ISSN 2074-1871 Уфимский математический журнал. Том 15. № 3 (2023). С. 109-120.

POINT SPECTRUM AND HYPERCYCLICITY PROBLEM FOR A CLASS OF TRUNCATED TOEPLITZ OPERATORS

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Abstract. Truncated Toeplitz operators are restrictions of usual Toeplitz operators onto model subspaces $K_\theta = H^2 \ominus \theta H^2$ of the Hardy space H^2, where θ is an inner function. In this note we study the structure of eigenvectors for a class of truncated Toeplitz operators and discuss an open problem whether a truncated Toeplitz operator on a model space can be hypercyclic, that is, whether there exists a vector with a dense orbit. For the classical Toeplitz operators on H^2 with antianalytic symbols a hypercyclicity criterion was given by G. Godefroy and J. Shapiro, while for Toeplitz operators with polynomial or rational antianalytic part some partial answers were obtained by the authors jointly with E. Abakumov and S. Charpentier.

We find point spectrum and eigenfunctions for a class of truncated Toeplitz operators with polynomial analytic and antianalytic parts. It is shown that the eigenvectors are linear combinations of reproducing kernels at some points such that the values of the inner function θ at these points have a polynomial dependence. Next we show that, for a class of model spaces, truncated Toeplitz operators with symbols of the form $\Phi(z) = a\bar{z} + b + cz$, where $|a| \neq |c|$, have complete sets of eigenvectors and, in particular, are not hypercyclic. Our main tool here is the factorization of functions in an associated Hardy space in an annulus. We also formulate several open problems.

Keywords: Hypercyclic operator, Toeplitz operator, model space, truncated Toeplitz operator.

Mathematics Subject Classification: 47A16, 47B35, 30H10

1. Introduction

A continuous linear operator T on a separable Banach (or Frechét) space X is said to be *hypercyclic* if there exists $x \in X$ such that the set $\{T^n x : n \in \mathbb{N}_0\}$ is dense in X (here $\mathbb{N}_0 = \{0, 1, 2, \dots\}$).

The first example of a hypercyclic operator in a Banach space setting was given by S. Rolewicz [15] who showed that the operator αB, where B is the backward shift and α is an arbitrary complex number with $|\alpha| > 1$, is hypercyclic on $\ell^p(\mathbb{N})$, $1 \leqslant p < \infty$. It was shown that many important classes of operators have this property. Among basic examples of hypercyclic operators were Toeplitz operators with antianalytic symbols.

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The work of A. Lishanskii in Section 2 was performed at the Saint Petersburg Leonhard Euler International Mathematical Institute and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-15-2022-287). The results of Section 3 were obtained with the support of the Russian Science Foundation project 19-11-00058P.

Submitted November 3, 2022.

1.1. Hypercyclic Toeplitz operators. As usual, $\mathbb{D}$ and $\mathbb{T}$ denote the unit disk and the unit circle, respectively, and H^2 stands for the Hardy space in $\mathbb{D}$. Recall that for a function $\psi \in L^\infty(\mathbb{T})$ the *Toeplitz operator* $T_\psi : H^2 \to H^2$ with the symbol ψ is defined as $T_\psi f = P_+(\psi f)$, where P_+ is the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. The following theorem, due to G. Godefroy and J. Shapiro [10], describes all hypercyclic Toeplitz operators with antianalytic symbols (i.e., $T_{\overline{\varphi}}$ where $\varphi \in H^\infty$).

Theorem. (G. Godefroy, J. Shapiro, 1991) *The Toeplitz operator $T_{\overline{\varphi}}$, where $\varphi \in H^\infty$, $\varphi \neq const$, is hypercyclic if and only if $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.*

It is well known that every Cauchy kernel $k_\lambda(z) = (1 - \bar{\lambda} z)^{-1}$, $|\lambda| < 1$, is an eigenvector for an antianalytic Toeplitz operator $T_{\overline{\varphi}}$ corresponding to the eigenvalue $\overline{\varphi(\lambda)}$. Then sufficiency of the condition $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ for hypercyclicity follows immediately from the Godefroy–Shapiro Criterion (see [10], [4, Corollary 1.10] or [11, Theorem 3.1]).

Analytic Toeplitz operators T_φ, $\varphi \in H^\infty$, are simply multiplication operators. It is clear that they cannot be hypercyclic.

In [18] S. Shkarin posed the problem to describe hypercyclic Toeplitz operators in terms of their symbols. This seems to be a difficult and, at the moment, widely open problem. Shkarin gave a necessary and sufficient condition for hypercyclicity in the case when the symbol is of the form $\Phi(z) = a\overline{z} + b + cz$ (i.e., with tridiagonal matrix): T_Φ is hypercyclic if and only if $|a| > |c|$ and

$$(\mathbb{C} \setminus \Phi(\mathbb{D})) \cap \mathbb{D} \neq \emptyset, \qquad (\mathbb{C} \setminus \Phi(\mathbb{D})) \cap \widehat{\mathbb{D}} \neq \emptyset. \tag{1.1}$$

Here $\widehat{\mathbb{D}} = \mathbb{C} \setminus \overline{\mathbb{D}}$ and Φ is extended to $\mathbb{C}$ as $\Phi(z) = \frac{a}{z} + b + cz$.

