ISSN 2074-1871 Уфимский математический журнал. Том 16. № 1 (2024). С. 111-125.

INVERSE PROBLEM FOR SUBDIFFUSION EQUATION WITH FRACTIONAL CAPUTO DERIVATIVE

R.R. ASHUROV, M.D. SHAKAROVA

Abstract. We consider an inverse problem on determining the right-hand side of the subdiffusion equation with the fractional Caputo derivative. The right-hand side of the equation has the form $f(x)g(t)$ and the unknown is the function $f(x)$. The condition $u(x,t_0) = \psi(x)$ is taken as the over-determination condition, where t_0 is some interior point of the considered domain and $\psi(x)$ is a given function. By the Fourier method we show that under certain conditions on the functions $g(t)$ and $\psi(x)$ the solution of the inverse problem exists and is unique. We provide an example showing the violation of the uniqueness of the solution of the inverse problem for some sign-changing functions $g(t)$. For such functions $g(t)$ we find necessary and sufficient conditions on the initial function and on the function from the over-determination condition, which ensure the existence of a solution to the inverse problem.

Keywords: subdiffusion equation, forward and inverse problems, the Caputo derivatives, Fourier method.

Mathematics Subject Classifications: 35R11, 34A12

1. Introduction

Given a fixed number $\rho \in (0,1]$, we consider the following initial-boundary value problem

$$\begin{cases} D_t^\rho u(x,t) - \Delta u(x,t) = F(x,t) \equiv f(x)g(t), \quad x \in \Omega, \quad t \in (0,T], \\ u(x,t)|_{\partial\Omega} = 0, \\ u(x,0) = \varphi(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

Here $f(x)$, $g(t)$ and $\varphi(x)$ are continuous functions in the domain $\Omega \subset \mathbb{R}^N$ and $D_t^\rho h(t)$ stands for the Caputo fractional derivative (see, for instance, [1])

$$D_t^\rho h(t) = \int_0^t \omega_{1-\rho}(t-s)\frac{d}{ds}h(s)ds, \qquad \omega_\rho(t) = \frac{t^{\rho-1}}{\Gamma(\rho)},$$

where $\Gamma(\rho)$ is the gamma function. If we first integrate and then differentiate, then we get the Riemann-Liouville derivative.

It should be noted that if $\rho = 1$, then both the Caputo derivative and the Riemann-Liouville derivative coincide with the classical first order derivative. Therefore, if $\rho = 1$, then problem (1.1) coincides with the usual initial-boundary value problem for the diffusion equation.

Problem (1.1) is also called *the forward problem*. The main purpose of this study is the inverse problem on determining the right-hand side of the equation, namely, the function $f(x)$.

R.R. Ashurov, M.D. Shakarova, Inverse problem for the subdiffusion equation with fractional Caputo derivative.

The authors acknowledge financial support from the Ministry of Innovative Development of the Republic of Uzbekistan, Grant No F-FA-2021-424.

Submitted November 02, 2022.

To solve the inverse problem, one needs an extra condition. Following A.I. Prilepko and A.B. Kostin [2] and K.B. Sabitov [3] (see also [4]), we consider the additional condition in the form:

$$u(x,t_0)=\psi(x), \qquad x\in\Omega, \tag{1.2}$$

where t_0 is a given fixed point of the segment $(0,T]$.

We call the initial-boundary value problem (1.1) together with the additional condition (1.2) *the inverse problem* on finding the part $f(x)$ of the right-hand side of the equation.

The authors usually impose an additional condition (1.2) at the final time $t_0=T$ (see, for instance, [5], [6] for classical diffusion equations and [7], [8] for subdiffusion equations). The meaning of taking condition (1.2) at t_0 is that in some cases the uniqueness of the solution of the inverse problem is violated if $t_0=T$ and by choosing t_0 it is possible to achieve uniqueness in these cases as well.

We are interested in *the classical solution* (we simply call it a solution) of the problems under consideration, i.e. such solutions that themselves and all the derivatives involved in the equation are continuous, moreover, all the given functions are continuous and the equation is obeyed at each point. As an example, let us give the definition of the solution to the inverse problem.

Definition 1.1. *A pair of functions $\{u(x,t),f(x)\}$ with the properties*

1. $D_t^\rho u(x,t),\Delta u(x,t)\in C(\overline{\Omega}\times(0.T])$,
2. $u(x,t)\in C(\overline{\Omega}\times[0.T])$,
3. $f(x)\in C(\overline{\Omega})$,

and satisfying conditions (1.1), (1.2) is called a solution *of the inverse problem.*

We note that in this definition the requirement of continuity in a closed domain of all derivatives of the solution appearing in (1.1) was proposed by O.A. Ladyzhenskaya [9]. The advantage of this choice is that the uniqueness of such a solution is proved quite simply, moreover, the solution found by the Fourier method satisfies the above conditions.

Inverse problems on determining the right hand side of various subdiffusion equations were studied by a number of authors due to the importance of such problems for applications. However, it should be immediately noted that for the abstract case of the source function $F(x,t)$ there is no general theory yet, see survey paper [10] and the references therein. In all known works, the split source function $F(x,t)\equiv f(x)g(t)$ is considered and the methods of investigation depend on whether $f(x)$ or $g(t)$ is unknown. It is somewhat more difficult to study the case when function $g(t)$ is unknown. For example, in papers [11] and [12] the questions of finding the non-stationary source function $g(t)$ were studied. It should be noted that in these papers the over-determination condition is taken in a fairly general form: $B[u(\cdot,t)]=\psi(t)$, where B is a linear continuous functional. In particular, one can take $u(x_0,t)$ or $\int_\Omega u(x,t)dx$ as $B[u(\cdot,t)]$. The determination of the unknown function $g(t)$ for subdiffusion equations was studied in the articles [10] and [13].

For subdiffusion and diffusion equations, the case $g(t)\equiv 1$ and the unknown is $f(x)$ was studied by many authors, see, for example, [14]–[20]. We mention only some of these articles.

Subdiffusion equations with an elliptic part as an ordinary differential expression were considered in papers [14],[15], [16]. The authors of papers [17], [18] studied subdiffusion equations, the elliptic part of which is the Laplace operator or a second-order differential operator. Paper [19] is devoted to study the inverse problem for a subdiffusion equation with the Caputo fractional derivative and an arbitrary elliptic self-adjoint differential operator. The authors of this paper proved the uniqueness and existance of a generalized solution. The case of the Riemann-Liouville derivative was considered in [20]. Here the uniqueness and existence of a classical solution were proved. In papers [14] and [18], the fractional derivative in the subdiffusion equation is a two-parameter generalized Hilfer fractional derivative.

In [21], the authors considered the inverse problem of simultaneous determination of the order of the Riemann-Liouville fractional derivative and the source function in subdiffusion equations. Using the classical Fourier method, the authors proved that the solution to this inverse problem exists and is unique.

In monograph by K.B. Sabitov [22] the solvability of forward and inverse problems for equations of mixed parabolic-hyperbolic type was studied.

