ISSN 2074-1871 Уфимский математический журнал. Том 15. № 3 (2023). С. 121-131.

ON INDIRECT REPRESENTABILITY OF FOURTH ORDER ORDINARY DIFFERENTIAL EQUATION IN FORM OF HAMILTON-OSTROGRADSKY EQUATIONS

S.A. BUDOCHKINA, T.H. LUU, V.A. SHOKAREV

Abstract. In the paper we solve the problem on the representability of a fourth order ordinary differential equation in the form of Hamilton-Ostrogradsky equations. Local bilinear forms play an essential role in the investigation of the potentiality property of the considered equation. It is well known that the problem of representing differential equations in the form of Hamilton-Ostrogradsky equations is closely related to the existence of a solution to the inverse problem of the calculus of variations, that is, for a given equation one needs to construct a functional-variational principle. To solve this problem, we first obtain necessary and sufficient conditions for the given equation to admit an indirect variational formulation relative to a local bilinear form and then construct the corresponding Hamilton-Ostrogradsky action. Note that the found conditions are analogous to the Helmholtz potentiality conditions for the given ordinary differential equation. We also define the structure of the considered equation with the potential operator and use the Ostrogradsky scheme to reduce the given equation to the form of Hamilton-Ostrogradsky equations.

It should be noted that applications and extensions of the work are the possibility to establish connections between invariance of the functional and first integrals of the given equation and to extend the proposed scheme to partial differential equations and systems of such equations.

Keywords: Local bilinear form, potential operator, Hamilton-Ostrogradsky action, Hamilton-Ostrogradsky equations.

Mathematics Subject Classification: 49N45, 70H05

1. Introduction

Classical Hamiltonian formalism is employed for solving various problems of mathematics, mechanics and physics. It plays an important role in determining some approaches to the investigation of motion of different complex systems, which is motivated by very effective methods for integration and qualitative investigation of equations of motion [11], [12], [13], [15]. For this reason, the solution to the problems associated with the representation of different types of equations and their systems in the form of Hamilton equations and their generalizations is of great interest, see, for instance, [2], [5], [20].

S.A. Budochkina, T.H. Luu, V.A. Shokarev, On indirect representability of fourth order ordinary differential equation in form of Hamilton-Ostrogradsky equations.

This publication has been partially supported by the RUDN University named after Patrice Lumumba, project no. 002092-0-000 and by the Ministry of Science and Higher Education of the Russian Federation (megagrant agreement No. 075-15-2022-1115).

Submitted April 26, 2022.

These issues are closely related to the inverse problem of the calculus of variations (IPCV) in the following statement: for a given equation, one needs to construct a functional such that its set of stationary points coincides with the set of solutions to this equation. There is a large number of works devoted to inverse problems of the calculus of variations: for ordinary differential equations and partial differential equations [3], [4], [7], [9], [19], [20], [26], [27], operator equations [6], [21], [22], differential-difference equations [8], [17], [18], stochastic differential equations [23], [24], [25], fractional differential equations [1], [10], [14], [28]. In these works, nonlocal bilinear forms were mainly used to solve the IPCV.

The representability of a given equation with a non-potential operator in the form of Hamilton equations can be a difficult problem. One of the ways for solving it is to use local bilinear forms.

The main aim of the paper is to establish connection between variationality of a fourth order ordinary differential equation and Hamilton-Ostrogradsky equations, i.e., to obtain the conditions of the indirect representability of the given equation in the form of the Lagrange-Ostrogradsky equation, to construct the corresponding functional-variational principle, to define the variational structure of the considered equation and to apply the Ostrogradsky scheme [16] for the representation of this equation in the form of Hamilton-Ostrogradsky equations. The method of constructing functionals developed in [3] is extended to a fourth order ordinary differential equation. Local bilinear forms play a significant role in our study.

In what follows we use the notation and terminology of [3], [20]. Let U, V be real linear normed spaces. The following definition and theorem will be employed in what follows.

