INEXISTENCE OF NON-PRODUCT HESSIAN RANK 1 AFFINELY HOMOGENEOUS HYPERSURFACES $H^n \subset \mathbb{R}^{n+1}$ IN DIMENSION $n \geqslant 5$

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Abstract. Equivalences under the affine group $\mathrm{Aff}(\mathbb{R}^3)$ of constant Hessian rank 1 surfaces $S^2 \subset \mathbb{R}^3$, sometimes called *parabolic*, were, among other objects, studied by Doubrov, Komrakov, Rabinovich, Eastwood, Ezhov, Olver, Chen, Merker, Arnaldsson, Valiquette. In particular, homogeneous models and algebras of differential invariants in various branches were fully understood.

Then what is about higher dimensions? We consider hypersurfaces $H^n \subset \mathbb{R}^{n+1}$ graphed as $\{u = F(x_1, \dots, x_n)\}$ whose Hessian matrix $(F_{x_i x_j})$, a relative affine invariant, is similarly of constant rank 1. *Are there homogeneous models?*

Complete explorations were done by the author on a computer in dimensions $n = 2, 3, 4, 5, 6, 7$. The first, expected outcome, was a complete classification of homogeneous models in dimensions $n = 2, 3, 4$ (forthcoming article, case $n = 2$ already known). The second, unexpected outcome, was that in dimensions $n = 5, 6, 7$, there are no affinely homogenous models except those that are affinely equivalent to a product of $\mathbb{R}^m$ with a homogeneous model in dimensions $2, 3, 4$.

The present article establishes such a non-existence result in every dimension $n \geqslant 5$, based on the production of a normal form for $\{u = F(x_1, \dots, x_n)\}$, under $\mathrm{Aff}(\mathbb{R}^{n+1})$ up to order $\leqslant n + 5$, valid in any dimension $n \geqslant 2$.

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1. Introduction

Let $n \geqslant 1$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $v \in \mathbb{R}$. We consider equivalences of local analytic hypersurfaces $H^n \subset \mathbb{R}^{n+1}_{x,u}$ and $K^n \subset \mathbb{R}^{n+1}_{y,v}$ graphed as:

$$u = F(x_1, \dots, x_n) \qquad \text{and} \qquad v = G(y_1, \dots, y_n),$$

under *affine transformations* $\Psi \colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$:

$$\begin{aligned} y_1 &= a_{1,1}\, x_1 + \cdots + a_{1,n}\, x_n + b_1\, u + \tau_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= a_{n,1}\, x_1 + \cdots + a_{n,n}\, x_n + b_n\, u + \tau_n, \\ v &= c_1\, x_1 + \cdots + c_n\, x_n + d\, u + \tau_0, \end{aligned} \tag{1.1}$$

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where the $(n+1)\times(n+1)$ linear-part $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ matrix is invertible, i.e., belongs to $\mathrm{GL}(n+1,\mathbb{R})$. The collection of all these transformations is the Lie transformation group $\mathrm{Aff}(\mathbb{R}^{n+1})$.

The Lie algebra $\mathfrak{aff}(\mathbb{R}^{n+1})$ of $\mathrm{Aff}(\mathbb{R}^{n+1})$ consists of the vector fields:

$$\begin{aligned} L \,=\, & T_1\,\frac{\partial}{\partial x_1}+\cdots+T_n\,\frac{\partial}{\partial x_n}+T_0\,\frac{\partial}{\partial u} \\ & +\Big(A_{1,1}\,x_1+\cdots+A_{1,n}\,x_n+B_1\,u\Big)\,\frac{\partial}{\partial x_1} \\ & +\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & +\Big(A_{n,1}\,x_1+\cdots+A_{n,n}\,x_n+B_n\,u\Big)\,\frac{\partial}{\partial x_n}+ \\ & \quad +\Big(C_1\,x_1+\cdots+C_n\,x_n+D\,u\Big)\,\frac{\partial}{\partial u}. \end{aligned}$$

A fixed hypersurface $H^n\subset\mathbb{R}^{n+1}$ possesses an affine symmetry group, which is a *local Lie group*,for background see [1, Chap. 3]:

$$\mathrm{Sym}(H)\,:=\,\big\{\Psi\in\mathrm{Aff}(\mathbb{R}^{n+1})\colon\;\Psi(H)\subset H\big\},$$

where "$\subset$" is understood up to shrinking H, and where the transformations Ψ are close to the identity. Then $\mathrm{Sym}(H)$ has a Lie algebra:

$$\mathrm{Lie}\;\mathrm{Sym}(H)\,=\,\mathfrak{sym}(H)\,:=\,\big\{L\colon\;L\big|_H\text{ tangent to }H\big\}.$$

Since all our considerations will be *local*, we can assume that everything takes place in some neighborhood of a fixed point $p_0\in H$; such neighborhood can be lessen finitely many times.

Definition 1.1. *The hypersurface H is said to be (locally) affinely homogeneous if:*

$$T_{p_0}H\,=\,\mathrm{Span}_{\mathbb{R}}\,\big\{L\big|_{p_0}\colon\;L\in\mathfrak{sym}(H)\big\}.$$

According to the Lie theory, the 1-parameter groups $p\longmapsto\exp(t\,L)(p)$ stabilize H, and $\mathrm{Sym}(H)$ is then locally transitive in a neighborhood of $p_0\in H$.

The problem of classifying all affinely homogeneous n-dimensional local analytic smooth submanifolds $H^n\subset\mathbb{R}^{n+c}$ is probably of infinite complexity. Even for $n=2=c$, it is not completed.

In the hypersurface case $c=1$ and in dimension $n=2$, the classification was completed two decades ago by Doubrov-Komrakov-Rabinovich [2], [3] and by Eeastwood-Ezhov [4], see also [5], [6] for a differential invariants perspective.

In dimension $n=3$, and codimension $c=1$, all *multiply transitive* models were classified in [7], while for the a *special* affine subgroup $\mathrm{Saff}(\mathbb{R}^{3+1})\subset\mathrm{Aff}(\mathbb{R}^{3+1})$ was treated and completed in the (unpublished) Ph.D. thesis of Marc Wermann [8], and the works of Eastwood-Ezhov [9, 10].

Jointly with Chen [11], the author has studied a so-called *parabolic surfaces* $H^2\subset\mathbb{R}^3$, the Hessian of which had a constant rank 1 (see also [6]). Somewhat analogous 5-dimensional CR structures of dimension 5 whose Levi form is of constant rank 1 were studied in [12], [13].

Problem 1.1. *Study algebras of differential invariants and classify homogenous models of constant Hessian rank* 1 *hypersurfaces $H^n\subset\mathbb{R}^{n+1}$.*

A similar problem can be formulated in the context of CR geometry, cf. [14].

In Winter 2021, using a computer, the author found all affinely homogenous Hessian rank 1 hypersurfaces $H^n\subset\mathbb{R}^{n+1}$ in dimensions $n=2,\,3,\,4$; these results are presented in a forthcoming paper [14]. Exploring then dimensions $n=5,6,7$, the author was surprised to realize that there are *no* homogenous models except the degenerate ones obtained by taking a product of $\mathbb{R}^m$ with a homogeneous hypersurface $H^{n-m}\subset\mathbb{R}^{n-m+1}$ so that $2\leqslant n-m\leqslant 4$.