We note some results obtained for the case $g(t) \not\equiv 1$. For classical diffusion equations, such an inverse problem was studied in detail, see the well-known monograph by S. Kabanikhin [23, Ch. 8] as well as [2], [3], [4], [5], [6]. Since the equation considered by us also covers the diffusion equation, we will dwell on these works in more detail at the end of Section 4.

In paper [24] the problem on finding function $f(x)$ for an abstract subdiffusion equation with the Caputo derivative was studied. To find the function $f(x)$, the authors used the following additional condition $\int_0^T u(t)d\mu(t) = u_T$.

M. Slodichka et al. [7] and [8] studied the uniqueness of a solution of the inverse problem for a subdiffusion equation, the elliptic part of which depends on time. It was proved that if function $g(t)$ is sign-definite, then the solution of the inverse problem is unique. It should be especially noted that in [8] the authors constructed an example of a function $g(t)$ that changes sign in the domain under consideration and this resulted in the loss of the uniqueness of the solution to the inverse problem.

It is well known that the considered inverse problem is ill-posed, i.e., the solution does not depend continuously on the given data. Therefore, in the works of some authors, various regularization methods were proposed for constructing an approximate solution of the inverse problem, see, for instnace, [25], [26]. In paper [25] the inverse problem for the fractional diffusion equation with the Riemann-Liouville derivative was considered. Assuming that solutions to the equation can be represented by a Fourier series, the authors applied the Tikhonov regularization method to find an approximate solution. Convergence estimates for exact and regularized solutions were presented for a priori and a posteriori rules for choosing parameters. In [26], similar questions were investigated for the stochastic fractional diffusion equation.

This work is devoted to the study of forward problem (1.1) and inverse problem (1.1), (1.2) on determining the right-hand side of the equation. Let us list the main results of this paper.

1) First, in Section 3, we prove the existence and uniqueness theorem for the forward problem (1.1) by using the Fourier method. We present conditions on the initial function $\varphi(x)$ and on the right-hand side of the equation that ensure the validity of the application of the Fourier method. Due to the fact that the elliptic part of the equation is the Laplace operator, the conditions on the functions $f(x)$ and $g(t)$ turned out to be easier to check than in the case of a general elliptic operator, see [27];

2) Then in Section 4, under a certain condition on function $g(t)$ (for example, the constant sign is sufficient), we prove the existence and uniqueness of a solution to the inverse problem. Further, we show that if this condition is violated, then for the existence of a solution to the inverse problem it is sufficient the functions in the initial condition and the over-determination condition to be orthogonal to some eigenfunctions of the Laplace operator with the Dirichlet condition;

3) An example of function $g(t)$ is constructed in Section 4, for which the condition noted above is not satisfied and, as a result, the inverse problem has more than one solution.

The following Section 2 is auxiliary and contains definitions and well-known assertions necessary for further presentation. The section Conclusions completes this work.

2. Preliminaries

In this auxiliary section we define fractional powers of a self-adjoint extension of the Laplace operator, formulate a lemma from book by Krasnoselskii et al. [28], a fundamental result by V.A. Il'in [29] about the convergence of the Fourier coefficients and indicate some needed properties of the Mittag-Leffler function.

We denote by $\{\lambda_k\}$ and $\{v_k(x)\}$ a set of positive eigenvalues and an associated complete system of orthonormal eigenfunctions in $L_2(\Omega)$ of the following spectral problem

$$\begin{cases} -\Delta v(x) = \lambda v(x), \quad x \in \Omega, \\ v(x)|_{\partial\Omega} = 0. \end{cases}$$

Let σ be an arbitrary real number. Consider an operator $\hat{A}^\sigma$ acting in $L_2(\Omega)$ as

$$\hat{A}^\sigma g(x) = \sum_{k=1}^{\infty} \lambda_k^\sigma g_k v_k(x), \qquad g_k = (g, v_k),$$

on the domain

$$D(\hat{A}^\sigma) = \left\{ g \in L_2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 < \infty \right\}.$$

On the elements of $D(\hat{A}^\sigma)$ we introduce the norm

$$\|g\|_\sigma^2 = \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 = \|\hat{A}^\sigma g\|^2.$$

Let A be the operator acting in $L_2(\Omega)$ as $Ag(x) = -\Delta g(x)$ on the domain $D(A) = \{g \in C^2(\Omega) : g(x) = 0, x \in \partial\Omega\}$, then by $\hat{A} \equiv \hat{A}^1$ we denote the self-adjoint extension of A in $L_2(\Omega)$.

In our reasoning the following lemma from the book Krasnoselskii et al. [28] plays an important role.

Lemma 2.1. *Let $\sigma > \frac{N}{4}$. Then operator $\hat{A}^{-\sigma}$ continuously maps the space $L_2(\Omega)$ into $C(\Omega)$, and moreover, the following estimate holds*

$$\|\hat{A}^{-\sigma} g\|_{C(\Omega)} \leqslant C \|g\|_{L_2(\Omega)}.$$

In order to prove the existence of solutions of forward and inverse problems, it is necessary to study the convergence of the following series:

$$\sum_{k=1}^{\infty} \lambda_k^\tau |h_k|^2, \quad \tau > \frac{N}{2}, \tag{2.1}$$

where h_k are the Fourier coefficients of function $h(x)$. In the case of integers τ, in fundamental paper [29] by V.A. Il'in, conditions were obtained for the convergence of such series in terms of the membership of the function $h(x)$ in the classical Sobolev spaces $W_2^k(\Omega)$. To formulate these conditions, we introduce the class $\hat{W}_2^1(\Omega)$ as the closure in the $W_2^1(\Omega)$-norm of the set of all continuously differentiable in Ω functions vanishing in the vicinity of the boundary of Ω.

The theorem of V.A. Il'in states that if the function $h(x)$ satisfies the conditions

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+1}(\Omega) \qquad \text{and} \qquad h(x), \Delta h(x), \ldots, \Delta^{\left[\frac{N}{4}\right]} h(x) \in \hat{W}_2^1(\Omega), \tag{2.2}$$

then scalar series (2.1) converges. Here $[a]$ denotes the integer part of a number a. Similarly, if in (2.1) we replace τ by $\tau + 2$, then the convergence conditions becomes

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+3}(\Omega) \qquad \text{and} \qquad h(x), \Delta h(x), \ldots, \Delta^{\left[\frac{N}{4}\right]+1} h(x) \in \hat{W}_2^1(\Omega). \tag{2.3}$$

For $0 < \rho < 1$ and an arbitrary complex number μ, let $E_{\rho,\mu}(z)$ denote the Mittag-Leffler function with two parameters of the complex argument z:

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}. \tag{2.4}$$

For $\mu = 1$ we have the classical Mittag-Leffler function $E_\rho(z) = E_{\rho,1}(z)$.

We recall some properties of the Mittag-Leffler functions, see, for instance, [30].