Definition 1.1. ([20]) *An operator $N : D(N) \subset U \to V$ is called potential on a set $D(N)$ relative to a local bilinear form $\Phi(u; \cdot, \cdot) : V \times V \to \mathbb{R}$ if there exists a Gâteaux differentiable functional $F_N : D(F_N) = D(N) \to \mathbb{R}$ such that*

$$\delta F_N[u, h] = \Phi(u; N(u), h) \qquad \forall u \in D(N),\ \forall h \in D(N'_u).$$

Theorem 1.1. ([20]) *Consider a Gâteaux differentiable operator $N : D(N) \subset U \to V$ and a local bilinear form $\Phi(u; \cdot, \cdot) : V \times V \to \mathbb{R}$ such that for all fixed elements $u \in D(N)$, $g, h \in D(N'_u)$ the function $\psi(\varepsilon) = \Phi(u + \varepsilon h; N(u + \varepsilon h), g)$ belongs to the class $C^1[0, 1]$. Then the operator N is potential on the convex set $D(N)$ relative to Φ if and only if*

$$\Phi\left(u; N'_u h, g\right) + \Phi'_u\left(h; N(u), g\right) = \Phi\left(u; N'_u g, h\right) + \Phi'_u\left(g; N(u), h\right) \tag{1.1}$$

for all $u \in D\left(N\right)$, $h, g \in D\left(N'_u\right)$. Under this condition the potential F_N is given by

$$F_N[u] = \int_0^1 \Phi(u_0 + \lambda(u - u_0); N(u_0 + \lambda(u - u_0)), u - u_0)\, d\lambda + F_N[u_0], \tag{1.2}$$

where u_0 is a fixed element of $D(N)$.

2. On indirect variational formulation of fourth order ordinary differential equation

We consider a fourth order ordinary differential equation

$$N(u) \equiv \sum_{i=1}^{4} a_i(t, u(t))u^{(i)}(t) + a_0(t, u(t)) = 0, \quad t \in (t_0, t_1). \tag{2.1}$$

Here $u = u(t)$ is an unknown function, $a_i \in C^i([t_0, t_1] \times T)$, $i = \overline{1, 4}$, $a_0 \in C^1([t_0, t_1] \times T)$ are given functions, $T \subseteq \mathbb{R}$, $a_4(t, u(t)) \neq 0$.

We define the domain of the operator N as follows:

$$D(N) = \left\{u \in U = C^4[t_0, t_1] : u(t_0) = u_1,\ u(t_1) = u_2,\ u'(t_0) = u'_1,\ u'(t_1) = u'_2\right\}. \tag{2.2}$$

We observe that $V = C[t_0, t_1]$ and

$$D(N'_u) = \left\{ h \in U = C^4[t_0, t_1] : h(t_0) = 0,\ h(t_1) = 0,\ h'(t_0) = 0,\ h'(t_1) = 0 \right\}.$$

We introduce a local bilinear form by

$$\Phi(u; v, g) = \int_{t_0}^{t_1} M(t, u(t)) v(t) g(t) dt, \tag{2.3}$$

where $M \in C^4([t_0, t_1] \times T)$, $M(t, u(t)) \neq 0$. We denote

$$M_i(t, u(t)) = M(t, u(t)) a_i(t, u(t)), \qquad i = \overline{0, 4}.$$

Theorem 2.1. *The operator N is potential on $D(N)$ relative to bilinear form (2.3) if and only if $M_i = M_i(t)$, $i = \overline{1, 4}$ and for all $t \in [t_0, t_1]$ the following conditions hold:*

$$M_3(t) = 2M'_4(t), \tag{2.4}$$

$$M_1(t) = M'_2(t) - M'''_4(t). \tag{2.5}$$

Proof. We have

$$N'_u h = \sum_{i=1}^{4} a'_{iu}(t, u(t)) h(t) u^{(i)}(t) + a'_{0u}(t, u(t)) h(t) + \sum_{i=1}^{4} a_i(t, u(t)) h^{(i)}(t).$$

