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ON SOLVABILITY OF SOME CLASSES OF URYSOHN NONLINEAR INTEGRAL EQUATIONS WITH NONCOMPACT OPERATORS

KH.A. KHACHATRYAN

Abstract. In present paper the classes of nonlinear integral equations with non completely continuous operators are considered.
It is assumed that conservative nonlinear operator of Wiener-Hopf-Hankell-Hammerstein type is a linear minorant to the initial Urysohn operator. The alternative theorems of the existence of positive solutions of above-mentioned class equations are proved. The asymptotic behavior of the obtained solutions at infinity is investigated. The article is finalized by the presentation of some examples arising in applications.

Keywords: Wiener-Hopf operator, eighen-value, limit of solution, one parameter family of positive solutions, asymptotic properties, Caratede'ory condition.

1. Introduction

The present work is devoted to the solvability and investigation of asymptotic behavior of solutions of the following classes nonlinear integral equations of Urysohn's type

$$\varphi(x) = \int_0^\infty U(x,t,\varphi(t))dt, \quad x \in \mathbb{R}^+ \tag{1.1}$$

in regard to unknown measurable real function $\varphi(x)$. Here $U(x,t,\tau)$ is defined on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ real function, satisfying the following conditions:
$a)$ There exists number $A > 0$, such that $U(x,t,\tau) \uparrow$ by τ on $[A,+\infty)$, for each fixed pairs $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$.
$b)$ The function $U(x,t,\tau)$ satisfies Caratede'ory condition on set $\Delta_A = \mathbb{R}^+ \times \mathbb{R}^+ \times [A,+\infty)$ by τ, i.e. for each fixed $\tau \in [A,+\infty)$, $U(x,t,\tau)$ is measurable by $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ and for almost all $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, $U(x,t,\tau)$ is continuous by τ on $[A,+\infty)$.
$c)$ Let $\overset{\circ}{K}(x)$ is defined on $\mathbb{R}$ and summerable function of the following structure:

$$\overset{\circ}{K}(x) = \int_a^b e^{-|x|s} d\sigma(s), \tag{1.2}$$

where

$$\sigma \uparrow [a,b), \quad 0 < a < b \leqslant +\infty, \quad 2\int_a^b \frac{d\sigma(s)}{s} = 1. \tag{1.3}$$

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Let ω is the measurable function on $\mathbb{R}$, and

$$\omega \in L_1(0,+\infty) \cap C(0,+\infty) \quad m_1(\omega) \equiv \int\limits_0^\infty x\omega(x)dx < +\infty, \tag{1.4}$$

$$\omega(x) \geq 0, \quad x \in [A,+\infty), \quad \omega \downarrow \text{ by } x \text{ on } [A,+\infty). \tag{1.5}$$

We also assume, that function $U(x,t,\tau)$ satisfies the following inequality: there exists non negative function $K^*(x)$, $\quad x \in \mathbb{R}^+$, $\quad K^*(x) < \overset{\circ}{K}(x)$, $\quad x \in \mathbb{R}^+$ and number $p > 1$, such that

$$U(x,t,\tau) \geq (\overset{\circ}{K}(x-t) - K^*(x+pt))(\tau - \omega(t+\tau)), \tag{1.6}$$

for all $(x,t,\tau) \in \Delta_A$.

Historically Urysohn equation was studied in case when limit of integration is finite, and corresponding Urysohn operator is completely continuous in considered banach spaces (see for example and [1]-[6]).

Equation (1.1), besides independent mathematical interest, has an important applications in different fields of mathematical physics (see [7]-[9]). In particular case, *i)* when $U(x,t,\tau) = \overset{\circ}{K}(x-t)(\tau - \omega(\tau+t))$ equation (1.1) was studied in works [10]-[12], *ii)* when in (1.6) $K^* \equiv 0$ equation (1.1) was considered in works [12]-[14].

In present work putting additional condition on function U the alternative theorems of existence of positive solution is proved. Asymptotic property of the obtained solutions at infinity is also investigated. At the end of the work the obtained results are illustrated by examples.

2. Auxiliary Facts and Denotations

Let E–be one of the following Banach spaces $L_p(0,+\infty)$, $1 \leqslant p < +\infty$, $M(0,+\infty)$, $C_M(0,+\infty)$, $C_0(0,+\infty)$. We denote by Ω the class of Wiener-Hopf integral operators: $\mathcal{K} \in \Omega$ if function $K \in L_1(\mathbb{R})$ exists, such that

$$(\mathcal{K}f)(x) = \int\limits_0^\infty K(x-t)f(t)dt, \quad f \in E. \tag{2.1}$$

Operators $\mathcal{K} \in \Omega$ act in Banach spaces E, still these operators are not completely continuous in E (see [15],[17]).

It is known that in each of space E, the norm of operators $\mathcal{K} \in \Omega$ is estimated (upper) by the mentioned below form.

$$\|\mathcal{K}\|_E \leqslant \int\limits_{-\infty}^{+\infty} |K(z)|dz. \tag{2.2}$$

We also introduce the following class of Henkell operators: $\mathcal{K}^* \in \Omega^*$ measurable function $K^* \in L_1(0,+\infty)$, $\quad m_1(K^*) \equiv \int\limits_0^\infty xK^*(x)dx < +\infty$, such that

$$(\mathcal{K}^*f)(x) = \int\limits_0^\infty K^*(x+pt)f(t)dt, \quad p \geq 1, \tag{2.3}$$

$$f \in E. \tag{2.4}$$

In contradistinction to Wiener-Hopf operators, the Henkell integral operators are completely continuous in E. Let $\Omega^{\pm} \subset \Omega$–are class of the following lower and upper Vollteryan type integral operators: $V_{\pm} \in \Omega^{+}$ if functions $v_{\pm} \in L_1(0,+\infty)$ exist such that

$$(V_- f)(x) = \int\limits_x^{\infty} v_-(t-x)f(t)dt, \quad (V_+ f)(x) = \int\limits_0^{x} v_+(x-t)f(t)dt, \quad f \in E. \tag{2.5}$$