Lemma 2.2. *For any $t \geqslant 0$ one has*

$$|E_{\rho,\mu}(-t)| \leqslant \frac{C}{1+t}, \tag{2.5}$$

where constant C is independent of t and μ.

Lemma 2.3. *(see [31]). The classical Mittag-Leffler function of the negative argument $E_\rho(-t)$ is monotonically decreasing function for all $0 < \rho < 1$ and*

$$0 < E_\rho(-t) < 1, \quad E_\rho(0) = 1.$$

Lemma 2.4. *(see [30, Eq. (2.30)] and [32, Lm. 4]). Let μ be an arbitrary complex number. Then the following asymptotic estimate holds*

$$\left| E_{\rho,\mu}(-t) - \frac{t^{-1}}{\Gamma(\mu-\rho)} \right| \leqslant \frac{C}{t^2}, \qquad t > 1,$$

where C is an absolute constant.

Lemma 2.5. *(see [31, Eq. (4.4.5)]). Let $\rho > 0$, $\mu > 0$ and $\lambda \in C$. Then for all positive t one has*

$$\int_0^t (t-\eta)^{\mu-1}\eta^{\rho-1}E_{\rho,\rho}(\lambda\eta^\rho)d\eta = t^{\mu+\rho-1}E_{\rho,\rho+\mu}(\lambda t^\rho). \tag{2.6}$$

3. Well-posedness of forward problem (1.1)

First we consider the following problem for a homogeneous equation

$$\begin{cases} D_t^\rho y(x,t) - \Delta y(x,t) = 0, & (x,t) \in \Omega \times (0,T], \\ y(x,t)|_{\partial\Omega} = 0, \\ y(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where $\varphi(x)$ is a given function.

Theorem 3.1. *Let function $\varphi(x)$ satisfy conditions (2.2). Then problem (3.1) has a unique solution:*

$$y(x,t) = \sum_{k=1}^{\infty} \varphi_k E_\rho(-\lambda_k t^\rho) v_k(x), \tag{3.2}$$

where φ_k are the Fourier coefficients of function $\varphi(x)$.

Proof. This theorem for a more general subdiffusion equation was proved in [27]. We only mention the main points of the proof.

Obviously, (3.2) is a formal solution to problem (3.1), see [1], [33]. Let us show that the operators $A = -\Delta$ and D_t^ρ can be applied term-by-term to series (3.2) and the resulting series converges uniformly in $(x,t) \in (\overline{\Omega} \times (0,T])$. If $S_j(x,t)$ is the partial sum of series (3.2), then

$$-\Delta S_j(x,t) = \sum_{k=1}^{j} \lambda_k \varphi_k E_\rho(-\lambda_k t^\rho) v_k(x).$$

Using the identity

$$\hat{A}^{-\sigma} v_k(x) = \lambda_k^{-\sigma} v_k(x),$$

with $\sigma > \frac{N}{4}$ and applying Lemma 2.1 for $g(x) = -\Delta S_j(x,t)$, we have

$$\| -\Delta S_j(x,t)\|^2_{C(\Omega)} = \left\| \sum_{k=1}^{j} \lambda_k \varphi_k E_\rho(-\lambda_k t^\rho) v_k(x) \right\|^2_{C(\Omega)} \leqslant C \sum_{k=1}^{j} \lambda_k^{2(\sigma+1)} |\varphi_k E_\rho(-\lambda_k t^\rho)|^2.$$

We apply estimates (2.5) to obtain

$$\| -\Delta S_j(x,t)\|^2_{C(\Omega)} \leqslant C \sum_{k=1}^{j} \frac{\lambda_k^{2(\sigma+1)} |\varphi_k|^2}{|1+\lambda_k t^\rho|^2} \leqslant C t^{-2\rho} \sum_{k=1}^{j} \lambda_k^{2\sigma} |\varphi_k|^2, \quad t > 0.$$

Therefore, if $\varphi(x)$ satisfies conditions (2.2), then $-\Delta y(x,t) \in C(\overline{\Omega} \times (0,T])$. From equation (3.1) one has $D_t^\rho y(x,t) = \Delta y(x,t)$, $t > 0$, and hence we get $D_t^\rho y(x,t) \in C(\overline{\Omega} \times (0,T])$.

The uniqueness of the solution follows from the completeness of the system $\{v_k(x)\}$ in $L_2(\Omega)$, see [20]. We only note that it is important here that the derivatives of the solution involved in the equation are continuous up to the boundary of domain Ω, see Definition 1.1. Nevertheless, below we give a proof of the uniqueness of a solution of the inverse problem in detail, see the proof of Theorem 4.1. □

Now we consider the following auxiliary initial-boundary value problem:

$$\begin{cases} D_t^\rho \omega(x,t) - \Delta \omega(x,t) = f(x) g(t), \ (x,t) \in \Omega \times (0,T], \\ \omega(x,t)|_{\partial\Omega} = 0, \\ \omega(x,0) = 0, \ x \in \Omega. \end{cases} \tag{3.3}$$

Theorem 3.2. *Let $f(x)$ satisfy conditions (2.2) and $g(t) \in C[0,T]$. Then problem (3.3) has a unique solution*

$$\omega(x,t) = \sum_{k=1}^{\infty} f_k \left[\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) g(t-\eta) d\eta \right] v_k(x). \tag{3.4}$$

where $f_k = (f, v_k)$.

Proof. Again, as in the previous theorem, (3.4) is a formal solution to problem (3.3), see [1], [33].

Let $S_j(x,t)$ be the partial sum of series (3.4) and $\sigma > \frac{N}{4}$. Repeating the above reasoning based on Lemma 2.1 and using the Parseval's identity and Lemma 2.5, we arrive at

$$\begin{aligned} \| -\Delta S_j(x,t)\|^2_{C(\Omega)} =& \left\| \sum_{k=1}^{j} \lambda_k f_k \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) g(t-\eta) d\eta v_k(x) \right\|^2_{C(\Omega)} \\ \leqslant& \left\| \hat{A}^{-\sigma} \sum_{k=1}^{j} \lambda_k^{\sigma+1} f_k \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) g(t-\eta) d\eta v_k(x) \right\|^2_{C(\Omega)} \\ \leqslant& \left\| \sum_{k=1}^{j} \lambda_k^{\sigma+1} f_k \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) g(t-\eta) d\eta v_k(x) \right\|^2_{L_2(\Omega)} \\ \leqslant& C \sum_{k=1}^{j} \left| \lambda_k^{\sigma+1} f_k \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) g(t-\eta) d\eta \right|^2 \\ \leqslant& C \sum_{k=1}^{j} \left[\lambda_k^{\sigma+1} |f_k| \max_{0 \leqslant t \leqslant T} |g(t)| \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) d\eta \right]^2 \end{aligned}$$

$$\leqslant C\sum_{k=1}^{j}\left[\lambda_k^{\sigma+1}|f_k|\max_{0\leqslant t\leqslant T}|g(t)|t^{\rho}E_{\rho,\rho+1}(-\lambda_k t^{\rho})\right]^2,\qquad t>0.$$