He then tackled to prove a *non-existence* result, which, incidentally, provides a complete classification (in the constant Hessian rank 1 branch). But the computational task appeared to be unexpectedly hard, and it took one year to write a detailed proof in general dimension $n \geqslant 5$.

The main result of this paper concerns dimensions $n \geqslant 5$, but several results are true for all $n \geqslant 2$, and will be useful for [15].

In the paper we may emphasize three statements. The first one appears as Theorem 13.1 below.

Theorem 1.1. *Let $H^n \subset \mathbb{R}^{n+1}$ be a local affinely homogeneous hypersurface having constant Hessian rank 1. Then there exists an integer $1 \leqslant n_H \leqslant n$ and affine coordinates $(x_1, \dots, x_n)$ in which:*

$$H^n = H^{n_H} \times \mathbb{R}^{n-n_H-1}_{x_{n_H+1},\dots,x_n}$$

is a product of an affinely homogeneous hypersurface $H^{n_H} \subset \mathbb{R}^{n_H+1}$ times a 'dumb' $\mathbb{R}^{n-n_H-1}$, and is graphed as:

$$u = \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \sum_{m=3}^{n_H} \Big(\frac{x_1^m x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j\geqslant 2\\ i+j=m+1}} \tfrac{1}{2}\, \frac{x_i x_j}{(i-1)!(j-1)!} + \mathrm{O}_{x_2,\dots,x_{m-1}}(3) \Big)$$
$$+ \sum_{m=n_H+2}^{\infty} E^m(x_1, \dots, x_{n_H}),$$

with graphing function $F = F(x_1, \dots, x_{n_H})$ independent of $x_{n_H+1}, \dots, x_n$.

We notice that the variables $(x_1, \dots, x_{n_H}, u)$ *are present* in such a graphed equation.

Of course, we are interested in the hypersurfaces for which $n_H = n$. Such hypersurfaces can be called *nondegenerate*, but we will not use such a terminology. With $n_H = n$, Theorem 1.1 shows the graphing function up to order $n+1$ included. Up to order $n+3$ included, we prove

Theorem 1.2. *In any dimension $n \geqslant 2$, every local hypersurface $H^n \subset \mathbb{R}^{n+1}$ having constant Hessian rank 1, and which is not affinely equivalent to a product of $\mathbb{R}^m$ $(1 \leqslant m \leqslant n)$ with a hypersurface $H^{n-m} \subset \mathbb{R}^{n-m+1}$, can be affinely normalized up to order $n+3$ as:*

$$u = \frac{x_1^2}{2} + \frac{x_1^2\, x_2}{2} + \sum_{m=3}^{n} \Big(\frac{x_1^m\, x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j\geqslant 2\\ i+j=m+1}} \tfrac{1}{2}\, \frac{x_i\, x_j}{(i-1)!(j-1)!} \Big)$$
$$+ F_{n+1,10\cdots 0} \frac{x_1^{n+1} x_2}{(n+1)!} + x_1^n \sum_{\substack{i,j\geqslant 2\\ i+j=n+2}} \tfrac{1}{2}\, \frac{x_i\, x_j}{(i-1)!(j-1)!}$$
$$+ F_{n+3,0\cdots 0} \frac{x_1^{n+3}}{(n+3)!} + F_{n+2,10\cdots 0} \frac{x_1^{n+2}\, x_2}{(n+2)!} + F_{n+2,0010\cdots 0} \frac{x_1^{n+2}\, x_4}{(n+2)!} + \cdots + F_{n+2,0\cdots 01} \frac{x_1^{n+2}\, x_n}{(n+2)!}$$
$$+ F_{n+1,10\cdots 0} \frac{x_1^{n+1}\, x_2\, x_2}{n!} + x_1^{n+1} \sum_{\substack{i,j\geqslant 2\\ i+j=n+3}} \tfrac{1}{2}\, \frac{x_i\, x_j}{(i-1)!(j-1)!}$$
$$+ \mathrm{O}_{x_2,\dots,x_n}(3) + \mathrm{O}_{x_1,x_2,\dots,x_n}(n+4).$$

Furthermore, linear $\mathrm{GL}(n+1, \mathbb{R})$ self-maps (fixing the origin) $\left(\begin{smallmatrix} y \\ v \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right)$ of such a hypersurface are necessarily weighted dilations of the form:

$$y_1 = \boldsymbol{c} x_1, \qquad y_2 = 0, \qquad y_3 = \frac{1}{\boldsymbol{c}} x_3, \qquad \dots, \qquad y_n = \frac{1}{\boldsymbol{c}^{n-2}}\, x_n, \qquad v = \boldsymbol{c}^2 u,$$

with $\boldsymbol{c} \in \mathbb{R}^*$.

In other words, the isotropy is at most one-dimensional. The more advanced Theorem 25.1 gives terms of orders $n+4$, $n+5$, which are more complicated but unfortunately necessary in order to establish our unexpected main result.

Theorem 1.3. *In any dimension $n \geqslant 5$, there are no affinely homogeneous constant Hessian rank 1 nondegenerate hypersurfaces $H^n \subset \mathbb{R}^{n+1}$.*

Here is the key reason why homogeneous models do not exist when $n \geqslant 5$. In Sections 26 and 27, we shall obtain the two equations **I** and **II** shown in Proposition 26.1, that will contradict homogeneity, at the infinitesimal level.

Readers could admit Theorem 1.2 or Theorem 25.1, and go directly to Sections 26 and 27, which are simple to read. Unfortunately, the core normalizations done in the previous sections are hard, require patience, and indurance.

All computations were done on a computer fully in dimensions $n = 2, 3, 4, 5, 6, 7$, during more than two months of exploration, from December 2020, to February 2021. Especially, all technical statements of this article were *constantly checked to be true* on a computer in dimensions $n = 5, 6, 7$. This acted as a guide to set up the general dimension by hand.

In fact, it happened to be unexpectedly hard to write by hand a detailed proof in general dimension, the computer was unable to do that! It was really necessary to normalize $\big\{u = F(x_1, \dots, x_n)\big\}$ up to order $n+5$ as stated in Theorem 25.1, because no contradiction occured in lower order $\leqslant n+4$ for the 'particular' dimensions $n = 5, 6, 7$.

With slightly harder computations, it can be shown that in any dimension $n \geqslant 2$ we always have

$$\dim \text{Lie Sym}(H) \leqslant 4.$$

Thus, when $\dim H = n \geqslant 5$, (infinitesimal) homogeneity cannot take place.

All statements hold for $\mathbb{C}$ instead of $\mathbb{R}$, with the same proofs.

2. Hessian matrix and its rank invariancy

We suppose that coordinates (x, u) are centered at $p_0 \in M$, so that p_0 is the origin. Denote its image by $q_0 := \Psi(p_0)$. We also assume that q_0 is the origin in $\mathbb{R}^{n+1}_{y,v}$, too. Then $\Psi(0) = 0$ forces $0 = \tau_1 = \cdots = \tau_n = \tau_0$ in (1.1), which becomes a general $\mathrm{GL}(\mathbb{R}^{n+1})$ transformation:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \\ v \end{bmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,n} & b_1 \\ c_1 & \cdots & c_n & d \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ u \end{bmatrix}. \tag{2.1}$$

Importantly, the basic assumption that $\Psi(H) \subset K$, namely that $0 = -v + G(y)$ when $\binom{y}{v} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\binom{x}{u}$ as written above *and when u is replaced by $F(x)$*, is expressed as the *fundamental equation:*

$$\begin{aligned} 0 = & -c_1 x_1 - \cdots - c_n x_n - d\, F(x_1, \dots, x_n) \\ & + G\Big(a_{1,1}x_1 + \cdots + a_{1,n}x_n + b_1 F(x_1, \dots, x_n), \\ & \qquad \ldots\ldots, a_{n,1}x_1 + \cdots + a_{n,n}x_n + b_n F(x_1, \dots, x_n)\Big), \end{aligned} \tag{2.2}$$

which holds identically in $\mathbb{C}\{x_1, \dots, x_n\}$.