Let kernel $\overset{\circ}{K}$ of Wiener-Hopf integral operator is given by formulae (1.6). From the results of the work [16] it follows that operator $I - \overset{\circ}{\mathcal{K}}$ permits the following factorization

$$I - \overset{\circ}{\mathcal{K}} = (I - V_+)(I - H)(I - V_+). \tag{2.6}$$

It means as an equality of integral operators acting in E. Here I–is an unite operator, $V_{\pm} \in \Omega^{\pm}$ are operators of the following simple structures:

$$(V_- f)(x) = a\int\limits_x^{\infty} e^{-a(t-x)}f(t)dt, \quad (V_+ f)(x) = a\int\limits_0^{x} e^{-a(x-t)}f(t)dt, \tag{2.7}$$

$f \in E$, and $H \in \Omega$, kernel of H is the form of

$$h(x) = \int\limits_a^{b} e^{-|x|s}\left(1 - \frac{a^2}{s^2}\right)d\sigma(s) \tag{2.8}$$

From (2.8) it follows that

$$h(x) \geq 0, \ x \in \mathbb{R}^+, \quad \int\limits_{-\infty}^{+\infty} h(x)dx = 1 - 2a^2\int\limits_a^{b} \frac{d\sigma(s)}{s^3} \equiv \rho < 1. \tag{2.9}$$

Taking into account (2.2) we state that operator $H \in \Omega$, in contradistinction to initial operator $\overset{\circ}{\mathcal{K}} \in \Omega$, is contractive in each spaces E with contraction coefficient $\rho < 1$.

We denote by $I + \Phi_{\pm}$ resolvent operators of Volteryan operators $I - V_{\pm}$ giving by (2.7). It is easy to check that

$$(\Phi_- \varphi)(x) = a\int\limits_x^{\infty} \varphi(t)dt, \quad (\Phi_+ f)(x) = a\int\limits_0^{x} f(t)dt$$

$$\varphi \in L_1(0,+\infty), \quad f \in E. \tag{2.10}$$

The following Lemma will be used in future:

Lemma 1. *Let $K^*(x)$ is the kernel of operator $\mathcal{K}^* \in \Omega^*$, satisfying condition $0 \leqslant K^*(x) < \overset{\circ}{K}(x)$. Then*

$$\Phi_- \mathcal{K}^* \in \Omega^*, \quad \mathcal{K}^* \Phi_+ \in \Omega^*.$$

Proof. For example we prove the second statement. For arbitrary function $f \in E$ we have

$$(\mathcal{K}^* \Phi_+)f(x) = a\int\limits_0^{\infty} K^*(x+pt)\int\limits_0^{t} f(\tau)d\tau dt.$$

Changing the order of integration in last formulae we get

$$(\mathcal{K}^* \Phi_+ f)(x) = a\int\limits_0^{\infty} f(\tau)\int\limits_{\tau}^{\infty} K^*(x+pt)dtd\tau =$$

$$= \frac{a}{p}\int_0^\infty f(\tau)\int_{x+p\tau}^\infty K^*(z)dzd\tau \equiv \int_0^\infty T^*(x+p\tau)f(\tau)d\tau,$$

where

$$T^*(x) = \frac{a}{p}\int_x^\infty K^*(\tau)d\tau. \tag{2.11}$$

To finish the proving of the theorem we have to show that

$$m_j(T^*) = \int_0^\infty x^j T^*(x)dx < +\infty, \quad j = 0, 1. \tag{2.12}$$

Let $r > 0$–is the arbitrary number. We estimate the integral

$$\int_0^r x^j T^*(x)dx = \frac{a}{p}\int_0^r x^j \int_x^r K^*(\tau)d\tau dx +$$

$$+\frac{a}{p}\int_0^r x^j \int_r^\infty K^*(\tau)d\tau dx \leqslant \frac{a}{p(j+1)}\left(\int_0^r \tau^{j+1}K^*(\tau)d\tau + \int_r^\infty \tau^{j+1}K^*(\tau)d\tau\right) =$$

$$= \frac{a}{p(j+1)}\int_0^\infty \tau^{j+1}K^*(\tau)d\tau < +\infty.$$

Thus lemma has been proved.

Corollary. From lemma 1 it immediately follows, that if K^* satisfy condition (1.6), then $\Phi_-\mathcal{K}^*\Phi_+ \in \Omega^*$.

Below using lemma 1 the operator $I - \overset{\circ}{\mathcal{K}} + \mathcal{K}^*$ we represent the form of products of five factors

$$I - \overset{\circ}{\mathcal{K}} + \mathcal{K}^* = (I - V_-)(I - H)(I - V_+) + \mathcal{K}^* = (I - V_-)(I - H + (I + \Phi_-)\mathcal{K}^*(I + \Phi_+))(I - V_+) =$$
$$= (I - V_-)(I - H + T)(I - V_+).$$

As operator $H \in \Omega$ is contractive in E, then $I - H$ permits the factorization (see [13]):

$$I - H = (I - U_-)(I - U_+), \tag{2.13}$$

where $U_\pm \in \Omega_\pm$ are contractive operators of the form:

$$(U_- f)(x) = \int_x^\infty u(t - x)f(t)dt,$$

$$(U_+ f)(x) = \int_0^x u(x - t)f(t)dt, \quad f \in E, \tag{2.14}$$

$$0 \leqslant u \in L_1(0, +\infty), \quad \gamma = \int_0^\infty u(\tau)d\tau < 1. \tag{2.15}$$

