z\in T_1} |P(z)|, \tag{1.1}$$

whereas if $P(z) \neq 0$ in D_1^+, then (1.1) can be replaced by

$$\max_{z\in T_1} |P'(z)| \geqslant \frac{n}{2} \max_{z\in T_1} |P(z)|. \tag{1.2}$$

Both these inequalities are sharp and equality in each case holds for the polynomials of the form $P(z) = az^n + b$, where $|a| = |b|$.

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Inequality (1.1) was conjectured by Erdös and latter it was verified by Lax [2], whereas inequality (1.2) is due to Turán [3]. Both these inequalities were generalized by Malik [4] as follows: if $P(z)$ is a polynomial of degree n, which does not vanish in D_k^-, where $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |P(z)| \tag{1.3}$$

and if does not vanish in $D_k^+, k \leqslant 1$, then

$$\max_{z\in T_1} |P'(z)| \geqslant \frac{n}{1+k} \max_{z\in T_1} |P(z)|. \tag{1.4}$$

These inequalities were refined and generalized by various authors (for the references see [5], [6], [7]) for the operators besides the ordinary derivative and in some cases underlying polynomials were replaced by a more general class of functions.

For a polynomial $P(z)$ of degree at most n, the polar derivative with respect to a point $\alpha \in \mathbb{C}$, denoted by $D_\alpha P(z)$, is defined as

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

Here $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{|\alpha|\to\infty} \frac{D_\alpha P(z)}{\alpha - z} = P'(z).$$

Aziz [8] extended inequality (1.1) to the polar derivative of a polynomial and proved that If $P(z)$ is a polynomial of degree n, which does not vanish in D_1^-, then for every real or complex number α with $|\alpha| \geqslant 1$ and for $z \in T_1 \cup D_1^-$

$$\max_{z\in T_1} |D_\alpha P(z)| \leqslant \frac{n}{2}(|\alpha z^{n-1}| + 1) \max_{z\in T_1} |P(z)|.$$

In the same paper Aziz proved that if $P(z)$ is a polynomial of degree n, which does not vanish in $D_k^-, k \geqslant 1$, then for every real or complex number α with $|\alpha| \geqslant 1$,

$$\max_{z\in T_1} |D_\alpha P(z)| \leqslant n\left(\frac{k+|\alpha|}{1+k}\right) \max_{z\in T_1} |P(z)|. \tag{1.5}$$

Shah [9] extended inequality (1.2) to the polar derivative and under the same assumption observed for every $\alpha \in \mathbb{C}$, with $|\alpha| \geqslant 1$ that

$$|zD_\alpha P(z)| \geqslant \frac{n}{2}(|\alpha| - 1)|P(z)|, \text{ for } z \in T_1.$$

These results were further extended and generalized in various ways by various authors (for references see [10], [11]).

R. P. Bose proposed to obtain Bernstein-type inequalities for the rational functions instead of polynomials and accordingly over the past few decades, many inequalities for rational functions were established and also used in rational approximation theory. In particular, during the last few decades Bernstein-type inequalities for polynomials were extended to a class of rational functions $\mathcal{R}_n$, where

$$\mathcal{R}_n = \mathcal{R}_n(\alpha_1, \ldots, \alpha_n) := \left\{ \frac{P(z)}{w(z)} : P \in \mathcal{P}_n, w(z) = \prod_{j=1}^{n}(z - \alpha_j) \right\}$$

with poles $\alpha_1, \alpha_2, \dots, \alpha_n$ and with finite limit at infinity. We observe that the Blaschke product $B \in \mathcal{R}_n$, where

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{\alpha_j} z}{z - \alpha_j} \right) = \frac{w^*(z)}{w(z)},$$

with

$$w^*(z) = z^n \overline{w\big(\frac{1}{\overline{z}}\big)} = \prod_{j=1}^{n} (1 - \overline{\alpha_j} z)$$

and satisfying $|B(z)| = 1$ for $z \in T_1$. Through out this paper we assume that all poles $\alpha_1, \alpha_2, \dots, \alpha_n$ lie in D_1^+.