Hypercyclicity of Toeplitz operators with the symbols of the form

$$\Phi(z) = R(\overline{z}) + \varphi(z),$$

where R is a polynomial or a rational function with poles outside $\overline{\mathbb{D}}$ and $\varphi \in H^\infty$, was studied in [1, 3]. It turned out that the valence of the meromorphic continuation of the symbol Φ, that is, $\Phi(z) = R\big(\frac{1}{z}\big) + \varphi(z)$, is crucial for hypercyclicity. Also it was shown that the hypercyclicity problem is closely related to cyclicity of certain families for some associated analytic Toeplitz operator. A complete description of hypercyclic Toeplitz operators in this class is not known, however, there is a number of conditions sufficient for hypercyclicity. For the operators with symbols of the form $\Phi(z) = a\overline{z} + \varphi(z)$ known necessary and sufficient conditions almost meet.

Theorem. ([3]) *Let $\Phi(z) = \frac{a}{z} + \varphi(z)$, where $\varphi \in H^\infty$, satisfy* (1.1).
1. If T_Φ is hypercyclic, then Φ is univalent in $\mathbb{D}$;
2. If $\varphi \in A(\mathbb{D})$ (disk-algebra) and Φ is injective in $\overline{\mathbb{D}}$, then T_Φ is hypercyclic.

Thus, the difference is only in the injectivity on the boundary. Also it should be noted that the second part of (1.1) (i.e. $(\mathbb{C} \setminus \Phi(\mathbb{D})) \cap \widehat{\mathbb{D}} \neq \emptyset$) is necessary for hypercyclicity of T_Φ, while for the first part only a weaker variant $(\mathbb{C} \setminus \Phi(\mathbb{D})) \cap \mathbb{T} \neq \emptyset$ is known to be necessary.

1.2. Truncated Toeplitz operators. Let $S^* f = T_{\overline{z}} f = \frac{f(z) - f(0)}{z}$ be the backward shift in H^2. By the classical theorem of Arne Beurling, any S^*-invariant subspace is of the form $K_\theta = H^2 \ominus \theta H^2$ for some inner function θ in $\mathbb{D}$. Subspaces K_θ are often referred to as *model spaces* due to their role in Nagy–Foiaş model for contractions.

Truncated Toeplitz operators (TTO) are restrictions of usual Toeplitz operators onto K_θ. Namely, given a space K_θ and $\psi \in L^\infty(\mathbb{T})$, define the operator $A_\psi : K_\theta \to K_\theta$ by the formula

$$A_\psi f = P_\theta(\psi f), \qquad f \in K_\theta,$$

where P_θ stands for the orthogonal projection from $L^2(\mathbb{T})$ onto K_θ. Formally, A_ψ depends also on θ, but we always assume θ to be fixed and do not include it into notations.

Even if some special cases of TTO-s were well studied for a long time (e.g., A_z is a model operator of Nagy–Foiaş theory and its functions $\varphi(A_z) = A_\varphi$, $\varphi \in H^\infty$), a systematic study of TTO-s was initiated by a seminal paper by D. Sarason [17]. Note that one can consider operators with unbounded symbols (assuming there exists a bounded extension of an operator defined on the dense subset of bounded functions in K_θ) and it may happen that a bounded TTO has no bounded symbol at all (see, e.g., [2]. In what follows we consider the case of bounded symbols only.

During the last decade the theory of truncated Toeplitz operators became an active field of research (see the surveys [7, 9] and the references therein). However, it seems that the spectral properties of TTO-s are far from being well understood. For instance, to the best of our knowledge, it is not known whether the point spectrum of a TTO can have a nonempty interior. For some classes of TTO-s Fredholmness and invertibility criteria were found by M. C. Câmara and J. R. Partington [6]. In particular, they gave a description of the point spectrum of a TTO with a rational symbol in case of the Hardy space in the half-plane.

A problem which apparently did not attract much attention yet is to understand the dynamics of TTO-s and in particular their hypercyclicity. In fact, it is not known whether hypercyclic TTO-s do exist.

Problem. *Do there exist hypercyclic truncated Toeplitz operators?*

Note that, since K_θ is invariant with respect to antianalytic Toeplitz operators, one has $A_{\bar\varphi} = T_{\bar\varphi}$ for $\varphi \in H^\infty$. Also, recall that the existence of an eigenvector for the adjoint operator is a trivial obstacle for hypercyclicity. Combining this, one comes to the following simple observation.

Proposition 1.1. *If an inner function θ has a zero in $\mathbb{D}$, then for any $\varphi \in H^\infty$ the truncated Toeplitz operators A_φ and $A_{\bar\varphi}$ are not hypercyclic.*

Proof. If $\theta(\lambda) = 0$, then $k_\lambda(z) = (1-\bar\lambda z)^{-1}$ belongs to K_θ, and $A_{\bar\varphi} k_\lambda = \overline{\varphi(\lambda)} k_\lambda$ whence $A_\varphi = A^*_{\bar\varphi}$ is not hypercyclic. Also,

$$A_\varphi\Big(\frac{\theta}{z-\lambda}\Big) = P_\theta\Big(\frac{\varphi\theta}{z-\lambda}\Big) = P_\theta\Big(\theta\frac{\varphi-\varphi(\lambda)}{z-\lambda} + \varphi(\lambda)\frac{\theta}{z-\lambda}\Big) = \varphi(\lambda)\frac{\theta}{z-\lambda}.$$

Hence, A_φ also has an eigenvector and so $A_{\bar\varphi}$ is not hypercyclic. □

Thus, if we are interested in hypercyclicity of TTO-s with analytic or antianalytic symbols, the only interesting case is the case of singular inner function θ. In this case the point spectrum is empty.