Taking into account (2.13) and lemma 2.1 we obtain

$$I - \overset{\circ}{\mathcal{K}} + \mathcal{K}^* = (I - V_-)(I - U_-)(I + W)(I - U_+)(I - V_+), \tag{2.16}$$

where

$$W \equiv (I - U_-)^{-1}T(I - U_+)^{-1} \in \Omega^*. \tag{2.17}$$

In next paragraph we'll study the construction of nontrivial monotonic solution and investigation of asymptotic properties of the following linear equation.

$$S(x) = \int_0^\infty \overset{\circ}{K}(x-t)S(t)dt - \int_0^\infty K^*(x+pt)S(t)dt. \tag{2.18}$$

3. Asymptotic Behavior of Nontrivial Solution of Equation (2.18)

Using factorization (2.16) the solution of equation (2.18) will be written in the form of

$$(I-V_-)(I-U_-)(I+W)(I-U_+)(I-V_+)S = 0. \tag{3.1}$$

The solution of equation (3.1) is equivalent to the sequence solation of the following coupled equations:

$$(I-V_-)S_0 = 0, \tag{3.2}$$

$$(I-U_-)S_1 = S_0, \tag{3.3}$$

$$(I+W)S_2 = S_1, \tag{3.4}$$

$$(I-U_+)S_3 = S_2, \tag{3.5}$$

$$(I-V_+)S = S_3. \tag{3.6}$$

The following two possibilities will be discussed separately.
a) $\varepsilon = -1$ is the eighen-value for the operator W
b) $\varepsilon = -1$ is not eighen-value for the operator W.
First we consider case *a*).

a) From the definition of operator V_- it immediately follows that $S_0(x) \equiv 1$ satisfies equation (3.2). Substituting in (3.3) we come to the following Volltera equation

$$S_1(x) = 1 + \int_x^\infty u(t-x)S_1(t)dt, \quad x \in \mathbb{R}^+. \tag{3.7}$$

In factorization (2.13) passing to equality of symbols at point 0 we obtain

$$1-\rho = (1-\gamma)^2, \quad \gamma \equiv \int_0^\infty u(\tau)d\tau < 1, \tag{3.8}$$

or

$$\gamma = 1 - \sqrt{1-\rho}. \tag{3.9}$$

By direct checking it is confirmed that function

$$S_1(x) = \frac{1}{\sqrt{1-\rho}} \tag{3.10}$$

satisfies equation (3.7).

Now let consider the equation (3.4)

$$S_2(x) = \frac{1}{\sqrt{1-\rho}} - \int_0^\infty W(x+pt)S_2(t)dt, \quad x \in \mathbb{R}^+. \tag{3.11}$$

As $\varepsilon = -1$ is not eighen-value for completely continuous operator $W \in \Omega^*$, then equation (3.11) has bounded solution $S_2(x)$.

Taking into account the well known inequality $||a| - |b|| \leqslant |a - b|$ and from the following simple estimation:

$$\left|\frac{1}{\sqrt{1-\rho}} - S_2(x)\right| \leqslant \sup_{x>0} |S_2(x)| - \frac{1}{p}\int_x^\infty W(\tau)d\tau \in L_1(0,+\infty)$$

we have

$$\frac{1}{\sqrt{1-\rho}} - S_2(x) \in L_1(0,+\infty) \tag{3.12}$$

$$\lim_{x\to\infty}\left|\frac{1}{\sqrt{1-\rho}} - S_2(x)\right| = \lim_{x\to\infty}\left|\frac{1}{\sqrt{1-\rho}} - |S_2(x)|\right| = 0. \tag{3.13}$$

Now we consider the equation (3.5):

$$S_3(x) = S_2(x) + \int_0^x u(x-t)S_3(t)dt, \quad x \in \mathbb{R}^+. \tag{3.14}$$

Denote

$$\frac{1}{1-\rho} - S_3(x) \equiv \psi(x), \quad x \in \mathbb{R}^+. \tag{3.15}$$

Then equation (3.14) takes the form of

$$\psi(x) = \frac{1}{1-\rho} - \frac{1}{1-\rho}\int_0^x u(\tau)d\tau - S_2(x) + \int_0^x u(x-t)\psi(t)dt. \tag{3.16}$$

Note that

$$g(x) \equiv \frac{1}{1-\rho} - \frac{1}{1-\rho}\int_0^x u(\tau)d\tau - S_2(x) \in L_1(0,+\infty)$$

$$\lim_{x\to\infty} g(x) = 0. \tag{3.17}$$

Really we get

$$|g(x)| \leqslant \left|\frac{1}{1-\rho} - S_2(x)\right| + \frac{1}{1-\rho}\int_x^\infty u(\tau)d\tau. \tag{3.18}$$

From (3.18) it follows that $\lim\limits_{x\to\infty} g(x) = 0$.

On the other hand in (2.13) passing from equality of integral operators to equality of kernels we get to Yengibaryan's nonlinear factorization equation:

$$u(x) = h(x) + \int_0^\infty u(t)u(x+t)dt, \quad x \in \mathbb{R}^+. \tag{3.19}$$