Lemma 2.2 implies

$$\|-\Delta S_j(x,t)\|_{C(\Omega)}\leqslant C\max_{0\leqslant t\leqslant T}|g(t)|f\|_{\sigma},\qquad t>0.$$

Hence, $-\Delta\omega(x,t)\in C(\overline{\Omega}\times(0,T])$ and in particular $\omega(x,t)\in C(\overline{\Omega}\times[0,T])$. Then from equation (3.3) one has

$$D_t^{\rho}S_j(x,t)=\Delta S_j(x,t)+\sum_{k=1}^{j}f_k g(t)v_k(x),\qquad t>0.$$

Therefore, from the above reasoning, we have $D_t^{\rho}\omega(x,t)\in C(\overline{\Omega}\times(0,T])$. The uniqueness of the solution follows from the completeness of the system $\{v_k(x)\}$ in $L_2(\Omega)$. □

We proceed to solving main problem (1.1). We note that if $y(x,t)$ and $\omega(x,t)$ are solutions of problems (3.1) and (3.3), respectively, then the function $u(x,t)=y(x,t)+\omega(x,t)$ is a solution to problem (1.1). Therefore, we can use the already proven assertions and obtain the following result.

Theorem 3.3. *Let $\varphi(x)$, $f(x)$ satisfy conditions (2.2) and $g(t)\in C[0,T]$. Then problem (1.1) has a unique solution*

$$u(x,t)=\sum_{k=1}^{\infty}\left[\varphi_k E_{\rho}(-\lambda_k t^{\rho})+f_k\int_0^t\eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^{\rho})g(t-\eta)d\eta\right]v_k(x).\tag{3.5}$$

4. Well-posedness of inverse problem (1.1), (1.2)

We apply additional condition (1.2) to equation (3.5) and denote by ψ_k the Fourier coefficients of function $\psi(x):\psi_k=(\psi,v_k)$. Then

$$\sum_{k=1}^{\infty}f_k b_{k,\rho}(t_0)v_k(x)=\sum_{k=1}^{\infty}\psi_k v_k(x)-\sum_{k=1}^{\infty}\varphi_k E_{\rho}(-\lambda_k t_0^{\rho})v_k(x),\tag{4.1}$$

where

$$b_{k,\rho}(t)=\int_0^t(t-s)^{\rho-1}E_{\rho,\rho}(-\lambda_k(t-s)^{\rho})g(s)ds.$$

From here, to find f_k, we obtain the following equation

$$f_k b_{k,\rho}(t_0)=\psi_k-\varphi_k E_{\rho}(-\lambda_k t_0^{\rho}).\tag{4.2}$$

Of course, the case $b_{k,\rho}(t_0)=0$ is critical. This can happen when $g(t)$ changes sign. The following example shows that for such $g(t)$ the uniqueness of the unknowns f_k can be violated, see also [8].

Example 1. We consider the following homogeneous inverse problem

$$\begin{cases}D_t^{\rho}u(x,t)-\Delta u(x,t)=f(x)g(t), & (x,t)\in\Omega\times(0,T],\\ u(x,t)|_{\partial\Omega}=0, & \\ u(x,0)=0, & x\in\Omega,\\ u(x,t_0)=0, & x\in\Omega.\end{cases}\tag{4.3}$$

Take any eigenfunction v of the Laplace operator subject to homogeneous Dirichlet boundary conditions, i.e. $-\Delta v = \lambda v$ with $v(x)|_{\partial\Omega} = 0$ and set $t_0 = 1$, $T(t) = t^\rho(1-t^b)$, $b > 0$. Then, $u(x,t) = T(t)v(x)$ satisfies problem (4.3) with

$$f(x) = v(x) \qquad \text{and} \qquad g(t) = D_t^\rho T(t) + \lambda T(t).$$

Then, besides the trivial solution $(u, f) = (0, 0)$ to problem (4.3), we also have the following non-trivial solution

$$u(x,t) = T(t)v(x), \qquad f(x) = v(x).$$

It can be shown easily that, for example, for the parameters $b = 0.1$ and $\rho = 0.5$, the function $g(t)$ changes its sign. Indeed, one has

$$g(t) = \frac{\rho B(\rho, 1-\rho)}{\Gamma(1-\rho)} - \frac{(b+\rho)t^\rho B(b+\rho, 1-\rho)}{\Gamma(1-\rho)} + \lambda t^\rho(1-t^b),$$

and

$$g(0) = 0.5\Gamma(0.5) = \frac{\sqrt{\pi}}{2} > 0,$$
$$g(1) = 0.5\Gamma(0.5) - \frac{0.6B(0.6, 0.5)}{\Gamma(0.5)} = \frac{\sqrt{\pi}}{2} - \frac{6\Gamma(0.6)}{\Gamma(1.1)} < 0.$$

We note that $g(t)$ does not belong to $C^1[0,T]$, see Lemma 4.3 below.

Let us divide the set of natural numbers $\mathbb{N}$ into two groups $K_{0,\rho}$ and K_ρ: $\mathbb{N} = K_\rho \cup K_{0,\rho}$, while the number k is assigned to $K_{0,\rho}$, if $b_{k,\rho}(t_0) = 0$, and if $b_{k,\rho}(t_0) \neq 0$, then this number is assigned to K_ρ. Note that for some t_0 the set $K_{0,\rho}$ can be empty, then $K_\rho = \mathbb{N}$. For example, if $g(t)$ is sign-preserving, then $K_\rho = \mathbb{N}$, for all t_0.

There arises a natural question about the size of set $K_{0,\rho}$, i.e., how many elements does $K_{0,\rho}$ contain? As the authors of paper [6] noted, at least for $\rho = 1$, the set $K_{0,1}$ can contain infinitely many elements. Indeed, in this case

$$b_{k,1}(t_0) = \int\limits_0^{t_0} e^{-\lambda_k(t_0-s)} g(s)ds,$$

and according to Muntz's theorem (see monograph by S. Kaczmarz and H. Steinhouse [34]), the set $K_{0,1}$ for some continuous functions $g(t)$ contains infinitely many elements, see also [35].

In the case of the diffusion equation, the criterion for the uniqueness of a solution of the inverse problem was studied in the papers cited above [2], [3], [4], [5], [6]. This criterion can be formulated as follows: the inverse problem has a unique solution if and only if

$$b_{k,1}(t_0) \neq 0. \tag{4.4}$$

From equation (4.2) for finding f_k it easily follows that the criterion for the uniqueness of the solution of the inverse problem for the subdiffusion equation has a similar form:

$$b_{k,\rho}(t_0) \neq 0. \tag{4.5}$$

Let us establish two-sided estimates for $b_{k,\rho}(t_0)$. First we suppose that $g(t)$ does not change sign, for the diffusion equation, i.e. for $b_{k,1}(t_0)$, see Sabitov et al. [3], [4]. Then $K_{0,\rho}$ is empty.