Problem. *Let θ be a singular inner function. Do there exist hypercyclic truncated Toeplitz operators with analytic or antianalytic symbols?*

One can start with the case of a sufficiently regular symbol φ, say in the disk-algebra $A(\mathbb{D})$ or even analytic in a neighborhood of $\overline{\mathbb{D}}$. In this case the spectrum of the operator A_φ is known to be $\varphi(\sigma(\theta))$ [14, Lecture III], where $\sigma(\theta)$ stands for the boundary spectrum of θ (the set of those $\zeta \in \mathbb{T}$ for which $\liminf_{z\to\zeta, z\in\mathbb{D}} |\theta(z)| < 1$). Thus, a necessary condition for hypercyclicity is that $\varphi(\sigma(\theta)) \cap \mathbb{T} \ne \emptyset$.

Problem. *Let $\theta(z) = \exp\big(\frac{z+1}{z-1}\big)$ be the simplest atomic singular inner function. Do there exist hypercyclic truncated Toeplitz operators with a symbol φ in H^∞ or in $A(\mathbb{D})$? This is equivalent to the following approximation problem: does there exist a function $f \in K_\theta$ such that*

$$\operatorname{Clos}\{\varphi^n f + \theta g : \ g \in H^2, n \in \mathbb{N}_0\} = H^2\,?$$

D. Sarason [16] showed that the operator $(I+V)^{-1}$, where V is the Volterra integration operator in $L^2(0,1)$, is unitarily equivalent to the operator $\frac{1}{2}(I+A_z)$ in K_θ with $\theta(z) = \exp\big(\frac{z+1}{z-1}\big)$.

In [13] it was shown that $I+V$ (and, thus, $(I+V)^{-1}$) is not supercyclic. We conclude that the truncated Toeplitz operator $\frac{1}{2}(I+A_z)$ is not supercyclic in K_θ.

2. Point spectrum of TTO-s with polynomial symbols

In this section we consider truncated Toeplitz operators with symbols of the form

$$\Phi(z)=\sum_{k=1}^{N}a_k z^{-k}+\sum_{l=0}^{M}c_l z^l, \qquad M,N\in\mathbb{N}, \qquad a_N,c_M\neq 0, \tag{2.1}$$

and describe their point spectrum under some mild restrictions. It turns out that the set of eigenvectors λ has a rather curious structure involving some polynomial dependence between the values of θ at the roots of $\Phi-\lambda$. The next theorem is very close to [6, Thm. 5.4] where the invertibility criterion for a TTO with a rational symbol is given in the case of the Hardy space in the half-plane. The proof in [6] is based on the Riemann–Hilbert problem methods, while our approach is completely elementary.

Recall that the mapping $f\to\tilde f=\bar z\bar f\theta$ in $L^2(\mathbb{T})$ is an involution on the space K_θ (considered as a subspace of $L^2(\mathbb{T})$). Reproducing kernel of K_θ at the point $\lambda\in\mathbb{D}$ is given by

$$k_\lambda^\theta(z)=P_\theta\Big(\frac{1}{1-\bar\lambda z}\Big)=\frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\bar\lambda z}.$$

Note that the conjugate kernel $\tilde k_\lambda^\theta=\widetilde{(k_\lambda^\theta)}$ is of the form

$$\tilde k_\lambda^\theta(z)=\frac{\theta(z)-\theta(\lambda)}{z-\lambda}.$$

Theorem 2.1. *Let Φ be given by (2.1) and let z_j, $j=1,\dots M+N$, be the zeros of $\Phi-\lambda$. Assume that all z_j are distinct and $|z_j|\neq 1$ for any j. Then λ is an eigenvalue of A_Φ if and only if there exist nonzero polynomials P_1,P_2 of degrees at most $M-1$ and $N-1$ respectively such that the following conditions hold:*

$$z_j^N P_1(z_j)\theta(z_j)+P_2(z_j)=0, \qquad |z_j|<1, \tag{2.2}$$

and

$$z_j^N P_1(z_j)+P_2(z_j)\overline{\theta(1/\bar z_j)}=0, \qquad |z_j|>1. \tag{2.3}$$

In this case the corresponding eigenfunction is given by

$$f=\sum_{|z_j|<1}\beta_j z_j^N P_1(z_j)\tilde k_{z_j}^\theta+\sum_{|z_j|>1}\beta_j P_2(z_j)k_{1/\bar z_j}^\theta$$

for some complex coefficients β_j.

Proof. We are looking for solutions of the equation $A_\Phi f=\lambda f$, $f\in K_\theta$. Denote by φ the analytic part of Φ, $\varphi(z)=\sum_{l=0}^{M}c_l z^l$. Note that $A_\Phi f=P_\theta(T_\Phi f)$, and

$$T_\Phi f=\varphi f+\sum_{k=1}^{N}a_k\frac{f-\sum_{j=0}^{k-1}\frac{f^{(j)}(0)}{j!}z^j}{z^k}.$$

Thus, $A_\Phi f=\lambda f$ is equivalent to the equation

$$\varphi f+\sum_{k=1}^{N}a_k\frac{f-\sum_{j=0}^{k-1}\frac{f^{(j)}(0)}{j!}z^j}{z^k}=\lambda f+\theta h,$$

where $h \in H^2$ or, equivalently,

$$(\Phi - \lambda)f = \theta h + \sum_{k=1}^{N} a_k \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^{j-k}.$$

Multiplying by z^N, we get

$$Qf = z^N \theta h + R, \tag{2.4}$$

where $Q = z^N(\Phi - \lambda)$ is a polynomial of degree $M + N$ with the zeros z_j and

$$R(z) = \sum_{k=1}^{N} a_k \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^{N-k+j}$$

is a polynomial of degree at most $N - 1$. Since z_j are distinct we can write

$$\frac{1}{Q(z)} = \sum_{j=1}^{M+N} \frac{\beta_j}{z - z_j} \tag{2.5}$$

for some coefficients β_j.