As

$$\nu^+ \equiv \int_0^\infty xh(x)dx < +\infty,$$

then from (3.19) we obtain

$$\int_0^\infty xu(x)dx \leqslant \frac{\nu^+}{1-\gamma}. \tag{3.20}$$

From estimation (3.20) it follows that $\int\limits_x^\infty u(\tau)d\tau \in L_1(0,+\infty)$. Therefore from (3.18) and formulae (3.12) it follows that g belongs to space $L_1(0,+\infty)$. Taking into account (3.8) from (3.17) we conclude that equation (3.16) in space $L_1(0,+\infty)$ has unique solution, besides

$$\lim_{x\to\infty}\psi(x)=0. \tag{3.21}$$

Therefore

$$\lim_{x\to\infty}\left(\frac{1}{1-\rho}-S_3(x)\right)=\lim_{x\to\infty}\left(\frac{1}{1-\rho}-|S_3(x)|\right)=0 \tag{3.22}$$

$$\frac{1}{1-\rho}-S_3(x)\in L_1(0,+\infty). \tag{3.23}$$

Finally solving equation (3.6) we come to the following formulae:

$$S(x)=S_3(x)+a\int\limits_0^x S_3(t)dt. \tag{3.24}$$

We have

$$|S(x)|\leqslant\frac{1}{1-\rho}(1+ax),\quad \lim_{x\to\infty}\frac{|S(x)|}{1+ax}=\frac{1}{1-\rho}. \tag{3.25}$$

b) In this case as a $S_0(x)$ we choose trivial solution of equation (3.2). Inserting it in (2.14) and using contractility of operator U_- we obtain $S_1(x)\equiv 0$. As $\varepsilon=-1$ is the eighen-value for a operator W, then homogeneous equation

$$S_2(x)=-\int\limits_0^\infty W(x+pt)S_2(t)dt,\quad x\in\mathbb{R}^+, \tag{3.26}$$

has nontrivial bounded solution. From estimation

$$|S_2(x)|\leqslant\sup_{x>0}|S_2(x)|\frac{1}{p}\int\limits_x^\infty W(\tau)d\tau$$

it follows that

$$S_2\in L_1(0,+\infty),\quad \lim_{x\to\infty}S_2(x)=0. \tag{3.27}$$

Let's pass to the consideration of equation (3.5). As $\gamma<1$, then from (3.27) follows that equation (3.5) in $L_1(0,+\infty)$, has in unique solution $S_3(x)\in L_1(0,+\infty)$, moreover

$$\lim_{x\to\infty}S_3(x)=0. \tag{3.28}$$

Thus solution of equation (3.6) has another asymptotic

$$S\in C_M(0,+\infty),\quad \lim_{x\to\infty}S(x)=\int\limits_0^\infty S_3(\tau)d\tau. \tag{3.29}$$

The following lemma holds

Lemma 2. *Let the conditions (1.2), (1.3) are fulfilled, and* $0\leqslant K^*(x)<\overset{\circ}{K}(x)$, $x\in\mathbb{R}$, $p\geq 1$, *then equation (2.18) has positive monotonic increasing solution, moreover*
a) *if* $\varepsilon=-1$ *is not eighen-value for operator* W, *then solution has asymptotic*

$$\lim_{x\to\infty}\frac{S^*(x)}{1+ax}=\frac{1}{1-\rho}$$

b) *if* $\varepsilon=-1$ *the eighen-value for operator* W, *then solution is the bounded function.*

It should be noted, that in both cases

$$inf\, S^*(x) > 0.$$

Proof. *a)* We consider the following iteration

$$S^{(n+1)}(x) = \int_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))S^{(n)}(t)dt, \tag{3.30}$$

$$S^{(0)}(x) = \frac{1}{1-\rho}(1+ax), \quad n = 0, 1, 2, \ldots .$$

First we prove that sequence of functions $\{S^{(n)}(x)\}_0^\infty$ monotonic decreases by n.

Really, we have

$$S^{(1)}(x) \leqslant \frac{1}{1-\rho}\int_0^\infty \overset{\circ}{K}(x-t)(1+at)dt =$$

$$= \frac{1}{1-\rho}\left(\int_{-\infty}^{+\infty} \overset{\circ}{K}(z)(1+a(x-z))dz - \int_x^\infty \overset{\circ}{K}(z)(1+a(x-z))dz\right) =$$

$$= \frac{1+ax}{1-\rho} - \int_x^\infty \overset{\circ}{K}(z)(1+a(x-z))dz \leqslant \frac{1}{1-\rho}(1+ax) = S^{(0)}(x),$$

because

$$(1+ax)\int_x^\infty \overset{\circ}{K}(z)dz \geq a\int_x^\infty \overset{\circ}{K}(z)zdz, \quad x \in \mathbb{R}^+.$$

We assume that $S^{(n)}(x) \leqslant S^{(n-1)}(x)$.
Taking into account that

$$K(x,t) \equiv \overset{\circ}{K}(x-t) - K^*(x+pt) > 0, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

from (3.30) we obtain $S^{(n+1)}(x) \leqslant S^{(n)}(x)$. Now we prove that $S^{(n)}(x) \geq |S(x)|$, where $S(x)$ is nontrivial solution of equation (2.18). For $n = 0$ inequality follows from (3.25). Assume that $S^{(n)}(x) \geq |S(x)|$ then from (3.30) we have

$$S^{(n+1)}(x) \geq \int_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))|S(t)|dt \geq$$

$$\geq \left|\int_0^\infty \overset{\circ}{K}(x-t) - K^*(x+pt)S(t)dt\right| = |S(x)|.$$

Therefore sequence $\{S^{(n)}(x)\}_0^\infty$ has pointwise limit

$$\lim_{n\to\infty} S^{(n)}(x) \equiv S^*(x) \geq 0, \quad S^*(x) \not\equiv 0.$$

From B. Levi's theorem (see [18]) it follows that $S^*(x)$ satisfies equation (2.18). Moreover, function $S^*(x)$ satisfies the following double inequalities:

$$|S(x)| \leqslant S^*(x) \leqslant \frac{1}{1-\rho}(1+ax). \tag{3.31}$$

Taking into account (3.25) from (3.31) we obtain

$$\lim_{x\to\infty}\frac{S^*(x)}{1+ax}=\frac{1}{1-\rho}. \tag{3.32}$$

Now we show that $S^*(x)\uparrow$ by x. First we convince that $S^{(n)}(x)\uparrow$ by x. In case when $n=0$ it is obvious. Assume that $S^{(n)}(x)\uparrow$ by x. Then for arbitrary $x_1,x_2>0,\quad x_1>x_2$ we have

$$S^{(n+1)}(x_1)-S^{(n+1)}(x_2)\geq$$

$$\geq\int_{-\infty}^{x_2}\overset{\circ}{K}(\tau)(S^{(n)}(x_1-\tau)-S^{(n)}(x_2-\tau))d\tau+\frac{1}{p}\int_{x_2}^{\infty}K^*(\tau)(S^{(n)}(\tau-x_2)-S^{(n)}(\tau-x_1))d\tau\geq 0,$$

i.e. $S^{(n+1)}(x)\uparrow$ by x. Therefore $S^*(x)\uparrow$ by x.