Lemma 4.1. *Let $g(t) \in C[0,T]$ and $g(t) \neq 0$, $t \in [0,T]$. Then there are constants $C_0, C_1 > 0$, depending on t_0, such that for all k:*

$$\frac{C_0}{\lambda_k} \leqslant |b_{k,\rho}(t_0)| \leqslant \frac{C_1}{\lambda_k}.$$

Proof. By virtue of the Weierstrass theorem, we have $|g(t)| \geqslant g_0 = const > 0$. We apply the mean value theorem and Lemma 2.5 to obtain

$$\begin{aligned}|b_{k,\rho}(t_0)| =& \left|\int\limits_0^{t_0} \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) g(t_0-\eta) d\eta\right| \\ =& |g(\xi_k)| t_0^\rho E_{\rho,\rho+1}(-\lambda_k t_0^\rho), \qquad \xi_k \in [0,t_0].\end{aligned}$$

It is easy to see that

$$E_{\rho,\rho+1}(-t) = t^{-1}(1 - E_\rho(-t)).$$

Therefore, using Lemma 2.3 and the estimate $|g(t)| \geqslant g_0$ one has

$$|b_{k,\rho}(t_0)| = |g(\xi_k)| \frac{1}{\lambda_k}(1 - E_\rho(-\lambda_k t_0^\rho)) \geqslant \frac{C_0}{\lambda_k}.$$

Finally Lemma 2.2 implies

$$|b_{k,\rho}(t_0)| \leqslant C \frac{|g(\xi_k)| t_0^\rho}{1 + \lambda_k t_0^\rho} \leqslant C \frac{\max\limits_{0\leqslant\xi\leqslant t_0} |g(\xi)|}{\lambda_k} \leqslant \frac{C_1}{\lambda_k}.$$

□

Theorem 4.1. *Let $\rho \in (0,1]$, $g(t) \in C[0,T]$ and $g(t) \neq 0$, $t \in [0,T]$. Moreover, let the function $\varphi(x)$ satisfy condition (2.2) and $\psi(x)$ satisfy condition (2.3). Then there exists a unique solution of inverse problem (1.1)-(1.2):*

$$f(x) = \sum_{k=1}^\infty \frac{1}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0^\rho)\right] v_k(x), \tag{4.6}$$

$$u(x,t) = \sum_{k=1}^\infty \varphi_k E_\rho(-\lambda_k t^\rho) v_k(x) + \sum_{k=1}^\infty \frac{b_{k,\rho}(t)}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0^\rho)\right] v_k(x). \tag{4.7}$$

For the diffusion equation ($\rho = 1$), this theorem is proved only in cases where Ω is an interval on $\mathbb{R}$, see [3], or a rectangle in $\mathbb{R}^2$, see [4]. This is a new theorem for subdiffusion equations ($\rho \in (0,1)$).

Proof. Since $b_{k,\rho}(t_0) \neq 0$ for all $k \in \mathbb{N}$, we get the following equations from (4.2):

$$f_k = \frac{1}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0^\rho)\right], \tag{4.8}$$

$$u_k(t) = \varphi_k E_\rho(-\lambda_k t^\rho) + \frac{b_{k,\rho}(t)}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0^\rho)\right]. \tag{4.9}$$

With these Fourier coefficients, we have the following series for the unknown functions $f(x)$ and $u(x,t)$:

$$f(x) = \sum_{k=1}^\infty \frac{1}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0^\rho)\right] v_k(x) = \sum_{k=1}^\infty [f_{k,1} + f_{k,2}] v_k(x), \tag{4.10}$$

$$u(x,t) = \sum_{k=1}^\infty \varphi_k E_\rho(-\lambda_k t^\rho) v_k(x) + \sum_{k=1}^\infty \frac{b_{k,\rho}(t)}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0^\rho)\right] v_k(x). \tag{4.11}$$

If $F_j(x)$ is the partial sums of series (4.10), then applying Lemma 2.1 as above we have

$$\|\hat{A}^{-\sigma} F_j(x)\|^2_{C(\Omega)} \leqslant \sum_{k=1}^j \lambda_k^{2\sigma} |f_{k,1} + f_{k,2}|^2 \leqslant 2 \sum_{k=1}^j \lambda_k^{2\sigma} f_{k,1}^2 + 2 \sum_{k=1}^j \lambda_k^{2\sigma} f_{k,2}^2 \equiv 2I_{1,j} + 2I_{2,j}, \tag{4.12}$$

where $\sigma > \frac{N}{4}$. Therefore by Lemma 4.1 one has

$$I_{1,j} \leqslant \sum_{k=1}^{j} \frac{\lambda_k^{2\sigma}}{|b_{k,\rho}(t_0)|^2}|\psi_k|^2 \leqslant C\sum_{k=1}^{j}\lambda_k^{\tau+2}|\psi_k|^2, \quad \tau = 2\sigma > \frac{N}{2}, \tag{4.13}$$

$$I_{2,j} \leqslant \sum_{k=1}^{j} \left|\frac{E_\rho(-\lambda_k t_0^\rho)}{b_{k,\rho}(t_0)}\right|^2 \lambda_k^{2\sigma}|\varphi_k|^2 \leqslant C\sum_{k=1}^{j}\lambda_k^{\tau}|\varphi_k|^2, \quad \tau = 2\sigma > \frac{N}{2}. \tag{4.14}$$

Thus, if $\varphi(x)$ satisfies conditions (2.2) and $\psi(x)$ satisfies conditions (2.3), then from estimates of $I_{i,j}$ and (4.12) we obtain $f(x) \in C(\overline{\Omega})$. Further, the fact that function $u(x,t)$ given by (4.11) is a solution to the inverse problem is proved exactly as the proof of Theorem 3.3. Here we also apply Lemma 4.1.

To prove the uniqueness of the solution, we assume the contrary. Let there exist two different solutions $\{u_1, f_1\}$ and $\{u_2, f_2\}$ satisfying inverse problem (1.1)-(1.2). We need to show that $u \equiv u_1 - u_2 \equiv 0$, $f \equiv f_1 - f_2 \equiv 0$. For $\{u, f\}$ we have the following problem:

$$\begin{cases} D_t^\rho u(x,t) - \Delta u(x,t) = f(x)g(t), \quad (x,t) \in \Omega \times (0,T], \\ u(x,t)|_{\partial\Omega} = 0, \\ u(x,0) = 0, \quad x \in \Omega, \\ u(x,t_0) = 0, \quad x \in \Omega, \quad t_0 \in (0,T]. \end{cases} \tag{4.15}$$

We take any solution $\{u, f\}$ and define $u_k = (u, v_k)$ and $f_k = (f, v_k)$. Then, due to the self-adjointness of the operator $-\Delta$ and the continuity of the derivatives of the solution up to the boundary of the domain Ω, we have

$$D_t^\rho u_k(t) = (D_t^\rho u, v_k) = (\Delta u, v_k) + f_k g(t) = (u, \Delta v_k) + f_k g(t) = -\lambda_k u_k(t) + f_k g(t).$$