One step further, by composing Ψ with two elementary affine transformations, we may assume that the tangent space $T_0 H^n = \{u = 0\}$ is horizontal, and that $T_0 K^n = \{v = 0\}$ as well. In other words, $F = \mathrm{O}_x(2)$ and $G = \mathrm{O}_y(2)$.

Thus, order 0 and order 1 terms are absent in the following two expansions:

$$u = F(x) = \sum_{\substack{\sigma_1\geqslant 0,\dots,\sigma_n\geqslant 0\\ \sigma_1+\cdots+\sigma_n\geqslant 2}} \frac{x_1^{\sigma_1}\cdots x_n^{\sigma_n}}{\sigma_1!\cdots\sigma_n!} F_{x_1^{\sigma_1}\cdots x_n^{\sigma_n}}(0),$$

$$v = G(y) = \sum_{\substack{\tau_1\geqslant 0,\dots,\tau_n\geqslant 0\\ \tau_1+\cdots+\tau_n\geqslant 2}} \frac{y_1^{\tau_1}\cdots y_n^{\tau_n}}{\tau_1!\cdots\tau_n!} G_{y_1^{\tau_1}\cdots y_n^{\tau_n}}(0).$$

Lemma 2.1. *The linear transformation* $\left(\begin{smallmatrix} y\\ v\end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix} x\\ u\end{smallmatrix}\right)$ *sends* $u = \mathrm{O}_x(2)$ *to* $v = \mathrm{O}_y(2)$ *if and only if* $0 = c_1 = \cdots = c_n$.

Proof. We write (2.2) modulo $\mathrm{O}_x(2)$:

$$0 \equiv -c_1\,x_1 - \cdots - c_n\,x_n - \mathrm{O}_x(2) + \mathrm{O}_x(2).$$

□

This is interpreted as a *group reduction:*

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} & b_1\\ \vdots & \ddots & \vdots & \vdots\\ a_{n,1} & \cdots & a_{n,n} & b_n\\ c_1 & \cdots & c_n & d \end{bmatrix}^{\mathbf{0}} \rightsquigarrow \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} & b_1\\ \vdots & \ddots & \vdots & \vdots\\ a_{n,1} & \cdots & a_{n,n} & b_n\\ \mathbf{0} & \cdots & \mathbf{0} & d \end{bmatrix}^{\mathbf{1}}.$$

We then necessarily have

$$\det\big(a_{i,j}\big) \neq 0 \neq d.$$

In what follows, we shall need an equivalent expansion gathering homogeneous terms of fixed order:

$$u = \sum_{m\geqslant 2} \tfrac{1}{m!} \sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} x_{i_1}\cdots x_{i_m}\,F_{x_{i_1}\cdots x_{i_m}}(0) =: \sum_{m\geqslant 2} F^m(x). \tag{2.3}$$

The proof of this expansion is left for the reader.

At order 2 we have:

$$u = \tfrac{1}{2}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} x_{i_1}x_{i_2}\,f_{i_1,i_2} + \mathrm{O}_x(3) \qquad\text{and}\qquad v = \tfrac{1}{2}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} y_{j_1}y_{j_2}\,g_{j_1,j_2} + \mathrm{O}_y(3),$$

where:

$$f_{i_1,i_2} := F_{x_{i_1}x_{i_2}}(0) = f_{i_2,i_1} \qquad\text{and}\qquad g_{j_1,j_2} := G_{y_{j_1}y_{j_2}}(0) = g_{j_2,j_1}.$$

With $x = {}^{\mathrm{T}}(x_1,\dots,x_n)$ being a column vector, we can abbreviate:

$$u = \tfrac{1}{2}\,{}^{\mathrm{T}}x\cdot F^{(2)}\cdot x + \mathrm{O}_x(3) \qquad\text{and}\qquad v = \tfrac{1}{2}\,{}^{\mathrm{T}}y\cdot G^{(2)}\cdot y + \mathrm{O}_y(3).$$

Definition 2.1. *At any point* $p\in H^n$*, the Hessian matrix is well-defined in any system of coordinates* (x,u) *centered at* $p=(0,0)$*, in which* H^n *is graphed as* $u=F(x)$ *with* $0=F(0)=F_{x_1}(0)=\cdots=F_{x_n}(0)$*:*

$$\mathrm{Hessian}(F) := \begin{bmatrix} F_{x_1x_1}(0) & \cdots & F_{x_1x_n}(0)\\ \vdots & \ddots & \vdots\\ F_{x_nx_1}(0) & \cdots & F_{x_nx_n}(0)\end{bmatrix} = \begin{bmatrix} f_{1,1} & \cdots & f_{1,n}\\ \vdots & \ddots & \vdots\\ f_{n,1} & \cdots & f_{n,n}\end{bmatrix} = F^{(2)}.$$

At first sight, this definition depends on coordinates, but a key invariancy lies behind.

Lemma 2.2. *The rank of the Hessian matrix* Hessian(F) *is independent of the affine coordinates* $(x_1,\dots,x_n,u)$ *centered at* p *in which* $0=F(0)=F_{x_1}(0)=\cdots=F_{x_n}(0)$.

Proof. We take another system of coordinates $(y_1, \dots, y_n, v)$ centered at $p = 0$ in which the hypersurface is graphed as $v = G(y)$ with $0 = G(0) = G_{y_1}(0) = \cdots = G_{y_n}(0)$, namely:

$$v \,=\, {}^{\mathrm{T}}y \cdot G^{(2)} \cdot y + \mathrm{O}_y(3).$$

Hence, there exists an invertible linear transformation of form (2.1) with $0 = c_1 = \cdots = c_n$ by Lemma 2.1, namely, $y = A \cdot x + B \cdot u$ and $v = d\,u$, which sends $u = F(x)$ to $v = G(y)$.