Below we show that

$$\zeta\equiv\inf_{x>0}S^*(x)>0. \tag{3.33}$$

Really, as $S^*(x)\geq 0,\quad S^*(x)\not\equiv 0$ then there exist at least point $x_0\geq 0$, such that

$$\alpha_0\equiv S^*(x_0)>0 \tag{3.34}$$

We fixe this point. Then from (2.18) we have

$$S^*(x)\geq\int_{x_0}^{\infty}(\overset{\circ}{K}(x-t)-K^*(x+pt))S^*(t)dt\geq$$

$$\geq\frac{\alpha_0}{p}\left(\int_{-\infty}^{-x_0}\overset{\circ}{K}(\tau)d\tau-\int_{x_0}^{\infty}K^*(\tau)d\tau\right)=\frac{\alpha_0}{p}\left(\int_{x_0}^{\infty}[\overset{\circ}{K}(\tau)-K^*(\tau)]d\tau\right)>0.$$

Therefore

$$\zeta\geq\frac{\alpha_0}{p}\int_{x_0}^{\infty}(\overset{\circ}{K}(\tau)-K^*(\tau))d\tau>0.$$

$b)$ In this case we consider the iteration:

$$S^{(n+1)}(x)=\int_{0}^{\infty}(\overset{\circ}{K}(x-t)-K^*(x+pt))S^{(n)}(t)dt, \tag{3.35}$$

$$S^{(0)}(x)\equiv\sup_{x>0}|S(x)|\equiv l_0,\quad n=0,1,2,\ldots.$$

and making analogous discussion we have proved statement $b)$ of lemma 2.
Lemma is proved.

4. Some A'priori Upper Estimations for Corresponding Linear Nonhomogeneous Equations

Let's consider the following nonhomogeneous integral equation:

$$f(x)=2\omega(x+A)+\int_{0}^{\infty}(\overset{\circ}{K}(x-t)-K^*(x+pt))f(t)dt,\quad x\in(0,+\infty), \tag{4.1}$$

where ω–satisfies conditions (1.4), (1.5).

Together with equation (4.1) we consider the following Wiener-Hopf integral equation

$$\widetilde{f}(x) = 2\omega(x+A) + \int_0^\infty \overset{\circ}{K}(x-t)\widetilde{f}(t)dt, \quad x \in \mathbb{R}^+. \tag{4.2}$$

Using factorization (2.6) and properties of function ω in work [17] has been proved, that equation (4.2) has nonnegative and bounded solution $\widetilde{f}(x)$.
We consider the following iteration:

$$f^{(n+1)}(x) = 2\omega(x+A) + \int_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))f^{(n)}(t)dt, \quad n = 0, 1, 2, \ldots, \tag{4.3}$$

$$f^{(0)}(x) = 2\omega(x+A), \quad x \in \mathbb{R}^+.$$

Using nonnegativity of kernel $K(x,t)$ by induction it is easy to check that $\{f^{(n)}(x)\}$ possesses the following properties

$$f^{(n)}(x) \uparrow \text{ by } n, \quad f^{(n)}(x) \geq 2\omega(x+A), \quad n = 0, 1, 2, \ldots, \tag{4.4}$$

$$f^{(n)}(x) \leqslant \widetilde{f}(x), \quad n = 0, 1, 2, \ldots .$$

Therefore there exists

$$\lim_{n\to\infty} f^{(n)}(x) = f(x) \leqslant \widetilde{f}(x) \tag{4.5}$$

and function $f(x)$ satisfies equation (4.1).

We introduce function $\lambda(x)$ defined on $(0, +\infty)$

$$\lambda(x) \equiv \lambda_c(x) = 1 - \frac{\omega(x + S_c(x))}{S_c(x)}, \quad x \in \mathbb{R}^+, \tag{4.6}$$

where

$$S_c(x) = cS^*(x), \tag{4.7}$$

$$c \in \Pi \equiv \left[\max\left(\frac{\max(æ, \gamma_0)}{\zeta}, A\right), +\infty\right) \tag{4.8}$$

Here c–is the fixed number, $\gamma_0 \geq A$–some number for

$$\omega(\gamma_0) < \gamma_0, \tag{4.9}$$

and

$$æ = \sup_{x\in\mathbb{R}^+} f(x). \tag{4.10}$$

We note, that if $c \in \Pi$, then

$$S_c(x) \geq æ. \tag{4.11}$$

The last inequality follows from the chain of inequalities:

$$S_c(x) = cS^*(x) \geq c\zeta \geq \max(æ, \gamma_0) \geq æ. \tag{4.12}$$

From (4.6) we immediately obtain the following properties of function $\lambda(x)$:

$$j_1) \quad 0 < 1 - \frac{\omega(\gamma_0)}{\gamma_0} \leqslant \lambda(x) \leqslant 1, \quad x \in \mathbb{R}^+, \tag{4.13}$$

$$j_2) \quad (1 - \lambda(x))x^j \in L_1(\mathbb{R}^+), \quad j = 0, 1, \tag{4.14}$$

$$j_3) \quad \lambda(x) \uparrow \text{ by } x \text{ and } \lim_{x\to\infty} \lambda(x) = 1. \tag{4.15}$$

The inequality (4.11) and properties of function $\lambda(x)$ will be of use in future.