In this case potentiality criterion (1.1) becomes

$$\begin{aligned}
&\int_{t_0}^{t_1} \Bigg(M(t, u(t)) \sum_{i=1}^{4} a'_{iu}(t, u(t)) h(t) u^{(i)}(t) g(t) + M(t, u(t)) a'_{0u}(t, u(t)) h(t) g(t) \\
&\qquad + M(t, u(t)) \sum_{i=1}^{4} a_i(t, u(t)) h^{(i)}(t) g(t) \Bigg) dt \\
&+ \int_{t_0}^{t_1} \Bigg(M'_u(t, u(t)) \sum_{i=1}^{4} a_i(t, u(t)) u^{(i)}(t) h(t) g(t) + M'_u(t, u(t)) a_0(t, u(t)) h(t) g(t) \Bigg) dt \\
&= \int_{t_0}^{t_1} \Bigg(M(t, u(t)) \sum_{i=1}^{4} a'_{iu}(t, u(t)) g(t) u^{(i)}(t) h(t) + M(t, u(t)) a'_{0u}(t, u(t)) g(t) h(t) \\
&\qquad + M(t, u(t)) \sum_{i=1}^{4} a_i(t, u(t)) g^{(i)}(t) h(t) \Bigg) dt \\
&+ \int_{t_0}^{t_1} \Bigg(M'_u(t, u(t)) \sum_{i=1}^{4} a_i(t, u(t)) u^{(i)}(t) g(t) h(t) + M'_u(t, u(t)) a_0(t, u(t)) g(t) h(t) \Bigg) dt
\end{aligned} \tag{2.6}$$

for all $u \in D(N)$, $h, g \in D(N'_u)$. Hence,

$$\int_{t_0}^{t_1} M(t, u(t)) \sum_{i=1}^{4} a_i(t, u(t)) h^{(i)}(t) g(t) dt = \int_{t_0}^{t_1} M(t, u(t)) \sum_{i=1}^{4} a_i(t, u(t)) g^{(i)}(t) h(t) dt$$

for all $u \in D(N)$, $h, g \in D(N'_u)$, or

$$\int_{t_0}^{t_1} \sum_{i=1}^{4} M_i(t,u(t))h^{(i)}(t)g(t)dt = \int_{t_0}^{t_1} \sum_{i=1}^{4} M_i(t,u(t))g^{(i)}(t)h(t)dt \tag{2.7}$$

for all $u \in D(N)$, $h, g \in D(N'_u)$. Integrating by parts and taking into consideration that $h, g \in D(N'_u)$, we get