Assume that λ is an eigenvector and so there exists a nontrivial $f \in K_\theta$ satisfying (2.4). If $|z_j| < 1$, then the right-hand side of (2.4) must vanish at z_j and we get

$$z_j^N \theta(z_j) h(z_j) + R(z_j) = 0.$$

Hence, we have

$$\begin{aligned} f(z) = \sum_{|z_j|<1} \beta_j \frac{z^N \theta(z) h(z) - z_j^N \theta(z_j) h(z_j)}{z - z_j} + \sum_{|z_j|<1} \beta_j \frac{R(z) - R(z_j)}{z - z_j} \\ + \sum_{|z_j|>1} \beta_j \frac{z^N \theta(z) h(z)}{z - z_j} + \sum_{|z_j|>1} \beta_j \frac{R(z)}{z - z_j}. \end{aligned} \tag{2.6}$$

For $|z_j| < 1$ we further have

$$\frac{z^N \theta(z) h(z) - z_j^N \theta(z_j) h(z_j)}{z - z_j} = \theta(z) \frac{z^N h(z) - z_j^N h(z_j)}{z - z_j} + z_j^N h(z_j) \frac{\theta(z) - \theta(z_j)}{z - z_j}.$$

Note that the first term is in θH^2, while the second belongs to K_θ.

Since the degree of R is at most $N - 1$, we have the Lagrange interpolation formula

$$\frac{R(z)}{Q(z)} = \sum_{j=1}^{M+N} \beta_j \frac{R(z)}{z - z_j} = \sum_{j=1}^{M+N} \beta_j \frac{R(z_j)}{z - z_j}. \tag{2.7}$$

Finally, note that for $|z_j| > 1$ we have

$$\frac{1}{z - z_j} = \frac{\theta(z)\overline{\theta(1/\bar{z}_j)}}{z - z_j} + \frac{1 - \theta(z)\overline{\theta(1/\bar{z}_j)}}{z - z_j},$$

where, again, the first term is in θH^2, while the second is in K_θ. Combining all these observations we conclude that $f = f_1 + \theta f_2$ where $f_1 \in K_\theta$ and $f_2 \in H^2$ are given by

$$\begin{aligned} f_1(z) &= \sum_{|z_j|<1} \beta_j z_j^N h(z_j) \frac{\theta(z) - \theta(z_j)}{z - z_j} + \sum_{|z_j|>1} \beta_j R(z_j) \frac{1 - \theta(z)\overline{\theta(1/\bar{z}_j)}}{z - z_j}, \\ f_2(z) &= \sum_{|z_j|<1} \beta_j \frac{z^N h(z) - z_j^N h(z_j)}{z - z_j} + \sum_{|z_j|>1} \beta_j \frac{R(z_j)\overline{\theta(1/\bar{z}_j)}}{z - z_j}. \end{aligned} \tag{2.8}$$

However, by our assumption, $f \in K_\theta$, whence $f_2 = 0$. In view of (2.5) this is equivalent to

$$\frac{z^N h(z)}{Q(z)} = \sum_{|z_j|<1} \beta_j \frac{z_j^N h(z_j)}{z - z_j} - \sum_{|z_j|>1} \beta_j \frac{R(z_j)\overline{\theta(1/\bar{z}_j)}}{z - z_j}. \tag{2.9}$$

Hence $z^N h$ is a polynomial of degree at most $M+N-1$ and since h is analytic at 0 we conclude that h is a polynomial of degree at most $M-1$. If we put $P_1 = h$ and $P_2 = R$, then the condition

$$z_j^N \theta(z_j) h(z_j) + R(z_j) = 0 \quad \text{for} \quad |z_j| < 1$$

is equivalent to (2.2). Comparing the residues at z_j with $|z_j| > 1$ in (2.9) we conclude that $z_j^N h(z_j) = -R(z_j)\overline{\theta(1/\bar{z}_j)}$ which is equivalent to (2.3).

To prove that conditions (2.2) and (2.3) are sufficient for λ to be an eigenvalue for A_Φ, we reverse the arguments. Assume that there exists P_1 and P_2 as in (2.2) and (2.3) and let $h = P_1$, $R = P_2$. Then the interpolation formula (2.9) holds and so the function f_2 in (2.8) is zero. Define f_1 by (2.8). Combining the formula for f_1 with (2.7) we see that $f = f_1 = f_1 + \theta f_2$ satisfies the equality (2.6). Using the fact that

$$z_j^N \theta(z_j) h(z_j) + R(z_j) = 0 \quad \text{for} \quad |z_j| < 1,$$

we finally conclude that $Qf = z^N h\theta + R$. Dividing by z^N and comparing the coefficients at negative powers it is easy to see that R must coincide with

$$\sum_{k=1}^{N} a_k \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^{N-k+j},$$

where a_k are the coefficients at z^{-1} in (2.1). □

Remark. Note that for some configurations of zeros with respect to the unit circle, the conditions (2.2) and (2.3) may be never satisfied. In particular, as is shown below, for a three-term TTO λ is never an eigenvalue if z_1 and z_2 lie in different components of $\mathbb{C} \setminus \mathbb{T}$.