Therefore, for u_k we obtain the Cauchy problem

$$D_t^\rho u_k(t) + \lambda_k u_k(t) = f_k g(t), \quad t > 0, \quad u_k(0) = 0,$$

and the additional condition

$$u_k(t_0) = 0.$$

If f_k is known, then the unique solution of the Cauchy problem has the form

$$u_k(t) = f_k \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) g(t-\eta) d\eta = f_k b_{k,\rho}(t).$$

Apply the additional condition to get

$$u_k(t_0) = f_k b_{k,\rho}(t_0) = 0.$$

Since $b_{k,\rho}(t_0) \neq 0$ for all $k \in \mathbb{N}$, then due to completeness of the set of eigenfunctions $\{v_k\}$ in $L_2(\Omega)$, we finally have $f(x) \equiv 0$ and $u(x,t) \equiv 0$. $\square$

Now consider the case when $g(t)$ changes sign. In this case, function $b_{k,\rho}(t_0)$ can become zero, and as a result, the set $K_{0,\rho}$ may turn out to be non-empty. Now we should consider separately the case of diffusion ($\rho = 1$) and subdiffusion ($0 < \rho < 1$) equations.

Lemma 4.2. *Let $\rho = 1$, $g(t) \in C^1[0,T]$ and $g(t_0) \neq 0$. Then there exists a number k_0 such that, starting from the number $k \geqslant k_0$, the following estimates hold:*

$$\frac{C_0}{\lambda_k} \leqslant |b_{k,1}(t_0)| \leqslant \frac{C_1}{\lambda_k}, \tag{4.16}$$

where constants C_0 and $C_1 > 0$ depend on k_0 and t_0.

Proof. By integrating by parts and the mean value theorem, we get

$$b_{k,1}(t_0) = \int\limits_0^{t_0} e^{-\lambda_k s} g(t_0 - s) ds = -\frac{1}{\lambda_k} g(t_0 - s) e^{-\lambda_k s}\Big|_0^{t_0} - \frac{1}{\lambda_k}\int\limits_0^{t_0} e^{-\lambda_k s} g'(t_0 - s) ds$$

$$=\frac{1}{\lambda_k}\big[g(t_0) - g(0)e^{-\lambda_k t_0}\big] + \frac{g'(\xi_k)}{\lambda_k^2}\big[e^{-\lambda_k t_0} - 1\big], \qquad \xi_k \in [0, t_0].$$

Therefore, there exists a constant C_0 such that the required lower bound holds. The upper estimate follows from the boundedness of function $g(t)$. □

Corollary 4.1. *If conditions of Lemma 4.2 are satisfied, then estimate (4.16) holds for all* $k \in K_1$.

Corollary 4.2. *If conditions of Lemma 4.2 are satisfied, then set* $K_{0,1}$ *has a finite number elements.*

In case of subdiffusion equation ($\rho \in (0, 1)$) we have

Lemma 4.3. *Let* $\rho \in (0, 1)$, $g(t) \in C^1[0, T]$ *and* $g(0) \neq 0$. *Then there exist numbers* $m_0 > 0$ *and* k_0 *such that, for all* $t_0 \leqslant m_0$ *and* $k \geqslant k_0$, *the following estimates hold:*

$$\frac{C_0}{\lambda_k} \leqslant |b_{k,\rho}(t_0)| \leqslant \frac{C_1}{\lambda_k}, \tag{4.17}$$

where constants C_0 *and* $C_1 > 0$ *depend on* m_0 *and* k_0.

Proof. Let $\rho \in (0, 1)$. Using equality (2.6) we integrate by parts, then apply the mean value theorem. Then we have

$$b_{k,\rho}(t_0) = \int\limits_0^{t_0} g(t_0 - s) s^{\rho-1} E_{\rho,\rho}(-\lambda_k s^\rho) ds = \int\limits_0^{t_0} g(t_0 - s) d\big[s^\rho E_{\rho,\rho+1}(-\lambda_k s^\rho)\big]$$

$$=g(t_0 - s) s^\rho E_{\rho,\rho+1}(-\lambda_k s^\rho)\Big|_0^{t_0} + \int\limits_0^{t_0} g'(t_0 - s) s^\rho E_{\rho,\rho+1}(-\lambda_k s^\rho) ds$$

$$=g(0)\, t_0^\rho\, E_{\rho,\rho+1}(-\lambda_k t_0^\rho) + g'(\xi_k)\int\limits_0^{t_0} s^\rho E_{\rho,\rho+1}(-\lambda_k s^\rho) ds, \qquad \xi_k \in [0, t_0].$$

For the last integral formula (2.6) implies

$$\int\limits_0^{t_0} s^\rho E_{\rho,\rho+1}(-\lambda_k s^\rho) ds = t_0^{\rho+1} E_{\rho,\rho+2}(-\lambda_k t_0^\rho).$$

We apply the asymptotic estimate of the Mittag-Leffler functions (Lemma 2.4) to get

$$b_{k,\rho}(t_0) = \frac{g(0)}{\lambda_k} + \frac{g'(\xi_k)}{\lambda_k} t_0 + O\bigg(\frac{1}{(\lambda_k t_0^\rho)^2}\bigg).$$

If $g(0) \neq 0$, then for sufficiently small t_0 and sufficiently large k we obtain the required lower estimate. This also implies the required upper bound. □

Corollary 4.3. *If conditions of Lemma 4.3 are satisfied, then estimate (4.17) holds for all* $t_0 \leqslant m_0$ *and* $k \in K_\rho$.

Corollary 4.4. *If conditions of Lemma 4.3 are satisfied and* t_0 *is sufficiently small, then set* $K_{0,\rho}$ *has a finite number elements.*

Theorem 4.1 proves the existence and uniqueness of a solution to the inverse problem (1.1)-(1.2) under condition $g(t) \in C[0,T]$ and $g(t) \neq 0$, $t \in [0,T]$, i.e., $g(t)$ does not change sign. In Example 1, we saw that if this condition is violated, then the uniqueness of the solution to problem (1.1)-(1.2) is violated. This naturally give rise to the questions: if $g(t)$ changes sign, is uniqueness always violated? What can be said about the existence of a solution? How many solutions can there be?

It should be emphasized that the answers to these questions were not known even for the classical diffusion equation (i.e. $\rho = 1$).

Lemmas 4.2 and 4.3 proved above allow us to answer these questions. Let us formulate the corresponding result.