Then we write fundamental equation (2.2) modulo $\mathrm{O}_x(3)$, observe that $\mathrm{O}_y(3) = \mathrm{O}_x(3)$ when $y = A \cdot x + B \cdot F(x) = \mathrm{O}_x(1)$ and we factorize:

$$\begin{aligned} 0 \,&=\, -v + G(y) \\ &\equiv\, -d\,F(x) + G\big(A \cdot x + B \cdot F(x)\big) \\ &\equiv\, -d\,\frac{1}{2}\,{}^{\mathrm{T}}x \cdot F^{(2)} \cdot x - \mathrm{O}_x(3) + \tfrac{1}{2}\,{}^{\mathrm{T}}\big[A \cdot x + B \cdot F(x)\big] \cdot G^{(2)} \cdot \big[A \cdot x + B \cdot F(x)\big] + \mathrm{O}_y(3) \\ &=\, \frac{1}{2}\,{}^{\mathrm{T}}x \cdot \Big(-\,d\,F^{(2)} + {}^{\mathrm{T}}A \cdot G^{(2)} \cdot A\Big) \cdot x + \mathrm{O}_x(3), \end{aligned}$$

and deduce, since $x \in \mathbb{R}^n$ is arbitrary, that:

$$d\,F^{(2)} \,=\, {}^{\mathrm{T}}A \cdot G^{(2)} \cdot A.$$

Finally, since $d \neq 0 \neq \det A$, we conclude that $\operatorname{rank} F^{(2)} = \operatorname{rank} G^{(2)}$. □

Suppose that $H = \{u = F(x)\}$ is given in coordinates (x, u) centered at $p_0 = 0$, with $0 = F(0) = F_{x_1}(0) = \cdots = F_{x_n}(0)$. At any other point $p \sim p_0$ close to the origin with $p = (x_{1,p}, \dots, x_{n,p}, u_p)$, where $u_p = F(x_p)$, we use the centered coordinates:

$$\begin{aligned} &y_1 \,:=\, x_1 - x_{1,p}, \qquad \dots\dots \qquad y_n \,:=\, x_n - x_{n,p}, \\ &v \,:=\, u - F(x_p) - F_{x_1}(x_p)\big(x_1 - x_{1,p}\big) - \cdots - F_{x_n}(x_p)\big(x_n - x_{n,p}\big). \end{aligned}$$

The new graphing function

$$G(y) \,:=\, F(x_p + y) - F(x_p) - F_{x_1}(x_p)\,y_1 - \cdots - F_{x_n}(x_p)\,y_n,$$

satisfies $G(0) = G_{y_1}(0) = \cdots = G_{y_n}(0)$, hence Definition 2.1 applies. But since the correction terms are of degree $\leqslant 1$ in $y_1, \dots, y_n$, they disappear after the second order differentiation:

$$\begin{bmatrix} G_{y_1y_1}(0) & \cdots & G_{y_1y_n}(0) \\ \vdots & \ddots & \vdots \\ G_{y_ny_1}(0) & \cdots & G_{y_ny_n}(0) \end{bmatrix} = \begin{bmatrix} F_{x_1x_1}(x_p) & \cdots & F_{x_1x_n}(x_p) \\ \vdots & \ddots & \vdots \\ F_{x_nx_1}(x_p) & \cdots & F_{x_nx_n}(x_p) \end{bmatrix}.$$

Lemma 2.3. *The Hessian of a graphed hypersurface $u = F(x_1, \dots, x_n)$ is a well defined matrix-valued function of x:*

$$\mathrm{Hessian}(F)(x) \,:=\, \begin{bmatrix} F_{x_1x_1}(x) & \cdots & F_{x_1x_n}(x) \\ \vdots & \ddots & \vdots \\ F_{x_nx_1}(x) & \cdots & F_{x_nx_n}(x) \end{bmatrix}$$

whose rank is invariant under affine equivalences (at pairs of points corresponding one to another).

3. Constant Hessian rank 1

We take a hypersurface $H^n \subset \mathbb{R}^{n+1}$ with $0 \in H^n$ and $T_0H^n = \{u = 0\}$ graphed as:

$$u \,=\, F(x) \,=\, \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_ix_j\,F_{x_ix_j}(0) + \mathrm{O}_x(3).$$

Its Hessian at the origin 0 is represented by the $n \times n$ matrix $\big(F_{x_ix_j}(0)\big)$.

Lemma 3.1. *Whenever $\big(F_{x_ix_j}(0)\big)$ is not the zero matrix, there exists an affine change of coordinates $y = A\,x$, $v = u$, making nonzero (after renaming $y =: x$):*

$$F_{x_1x_1}(0) \neq 0.$$

Proof. If there exists an index i_* with $F_{x_{i_*}x_{i_*}}(0) \neq 0$, simply permute affine coordinates to set $x_1 := x_{i_*}$.

Assume therefore that $0 = F_{x_ix_i}(0)$ for all $i = 1, \dots, n$. By assumption, there exist i_* and $j_* \neq i_*$ such that $F_{x_{i_*}x_{j_*}}(0) \neq 0$. We rename $x_1 := x_{i_*}$ to have $0 \neq F_{x_1x_{j_*}}(0)$ and we change $j_* \geqslant 2$ to be the smallest satisfying $F_{x_1x_{j_*}}(0) \neq 0$. We also abbreviate $f_{i,j} := F_{x_ix_j}(0)$. Hence, $f_{1,j_*} \neq 0$.

Since all diagonal terms are zero, only $\sum_{i<j}$ remains, and we can expand modulo $\mathrm{O}_x(3)$:

$$\begin{aligned} u \equiv \sum_{i<j} x_ix_j\, f_{i,j} &= \sum_{\substack{i<j\\ j\leqslant j_*}} x_ix_j\, f_{i,j} + \sum_{\substack{i<j\\ j_*<j}} x_ix_j\, f_{i,j} \\ &= x_1x_{j_*}\, f_{1,j_*} + \sum_{j_*<j} x_1x_j\, f_{1,j} + \sum_{\substack{2\leqslant i<j\\ j\leqslant j_*}} x_ix_j\, f_{i,j} + \sum_{\substack{2\leqslant i<j\\ j_*<j}} x_ix_j\, f_{i,j}. \end{aligned}$$

We let $x_{j_*} := y_1 + y_{j_*}$, while $x_j := y_j$ for $j \neq j_*$ and $u := v$, whence $\mathrm{O}_x(3) = \mathrm{O}_y(3)$, so that the first monomial becomes

$$y_1\,\big(y_1 + y_{j_*}\big)\, f_{1,j_*},$$

with a nonzero coefficient for y_1y_1. The three remaining sums cannot incorporate y_1y_1. Thus, the new graph $v = G(y)$ satisfies $G_{y_1y_1}(0) = 2\, f_{1,j_*} \neq 0$. □

4. Independent and border-dependent jets

Up to the end of the article, we shall assume that

$$F_{x_1x_1}(0) \neq 0.$$

Also, our main assumption is that the Hessian matrix $\big(F_{x_ix_j}(x)\big)$ has constant rank 1 for all $x \sim 0$ in some neighborhood of the origin.

It is elementary to verify that, for constants $\varphi_{i,j} \in \mathbb{R}$ with $\varphi_{1,1} \neq 0$,

$$1 = \operatorname{rank} \begin{bmatrix} \varphi_{1,1} & \varphi_{1,2} & \cdots & \varphi_{1,n} \\ \varphi_{2,1} & \varphi_{2,2} & \cdots & \varphi_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n,1} & \varphi_{n,2} & \cdots & \varphi_{n,n} \end{bmatrix} \qquad \Longleftrightarrow \qquad 0 = \begin{vmatrix} \varphi_{1,1} & \varphi_{1,j} \\ \varphi_{i,1} & \varphi_{i,j} \end{vmatrix} \qquad \forall\, 2 \leqslant i,j \leqslant n.$$

Main assumption. *For all $x \sim 0$ in some neighborhood of the origin:*

$$F_{x_ix_j}(x) \equiv \frac{F_{x_1x_i}(x)\, F_{x_1x_j}(x)}{F_{x_1x_1}(x)} \qquad \forall\, i,j = 2, \dots, n. \tag{4.1}$$

By differentiating this identity with respect to $x_1, x_2, \dots, x_n$, by induction it is easy to prove that every derivative $F_{x_1^\tau x_2^{i_2}\cdots x_n^{i_n}}(x)$ with $i_2 + \cdots + i_n \geqslant 2$ and arbitrary $\tau \in \mathbb{N}$ is expressed as a polynomial in the derivatives $F_{x_1^{i_1'}x_2^{i_2'}\cdots x_n^{i_n'}}(x)$ with $i_1' \leqslant 1$ divided by a certain power $\big(F_{x_1x_1}(x)\big)^*$. In fact, we shall need to know what formulae hold only for $i_2 + \cdots + i_n = 2$.