Remark. The above results remain true if there exist zeros of $\Phi - \lambda$ with $|z_j| = 1$, but $z_j \notin \sigma(\theta)$ and so θ is analytic in a neighborhood of z_j. Such z_j can be included in any of the conditions (2.2) or (2.3) since in this case $\theta(z_j) = 1/\overline{\theta(1/\bar{z}_j)}$. The same is true if $z_j \in \sigma(\theta)$ but $|\theta(z_j)| = 1$ and θ has a finite angular derivative at z_j (z_j is a Julia–Carathéodory point for θ) and so the space K_θ contains the reproducing kernel at z_j. It is not, however, clear to us whether λ can be an eigenvalue if one of z_j lies on $\mathbb{T}$ and is not a Julia–Carathéodory point for θ.

3. Three-term truncated Toeplitz operators

In this section we consider the case when

$$\Phi(z) = a\bar{z} + b + cz, \qquad z \in \mathbb{T}.$$

In this case the formulation of Theorem 2.1 will be substantially simplified.

Corollary 3.1. *Let*

$$\Phi(z) = \frac{a}{z} + b + cz, \qquad a, c \neq 0,$$

and assume that the zeros z_1, z_2 of the function $\Phi - \lambda$ are distinct and satisfy $|z_j| \neq 1$, $j = 1, 2$. Then λ is an eigenvalue for A_Φ if and only if either

1. $|z_1|, |z_2| < 1$ and $z_1\theta(z_1) = z_2\theta(z_2)$,

or

2. $|z_1|, |z_2| > 1$ and $\frac{1}{z_1}\theta\left(\frac{1}{\bar{z}_1}\right) = \frac{1}{z_2}\theta\left(\frac{1}{\bar{z}_2}\right)$.

In Case 1 the eigenfunction of A_Φ corresponding to λ is given by

$$f_\lambda = z_1 \tilde{k}^\theta_{z_1} - z_2 \tilde{k}^\theta_{z_2}, \tag{3.1}$$

while in Case 2

$$f_\lambda = \frac{1}{\bar{z}_1} k^\theta_{1/\bar{z}_1} - \frac{1}{\bar{z}_2} k^\theta_{1/\bar{z}_2}.$$

Proof. Since in this case $M = N = 1$, the polynomials P_1 and P_2 in (2.2) and (2.3) are just constants. If both $z_1, z_2 \in \mathbb{D}$, we have

$$P_1 z_1 \theta(z_1) + P_2 = P_1 z_2 \theta(z_2) + P_2 = 0$$

which implies $z_1\theta(z_1) = z_2\theta(z_2)$. Analogously, if $z_1, z_2 \in \widehat{\mathbb{D}}$, then

$$P_1 z_1 + P_2 \overline{\theta(1/\overline{z}_1)} = P_1 z_2 + P_2 \overline{\theta(1/\overline{z}_2)} = 0,$$

whence $z_1^{-1}\overline{\theta(1/\overline{z}_1)} = z_2^{-1}\overline{\theta(1/\overline{z}_2)}$.

Finally, consider the case $|z_1| < 1$, $|z_2| > 1$. Then

$$P_1 z_1 \theta(z_1) + P_2 = 0 = P_1 z_2 + P_2 \overline{\theta(1/\overline{z}_2)},$$

whence

$$z_1 \theta(z_1) \overline{\theta(1/\overline{z}_2)} = z_2,$$

an impossible equality since its left-hand side in $\mathbb{D}$ and $|z_2| > 1$.

Conversely, equalities $z_1\theta(z_1) = z_2\theta(z_2)$ and, respectively,

$$\frac{1}{\bar{z}_1}\theta\Big(\frac{1}{\bar{z}_1}\Big) = \frac{1}{\bar{z}_2}\theta\Big(\frac{1}{\bar{z}_2}\Big)$$

imply the existence of the polynomials P_1 and P_2 of degree zero (nonzero constants) such that (2.2) and (2.3) are satisfied and so λ is an eigenvector by Theorem 2.1.

The form of the eigenfunctions follows easily from the calculations of Theorem 2.1. □

We excluded the case of multiple zeros of $\Phi - \lambda$ to avoid uninteresting technicalities. In this case our polynomial dependencies (2.2) and (2.3) will involve also the values of the derivatives of θ at the multiple zeros. However, in the case of symbols of the form $\Phi(z) = \frac{a}{z} + b + cz$ the answer is easy.

Corollary 3.2. *Let*

$$\Phi(z) = \frac{a}{z} + b + cz, \qquad a, c \neq 0,$$

and assume that $\Phi - \lambda$ has a zero z_0 in $\mathbb{D}$ of multiplicity 2. Then λ is an eigenvalue for A_Φ if and only if $z_0\theta'(z_0) + \theta(z_0) = 0$.

Note that if z_1, z_2 are zeros of $z(\Phi(z) - \lambda) = cz^2 + (b-\lambda)z + a$, then $z_1 z_2 = a/c$. Put $\beta = a/c$. If $|\beta| < 1$ then λ can be an eigenvalue only when $|z_1|, |z_2| < 1$; if $|\beta| > 1$, then a necessary condition is that $|z_1|, |z_2| > 1$. Finally, in the case when $|\beta| = 1$, λ is not an eigenvalue unless $|z_1| = |z_2| = 1$.

We now address the question about existence of λ, z_1 and z_2 satisfying the conditions of Corollary 3.1. We will show that under some (apparently, rather mild) restriction on the function θ we do not only have infinitely many solutions, but, moreover, the eigenvectors of a tridiagoanal TTO are complete in K_θ. To formulate our results we will need some notions from the theory of Smirnov classes $E^p(G)$ in general domains.