Li, Mohapatra and Rodriguez [12] obtained Bernstein-type inequalities for rational functions $r \in \mathcal{R}_n$ with prescribed poles $\alpha_1, \alpha_2, \dots, \alpha_n$ replacing z^n by $B(z)$. In particular, they proved that if all the zeros of $r \in \mathcal{R}_n$ lie in $T_1 \cup D_1^+$, then for $z \in T_1$

$$|r'(z)| \leqslant \frac{1}{2} |B'(z)| |r(z)|. \tag{1.6}$$

The result is sharp and equality holds for the rational function

$$r(z) = aB(z) + b, |a| = |b| = 1.$$

In the same paper they proved that if all the zeros of a rational function $r \in \mathcal{R}_n$ lie in $T_1 \cup D_1^-$, then for $z \in T_1$

$$|r'(z)| \geqslant \frac{1}{2} |B'(z)| |r(z)|.$$

The result is sharp and equality holds for the rational function

$$r(z) = aB(z) + b \quad \text{with} \quad |a| = |b| = 1.$$

These results were further improved and generalized in various ways from time to time, see [13], [14], [5]. In this paper we prove some results which generalize the known inequalities for rational functions and thereby deduce generalizations of the known estimates for the maximum modulus of the polar derivative as well as the derivative of a polynomial on the disk.

2. Main results

We first prove the following comparison inequality, which gives a rational analogue of a result due to Dewan et al. [15].

Theorem 2.1. *If $r \in \mathcal{R}_n$ has all zeros in $T_k \cup D_k^-, k \leqslant 1$, then for every β with $|\beta| \leqslant 1$, and for $z \in T_1$, we have*

$$\begin{aligned}&\left| zr'(z) + \frac{\beta}{2} \left(|B'(z)| + \frac{2m - n(1+k)}{1+k} \right) r(z) \right| \\ &\qquad \geqslant \left| \left(1 + \frac{\beta}{2}\right) |B'(z)| + \frac{\beta}{2} \frac{2m - n(1+k)}{1+k} \right| \inf_{z \in T_k} |r(z)|.\end{aligned}$$

Here $m \leqslant n$ denotes the number of zeros of $r(z)$.

Remark 2.1. *For $k = 1$ and $m = n$, Theorem 2.1 reduces to a result due to Hans, et al.* [16, Thm. 1].

If in Theorem 2.1 we assume that $r(z)$ has a pole of order n at $z=\alpha, |\alpha| \geqslant 1$, then we can write

$$r(z)=\frac{P(z)}{(z-\alpha)^n},$$

so that

$$r'(z)=\frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}.$$

Also we have in this case

$$B(z)=\prod_1^n\left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)=\left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^n.$$

This gives

$$B'(z)=\frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}.$$

Using these facts, we immediately get from Theorem 2.1 for $z \in T_1$

$$\begin{aligned}&\left|\frac{-zD_\alpha P(z)}{(z-\alpha)^{n+1}}+\frac{\beta}{2}\left(\frac{2m-n(1+k)}{1+k}+\left|\frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}\right|\right)\frac{P(z)}{(z-\alpha)^n}\right|\\ &\geqslant\left|\left(1+\frac{\beta}{2}\right)\left|\frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}\right|+\frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right|\min_{z\in T_k}\frac{|P(z)|}{|z-\alpha|^n}.\end{aligned}$$

This gives

$$\begin{aligned}&\left|\frac{-zD_\alpha P(z)}{z-\alpha}+\frac{\beta}{2}\left(\frac{2m-n(1+k)}{1+k}+\left|\frac{n(|\alpha|^2-1)}{(z-\alpha)^2}\right|\right)P(z)\right|\\ &\geqslant\left|\left(1+\frac{\beta}{2}\right)\left|\frac{n(|\alpha|^2-1)}{(z-\alpha)^2}\right|+\frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right||z-\alpha|^n\min_{z\in T_k}\frac{|P(z)|}{|z-\alpha|^n}.\end{aligned}$$

Therefore from Theorem 2.1, we have the following corollary.

Corollary 2.1. *If all zeros of a polynomial $P(z)$ lie in $T_k \cup D_k^-$, then for every α with $|\alpha| \geqslant 1$ and β with $|\beta| \leqslant 1$, we have for $z \in T_1$*

$$\begin{aligned}&\left|\frac{-zD_\alpha P(z)}{z-\alpha}+\frac{\beta}{2}\left(\frac{2m-n(1+k)}{1+k}+\left|\frac{n(|\alpha|^2-1)}{(z-\alpha)^2}\right|\right)P(z)\right|\\ &\geqslant\left|\left(1+\frac{\beta}{2}\right)\left|\frac{n(|\alpha|^2-1)}{(z-\alpha)^2}\right|+\frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right||z-\alpha|^n\min_{z\in T_k}\frac{|P(z)|}{|z-\alpha|^n}.\end{aligned}$$

If in Corollary 2.1 we let $|\alpha| \to \infty$, then we have the following statement.