Terminology 4.1. • *The independent jets are the derivatives with $i_1 = 0$ or $i_1 = 1$:*

$$F_{x_2^{i_2}\cdots x_n^{i_n}}(x), \qquad F_{x_1x_2^{i_2}\cdots x_n^{i_n}}(x).$$

• *The border-dependent jets are the derivatives:*

$$F_{x_1^\tau x_ix_j}(x), \qquad \tau \geqslant 0, \qquad i,j = 2, \dots, n.$$

The way how these border-dependent jets $F_{x_1^\tau x_i x_j}$ are expressed in terms of the independent jets can be seen by differentiating (4.1) ν times with respect to x_1.

We shall also employ the following abbreviations:

$$x' := (x_2, \ldots, x_n), \qquad f_{i_1,i_2,\ldots,i_n} := F_{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}}(0).$$

5. Normalization at order 2

Assuming that $f_{1,1} \neq 0$, we begin with the identity

$$u = F(x) = \tfrac{1}{2} \sum_i \sum_j f_{i,j}\, x_i x_j + \mathrm{O}_x(3).$$

Changing u to $-u$ if needed, we can assume $f_{1,1} > 0$. The plain dilation $y_1 := \sqrt{f_{1,1}}\, x_1$ makes $f_{1,1} = 1$, namely, using again the letters x, u:

$$u = \tfrac{1}{2}\, x_1^2 + \sum_{2\leqslant j\leqslant n} x_1 x_j\, f_{1,j} + \tfrac{1}{2} \sum_{2\leqslant i,j\leqslant n} x_i x_j\, f_{i,j} + \mathrm{O}_x(3).$$

We should say that our Main assumption is (implicitly) assumed in all statements.

Assertion 5.1. *There exists an affine change of coordinates which normalizes* $u = F(x)$ *to*

$$u = \tfrac{1}{2}\, x_1^2 + \mathrm{O}_x(3).$$

Proof. We first collect all monomials incorporating x_1:

$$u = \frac{1}{2} \Big(\underbrace{x_1 + \sum_{2\leqslant j\leqslant n} x_j\, f_{1,j}}_{=:\ \text{new}\ x_1} \Big)^2 - \frac{1}{2} \Big(\sum_{2\leqslant j\leqslant n} x_j\, f_{1,j} \Big)^2 + \frac{1}{2} \sum_{2\leqslant i,j\leqslant n} x_i x_j\, f_{i,j} + \mathrm{O}_x(3),$$

and we get with a modified $f_{i,j}$ that

$$u = \frac{1}{2}\, x_1^2 + \frac{1}{2} \sum_{2\leqslant i,j\leqslant n} x_i x_j\, f_{i,j} + \mathrm{O}_x(3).$$

This implies:

$$0 = F_{x_1 x_j}(0) \quad \text{for all} \quad 2 \leqslant j \leqslant n.$$

Finally, (4.1) yields:

$$F_{x_i x_j}(0) = 0 \quad \text{for all} 2 \leqslant i,\, j \leqslant n.$$

□

Next, starting from $u = \frac{1}{2}\, x_1^2 + \mathrm{O}_x(3)$, the goal is to normalize order 3 terms. But before increasing the order, we must *conserve* or *stabilize* the normalization at order 2. This conducts to *group reduction*:

$$\overset{\mathrm{GL}(n+1,\mathbb{R})}{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \ddots & \vdots & \vdots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & b_n \\ c_1 & c_2 & \cdots & c_n & d \end{bmatrix}^{\mathbf{0}}} \rightsquigarrow \overset{G^1_{\mathrm{stab}}}{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \ddots & \vdots & \vdots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & b_n \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & d \end{bmatrix}^{\mathbf{1}}}$$

$$\rightsquigarrow \overset{G^2_{\mathrm{stab}}}{\begin{bmatrix} a_{1,1} & \mathbf{0} & \cdots & \mathbf{0} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \ddots & \vdots & \vdots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & b_n \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{2}}}.$$

Lemma 5.1. *The subgroup G^2_{stab} of G^1_{stab} which sends $u = \frac{1}{2}\,x_1^2 + \mathrm{O}_x(3)$ to $v = \frac{1}{2}\,y_1^2 + \mathrm{O}_y(3)$ consists of matrices in G^1_{stab} with:*

$$0 \,=\, a_{1,2} \,=\, \cdots \,=\, a_{1,n} \qquad \textit{and} \qquad d \,=\, a_{1,1}^2.$$

Proof. We rewrite (2.2) modulo $\mathrm{O}_x(3)$ and we equate the coefficients at $x_1x_2, \ldots, x_1x_n$ to zero:

$$\begin{aligned} 0 \,&=\, -\, v + \frac{1}{2}\,y_1^2 + \mathrm{O}_y(3) \\ &=\, -\, d\,u + \frac{1}{2}\,\big(a_{1,1}\,x_1 + a_{1,2}\,x_2 + \cdots + a_{1,n}\,x_n + b_1\,u\big)^2 + \mathrm{O}_x(3) \\ &\equiv\, -\, d\,\frac{1}{2}\,x_1^2 - \mathrm{O}_x(3) + \frac{1}{2}\,a_{1,1}^2\,x_1^2 + a_{1,1}\,x_1\,\big(a_{1,2}\,x_2 + \cdots + a_{1,n}\,x_n + \mathrm{O}_x(2)\big) \\ &\quad + \big(a_{1,2}\,x_2 + \cdots + a_{1,n}\,x_n + \mathrm{O}_x(2)\big)^2 + \mathrm{O}_x(3). \end{aligned}$$

□

6. Normalization at order 3

We proceed to the order 3 terms:

$$u \,=\, \frac{1}{2}\,x_1^2 + \sum_{\substack{\sigma_1,\sigma_2,\sigma_3\geqslant 0\\ \sigma_1+\sigma_2+\sigma_3=3}} \frac{x_1^{\sigma_1}x_2^{\sigma_2}x_3^{\sigma_3}}{\sigma_1!\,\sigma_2!\,\sigma_3!}\,f_{\sigma_1,\sigma_2,\sigma_3} + \mathrm{O}_x(4).$$