$$\begin{aligned}
\int_{t_0}^{t_1} \sum_{i=1}^{4} M_i(t,u(t))h^{(i)}(t)g(t)dt = \int_{t_0}^{t_1} h(t)\Big(& - M'_{1t}(t,u(t))g(t) - M'_{1u}(t,u(t))u'(t)g(t) \\
& - M_1(t,u(t))g'(t) + M''_{2tt}(t,u(t))g(t) \\
& + 2M''_{2tu}(t,u(t))u'(t)g(t) + M''_{2uu}(t,u(t))(u'(t))^2 g(t) \\
& + M'_{2u}(t,u(t))u''(t)g(t) + 2M'_{2t}(t,u(t))g'(t) \\
& + 2M'_{2u}(t,u(t))u'(t)g'(t) + M_2(t,u(t))g''(t) \\
& - M'''_{3ttt}(t,u(t))g(t) - 3M'''_{3ttu}(t,u(t))u'(t)g(t) \\
& - 3M''_{3tt}(t,u(t))g'(t) - 3M'''_{3tuu}(t,u(t))(u'(t))^2 g(t) \\
& - 3M''_{3tu}(t,u(t))u''(t)g(t) - 6M''_{3tu}(t,u(t))u'(t)g'(t) \\
& - M'''_{3uuu}(t,u(t))(u'(t))^3 g(t) - 3M''_{3uu}(t,u(t))u'(t)u''(t)g(t) \\
& - 3M''_{3uu}(t,u(t))(u'(t))^2 g'(t) - M'_{3u}(t,u(t))u'''(t)g(t) \\
& - 3M'_{3u}(t,u(t))u''(t)g'(t) - 3M'_{3t}(t,u(t))g''(t) \\
& - 3M'_{3u}(t,u(t))u'(t)g''(t) - M_3(t,u(t))g'''(t) \\
& + M^{(4)}_{4tttt}(t,u(t))g(t) + 4M'''_{4ttt}(t,u(t))g'(t) \\
& + 4M^{(4)}_{4tttu}(t,u(t))u'(t)g(t) + 6M^{(4)}_{4ttuu}(t,u(t))(u'(t))^2 g(t) \\
& + 6M'''_{4ttu}(t,u(t))u''(t)g(t) + 12M'''_{4ttu}(t,u(t))u'(t)g'(t) \\
& + 6M''_{4tt}(t,u(t))g''(t) + 4M^{(4)}_{4tuuu}(t,u(t))(u'(t))^3 g(t) \\
& + 12M'''_{4tuu}(t,u(t))u'(t)u''(t)g(t) + 12M'''_{4tuu}(t,u(t))(u'(t))^2 g'(t) \\
& + 4M''_{4tu}(t,u(t))u'''(t)g(t) + 12M''_{4tu}(t,u(t))u''(t)g'(t) \\
& + 12M''_{4tu}(t,u(t))u'(t)g''(t) + M^{(4)}_{4uuuu}(t,u(t))(u'(t))^4 g(t) \\
& + 6M'''_{4uuu}(t,u(t))(u'(t))^2 u''(t)g(t) + 4M'''_{4uuu}(t,u(t))(u'(t))^3 g'(t) \\
& + 3M''_{4uu}(t,u(t))(u''(t))^2 g(t) + 4M''_{4uu}(t,u(t))u'(t)u'''(t)g(t) \\
& + 12M''_{4uu}(t,u(t))u'(t)u''(t)g'(t) + 6M''_{4uu}(t,u(t))(u'(t))^2 g''(t) \\
& + M'_{4u}(t,u(t))u^{(4)}(t)g(t) + 4M'_{4u}(t,u(t))u'''(t)g'(t) \\
& + 6M'_{4u}(t,u(t))u''(t)g''(t) + 4M'_{4t}(t,u(t))g'''(t) \\
& + 4M'_{4u}(t,u(t))u'(t)g'''(t) + M_4(t,u(t))g^{(4)}(t)\Big)\, dt.
\end{aligned}$$

Therefore, it follows from (2.7) that

$$\begin{aligned}
& - M'_{1t}(t,u(t)) - M'_{1u}(t,u(t))u'(t) + M''_{2tt}(t,u(t)) + 2M''_{2tu}(t,u(t))u'(t) \\
& + M''_{2uu}(t,u(t))(u'(t))^2 + M'_{2u}(t,u(t))u''(t) - M'''_{3ttt}(t,u(t)) - 3M'''_{3ttu}(t,u(t))u'(t) \\
& - 3M'''_{3tuu}(t,u(t))(u'(t))^2 - 3M''_{3tu}(t,u(t))u''(t) - M'''_{3uuu}(t,u(t))(u'(t))^3
\end{aligned}$$