Theorem 4.2. *Let $g(t) \in C^1[0,T]$, function $\varphi(x)$ satisfy condition (2.2) and $\psi(x)$ satisfy condition (2.3). Further, we assume that for $\rho = 1$ the conditions of Lemma 4.2 are satisfied, and for $\rho \in (0,1)$, the conditions of Lemma 4.3 are satisfied and t_0 is sufficiently small.*

1) If set $K_{0,\rho}$ is empty, i.e. $b_{k,\rho}(t_0) \neq 0$, for all k, then there exists a unique solution of the inverse problem (1.1)-(1.2):

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0^\rho)\right] v_k(x), \tag{4.18}$$

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k E_\rho(-\lambda_k t^\rho) v_k(x) + \sum_{k=1}^{\infty} \frac{b_{k,\rho}(t)}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0^\rho)\right] v_k(x). \tag{4.19}$$

2) If set $K_{0,\rho}$ is not empty, then for the existence of a solution to the inverse problem, it is necessary and sufficient that the following conditions

$$\psi_k = \varphi_k E_\rho(-\lambda_k t_0^\rho), \qquad k \in K_{0,\rho}, \tag{4.20}$$

be satisfied. In this case, the solution to the problem (1.1)-(1.2) exists but is not unique:

$$f(x) = \sum_{k\in K_\rho} \frac{1}{b_{k,\rho}(t_0)} \left[\psi_k - \varphi_k E_\rho(-\lambda_k t_0^\rho)\right] v_k(x) + \sum_{k\in K_{0,\rho}} f_k v_k(x), \tag{4.21}$$

$$u(x,t) = \sum_{k=1}^{\infty} \big[\varphi_k E_\rho(-\lambda_k t^\rho) + f_k\big] v_k(x), \tag{4.22}$$

where f_k, $k \in K_{0,\rho}$, are arbitrary real numbers.

Proof. The proof of the first part of the theorem is completely analogous to the proof of Theorem 4.1. As regards the proof of the second part of the theorem, we note the following.

If $k \in K_\rho$, then again from (4.2) we have (4.8) and (4.9).

If $k \in K_{0,\rho}$, i.e. $b_{k,\rho}(t_0) = 0$, then the solution of equation (4.2) with respect to f_k exists if and only if the conditions (4.20) are satisfied. In this case, the solution of the equation can be arbitrary numbers f_k. As shown above (see Corollaries 4.2 and 4.4), under the conditions of the theorem, the set $K_{0,\rho}$, $\rho \in (0,1]$, contains finitely many elements. □

Note that condition (4.20) is rather difficult to verify. Given relation $E_\rho(-t) \neq 0$, $t > 0$ (see Lemma 2.3), one can replace this condition with a simpler condition.

Remark 4.1. *For conditions (4.20) to be satisfied, it suffices that the following orthogonality conditions hold:*

$$\varphi_k = (\varphi, v_k) = 0, \;\; \psi_k = (\psi, v_k) = 0, \;\; k \in K_{0,\rho}.$$

In other words, if the symbol H_0 denotes a subspace of $L_2(\Omega)$ spanned by a linear combination of eigenfunctions $v_k(x)$, $k \in K_{0,\rho}$ then in order for conditions (4.20) to be satisfied, it is sufficient that φ and ψ to be orthogonal to H_0.

Let us briefly mention some known results on inverse problems for the diffusion equation (i.e., $\rho = 1$). In work by D.G. Orlovskii [5] abstract diffusion equations in Banach and Hilbert spaces were considered. In the case of a Hilbert space, the elliptic part of the equation is self-adjoint, and the found uniqueness criterion is similar to (4.5). A condition on the function $b_{k,1}(T)$ is found, which ensures the existence of a generalized solution (note that here condition (4.5) is given at the point $t_0 = T$).

In I.V. Tikhonov, Yu.S. Éidel'man [6], abstract diffusion equations in Banach and Hilbert spaces are also considered. In the case of a self-adjoint elliptic part, the uniqueness criterion coincides with (4.5). It is shown that if we consider equations in a Banach space, then condition (4.5) is not a criterion, and an addition to (4.5) is found that turns (4.5) into a uniqueness criterion for equations with a non-conjugate elliptic part.

The elliptic part of the diffusion equation in work A.I. Prilepko, A.B. Kostin [2] is a second-order differential expression. Both non-self-adjoint and self-adjoint elliptic parts are considered. In this paper, $g(t)$ also depends on the spatial variable: $g(t) := g(x,t)$. In the case of a self-adjoint elliptic part, the authors succeeded to find a criterion for the uniqueness of the generalized solution of the inverse problem: the solution is unique if and only if the system

$$w_k(x) = v_k(x) \int\limits_0^{t_0} g(x,t) e^{-\lambda_k(t_0-t)} dt, \quad k = 1, 2, \cdots$$

is complete in $L_2(\Omega)$. It is easy to see that if $g(x,t)$ is independent of x, then this criterion coincides with (4.5). It should be emphasized that the Fourier method is not applicable to the equation considered in this paper.

The closest to our research are works by K.B. Sabitov and A.R. Zaynullov [3] and [4]. We borrowed some ideas from these works. In work [3] the elliptic part of the equation is u_{xx} defined on an interval (in [4] this was the Laplace operator on the rectangle). Having considered the over-determination condition in form (1.2), it is shown that the criterion for the uniqueness of the classical solution is (4.5). When condition (4.5) is satisfied, a classical solution is constructed by the Fourier method. We note that the existence of a classical solution was not discussed in the works listed above.

5. Conclusion

In this paper, we consider the subdiffusion equation with a fractional derivative of order $\rho \in (0, 1]$, and take the Laplass operator as the elliptic part. The right-hand side of the equation has the form $f(x)g(t)$, where $g(t)$ is a given function and the inverse problem of determining function $f(x)$ is considered. Following works [2] and [3], the over-determination condition is taken in a more general form. It is proved that the criterion for the uniqueness of the classical solution of the inverse problem for the subdiffusion equations coincides with the analogous condition for the diffusion equations.

In the case when this condition is not satisfied, a necessary and sufficient condition for the existence of a classical solution is found and all solutions of the inverse problem are constructed using the classical Fourier method. Note that if $g(t)$ changes sign, then it is only known (see [6]) that the set $K_{0,1}$ can contain infinitely many elements. In Corollaries 4.2 and 4.4, exact conditions are found that guarantee the finiteness of the number of elements $K_{0,\rho}$, $\rho \in (0, 1]$ for sign-variables $g(t)$. We emphasize that all the results listed in this paragraph are also new for the classical diffusion equation.

The results of this work can be generalized to more general subdiffusion equations by replacing the Laplace operator in (1.1) with a high-order self-adjoint elliptic operator with variable coefficients. At the same time, instead of the result of V.A. Il'in, similar results by Sh.A. Alimov [36] should be used for a general elliptic operator.

6. Acknowledgement

The authors are grateful to Sh.A. Alimov for discussions of these results.