Now the following more general nonhomogeneous integral equation is considered:

$$Q(x) = 2\omega(x + S_c(x)) + \lambda(x) \int\limits_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))Q(t)dt, \quad x \in \mathbb{R}^+, \tag{4.16}$$

in regard to function $Q(x)$.

We introduce the following iteration

$$Q^{(n+1)}(x) = 2\omega(x + S_c(x)) + \lambda(x) \int\limits_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))Q^{(n)}(t)dt, \tag{4.17}$$

$$n = 0, 1, 2, \ldots, \quad Q^{(0)}(x) \equiv 2\omega(x + S_c(x)), \quad x \in \mathbb{R}^+.$$

From estimation (3.16) it follows that $S_c(x) \geq \gamma_0 \geq A$, therefore

$$\omega(x + S_c(x)) \leqslant \omega(x + A), \quad x \in \mathbb{R}^+. \tag{4.18}$$

Using inequalities (4.18), (4.13) by induction it is easy to check that

$$Q^{(n)}(x) \uparrow \text{ by } n, \quad Q^{(n)}(x) \leqslant f(x), \quad n = 0, 1, 2, \ldots, \tag{4.19}$$

$$Q^{(n)}(x) \geq 2\omega(x + S_c(x)), \quad n = 0, 1, 2, \ldots.$$

Thus there exists the limit of sequence $\{Q^{(n)}(x)\}_0^\infty$:

$$\lim_{n\to\infty} Q^{(n)}(x) = Q(x) \leqslant f(x). \tag{4.20}$$

From B. Levi's theorem it follows that $Q(x)$ is the solution of equation (4.16).
Finally we get to the following chain of inequalities:

$$Q(x) \leqslant f(x) \leqslant \widetilde{f}(x), \quad x \in \mathbb{R}^+. \tag{4.21}$$

5. On Solution of Corresponding Homogeneous Equation

By direct checking it is easy to convince that the function

$$\widetilde{E}_c = 2S_c(x) - Q(x), \quad c \in \Pi \tag{5.1}$$

satisfies the following homogeneous equation:

$$E(x) = \lambda(x) \int\limits_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))E(t)dt, \quad x \in \mathbb{R}^+. \tag{5.2}$$

Note that

$$\widetilde{E}_c(x) \geq S_c(x), \quad c \in \Pi, \quad x \in \mathbb{R}^+. \tag{5.3}$$

Really, from (4.11), (4.21) it follows that

$$\widetilde{E}_c(x) \geq 2S_c(x) - f(x) \geq S_c(x). \tag{5.4}$$

Consider the following iteration:

$$E^{(n+1)}(x) = \lambda(x) \int\limits_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))E^{(n)}(t)dt,$$

$$E^{(0)}(x) = 2S_c(x), \quad n = 0, 1, 2, \ldots, \quad x \in \mathbb{R}^+, \quad c \in \Pi. \tag{5.5}$$

By induction it is easy to check that

$$\bullet \quad E^{(n)}(x) \downarrow \text{ by } n, \quad \bullet \quad E^{(n)}(x) \geq \widetilde{E}_c(x), \quad n = 0, 1, 2, \ldots, \tag{5.6}$$

$$\bullet \quad E^{(n)}(x) \leqslant 2\lambda(x)S_c(x), \quad n = 1, 2, 3 \ldots \tag{5.7}$$

From (5.6) it immediately follows that there exists limit of sequence $\{E^{(n)}(x)\}_0^\infty$, i.e. $\lim\limits_{n\to\infty} E^{(n)}(x) = E(x)$. Moreover we have the following chain of inequalities:

$$S_c(x) \leqslant \widetilde{E}(x) \leqslant E(x) \leqslant 2\lambda(x)S_c(x), \quad x \in \mathbb{R}^+. \tag{5.8}$$

It should be noted that if $E(x)$ satisfies equation (5.2) and inequality (5.8), then the function

$$Y(x) = \frac{E(x)}{\lambda(x)}, \quad x \in \mathbb{R}^+ \tag{5.9}$$

will satisfy the equation

$$Y(x) = \int_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))\lambda(t)Y(t)dt \tag{5.10}$$

and inequality

$$S_c(x) \leqslant \widetilde{E}(x) \leqslant E(x) \leqslant Y(x) \leqslant 2S_c(x). \tag{5.11}$$

Using (5.1), (5.11), (4.7) in case $a)$ we obtain

$$\lim_{x\to\infty} \frac{Y(x)}{1+ax} = 2\lim_{x\to\infty} \frac{S_c(x)}{1+ax} = \frac{2c}{1-\rho}, \tag{5.12}$$

and in case $b)$ $\quad Y \in M(0,+\infty)$.

6. One Parameter Family of Positive Solutions for One Class Wiener-Hopf-Hammershtein Nonlinear Integral Equation

We consider the following class Wiener-Hopf-Hammershtein type nonlinear integral equation.

$$N(x) = \int_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))(N(t) - \omega(t+N(t)))dt, \quad x \in \mathbb{R}^+ \tag{6.1}$$

in regard to unknown function $N(x)$.

We introduce the following special iteration:

$$N^{(m+1)}(x) = \int_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))(N^{(m)}(t) - \omega(t+N^{(m)}(t)))dt,$$

$$N^{(0)}(x) = 2S_c(x), \quad m = 0,1,2,\dots, \quad x \in \mathbb{R}^+, \quad c \in \Pi.$$

By induction it is easy to prove the truthfulness of the following statements.