$$
\begin{aligned}
&- 3M''_{3uu}(t,u(t))u'(t)u''(t) - M'_{3u}(t,u(t))u'''(t) + M^{(4)}_{4tttt}(t,u(t)) + 4M^{(4)}_{4tttu}(t,u(t))u'(t) \\
&+ 6M^{(4)}_{4ttuu}(t,u(t))(u'(t))^2 + 6M'''_{4ttu}(t,u(t))u''(t) + 4M^{(4)}_{4tuuu}(t,u(t))(u'(t))^3 \\
&+ 12M'''_{4tuu}(t,u(t))u'(t)u''(t) + 4M''_{4tu}(t,u(t))u'''(t) + M^{(4)}_{4uuuu}(t,u(t))(u'(t))^4 \\
&+ 6M'''_{4uuu}(t,u(t))(u'(t))^2u''(t) + 4M''_{4uu}(t,u(t))u'(t)u'''(t) \\
&+ 3M''_{4uu}(t,u(t))(u''(t))^2 + M'_{4u}(t,u(t))u^{(4)}(t) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&- 2M_1(t,u(t)) + 2M'_{2t}(t,u(t)) + 2M'_{2u}(t,u(t))u'(t) - 3M''_{3tt}(t,u(t)) - 6M''_{3tu}(t,u(t))u'(t) \\
&- 3M''_{3uu}(t,u(t))(u'(t))^2 - 3M'_{3u}(t,u(t))u''(t) + 4M'''_{4ttt}(t,u(t)) + 12M'''_{4ttu}(t,u(t))u'(t) \\
&+ 12M'''_{4tuu}(t,u(t))(u'(t))^2 + 12M''_{4tu}(t,u(t))u''(t) + 4M'''_{4uuu}(t,u(t))(u'(t))^3 \\
&+ 12M''_{4uu}(t,u(t))u'(t)u''(t) + 4M'_{4u}(t,u(t))u'''(t) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&- M'_{3t}(t,u(t)) - M'_{3u}(t,u(t))u'(t) + 2M''_{4tt}(t,u(t)) + 4M''_{4tu}(t,u(t))u'(t) \\
&+ 2M''_{4uu}(t,u(t))(u'(t))^2 + 2M'_{4u}(t,u(t))u''(t) = 0, \\
&- M_3(t,u(t)) + 2M'_{4t}(t,u(t)) + 2M'_{4u}(t,u(t))u'(t) = 0.
\end{aligned}
$$

This yields that $M_i = M_i(t)$, $i = \overline{1,4}$, and

$$- M_3(t) + 2M'_4(t) = 0, \tag{2.8}$$

$$- 2M_1(t) + 2M'_2(t) - 3M''_3(t) + 4M'''_4(t) = 0, \tag{2.9}$$

$$- M'_1(t) + M''_2(t) - M'''_3(t) + M^{(4)}_4(t) = 0. \tag{2.10}$$

We observe that conditions (2.8)–(2.10) are reduced to (2.4) and (2.5) and this completes the proof. □

3. Construction of Hamilton-Ostrogradsky action

Theorem 3.1. *If the operator N is potential on $D(N)$ relative to bilinear form (2.3), then the corresponding Hamilton-Ostrogradsky action is given by*

$$F_N[u] = \int_{t_0}^{t_1} \left[\frac{1}{2}\left(M''_4(t) - M_2(t)\right)(u'(t))^2 + \frac{1}{2}M_4(t)(u''(t))^2 + B_M(t,u(t))\right] dt, \tag{3.1}$$

where

$$B_M(t,u(t)) = \int_0^1 M_0(t,\widetilde{u}(t,\lambda))(u(t)-u_0(t))d\lambda + B_M(t,u_0(t)), \tag{3.2}$$

$$\widetilde{u}(t,\lambda) = u_0(t) + \lambda(u(t)-u_0(t)),$$

$u_0 = u_0(t)$ is a fixed element of $D(N)$, $B_M \in C^2([t_0,t_1]\times T)$.

Proof. Using formula (1.2) and potentiality conditions (2.4), (2.5), we get:

$$
\begin{aligned}
F_N[u] - F_N[u_0] = \int_{t_0}^{t_1}\int_0^1 \big(&M_0(t,\widetilde{u}(t,\lambda))(u(t)-u_0(t)) + M_1(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t)) \\
&+ M_2(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t)) + M_3(t)\widetilde{u}'''_{ttt}(t,\lambda)(u(t)-u_0(t))
\end{aligned}
$$