In the sum, we pick the monomial $\frac{1}{6}\,x_1^3\,f_{1,1,1}$. We recall $x' = (x_2, \ldots, x_n)$. The remaining cubic terms are of the form $x_1^2A(x') + x_1B(x') + C(x')$. Hence, they are $\mathrm{O}_{x'}(1)$. Since they are cubic, they are products of the form $\mathrm{O}_{x'}(1)\,\mathrm{O}_x(2)$. Thus,

$$\begin{aligned} u \,&=\, F(x) \,=\, \tfrac{1}{2}\,x_1^2 + \tfrac{1}{6}\,x_1^3\,f_{1,1,1} + \mathrm{O}_{x'}(1)\,\mathrm{O}_x(2) + \mathrm{O}_x(4), \\ v \,&=\, G(y) \,=\, \tfrac{1}{2}\,y_1^2 + \tfrac{1}{6}\,y_1^3\,g_{1,1,1} + \mathrm{O}_{y'}(1)\,\mathrm{O}_y(2) + \mathrm{O}_y(4), \end{aligned}$$

Lemma 6.1. *One can normalize $g_{1,1,1} := 0$.*

Proof. With free $b_1 \in \mathbb{R}$, use the map belonging to G^2_{stab}:

$$y_1 := x_1 + b_1\,u, \qquad y_2 := x_2, \qquad \ldots\ldots, \qquad y_n := x_n, \qquad v := u.$$

Since $y' = x'$, hence $\mathrm{O}_{y'}(1)\,\mathrm{O}_y(2) = \mathrm{O}_{x'}(1)\,\mathrm{O}_x(2)$, and fundamental equation (2.2) reads:

$$\begin{aligned} 0 \,&\equiv\, -\, \frac{1}{2}\,x_1^2 - \tfrac{1}{6}\,f_{1,1,1}\,x_1^3 - \mathrm{O}_{x'}(1)\,\mathrm{O}_x(2) - \mathrm{O}_x(4) \\ &\quad + \frac{1}{2}\,\bigg(x_1 + b_1\frac{1}{2}\,x_1^2 + \mathrm{O}_x(3)\bigg)^2 + \frac{1}{6}\,g_{1,1,1}\,\big(x_1 + \mathrm{O}_x(2)\big)^3 + \mathrm{O}_{x'}(1)\,\mathrm{O}_x(2) + \mathrm{O}_x(4) \\ &\equiv \frac{1}{6}\,x_1^3\,\big[-\,f_{1,1,1} + 3\,b_1 + g_{1,1,1}\big] + \mathrm{O}_{x'}(1)\,\mathrm{O}_x(2) + \mathrm{O}_x(4). \end{aligned}$$

No monomial x_1^3 can appear in remainders. Hence, the coefficient at x_1^3 must vanish. This means that $g_{1,1,1} := f_{1,1,1} - 3\,b_1$ necessarily. But since b_1 is a free parameter in the affine transformation, we can choose $b_1 := \frac{1}{3}\,f_{1,1,1}$ to normalize $g_{1,1,1} := 0$. □

To normalize further, we can restart from this $v = G(y)$ having $g_{1,1,1} = 0$, call it $u = F(x)$ with $f_{1,1,1} = 0$, and again normalize the new target $v = G(y)$ with $g_{1,1,1} = 0$. In other words, both hypersurfaces are normalized similarly (as always):

$$\begin{aligned} u \,&=\, F(x) \,=\, \frac{1}{2}\,x_1^2 + 0 + \mathrm{O}_{x'}(1)\,\mathrm{O}_x(2) + \mathrm{O}_x(4), \\ v \,&=\, G(y) \,=\, \frac{1}{2}\,y_1^2 + 0 + \mathrm{O}_{y'}(1)\,\mathrm{O}_y(2) + \mathrm{O}_y(4). \end{aligned}$$

Furthermore, before taking account of the normalizations $f_{1,1,1} := 0$ and $g_{1,1,1} := 0$, remind that the current stability group $G^2_{\rm stab}$ is:

$$\begin{bmatrix} a_{1,1} & \mathbf{0} & \cdots & \mathbf{0} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \ddots & \vdots & \vdots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & b_n \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{2}}.$$

This being a subgroup of $\mathrm{GL}(n+1, \mathbb{R})$, its block-trigonal determinant must be nonzero, whence:

$$a_{1,1} \neq 0 \neq \begin{vmatrix} a_{2,2} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{vmatrix}.$$

Next, let cubic terms of the form $x_1^2\,\mathrm{O}_{x'}(1)$ appear:

$$u = F(x) = \frac{1}{2}\,x_1^2 + 0 + \frac{1}{2}\,x_1^2\,\big(\varphi_2\,x_2 + \cdots + \varphi_n\,x_n\big) + x_1\,\mathrm{O}_{x'}(2) + \mathrm{O}_{x'}(3) + \mathrm{O}_x(4).$$

Lemma 6.2. *The remaining cubic terms $x_1\,\mathrm{O}_{x'}(2) + \mathrm{O}_{x'}(3) \equiv 0$ are zero.*

Proof. It suffices to show that

$$0 \overset{?}{=} F_{x_1x_ix_j}(0) \quad \text{for all} \quad 2 \leqslant i,\, j \leqslant n, \tag{6.1}$$

$$0 \overset{?}{=} F_{x_ix_jx_k}(0) \quad \text{for all} \quad 2 \leqslant i,\, j,\, k \leqslant n. \tag{6.2}$$

By previous normalizations, we have $0 = F_{x_1x_i}(0)$ for all $i = 2, \dots, n$. Differentiating (4.1) with respect to x_1 and to x_k, we get the following vanishings:

$$F_{x_1x_ix_j}(0) = \frac{F_{x_1x_1x_i}(0)\,\underline{F_{x_1x_j}(0)}_{\circ} + \underline{F_{x_1x_i}(0)}_{\circ}\,F_{x_1x_1x_j}(0)}{F_{x_1x_1}(0)} - \frac{\underline{F_{x_1x_i}(0)}_{\circ}\,\underline{F_{x_1x_j}(0)}_{\circ}\,F_{x_1x_1x_1}(0)}{F_{x_1x_1}(0)\,F_{x_1x_1}(0)} = 0,$$

$$F_{x_ix_jx_k}(0) = \frac{F_{x_1x_ix_k}(0)\,\underline{F_{x_1x_j}(0)}_{\circ} + \underline{F_{x_1x_i}(0)}_{\circ}\,F_{x_1x_jx_k}(0)}{F_{x_1x_1}(0)} - \frac{\underline{F_{x_1x_i}(0)}_{\circ}\,\underline{F_{x_1x_j}(0)}_{\circ}\,F_{x_1x_1x_k}(0)}{F_{x_1x_1}(0)\,F_{x_1x_1}(0)} = 0.$$

□

Thus, the two hypersurfaces are

$$u = F(x) = \frac{1}{2}\,x_1^2 + 0 + \frac{1}{2}\,x_1^2\,\big(\varphi_2\,x_2 + \cdots + \varphi_n\,x_n\big) + \mathrm{O}_x(4).$$

$$v = G(y) = \frac{1}{2}\,y_1^2 + 0 + \frac{1}{2}\,y_1^2\,\big(\psi_2\,y_2 + \cdots + \psi_n\,y_n\big) + \mathrm{O}_y(4).$$

Lemma 6.3. *The property $0 = \varphi_2 = \cdots = \varphi_n$ is equivalent to $0 = \psi_2 = \cdots = \psi_n$.*

Proof. The general map of $G^2_{\rm stab}$ which stabilizes the normalization up to order 2 is

$$\begin{aligned} y_1 &= a_{1,1}\,x_1 + b_1\,u, \\ y_2 &= a_{2,1}\,x_1 + a_{2,2}\,x_2 + \cdots + a_{2,n}\,x_n + b_2\,u, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= a_{n,1}\,x_1 + a_{n,2}\,x_2 + \cdots + a_{n,n}\,x_n + b_n\,u, \\ v &= a_{1,1}^2\,u. \end{aligned}$$