Corollary 2.2. *If all zeros of a polynomial $P(z)$ lie in $T_k \cup D_k^-$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, we have for $z \in T_1$*

$$\left|zP'(z)+\frac{m\beta}{1+k}P(z)\right|\geqslant\left|n+\frac{m\beta}{1+k}\right|\min_{z\in T_k}|P(z)|.$$

By taking $m=n, k=1$ in Corollary 2.2, we get a result due to Dewan et al. [15].

We next prove the following generalization of a result due to Li [13].

Theorem 2.2. *Let $r, s \in \mathcal{R}_n$ and assume that all zeros of $s(z)$ lie in $T_k \cup D_k^-, k \leqslant 1$. If*

$$|r(z)| \leqslant |s(z)|, \text{for } z \in T_1, \tag{2.1}$$

then for every real or complex number β with $|\beta| \leqslant 1$ and for $z \in T_1$

$$\begin{aligned}&\left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)r(z)\right| \\ &\qquad\leqslant \left|zs'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)s(z)\right|,\end{aligned}$$

where $m \leqslant n$ are the zeros of $s(z)$.

Remark 2.2. *A result recently proved by Mir* [14, Thm. 3] *follows from Theorem 2.1 once we take $m = n$. Also for $k = 1$ and $m = n$, Theorem 2.2 reduces to a result due to Hans, et al.*[16, Thm. 2].

If $s(z) = B(z)\|r\|$, where $\|r\| := \sup_{z \in T_1} |r(z)|$, then from Theorem 2.2 we get the following corollary.

Corollary 2.3. *Let $r \in \mathcal{R}_n$, then for every real or complex β with $|\beta| \leqslant 1$*

$$\begin{aligned}&\left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)r(z)\right| \\ &\qquad\leqslant \left|\left(1 + \frac{\beta}{2}\right)|B'(z)| + \frac{\beta}{2}\frac{2m - n(1+k)}{1+k}\right| \sup_{z \in T_1} |r(z)|.\end{aligned}$$

We also prove the following theorem.

Theorem 2.3. *If $r \in \mathcal{R}_n$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$,*

$$\begin{aligned}&\left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)r(z)\right| \\ &\quad+ \left|zr^{*'}(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)r^*(z)\right| \\ &\quad\leqslant \left(\left|\left(1 + \frac{\beta}{2}\right)|B'(z)| + \frac{\beta}{2}\frac{2m - n(1+k)}{1+k}\right|\right. \\ &\qquad\left.+ \left|\frac{\beta}{2}\right|\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)\right) \sup_{z \in T_1} |r(z)|,\end{aligned} \tag{2.2}$$

for $z \in T_1$, where $r^*(z) = B(z)\overline{r\big(\frac{1}{\overline{z}}\big)}$.

Remark 2.3. *For $k = 1$ and $m = n$, Theorem 2.3 reduces to a result due to Hans, et al.*[16, Thm. 3].

We consider $r \in \mathcal{R}_n$ having all its zeros in $T_k \cup D_k^+, k \leqslant 1$ and let $m' = \inf_{z \in T_k} |r(z)|$, then $m' \leqslant |r(z)|$ for all $z \in T_k$. By Rouche's theorem for some complex δ with $|\delta| < 1$ all the zeros of $R(z) = r(z) - \delta m'$ lie in $T_k \cup D_k^+$ and therefore all the zeros of

$$S(z) = B(z)\overline{R\big(\frac{1}{\overline{z}}\big)} = r^*(z) - \overline{\delta} m' B(z)$$

lie in $T_k \cup D_k^-$. Therefore from Theorem 2.2 we get

$$
\begin{aligned}
&\left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)(r(z) - \delta m')\right| \\
&\leqslant \left|z(r^{*'}(z) - \overline{\delta}B'(z)m') + \frac{\beta}{2}\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)(r^*(z) - \overline{\delta}B(z)m')\right|.
\end{aligned}
$$