$$+ M_4(t)\widetilde{u}^{(4)}_{tttt}(t,\lambda)(u(t)-u_0(t))\big)\,d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \big(M_0(t,\widetilde{u}(t,\lambda))(u(t)-u_0(t)) + M_1(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))$$
$$- M_2'(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t)) - M_2(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))'$$
$$- M_3'(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t)) - M_3(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))'$$
$$- M_4'(t)\widetilde{u}'''_{ttt}(t,\lambda)(u(t)-u_0(t)) - M_4(t)\widetilde{u}'''_{ttt}(t,\lambda)(u(t)-u_0(t))'\big)\,d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \big(M_0(t,\widetilde{u}(t,\lambda))(u(t)-u_0(t)) + M_1(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))$$
$$- M_2'(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t)) - M_2(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))'$$
$$+ M_3''(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t)) + M_3'(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))'$$
$$- M_3(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))' + M_4''(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))$$
$$+ 2M_4'(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))' + M_4(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))''\big)\,d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \big(M_0(t,\widetilde{u}(t,\lambda))(u(t)-u_0(t)) + M_1(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))$$
$$- M_2'(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t)) - M_2(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))'$$
$$+ M_3''(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t)) + M_3'(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))'$$
$$- M_3(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))' - M_4'''(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))$$
$$- M_4''(t)\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))' + 2M_4'(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))'$$
$$+ M_4(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))''\big)\,d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \big(\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))\big(M_1(t) - M_2'(t) + M_3''(t) - M_4'''(t)\big)$$
$$+ M_0(t,\widetilde{u}(t,\lambda))(u(t)-u_0(t))$$
$$+ \widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))'\big(-M_2(t) + M_3'(t) - M_4''(t)\big)$$
$$+ \widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))'\big(-M_3(t) + 2M_4'(t)\big)$$
$$+ M_4(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))''\big)\,d\lambda dt$$

$$= \int_{t_0}^{t_1}\int_0^1 \big(\widetilde{u}'_t(t,\lambda)(u(t)-u_0(t))'\,(M_4''(t) - M_2(t))$$
$$+ M_4(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))'' + M_0(t,\widetilde{u}(t,\lambda))(u(t)-u_0(t))\big)\,d\lambda dt.$$

We observe that

$$\int_0^1 M_4(t)\widetilde{u}''_{tt}(t,\lambda)(u(t)-u_0(t))''d\lambda = \int_0^1 M_4(t)\widetilde{u}''_{tt}(t,\lambda)\widetilde{u}'''_{tt\lambda}(t,\lambda)d\lambda$$
$$= \frac{1}{2}\int_0^1 \frac{\partial}{\partial\lambda}\left(M_4(t)(\widetilde{u}''_{tt}(t,\lambda))^2\right)d\lambda$$

$$= \frac{1}{2}M_4(t)(u''(t))^2 - \frac{1}{2}M_4(t)(u_0''(t))^2,$$

$$\int_0^1 \left(M_4''(t) - M_2(t)\right)\widetilde{u}_t'(t,\lambda)(u(t)-u_0(t))'d\lambda = \int_0^1 \left(M_4''(t) - M_2(t)\right)\widetilde{u}_t'(t,\lambda)\widetilde{u}_{t\lambda}''(t,\lambda)d\lambda$$
$$= \frac{1}{2}\int_0^1 \frac{\partial}{\partial\lambda}\Big(\left(M_4''(t) - M_2(t)\right)\left(\widetilde{u}_t'(t,\lambda)\right)^2\Big)d\lambda$$
$$= \frac{1}{2}\left(M_4''(t) - M_2(t)\right)\left(u'(t)\right)^2 - \frac{1}{2}\left(M_4''(t) - M_2(t)\right)\left(u_0'(t)\right)^2.$$

Therefore,

$$F_N[u] - F_N[u_0] = \int_{t_0}^{t_1}\Bigg(\frac{1}{2}\left(M_4''(t) - M_2(t)\right)\left(u'(t)\right)^2 - \frac{1}{2}\left(M_4''(t) - M_2(t)\right)\left(u_0'(t)\right)^2 + \frac{1}{2}M_4(t)\left(u''(t)\right)^2 - \frac{1}{2}M_4(t)\left(u_0''(t)\right)^2 + \int_0^1 M_0(t,\widetilde{u}(t,\lambda))(u(t)-u_0(t))\,d\lambda\Bigg)dt.$$