3.1. Smirnov classes in multiconnected domains. Let G be a simply-connected domain in $\mathbb{C}$. Denote by $H^\infty(G)$ the class of all bounded analytic functions in G. The *Smirnov class* $E^p(G)$, $0 < p < \infty$, consists of those functions f analytic in G for which $f(\varphi(z))(\varphi'(z))^{1/p} \in H^p$, where H^p is the usual Hardy space in $\mathbb{D}$ and φ is some (any) conformal map of $\mathbb{D}$ onto G. An equivalent definition describes $E^p(G)$ as the set of those functions f for which there exists a sequence of rectifiable Jordan curves Γ_n tending to the boundary (i.e., Γ_n surrounds each compact subset of G for sufficently large n) with the property

$$\sup_n \int_{\Gamma_n} |f(z)|^p |dz| < \infty.$$

For the theory of Smirnov classes see, e.g., [8, Ch. 10].

A simply-connected domain G with rectifiable boundary is said to be a *Smirnov domain* if φ' is an outer function (as above, φ is some conformal map of $\mathbb{D}$ onto G). In this case we say that a function $f \in E^p(G)$ is *outer* if $f(\varphi(z))(\varphi'(z))^{1/p}$ is an outer function in H^p and we say that f *has no singular inner factor* if $f(\varphi(z))(\varphi'(z))^{1/p} = B(z)F(z)$, where B is a Blaschke product and F is an outer function in $\mathbb{D}$.

We will need to define similar objects for the case of an annulus. For general multiconnected domains G factorization theory in $E^p(G)$ was developed by D. Khavinson in [12] (see, also, [5]). We will however use the following simple equivalent description of these classes. For $\beta \in \mathbb{D}$, $\beta \neq 0$, consider the annulus

$$R_\beta = \{|\beta| < |z| < 1\}.$$

For $\alpha \in \mathbb{T}$ let $R_\beta^\alpha = R_\beta \setminus \{r\alpha : \ |\beta| < r < 1\}$ be the annulus with a cut. Then $f \in E^p(R_\beta)$ if and only if for any $\alpha \in \mathbb{T}$ and a conformal mapping $\varphi : \mathbb{D} \to R_\beta^\alpha$ the function $f(\varphi(z))(\varphi'(z))^{1/p}$ is in H^p. Analogously, we say that $f \in E^p(R_\beta)$ has no singular inner factor in R_β if $f(\varphi(z))(\varphi'(z))^{1/p}$ has no singular inner factor in $\mathbb{D}$ for any α and φ. Obviously, it is sufficient to take only two different values of α.

3.2. Completeness of eigenvectors of a tridiagonal TTO. Let $\Phi(z) = \frac{a}{z} + b + cz$, $a, c \neq 0$, and let $\beta = a/c$. If $|\beta| < 1$ we put

$$\Psi(z) = z\theta(z) - \frac{\beta}{z}\theta\Big(\frac{\beta}{z}\Big), \tag{3.2}$$

while for $|\beta| > 1$ put

$$\Psi(z) = z\theta(z) - \frac{1}{\bar\beta z}\theta\Big(\frac{1}{\bar\beta z}\Big). \tag{3.3}$$

As before, let $R_\beta = \{z : \ |\beta| < |z| < 1\}$ for $|\beta| < 1$ and put $R_\beta = \{z : \ |\beta|^{-1} < |z| < 1\}$ for $|\beta| > 1$. Then the function Ψ is analytic in the respective choice of the annulus R_β and, moreover, $\Psi \in H^\infty(R_\beta)$.

Theorem 3.1. *Let $\Phi(z) = \frac{a}{z} + b + cz$, $a, c \neq 0$, and $|\beta| = |c/a| \neq 1$. Assume that the function Ψ defined by* (3.2) *or* (3.3) *has no singular inner factor in R_β. Then the set of eigenvectors of A_Φ is complete in K_θ.*

Proof. We consider the case $|\beta| < 1$, the case $|\beta| > 1$ is analogous. Assume that $f \in K_\theta$ is orthogonal to all eigenfunctions f_λ given by (3.1). Note that $(\tilde f, \tilde g) = (g, f)$ for $f, g \in K_\theta$. Then we have

$$0 = (f_\lambda, f) = z_1(\tilde k^\theta_{z_1}, f) - z_2(\tilde k^\theta_{z_2}, f) = z_1(\tilde f, k^\theta_{z_1}) - z_2(\tilde f, k^\theta_{z_2}) = z_1\tilde f(z_1) - z_2\tilde f(z_2).$$

Recall that $z_1 z_2 = \beta$ and $z_1\theta(z_1) = z_2\theta(z_2)$. Consider the function

$$F(z) = z\tilde f(z) - \frac{\beta}{z}\tilde f\Big(\frac{\beta}{z}\Big).$$

Then we conclude that $F(z) = 0$ whenever $z \in R_\beta$ and $\Psi(z) = 0$, whence F/Ψ is analytic in R_β. This is true even if Ψ has a multiple zero, since in this case it is not difficult to show that F will have the zero of at least the same multiplicity.

It is clear that $F \in E^2(R_\beta)$. Let us show, using the assumption that Ψ has no singular inner factor in R_β, that $F/\Psi \in E^2(R_\beta)$. First of all note that for almost all $z \in \mathbb{T}$

$$|\Psi(z)| \geqslant |\theta(z)| - |\beta| = 1 - |\beta|,$$

a similar estimate holds for $|z| = |\beta|$.