This in particular gives for $z \in T_1$

$$
\begin{aligned}
&\left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)r(z) - \frac{\beta}{2}\delta m'\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)\right| \\
&\leqslant \left|zr^{*'}(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)r^*(z) - \overline{\delta}zB'(z)m' \right.\\
&\qquad \left. - \frac{\beta}{2}\overline{\delta}B(z)m'\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)\right|.
\end{aligned}
$$

Now dividing both sides by $B(z)$ and using the fact that for $z \in T_1$

$$
|B(z)| = 1, \qquad \frac{zB'(z)}{B(z)} = |B'(z)|,
$$

we get

$$
\begin{aligned}
&\left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)r(z) - \frac{\beta}{2}\delta m'\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)\right| \\
&\leqslant \left|zr^{*'}(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)r^*(z) - \left(|B'(z)|\left(1+\frac{\beta}{2}\right) + \frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right)m'\overline{\delta}\right|.
\end{aligned}
$$

Choosing suitably the argument of δ in right hand side of above inequality and letting $|\delta| \to 1$, we get

$$
\begin{aligned}
&\left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)r(z)\right| \\
&\leqslant \left|zr^{*'}(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)r^*(z)\right| \\
&\quad - \left(\left|\left(1+\frac{\beta}{2}\right)|B'(z)| + \frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right| - \frac{|\beta|}{2}\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)\right)m'.
\end{aligned} \tag{2.3}
$$

Now combining (2.2) and (2.3), we have the following corollary.

Corollary 2.4. *Let $r \in \mathcal{R}_n$ has all zeros in $T_k \cup D_k^+, k \leqslant 1$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, we have for $z \in T_1$*

$$\left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)r(z)\right|$$
$$\leqslant \frac{1}{2}\left(\left|\left(1+\frac{\beta}{2}\right)|B'(z)| + \frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right| + \left|\frac{\beta}{2}\right|\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)\right)\|r\|$$
$$- \frac{1}{2}\left(\left|\left(1+\frac{\beta}{2}\right)|B'(z)| + \frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right| - \frac{|\beta|}{2}\left(|B'(z)| + \frac{2m-n(1+k)}{1+k}\right)\right)m'.$$

By taking $m = n, k = 1$ and $\beta = 0$, Corollary 2.4 reduces to a result due to Aziz and Shah [19, Thm. 3].

If we assume that $r(z)$ has a pole of order n at $z = \alpha, |\alpha| \geqslant 1$, then

$$r(z) = \frac{P(z)}{(z-\alpha)^n},$$

so that

$$r'(z) = \frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}.$$

Using the same procedure as in Corollary 2.1, we get from Corollary 2.4 for $z \in T_1$ the following statement.

Corollary 2.5. *Let $P \in \mathcal{P}_n$ has all zeros in $T_k \cup D_k^+, k \leqslant 1$, then for any α with $|\alpha| \geqslant 1$ and β with $|\beta| \leqslant 1$, we have*

$$\left|\frac{-zD_\alpha P(z)}{(z-\alpha)^{n+1}} + \frac{\beta}{2}\left(\frac{2m-n(1+k)}{1+k} + \left|\frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}\right|\right)\frac{P(z)}{(z-\alpha)^n}\right|$$
$$\leqslant \frac{1}{2}\left(\left|\left(1+\frac{\beta}{2}\right)\left|\frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}\right| + \frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right|\right.$$
$$\left. + \left|\frac{\beta}{2}\right|\left(\left|\frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}\right| + \frac{2m-n(1+k)}{1+k}\right)\right)\max_{z\in T_1}\left|\frac{P(z)}{(z-\alpha)^n}\right|$$
$$- \frac{1}{2}\left(\left|\left(1+\frac{\beta}{2}\right)\left|\frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}\right| + \frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right|\right.$$
$$\left. - \frac{|\beta|}{2}\left(\left|\frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}\right| + \frac{2m-n(1+k)}{1+k}\right)\right)\min_{z\in T_k}\left|\frac{P(z)}{(z-\alpha)^n}\right|.$$

In Corollary 2.5, if we make $|\alpha| \to \infty$ and on simplification, we get the following corollary.