- $N^{(m)}(x) \downarrow$ by m, • $N^{(m)}(x) \geq Y(x), \quad m = 0,1,2,\dots,$ (6.2)
- if $c_1, c_2 \in \Pi, \quad c_1 > c_2 -$ arbitrary numbers, then

$$N^{(m)}_{c_1}(x) - N^{(m)}_{c_2}(x) \geq 2(S_{c_1}(x) - S_{c_2}(x)) \geq 2(c_1 - c_2)\zeta, \quad m = 0,1,2,\dots \tag{6.3}$$

For example we prove (6.3). In case when $m = 0$ it is obvious. Assuming that inequality is true for $m \in N$, we prove it for $m+1$: We have

$$N^{(m+1)}_{c_1}(x) - N^{(m+1)}_{c_2}(x) = \int_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))(N^{(m)}_{c_1}(t) - N^{(m)}_{c_2}(t))dt +$$

$$+\int_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))(\omega(t+N^{m}_{c_2}(t)) - \omega(t+N^{(m)}_{c_1}(t)))dt \geq 2(S_{c_1}(x) - S_{c_2}(x)) \geq 2(c_1-c_2)\zeta.$$

From (6.2) it follows that sequence has pointwise limit

$$\lim_{m\to\infty} N^m(x) = N(x) \geq Y(x), \tag{6.4}$$

and

$$S_c(x) \leqslant \widetilde{E}(x) \leqslant E(x) \leqslant Y(x) \leqslant N(x) \leqslant 2S_c(x), \quad x \in \mathbb{R}^+. \tag{6.5}$$

Using B.Levi's theorem we conclude that $N(x)$ satisfies the equation (6.1). In inequality (6.3) passing $m \to \infty$, we conclude that for different values of parameter $c \in \Pi$ correspond different solutions of equation (6.1). Using (5.12) and taking into account (6.5) for case a) we obtain.

$$\lim_{x\to\infty} \frac{N_c(x)}{1+ax} = \frac{2c}{1-\rho}, \quad c \in \Pi, \tag{6.6}$$

and in case b) we have $N_c(x) \in M(0,+\infty)$.
Thus we come to the following result.

Theorem 1. *Let the conditions (1.2)-(1.5) are fulfilled, and* $0 \leqslant K^*(x) < \overset{\circ}{K}(x), \quad p \geq 1$. *Then equation (6.1) possesses one parameter family of positive solutions* $\{N_c(x)\}_{c\in\Pi}$ *besides*
a) if $\varepsilon = -1$ *is the eighen-value for operator* W, *then functions* $N_c(x)$ *have asymptotic (6.6)*
b) if $\varepsilon = -1$ *is not eighen-value for operator* W, *then* $N_c(x) \in M(0,+\infty)$.

7. Solvability of Basic Equation (1.1)

First we consider the case a). The following theorem holds.

Theorem 2. *Let conditions (1.2)-(1.6) are fulfilled and there exists a number* $\delta \geq \frac{2}{1-\rho}\left(\max\left(\frac{\max(æ,\gamma_0)}{\zeta}, A\right)\right)$ *such that*

$$\frac{1}{1+ax}\int_0^\infty U(x,t,\delta(1+ax))dt \leqslant \delta, \quad x \in \mathbb{R}^+. \tag{7.1}$$

Then if $\varepsilon = -1$ *is the eighen-value for operator* W, *then equation (1.1) has positive solution* $\varphi_\delta(x)$ *with asymptotic*

$$\lim_{x\to\infty} \frac{\varphi_\delta(x)}{1+ax} = \delta.$$

Proof. Let's consider the following iteration to equation (1.1):

$$\varphi^{(n+1)}(x) = \int_0^\infty U(x,t,\varphi^{(n)}(t))dt, \quad \varphi^{(0)}(x) = \delta(1+ax), \tag{7.2}$$

$$n = 0,1,2,\dots, \quad x \in \mathbb{R}^+.$$

From definition of number δ it follows that

$$c^* = \frac{\delta(1-\rho)}{2} \in \Pi. \tag{7.3}$$

Therefore from theorem 1 it follows that there is positive solution of equation (6.1) with asymptotic

$$N_{c^*}(x) = \delta + \delta a x + o(x), \quad x \to \infty. \tag{7.4}$$

Note that

$$N_{c^*}(x) \leqslant 2S_{c^*}(x) \leqslant \frac{2c^*(1+ax)}{1-\rho} = \delta(1+ax) \tag{7.5}$$

$$N_{c^*}(x) \geq S_{c^*}(x) \geq c^*\zeta \geq \begin{cases} \max(æ,\gamma_0) \geq \gamma_0 \geq A, & \zeta < 1, \\ A\zeta \geq A, & \zeta > 1. \end{cases}$$

Thus

$$A \leqslant N_{c^*}(x) \leqslant \delta(1+ax), \quad \lim_{x\to\infty} \frac{N_{c^*}(x)}{1+ax} = \delta. \tag{7.6}$$

Below by induction we prove that

$$N_{c^*}(x) \leqslant \varphi^{(n)}(x) \leqslant \delta(1+ax), \quad n = 0, 1, 2 \dots. \tag{7.7}$$

If $n = 0$ then estimation (7.7) follows from (7.6) and (7.2). Let (7.7) is true for $n = m \in N$. We prove it for $n = m + 1$. From (7.1) it follows

$$\varphi^{(m+1)}(x) \leqslant \int\limits_0^\infty U(x,t,\delta(1+at))dt \leqslant \delta(1+ax)$$

and from condition (1.6) we obtain

$$\varphi^{(m+1)}(x) \geq \int\limits_0^\infty U(x,t,N_{c^*}(t))dt \geq \int\limits_0^\infty (\overset{\circ}{K}(x-t) - K^*(x+pt))(N_{c^*}(t) - \omega(t+N_{c^*}(t)))dt = N_{c^*}(x).$$

It is easy also to convince, that

$$\varphi^{(n)}(x) \downarrow \text{ by } n. \tag{7.8}$$

Therefore sequence of functions $\{\varphi^{(n)}(x)\}_0^\infty$ has limit

$$\lim_{n\to\infty} \varphi^{(n)}(x) = \varphi(x), \tag{7.9}$$

besides

$$N_{c^*}(x) \leqslant \varphi(x) \leqslant \delta(1+ax), \quad x \in \mathbb{R}^+ \tag{7.10}$$

and $\varphi(x)$ satisfies the basic equation (1.1). From (7.10) and (7.6) we conclude that

$$\lim_{x\to\infty} \frac{\varphi_\delta(x)}{1+ax} = \delta.$$

Theorem is proved.