Fix some $\alpha \in \mathbb{T}$ such that $|\theta(r\alpha)| \to 1$ as $r \to 1-$ and $\left|\theta\left(\frac{\beta}{r\alpha}\right)\right| \to 1$ as $r \to |\beta|+$, and consider the annulus with a cut R_β^α. For a conformal map φ of $\mathbb{D}$ onto R_β^α we have $(F \circ \varphi) \cdot (\varphi')^{1/2} \in H^2$ and so $(F \circ \varphi) \cdot (\varphi')^{1/2} = IF_o$ where I is an inner function and F_o is an outer function. Also, $\Psi \circ \varphi = B\Psi_o$ for a Blaschke product B and an outer function Ψ_o. Since F/Ψ is analytic in R_β^α we conclude that B divides I. Thus,

$$\frac{(F \circ \varphi)}{\Psi \circ \varphi} \cdot (\varphi')^{1/2} = JH$$

for some inner function J and an outer function H. Now it is sufficient to show that $H \in L^2(\mathbb{T})$. We have $1/\Psi \in L^\infty(\partial R_\beta)$. By our choice of α there exists $\varepsilon, \delta > 0$ such that $|\Psi(r\alpha)| \geqslant \varepsilon$ when $r \in (|\beta|, |\beta| + \delta) \cup (1 - \delta, 1)$. Since $(F \circ \varphi) \cdot (\varphi')^{1/2} \in L^2(\partial R_\beta^\alpha)$ we have

$$H \in L^2\big(\varphi^{-1}(\partial R_\beta \cup \{r\alpha : r \in (|\beta|, |\beta| + \delta) \cup (1 - \delta, 1)\})\big).$$

The functions $(F/\Psi) \circ \varphi$ and φ' are obviously bounded on $\varphi^{-1}\big(\{r\alpha : r \in (|\beta| + \delta, 1 - \delta)\}\big)$. Thus, $H \in H^2$ and we conclude that $F/\Psi \in E^2(R_\beta^\alpha)$ for almost all α, and so $F/\Psi \in E^2(R_\beta)$.

Recall that any function $f \in K_\theta$ has a meromorhic pseudocontinuation to $\widehat{\mathbb{D}} = \{|z| > 1\}$ such that its nontangential boundary values on $\mathbb{T}$ taken from $\widehat{\mathbb{D}}$ coincide with its boundary values in $\mathbb{D}$ and also f/θ belongs to the Hardy space $H^2(\widehat{\mathbb{D}})$ (which is the same as $E^2(\widehat{\mathbb{D}})$). We claim that $F/\Psi \in H^2(\widehat{\mathbb{D}})$. Indeed,

$$\frac{F(z)}{\Psi(z)} = \frac{\frac{f(z)}{\theta(z)} - \frac{\beta}{z^2\theta(z)} f\left(\frac{\beta}{z}\right)}{1 - \frac{\beta}{z^2\theta(z)}\theta\left(\frac{\beta}{z}\right)},$$

and the function in the denominator is bounded away from zero. Analogously, one shows that $F/\Psi \in H^2(\{|z| < |\beta|\})$. Note also that $F(z)/\Psi(z) \to 0$ as $|z| \to \infty$ or $|z| \to 0$. Here we use the fact that $f/\theta \in H^2(\widehat{\mathbb{D}})$ and, therefore, $f(z)/\theta(z) \to 0$ as $|z| \to \infty$.

The nontangential boundary values of F/Ψ taken from inside of R_β coincide with the boundary values taken from the outer domains $\{|z| < |\beta|\}$ and $\{|z| > 1\}$ almost everywhere. Applying the Cauchy formula over the contours approaching the boundary from the opposite sides and passing to the limit (which is possible for functions in the classes E^2) we see that F/Ψ has an analytic extension across the circles $\{|z| = |\beta|\}$ and $\{|z| = 1\}$. Thus, F/Ψ is an entire function tending to zero at infinity. We conclude that $F/\Psi \equiv 0$, whence

$$z\tilde{f}(z) = \frac{\beta}{z}\tilde{f}\Big(\frac{\beta}{z}\Big), \qquad |\beta| < |z| < 1.$$

Since the right-hand side is analytic in $|z| > |\beta|$, it follows that $\tilde{f}$ has an analytic continuation to $\mathbb{C}$ with $|\tilde{f}(z)| = O(|z|^{-1})$, $|z| \to \infty$, and so $f \equiv 0$. □

We see that in the conditions of Theorem 3.1 the operator A_Φ has many eigenvectors whence $A_{\bar{\Phi}}$ is not hypercyclic.

Now let us consider in more detail the case when the boundary spectrum of θ is one point, e.g., $\sigma(\theta) = \{1\}$. In this case θ is meromorphic in $\mathbb{C} \setminus \{1\}$, while the function Ψ will be meromorphic in $\mathbb{C} \setminus \{1, \beta\}$ (here we again consider the case $|\beta| < 1$). Let R_β^α be the annulus with a cut which does not touch 1 and β, and let φ be the corresponding conformal map. Then the function $\Psi \circ \varphi$ in $\mathbb{D}$ can have no singular inner factors except the atomic singular functions $\exp\big(a_j \frac{z+\zeta_j}{z-\zeta_j}\big)$, $j = 1, 2$, where $a_j > 0$ and $\zeta_j \in \mathbb{T}$ are such that $\varphi(\zeta_1) = 1$ and $\varphi(\zeta_2) = \beta$. Indeed, if $z \in \mathbb{T}$, $z \neq \zeta_1, \zeta_2$, and z belongs to the support of the singular measure from the inner factor of $\Psi \circ \varphi$, then there exists a sequence $z_n \to z$, $z_n \in \mathbb{D}$, such that $\Psi(\varphi(z_n)) = o((z_n - z)^m)$ for any $m \in \mathbb{N}$, which contradicts the fact that Ψ is analytic at $\varphi(z)$ (note that the conformal map φ is itself analytic in a neighborhood of 1 and β).