Corollary 2.6. *Let $P \in \mathcal{P}_n$ has all zeros in $T_k \cup D_k^+, k \leqslant 1$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and for $z \in T_1$ the following inequality holds:*

$$\left|zP'(z) + \frac{m\beta}{1+k}P(z)\right| \leqslant \frac{1}{2}\left(\left(n + \frac{1}{2}(m\beta + n|\beta|)\right)\max_{z\in T_1}|P(z)|\right.$$
$$\left. - \left(n + \frac{1}{2}(m\beta - n|\beta|)\right)\min_{z\in T_k}|P(z)|\right).$$

By taking $\beta = 0, m = n$ and $k = 1$, Corollary 2.6 reduces to a result due to Aziz and Dawood [17, Thm. 2] .

3. Lemmas

For the proof of these results we need the following lemmas. The first lemma is due to Li, Mohapatra and Rodgriguez [12].

Lemma 3.1. *Let A and $B \neq 0$ be two complex numbers, then $|A| \geqslant |B|$ if and only if $A \neq vB$ for any complex number v with $|v| < 1$.*

The next lemma which we also require is due to Aziz and Shah [18].

Lemma 3.2. *Suppose $r \in \mathcal{R}_n$ be such that all the zeros of $r(z)$ lie in $T_k \cup D_k^-$, $k \leqslant 1$, then for $z \in T_1$,*

$$|r'(z)| \geqslant \frac{1}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)|r(z)|, \tag{3.1}$$

where m is the number of zeros of r, with $m \leqslant n$.

4. Proofs of Theorems

Proof of Theorem 2.1. If $r(z)$ has any zero on T_k, then $\inf\limits_{z\in T_k} |r(z)| = 0$ and the statement becomes trivial. Now we assume that $r(z)$ has all zeros in D_k^-. If $m' = \inf\limits_{z\in T_k} |r(z)|$, then $m' > 0$ and $|r(z)| \geqslant m'$ or $z \in T_k$. Also $|B(z)| \leqslant 1$ for $z \in T_1$ (see [5, p.40]), therefore $|B(z)| \leqslant 1$ for $z \in T_k, k \leqslant 1$. Hence, for each complex δ with $|\delta| < 1$, by the Rouché theorem we see that

$$F(z) = r(z) - \delta m' B(z)$$

has all zeros in D_k^-. Applying Lemma 3.2 to the rational function $F \in \mathcal{R}_n$, we get

$$|zF'(z)| \geqslant \frac{1}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)|F(z)|.$$

Since $F(z) = r(z) - \delta m' B(z) \neq 0$ in $T_k \cup D_k^+$, therefore for each complex number β such $|\beta| \leqslant 1$ by Lemma 3.1 we get:

$$\begin{aligned} T(z) :=& z\big(r'(z) - \delta m' B'(z)\big) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)(r(z) - \delta m' B(z)) \\ =& zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)r(z) \\ &- \delta\left\{zB'(z) + \frac{\beta}{2}B(z)\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)\right\}m' \neq 0 \end{aligned}$$

in $T_k \cup D_k^+$. In particular, for $|\delta| < 1$ we obtain

$$\left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)r(z)\right| \geqslant \left|\left(1 + \frac{\beta}{2}\right)|B'(z)| + \frac{\beta}{2}\frac{2m - n(1+k)}{1+k}\right| \inf_{z\in T_k} |r(z)|.$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Since $s(z)$ has all its zeros in $T_k \cup D_k^-$, $k \leqslant 1$ and $|r(z)| < |s(z)|$ for $z \in T_1$, by Rouché theorem for $|\lambda| < 1$ we see that $\lambda r(z) + s(z)$ has the same number of zeros in $T_k \cup D_k^-$ as $s(z)$. Hence, applying Lemma 3.2, for $z \in T_1$ we get

$$|z(\lambda r'(z) + s'(z))| \geqslant \frac{1}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)|(\lambda r(z) + s(z))|.$$

Since $(\lambda r(z) + s(z))$ has no zero in $T_k \cup D_k^+$, by using Lemma 3.1 for every real or complex β with $|\beta| \leqslant 1$ we get

$$z(\lambda r'(z) + s'(z)) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)(\lambda r(z) + s(z)) \neq 0.$$

This implies that

$$\begin{aligned}
&\lambda\left(zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)r(z)\right) \\
&\qquad \neq -\left(zs'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)s(z)\right).
\end{aligned}$$