СПИСОК ЛИТЕРАТУРЫ

1. A.V Pskhu. *Fractional Differential Equations.* Nauka, Moscow (2005) [in Russian].
2. A.I. Prilepko, A.B. Kostin. *On certain inverse problems for parabolic equations with final and integral observation* // Mat. Sb. **183**:4, 49-68 (1992). [Russ. Acad. Sci. Sb. Math. **75**:2, 473–490 (1993).]
3. K.B. Sabitov, A.R. Zaynullov. *On the theory of the known inverse problems for the heat transfer equation* // Uchenye Zapiski Kazanskogo Univ. Ser. Fiz.-Matem. Nauki. **161**:2, 274-291 (2019). (in Russian).
4. K.B. Sabitov, A.R. Zaynullov. *Inverse problems for a two-dimensional heat equation with unknown right-hand side* // Russian Math. **65**:3, 75–88 (2021).
5. D.G. Orlovskii. *Determination of a parameter of an evolution equation* // Differ. Uravn. **26**:9, 1614-1621 (1990). [Diff. Equat. **26**:9, 1201–1207 (1990).]
6. I.V. Tikhonov, Yu.S. Éidel'man. *Uniqueness criterion in an inverse problem for an abstract differential equation with nonstationary inhomogeneous term* // Matem. Zamet. **77**:2, 273-290 (2005). [Math. Notes. **77**:2, 246–262 (2005).]
7. M. Slodichka. *Uniqueness for an inverse source problem of determining a space-dependent source in a non-autonomous time-fractional diffusion equation* // Frac. Calc. Appl. Anal. **23**:6, 1703–1711 (2020).
8. M. Slodichka, K. Sishskova, V. Bockstal. *Uniqueness for an inverse source problem of determining a space dependent source in a time-fractional diffusion equation* // Appl. Math. Lett. **91**, 15-21 (2019).
9. O.A. Ladyzhenskaya. *Mixed problem for a hyperbolic equation.* Gostexizdat, Moscow (1953). (in Russian.)
10. Y. Liu, Z. Li, M. Yamamoto. *Inverse problems of determining sources of the fractional partial differential equations* // in "Fractional Differential Equations. V. 2." Eds. A. Kochubei, Yu. Luchko, De Gruyter. 411–430 (2019).
11. R. Ashurov, M. Shakarova. *Time-dependent source identification problem for fractional Schrödinger type equations* // Lobachevskii J. Math. **42**:3, 517-525 (2022).
12. R. Ashurov, M. Shakarova. *Time-dependent source identification problem for a fractional Schrödinger equation with the Riemann-Liouville derivative* // Ukrainian Math. J. **75**, 997–1015 (2023).
13. K. Sakamoto, M. Yamamoto. *Initial value boundary value problems for fractional diffusion-wave equations and applications to some inverse problems* // J. Math. Anal. Appl. **382**:1, 426–447 (2011).
14. K. Furati, O. Iyiola, M. Kirane. *An inverse problem for a generalized fractional diffusion* // Appl. Math. Comp. **249**, 24–31 (2014).
15. M. Kirane, A. Malik. *Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time* // Appl. Math. Comp. **218**, 163-170 (2011).
16. M. Kirane, B. Samet, B. Torebek. *Determination of an unknown source term and the temperature distribution for the subdiffusion equation at the initial and final data* // Electr. J. Diff. Equat. **217**, 1–13 (2017).
17. Z. Li, Y. Liu, M. Yamamoto. *Initial-boundary value problem for multi-term time-fractional diffusion equation with positive constant coefficients* // Appl. Math. Comp. **257**, 381–397 (2015).
18. S. Malik, S. Aziz. *An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions* // Comp. Math. Appl. **73**:12, 2548–2560 (2017).
19. M. Ruzhansky, N. Tokmagambetov, B.T. Torebek. *Inverse source problems for positive operators I: Hypoelliptic diffusion and subdiffusion equations* // J. Inverse Ill-Possed Probl. **27**:6, 891–911 (2019).

20. R.R. Ashurov, A.T. Mukhiddinova. *Inverse problem of determining the heat source density for the subdiffusion equation* // Diff. Equat. **56**:12, 1550–1563 (2020).
21. R.R. Ashurov, Yu.E. Fayziev. *Determination of fractional order and source term in a fractional subdiffusion equation* // Eurasian Math. J. **13**:1, 19-31 (2022).
22. K.B. Sabitov. *Direct and inverse problems for of mixed type parabolic-hyperbolic equations.* Nauka, Moscow (2016) [in Russian].
23. S.I. Kabanikhin. *Inverse and Ill-Posed Problems. Theory and Applications.* De Gruyter, Berlin (2011).
24. V.E. Fedorov, A.V. Nagumanova. *Inverse problem for evolutionary equation with the Gerasimov-Caputo fractional derivative in the sectorial case* // The Bulletin of Irkutsk State University, Series Mathematics. **28**, 123–137 (2019). (in Russian).
25. S. Liu, F. Sun, L. Feng. *Regularization of inverse source problem for fractional diffusion equation with Riemann-Liouville derivative* // Comp. Appl. Math. **40**, id 112 (2021).
26. P. Niu, T. Helin, Z. Zhang. *An inverse random source problem in a stochastic fractional diffusion equation* // Inver. Probl. **36**:4, id 045002 (2020).
27. R. Ashurov, A. Mukhiddinova. *Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator* // Lobachevskii J. Math. **42**:3, 517–525 (2021).
28. M.A. Krasnoselskii, P.P Zabreyko, E.I. Pustilnik, P.E. Sobolevski. *Integral operators in the spaces of integrable functions.* Nauka, Moscow (1966) (in Russian).
29. V.A. Il'in. *On the solvability of mixed problems for hyperbolic and parabolic equations* // Russ. Math. Surv. **15**:2, 97–154 (1960).
30. M.M. Dzherbashian. *Integral Transforms and Representation of Functions in the Complex Domain.* Nauka, Moscow (1966) (in Russian).
31. R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogozin. *Mittag-Leffler Functions, Related Topics and Applications.* Springer, Berlin (2014).
32. R. Ashurov, R. Zunnunov. *Intial-boundary value and inverse problems for subdiffusion equations in* $\mathbb{R}^N$ // Fract. Diff. Calc. **10**:2, 291–306 (2020).
33. R. Ashurov, A. Cabada, B. Turmetov. *Operator method for construction of solutions of linear fractional differential equations with constant coefficients* // Fract. Calc. Appl. Anal. 1, 229–252 (2016).
34. S. Kaczmarz, H. Steinhaus. *Theorie der Orthogonalreihen*, Seminarium Matematyczne Uniwersytetu Warszawskiego, Warszawa (1935).
35. A.F. Leont'ev. *Exponential series*, Nauka, Moscow (1976). (in Russian).
36. Sh.A. Alimov, V.A. Il'in, E.M. Nikishin. *Convergence problems of multiple trigonometrical series and spectral decompositions. I* // Russ. Math. Surv. **31**:6, 28–83 (1976).

Ravshan Radjabovich Ashurov,
Institute of Mathematics, Uzbekistan Academy of Science,
Student Town str.,
100174, Tashkent, Uzbekistan
University of Tashkent for Applied Sciences,
Gavhar str. 1,
100149, Tashkent, Uzbekistan
E-mail: `ashurovr@gmail.com`

Marjona Dilshod qizi Shakarova,
Institute of Mathematics, Uzbekistan Academy of Science,
Student Town str.,
100174, Tashkent, Uzbekistan
E-mail: `shakarova2104@gmail.com`