Therefore, fundamental equation (2.2) reads as

$$\begin{aligned}0 \equiv &- a_{1,1}^2 \left[\frac{1}{2}\,x_1^2 + \frac{1}{2}\,x_1^2\,\big(\varphi_2\,x_2 + \cdots + \varphi_n\,x_n\big) + \mathrm{O}_x(4)\right]\\ &+ \frac{1}{2}\,\big(a_{1,1}\,x_1 + b_1\,\frac{1}{2}\,x_1^2 + \mathrm{O}_x(3)\big)^2\\ &+ \frac{1}{2}\,\big(a_{1,1}\,x_1 + \mathrm{O}_x(2)\big)^2 \Big[\sum_{2\leqslant j\leqslant n} \psi_j\,\big(a_{j,1}\,x_1 + a_{j,2}\,x_2 + \cdots + a_{j,n}\,x_n + \mathrm{O}_x(2)\big)\Big] + \mathrm{O}_x(4).\end{aligned}$$

Of course, the coefficient at x_1^2 is zero. Next, picking the coefficients at x_1^3 and at $x_1^2x_2$, ..., $x_1^2x_n$, we get:

$$\begin{aligned}b_1 &= -\,a_{1,1} \sum_{2\leqslant j\leqslant n} \psi_j\,a_{j,1},\\ \varphi_2 &= \sum_{2\leqslant j\leqslant n} \psi_j\,a_{j,2}, \qquad \ldots\ldots\ldots, \qquad \varphi_n = \sum_{2\leqslant j\leqslant n} \psi_j\,a_{j,n}.\end{aligned} \tag{6.3}$$

Since the $(n-1)\times(n-1)$ matrix $(a_{j,k})$ is invertible, we have $\varphi = {}^{\mathrm{T}}a \cdot \psi$, hence $\varphi = 0$ if and only if $\psi = 0$. □

Before we discuss the two distinct affinely invariant cases $\varphi = 0$ and $\varphi \neq 0$, we must examine infinitesimal affine automorphisms.

7. Tangency at order 2

We take a hypersurface normalized at order 2:

$$u = \frac{1}{2}\,x_1^2 + \mathrm{O}_x(3) = F(x).$$

A general affine vector field reads as

$$\begin{aligned}L = &\Big(T_1 + A_{1,1}\,x_1 + \cdots + A_{1,n}\,x_n + B_1\,u\Big)\,\frac{\partial}{\partial x_1}\\ &+ \Big(T_2 + A_{2,1}\,x_1 + \cdots + A_{2,n}\,x_n + B_2\,u\Big)\,\frac{\partial}{\partial x_2}\\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &+ \Big(T_n + A_{n,1}\,x_1 + \cdots + A_{n,n}\,x_n + B_n\,u\Big)\,\frac{\partial}{\partial x_n}\\ &+ \Big(T_0 + C_1\,x_1 + \cdots + C_n\,x_n + D\,u\Big)\,\frac{\partial}{\partial u},\end{aligned}$$

is tangent to the hypersurface $\{u = F(x)\}$ if and only if:

$$L\big(-\,u + F(x)\big)\Big|_{u:=F(x)} \equiv 0 \tag{7.1}$$

identically in $\mathbb{C}\{x_1, \ldots, x_n\}$. By neglecting the terms of order $\geqslant 2$, this equation gives:

$$\begin{aligned}0 \equiv &-\,T_0 - C_1\,x_1 - C_2\,x_2 - \cdots - C_n\,x_n - D\,\mathrm{O}_x(2)\\ &+ \big(T_1 + \mathrm{O}_x(1)\big)\,\big(x_1 + \mathrm{O}_x(2)\big)\\ &+ \big(T_2 + \mathrm{O}_x(1)\big)\,\mathrm{O}_x(2)\\ &+ \cdots\cdots\cdots\cdots\\ &+ \big(T_n + \mathrm{O}_x(1)\big)\,\mathrm{O}_x(2),\end{aligned}$$

whence:

$$0 = T_0, \qquad C_1 = T_1, \qquad C_2 = 0, \qquad \ldots\ldots, \qquad C_n = 0.$$

Hence,

$$
\begin{aligned}
L = & \Big(T_1 + A_{1,1}\,x_1 + \cdots + A_{1,n}\,x_n + B_1\,u\Big)\,\frac{\partial}{\partial x_1} \\
& + \Big(T_2 + A_{2,1}\,x_1 + \cdots + A_{2,n}\,x_n + B_2\,u\Big)\,\frac{\partial}{\partial x_2} \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \Big(T_n + A_{n,1}\,x_1 + \cdots + A_{n,n}\,x_n + B_n\,u\Big)\,\frac{\partial}{\partial x_n} \\
& + \Big(\qquad T_1\,x_1 \qquad\qquad\qquad + D\,u\Big)\,\frac{\partial}{\partial u}.
\end{aligned}
\tag{7.2}
$$

8. Product case $H^n \cong H^1 \times \mathbb{R}^{n-1}$

We first examine the *affinely invariant* case $\varphi_2 = \cdots = \varphi_n = 0$:

$$
u = \frac{1}{2}\,x_1^2 + 0 + \mathrm{O}_x(4).
$$

Notice then that (6.3) becomes $b_1 = 0$.

Lemma 8.1. *If such a hypersurface is affinely homogeneous, then $F(x) = F(x_1)$ is independent of $x_2, \dots, x_n$.*

Proof. We examine tangency equation (7.1) using L from (7.2) modulo $\mathrm{O}_x(3)$:

$$
\begin{aligned}
0 \equiv & -\,T_1\,x_1 - D\,\left(\frac{1}{2}\,x_1^2 + \mathrm{O}_x(4)\right) \\
& + \Big(T_1 + A_{1,1}\,x_1 + A_{1,2}\,x_2 + \cdots + A_{1,n}\,x_n + \mathrm{O}_x(2)\Big)\,\big(x_1 + \mathrm{O}_x(3)\big) \\
& + \Big(T_2 + \mathrm{O}_x(1)\Big)\,\mathrm{O}_x(3) \\
& + \cdots\cdots\cdots\cdots \\
& + \Big(T_n + \mathrm{O}_x(1)\Big)\,\mathrm{O}_x(3).
\end{aligned}
$$

The coefficients at x_1^2, of x_1x_2, …, of x_1x_n must vanish and this gives:

$$
D = 2\,A_{1,1}, \qquad A_{1,2} = 0, \qquad \dots\dots, \qquad A_{1,n} = 0.
$$

Thus,

$$
\begin{aligned}
L = & \Big(T_1 + A_{1,1}\,x_1 \qquad\qquad\qquad\qquad + B_1\,u\Big)\,\frac{\partial}{\partial x_1} \\
& + \Big(T_2 + A_{2,1}\,x_1 + A_{2,2}\,x_2 + \cdots + A_{2,n}\,x_n + B_2\,u\Big)\,\frac{\partial}{\partial x_2} \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \Big(T_n + A_{n,1}\,x_1 + A_{n,2}\,x_2 + \cdots + A_{n,n}\,x_n + B_n\,u\Big)\,\frac{\partial}{\partial x_n} \\
& + \Big(T_1\,x_1 \qquad\qquad\qquad\qquad + 2\,A_{1,1}\,u\Big)\,\frac{\partial}{\partial u}.
\end{aligned}
$$