In particular, for eachλ with $|\lambda| < 1$ and for $z \in T_1$ we get

$$\left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)r(z)\right| \leqslant \left|zs'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)s(z)\right|.$$

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Let $M = \sup\limits_{z \in T_1} |r(z)|$, then $|r(z)| \leqslant M$ for $z \in T_1$. Therefore, for $|\gamma| > 1$, the function

$$H(z) = r(z) - \gamma M$$

has no zero in D_k^-. Hence, the function

$$G(z) = B(z)\overline{H\big(\frac{1}{\overline{z}}\big)} = r^*(z) - \overline{\gamma}B(z)M$$

has all its zeros in $T_k \cup D_k^-$. Applying Theorem 2.2 to the rational function $G(z)$ which has m zeros and n poles, we get

$$\left|zH'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)H(z)\right| \leqslant \left|zG'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)G(z)\right|.$$

This gives

$$\begin{aligned}
&\left|zr'(z) + \frac{\beta}{2}r(z)|B'(z)| + \frac{2m - n(1+k)}{1+k}\frac{\beta}{2}r(z) - \gamma\frac{\beta}{2}\left(|B'(z)| + \frac{2m - n(1+k)}{1+k}\right)M\right| \\
&\leqslant \left|zr^*(z) + \frac{\beta}{2}r^*(z)|B'(z)| + \frac{2m - n(1+k)}{1+k}\frac{\beta}{2}r^*(z)\right. \qquad (4.1) \\
&\leqslant \left.\left| - \overline{\gamma}\left(zB'(z) + \frac{\beta}{2}\left(|B'(z)|B(z) + \frac{2m - n(1+k)}{1+k}B(z)\right)M\right.\right|.
\end{aligned}$$

Now choosing suitably the argument of γ, which is possible by Corollary 2.3, from inequality (4.1) by using triangle inequality in the left hand side we find

$$\begin{aligned}&\left|zr'(z)+\frac{\beta}{2}r(z)|B'(z)|+\frac{2m-n(1+k)}{1+k}\frac{\beta}{2}r(z)\right|-|\gamma|\left|\frac{\beta}{2}\right|\left||B'(z)|+\frac{2m-n(1+k)}{1+k}\right|M\\&\leqslant\left|zB'(z)+\frac{\beta}{2}\left(zB'(z)+\frac{2m-n(1+k)}{1+k}B(z)\right)\right|M\\&-|\gamma|\left|zr^{*'}(z)+\frac{\beta}{2}|B'(z)|r^*(z)+\frac{\beta}{2}\frac{2m-n(1+k)}{1+k}r^*(z)\right|.\end{aligned}$$

Letting $|\gamma|\to 1$, we get

$$\begin{aligned}&\left|zr'(z)+\frac{\beta}{2}r(z)|B'(z)|+\frac{2m-n(1+k)}{1+k}\frac{\beta}{2}r(z)\right|\\&+\left|zr^{*'}(z)+\frac{\beta}{2}|B'(z)|r^*(z)+\frac{\beta}{2}\frac{2m-n(1+k)}{1+k}r^*(z)\right|\\&\leqslant\left(\left|\frac{\beta}{2}\right|\left||B'(z)|+\frac{2m-n(1+k)}{1+k}\right|+\left||B'(z)|+\frac{\beta}{2}\left(|B'(z)|+\frac{2m-n(1+k)}{1+k}\right)\right|\right)M,\end{aligned}$$

for $z\in T_1$. This allows to conclude that, for $z\in T_1$,

$$\begin{aligned}&\left|zr'(z)+\frac{\beta}{2}\left(|B'(z)|+\frac{2m-n(1+k)}{1+k}\right)r(z)\right|\\&+\left|zr^{*'}(z)+\frac{\beta}{2}\left(|B'(z)|+\frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right)r^*(z)\right|\\&\leqslant\left(\left|\frac{\beta}{2}\right|\left||B'(z)|+\frac{2m-n(1+k)}{1+k}\right|+\left|\left(1+\frac{\beta}{2}\right)|B'(z)|+\frac{\beta}{2}\frac{2m-n(1+k)}{1+k}\right|\right)M.\end{aligned}$$

This completes the proof of Theorem 2.3. □

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