Therefore, it is easy to see that Ψ has no singular inner factor in R_β if and only if

$$\limsup_{r \to 1-} (1 - r) \log |\Psi(r)| = 0. \tag{3.4}$$

Problem. *Is it true that the function $\Psi(z) = z\theta(z) - \frac{\beta}{z}\theta\Big(\frac{\beta}{z}\Big)$ has no singular inner factor in R_β for any inner function θ?*

We believe that the answer to this question is positive, at least, for a wide class of inner functions.

In the case when $\sigma(\theta) = \{1\}$ one can make several observations. They show that the case of a (possible) singular inner factor in the factorization of Ψ is exceptional and can happen only rarely. Also, the assumption that Ψ has a singular inner factor implies important restrictions on the zeros of θ and prohibits them to approach to 1 nontangentially.

Corollary 3.3. *Let θ be an inner function such that $\sigma(\theta) = \{1\}$. Then*

1. For any $a \in \mathbb{C}$, $a \neq 0$, there exist at most one value of $c \in \mathbb{C}$, $c \neq 0$, $|c| \neq |a|$, such that the function Ψ given by (3.2) *or* (3.3) *has a singular inner factor.*

Thus, for all values of $c \in \mathbb{C}$, $c \neq 0$, $|c| \neq |a|$, except at most one, the set of eigenvectors of the operator A_Φ is complete, and for all values of $c \in \mathbb{C}$, $c \neq 0$, $|c| \neq |a|$, except at most one, the operator A_Φ is not hypercyclic.

2. If θ has an atomic singular factor $\exp\big(a\frac{z+1}{z-1}\big)$, $a > 0$, then Ψ given by (3.2) *or* (3.3) *has no singular inner factor for any $\beta \neq 0$ with $|\beta| \neq 1$.*

3. If for some β the function Ψ has a singular inner factor, then there exist $a > 0$, $m \in \mathbb{N}$ and $c > 0$ such that all zeros of θ, maybe except of a finite number, lie in the domain

$$\Big\{z \in \mathbb{D} : \ \Big|\exp\Big(a\frac{1+z}{1-z}\Big)\Big| \geqslant c|z-1|^m\Big\}. \tag{3.5}$$

Proof. 1. Assume that there exist $\beta_1, \beta_2 \neq 0$, $|\beta_1|, |\beta_2| < 1$ and $\beta_1 \neq \beta_2$ such that

$$\Psi_{\beta_1}(z) = z\theta(z) - \frac{\beta_1}{z}\theta\Big(\frac{\beta_1}{z}\Big)$$

and

$$\Psi_{\beta_2}(z) = z\theta(z) - \frac{\beta_2}{z}\theta\Big(\frac{\beta_2}{z}\Big)$$

have singular inner factors. Then, by (3.4),

$$|\Psi_{\beta_j}(r)| \leqslant \exp\Big(-\frac{a}{1-r}\Big) \qquad \text{as } r \to 1-,$$

for some $a > 0$, $j = 1, 2$. Then the function

$$\frac{\beta_1}{z}\theta\Big(\frac{\beta_1}{z}\Big) - \frac{\beta_2}{z}\theta\Big(\frac{\beta_2}{z}\Big)$$

tends to zero faster than any power as $z = r \to 1-$. Since this function is analytic at $z = 1$, it is identically zero, which easily leads to a contradiction. The cases when $|\beta_1|, |\beta_2| > 1$ or $|\beta_1| < 1, |\beta_2| > 1$ are analogous.

Now, if a is fixed, then there exists at most one c as above such that Ψ_β has a singular inner factor for $\beta = c/a$. Thus, for all values of $c \in \mathbb{C}$, $c \neq 0$, $|c| \neq |a|$, except at most one, the set of eigenvectors of A_Φ is complete. Also, $A_{\bar\Phi}$ has a complete set of eigenvectors (and so A_Φ is not hypercyclic) for all values of c except at most one.

Statement 2 follows by the same argument as Statement 1. To prove Statement 3, assume that Ψ has a singular inner factor and fix a conformal mapping φ of the annulus R^α_β with a cut at $\alpha \neq 1$ such that $\varphi(1) = 1$. By (3.4), for some $a, \tilde{a} > 0$, we have

$$|\Psi(z)| \leqslant \left| \exp\left(a \frac{\varphi^{-1}(z)+1}{\varphi^{-1}(z)-1} \right) \right| \leqslant \left| \exp\left(\tilde{a} \frac{z+1}{z-1} \right) \right|$$

as $z \to 1$ inside some Stolz angle at 1; we use here that φ is analytic in a neighborghood of 1 and preserves the angles. On the other hand, $\Psi(z) = \frac{\beta}{z}\theta\big(\frac{\beta}{z}\big)$ at the points where $\theta(z) = 0$. However, $\frac{\beta}{z}\theta\big(\frac{\beta}{z}\big) \sim C(z-1)^m$, $z \to 1$, for some $C \neq 0$ and $m \in \mathbb{N}_0$ due to analyticity at $z = 1$. Since the zeros of θ tend to 1, all of them, maybe except of a finite number, lie in a domain of the form (3.5). □

It follows from Corollary 3.3 that in the case when $\sigma(\theta) = \{1\}$ and θ has an atomic singular factor or infinitely many zeros approaching the point 1 nontangentially all tridiagonal operators A_Φ with $|a| \neq |c|$ have complete sets of eigenvectors.

Acknowledgements

The authors are grateful to Dmitry Khavinson, Jonathan Partington and Ilia Zlotnikov for useful discussions and helpful remarks.

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