Taking into consideration (3.2), we obtain Hamilton-Ostrogradsky action (3.1). The proof is complete. □

4. On variational structure of equation (2.1)

Theorem 4.1. *The operator N is potential on $D(N)$ relative to bilinear form (2.3) if and only if equation (2.1) reads as*

$$N(u) \equiv a_4(t,u(t))u^{(4)}(t) + \frac{2M_4'(t)}{M(t,u(t))}u'''(t) + a_2(t,u(t))u''(t) + \frac{M_2'(t) - M_4'''(t)}{M(t,u(t))}u'(t) + \frac{(B_M)_u'(t,u(t))}{M(t,u(t))} = 0. \tag{4.1}$$

Proof. Let us find a variation of Hamilton-Ostrogradsky action (3.1). We have

$$\delta F_N[u,h] = \int_{t_0}^{t_1}\Big(\left(M_4''(t) - M_2(t)\right)u'(t)h'(t) + M_4(t)u''(t)h''(t) + (B_M)_u'(t,u(t))h(t)\Big)dt$$
$$= \int_{t_0}^{t_1}\Big(-M_4'''(t)u'(t)h(t) - M_4''(t)u''(t)h(t) + M_2'(t)u'(t)h(t) + M_2(t)u''(t)h(t) - M_4'(t)u''(t)h'(t) - M_4(t)u'''(t)h'(t) + (B_M)_u'(t,u(t))h(t)\Big)dt$$
$$= \int_{t_0}^{t_1}\Big(-M_4'''(t)u'(t)h(t) - M_4''(t)u''(t)h(t) + M_2'(t)u'(t)h(t) + M_2(t)u''(t)h(t) + M_4''(t)u''(t)h(t) + 2M_4'(t)u'''(t)h(t) + M_4(t)u^{(4)}(t)h(t) + (B_M)_u'(t,u(t))h(t)\Big)dt$$

$$= \int_{t_0}^{t_1} M(t,u(t))\big(a_4(t,u(t))u^{(4)}(t) + \frac{2M_4'(t)}{M(t,u(t))}u'''(t) + a_2(t,u(t))u''(t)$$

$$+ \frac{M_2'(t) - M_4'''(t)}{M(t,u(t))}u'(t) + \frac{(B_M)_u'(t,u(t))}{M(t,u(t))}\big)\, h(t)dt = \Phi(u; N(u), h)$$

for all $u \in D(N)$, $h \in D(N_u')$. Hence, equation (2.1) is represented in form (4.1). On the other hand, equation (4.1) is derived from the stationarity condition of Hamilton-Ostrogradsky action (3.1). It means that conditions (2.4), (2.5) must be satisfied. □

5. On connection between variationality of equation (2.1) and Hamilton-Ostrogradsky equations

Theorem 5.1. *If operator N (2.1) is potential on $D(N)$ (2.2) relative to bilinear form (2.3) then equation (2.1) is represented in the form of Hamilton-Ostrogradsky equations*

$$\begin{aligned} &u'(t) = u'(t), \qquad u''(t) = \frac{p_2(t)}{M_4(t)}, \\ &p_1'(t) = (B_M)_u'(t,u(t)), \qquad p_2'(t) = (M_4''(t) - M_2(t))u'(t) - p_1(t), \end{aligned} \tag{5.1}$$

where

$$p_1(t) = (M_4''(t) - M_2(t))u'(t) - M_4'(t)u''(t) - M_4(t)u'''(t), \qquad p_2(t) = M_4(t)u''(t).$$

Proof. From (3.1) we obtain the Lagrangian

$$L[t,u(t),u'(t),u''(t)] = \frac{1}{2}\left(M_4''(t) - M_2(t)\right)(u'(t))^2 + \frac{1}{2}M_4(t)(u''(t))^2 + B_M(t,u(t)).$$