Analogously is proved the following:

Theorem 3. *We assume that conditions (1.2)-(1.6) are fulfilled and there exists a number* $\eta \geq 2l_0 \max\left(\frac{\max(æ, \gamma_0)}{\zeta}, A\right)$, *such that*

$$\int\limits_0^\infty U(x,t,\eta)dt \leqslant \eta, \quad x \in \mathbb{R}^+, \tag{7.11}$$

where $l_0 = \sup\limits_{x>0} |S^*(x)|$. *If* $\varepsilon = -1$ *is eighen-value for operator* W, *then equation (1.1) has positive and bounded solution* $\varphi_\eta(x)$, *besides* $\varphi_\eta(x) \leqslant \eta, \quad x \in \mathbb{R}^+$.

8. Some Examples of Function $U(x,t,\tau)$

For the following class of function $U(x,t,\tau)$ the conditions of formulated theorem 2 are fulfilled.

1) $U(x,t,\tau) = (\overset{\circ}{K}(x-t) - K^*(x+pt))(\tau - \omega(t+\tau))$,

2) $U(x,t,\tau) = \overset{\circ}{K}(x-t)Q(t,\tau)$, where $Q(t,\tau)-$ is defined on $\mathbb{R}^+ \times \mathbb{R}$ real and measurable function, satisfying conditions

- there exist number $A_1 > 0$, such that $\tau - \omega(t+\tau) \leqslant Q(t,\tau) \leqslant \tau, \quad (t,\tau) \in \mathbb{R}^+ \times [A_1, +\infty)$,
- $Q(t,\tau) \uparrow$ by τ on $[A_0, +\infty)$, for each $t > 0$ and some $A_0 > 0$.

- $Q(t,\tau) \in Carat(\mathbb{R}^+ \times [A, +\infty))$, $A = \max(A_0, A_1)$.

As $Q(t,\tau)$ we can take, for example the following functions

$$Q(t,\tau) = (\tau^2 - \tau\omega(t+\tau))^{\frac{1}{2}}, \quad Q(t,\tau) = \frac{2(\tau^2 - \omega(t+\tau)\tau)}{2\tau - \omega(t+\tau)}$$

The following particular types of function $U(x,t,\tau)$ are the examples for theorem 3.

3) $U(x,t,\tau) = \widetilde{K}(x,t)(G_0(\tau) - \omega(t+\tau))$, where

$\widetilde{K}(x,t) \geq \overset{\circ}{K}(x-t) - K^*(x+pt)$, $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, and $\int\limits_0^\infty \widetilde{K}(x,t)dt \leqslant 1$, $x \in \mathbb{R}^+$

and $G_0 \in C[A,\eta]$, $G_0(x) \geq x$, $x \in [A,\eta]$, $G_0 \uparrow [A,\eta]$ and $G_0(\eta) = \zeta$.

4) $U(x,t,\tau) = \widetilde{K}(x,t)(G_0(\tau) - \omega_0(t,\tau))$,

where $\omega_0(t,\tau)$–is real function defined on set $\mathbb{R}^+ \times \mathbb{R}$ and satisfying condition

$$0 \leqslant \omega_0(t,\tau) \leqslant \omega(t+\tau), \quad (t,\tau) \in \mathbb{R}^+ \times [A_1, +\infty).$$

- $\omega_0(t,\tau) \downarrow$ by τ on $[A_0, +\infty)$ for each $t > 0$,
- $\omega_0 \in Carat(\mathbb{R}^+ \times [A_1, +\infty))$, $A = \max(A_0, A_1)$.

Below we reduce simple example of nonlinear integral equation with concrete mentioned one parameter family of positive solutions. Let

$$U(x,t,\tau) = K_0(x-t)(\tau - \omega_0(\tau+t)), \tag{8.1}$$

$$K_0(x) = \frac{1}{2}e^{-|x|},\ x \in \mathbb{R},\ \omega_0(z) = e^{-z},\ z \in \mathbb{R}^+. \tag{8.2}$$

We have $\rho = 0$, $a = 1$, æ $= 4$, $\gamma_0 = (0, \frac{1}{2}]$, $\zeta = 1$, $A = 0$. Then from (4.8) we obtain the following set of parameter

$$c \in [4, +\infty). \tag{8.3}$$

Taking into account (8.1), (8.2), the equation (1.1) is reduced to the following nonlinear differential equation

$$\varphi''(x) = e^{-(x+\varphi(x))}. \tag{8.4}$$

One parameter family of positive solutions has a form

$$\varphi_c(x) = \ln\frac{[2(c+1)^2 e^{(c+1)x} + 1]^2}{4(c+1)^4 e^{(c+1)x}} - x > 0, \quad c \in [4, +\infty), \tag{8.5}$$

with asymptotic $\lim\limits_{x\to\infty} \frac{\varphi_c(x)}{1+x} = c$.

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Khachatur Khachatryan,
Institute of Mathematics National Academy of Sciences,
Bagramyan st., 24b,
0019, Yerevan, Armenia
E-mail: `khach82@rambler.ru`