Then

$$p_2(t) = \frac{\partial L}{\partial u''(t)} = M_4(t)u''(t) \tag{5.2}$$

and

$$u''(t) = \frac{p_2(t)}{M_4(t)}.$$

Hence, the Hamiltonian is

$$\begin{aligned} H[t,u(t),u'(t),p_1(t),p_2(t)] =& p_1(t)u'(t) + \Big(p_2(t)u''(t) - L[t,u(t),u'(t),u''(t)]\Big)\Big|_{u''(t)=\frac{p_2(t)}{M_4(t)}} \\ =& p_1(t)u'(t) + \Big(p_2(t)u''(t) - \frac{1}{2}(M_4''(t) - M_2(t))(u'(t))^2 \\ &- \frac{1}{2}M_4(t)(u''(t))^2 - B_M(t,u(t))\Big)\Big|_{u''(t)=\frac{p_2(t)}{M_4(t)}} \\ =& p_1(t)u'(t) + \frac{1}{2}\frac{p_2^2(t)}{M_4(t)} - \frac{1}{2}(M_4''(t) - M_2(t))(u'(t))^2 - B_M(t,u(t)). \end{aligned}$$

We have

$$\frac{\partial H}{\partial p_2(t)} = \frac{p_2(t)}{M_4(t)},$$

that is,

$$\frac{\partial H}{\partial p_2(t)} = u''(t).$$

Then

$$\frac{\partial H}{\partial p_1(t)} = u'(t), \qquad \frac{\partial H}{\partial u(t)} = -(B_M)'_u(t, u(t)), \qquad \frac{\partial H}{\partial u'(t)} = p_1(t) - (M_4''(t) - M_2(t))u'(t).$$

We note that for the Hamilton-Ostrogradsky action

$$F_N[u] = \int_{t_0}^{t_1} L[t, u(t), u'(t), u''(t)]dt$$

the corresponding Lagrange-Ostrogradsky equation is

$$N(u) \equiv \frac{\partial L}{\partial u(t)} - \frac{d}{dt}\frac{\partial L}{\partial u'(t)} + \frac{d^2}{dt^2}\frac{\partial L}{\partial u''(t)} = 0,$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u'(t)} - \frac{d}{dt}\frac{\partial L}{\partial u''(t)}\right) = \frac{\partial L}{\partial u(t)}. \tag{5.3}$$

For

$$p_1(t) = \frac{\partial L}{\partial u'(t)} - \frac{d}{dt}\frac{\partial L}{\partial u''(t)} = (M_4''(t) - M_2(t))u'(t) - M_4'(t)u''(t) - M_4(t)u'''(t) \tag{5.4}$$

from (5.3) we get

$$p_1'(t) = \frac{\partial L}{\partial u(t)},$$

i.e.

$$p_1'(t) = (B_M)'_u(t, u(t)).$$

Thus,

$$p_1'(t) = -\frac{\partial H}{\partial u(t)}.$$

For $p_2(t)$ (5.2) from (5.4) we have

$$p_2'(t) = \frac{\partial L}{\partial u'(t)} - p_1(t) = (M_4''(t) - M_2(t))u'(t) - p_1(t),$$

that is,

$$p_2'(t) = -\frac{\partial H}{\partial u'(t)}.$$

Hence,

$$u'(t) = \frac{\partial H}{\partial p_1(t)}, \qquad u''(t) = \frac{\partial H}{\partial p_2(t)}, \qquad p_1'(t) = -\frac{\partial H}{\partial u(t)}, \qquad p_2'(t) = -\frac{\partial H}{\partial u'(t)}.$$

It means that equation (2.1) with the potential operator N is represented in the form of Hamilton-Ostrogradsky equations (5.1). □

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Svetlana Aleksandrovna Budochkina,
RUDN University,
Miklukho-Maklaya str. 6,
117198, Moscow, Russian Federation
E-mail: `budochkina-sa@rudn.ru`

Thi Huyen Luu,
RUDN University,
Miklukho-Maklaya str. 6,
117198, Moscow, Russian Federation
E-mail: `luuthihuyen250393@gmail.com`

Vladimir Andreevich Shokarev,
RUDN University,
Miklukho-Maklaya str. 6,
117198, Moscow, Russian Federation
E-mail: `vova.shokarev@mail.ru`