Next, let order 4 terms appear:

$$
u = \frac{1}{2}\,x_1^2 + 0 + F^4(x) + \mathrm{O}_x(5), \qquad \text{where} \qquad F^4(x) := \sum_{\substack{\sigma_1,\dots,\sigma_n \geqslant 0 \\ \sigma_1+\cdots+\sigma_n = 4}} \frac{x_1^{\sigma_1}\cdots x_n^{\sigma_n}}{\sigma_1!\,\cdots\,\sigma_n!}\,f_{\sigma_1,\dots,\sigma_n}.
$$

Lemma 8.2. *This $F^4(x) = F^4(x_1)$ is independent of $(x_2, \dots, x_n)$.*

Proof. We write and examine the tangency equation up to order 3:

$$\begin{aligned}
0 \equiv & - T_1\, x_1 - 2\, A_{1,1} \left(\frac{1}{2}\, x_1^2 + \mathrm{O}_x(4)\right) \\
& + \left(T_1 + A_{1,1}\, x_1 + B_1 \left(\frac{1}{2}\, x_1^2 + \mathrm{O}_x(4)\right)\right) \big(x_1 + F_{x_1}^4 + \mathrm{O}_x(4)\big) \\
& + \big(T_2 + \mathrm{O}_x(1)\big) \big(F_{x_2}^4 + \mathrm{O}_x(4)\big) \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \\
& + \big(T_n + \mathrm{O}_x(1)\big) \big(F_{x_n}^4 + \mathrm{O}_x(4)\big).
\end{aligned}$$

This equation considered modulo $\mathrm{O}_x(4)$ is a polynomial in $(x_1, x_2, \dots, x_n)$ of degree $\leqslant 3$ which must vanish identically:

$$0 \equiv T_1\, F_{x_1}^4 + B_1\, \tfrac{1}{2}\, x_1^3 + T_2\, F_{x_2}^4 + \cdots\cdots + T_n\, F_{x_n}^4. \tag{8.1}$$

Notation. [Coefficient picking] *For $(\tau_1, \tau_2, \dots, \tau_n) \in \mathbb{N}^n$, given a converging power series:*

$$E(x) = E(x_1, x_2, \dots, x_n) = \sum_{\sigma_1, \sigma_2, \dots, \sigma_n \geqslant 0} \frac{x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n}}{\sigma_1!\, \sigma_2!\, \cdots\, \sigma_n!}\, E_{x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n}}(0),$$

we denote

$$\big[x_1^{\tau_1} x_2^{\tau_2} \cdots x_n^{\tau_n}\big]\big(E(x)\big) := \frac{1}{\tau_1!\, \tau_2!\, \cdots\, \tau_n!}\, E_{x_1^{\tau_1} x_2^{\tau_2} \cdots x_n^{\tau_n}}(0).$$

Therefore, in (8.1), the coefficients at *all* monomials $x_1^{\tau_1} x_2^{\tau_2} \cdots x_n^{\tau_n}$ with $\tau_1 + \tau_2 + \cdots + \tau_n = 3$ must vanish.

Disregarding the monomial x_1^3 in order not to let B_1 intervene, i.e., taking all $\tau_1 + \tau_2 + \cdots + \tau_n = 3$ with $\tau_1 \neq 3$, we get:

$$\begin{aligned}
0 = & T_1 \left(\big[x_1^{\tau_1} x_2^{\tau_2} \cdots x_n^{\tau_n}\big]\big(F_{x_1}^4\big)\right) \\
& + T_2 \left(\big[x_1^{\tau_1} x_2^{\tau_2} \cdots x_n^{\tau_n}\big]\big(F_{x_2}^4\big)\right) \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \\
& + T_n \left(\big[x_1^{\tau_1} x_2^{\tau_2} \cdots x_n^{\tau_n}\big]\big(F_{x_n}^4\big)\right).
\end{aligned}$$

The following goes almost without saying.

Observation 8.1. [Transitivity] *Since $T_0 = 0$, at the origin $0 = (0, 0, \dots, 0) \in \mathbb{R}_{x,u}^{n+1}$, the value of L is*

$$L\big|_0 = T_1\, \frac{\partial}{\partial x_1}\Big|_0 + T_2\, \frac{\partial}{\partial x_2}\Big|_0 + \cdots + T_n\, \frac{\partial}{\partial x_n}\Big|_0,$$

and since also

$$T_0 H^n = \mathrm{Span}\left(\frac{\partial}{\partial x_1}\Big|_0, \frac{\partial}{\partial x_2}\Big|_0, \dots, \frac{\partial}{\partial x_n}\Big|_0\right),$$

linear relation between $T_1, T_2, \dots, T_n$ can hold only for $H^n \subset \mathbb{R}^{n+1}$ being affinely homogeneous.

Consequently, the coefficients of T_1, of T_2, …, of T_n above must all vanish. Restricting attention to $\tau_1 = 0$, and taking all $(\tau_2, \dots, \tau_n) \in \mathbb{N}^{n-1}$ with $\tau_2 + \cdots + \tau_n = 3$, we obtain:

$$\begin{aligned}
0 = & T_1 \left(\big[x_2^{\tau_2} \cdots x_n^{\tau_n}\big]\big(F_{x_1}^4\big)\right) \\
& + T_2 \left(\big[x_2^{\tau_2} \cdots x_n^{\tau_n}\big]\big(F_{x_2}^4\big)\right) \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \\
& + T_n \left(\big[x_2^{\tau_2} \cdots x_n^{\tau_n}\big]\big(F_{x_n}^4\big)\right),
\end{aligned}$$

whence:

$$0 \,=\, \big[x_2^{\tau_2}\cdots x_n^{\tau_n}\big]\big(F^4_{x_2}\big), \qquad \ldots\ldots, \qquad \big[x_2^{\tau_2}\cdots x_n^{\tau_n}\big]\big(F^4_{x_n}\big),$$

which is equivalent to:

$$0 \,\equiv\, F^4_{x_2}, \qquad \ldots\ldots, \qquad 0 \,\equiv\, F^4_{x_n}.$$

Thus, $F^4(x)$ is independent of $x_2, \ldots, x_n$, and can be denoted by $E^4(x_1)$. □

We can therefore let homogeneous terms of order 5 appear:

$$u \,=\, \frac{1}{2}\,x_1^2 + 0 + E^4(x_1) + F^5(x) + \mathrm{O}_x(6).$$

We argue by induction. For $m \geqslant 5$ we assume:

$$u \,=\, \frac{1}{2}\,x_1^2 + 0 + E^4(x_1) + \cdots + E^{m-1}(x_1) + F^m(x) + \mathrm{O}_x(m+1).$$

To complete the proof of the lemma, we should show that $F^m(x) \equiv E^m(x_1)$ is independent of $x_2, \ldots, x_n$.

Using L from (8.1), we write and examine tangency equations up to order $m-1$:

$$\begin{aligned}
0 \,\equiv\, & -\,T_1\,x_1 - 2\,A_{1,1}\Big(\tfrac{1}{2}\,x_1^2 + E^4 + \cdots + E^{m-1} + \mathrm{O}_x(m)\Big) \\
& + \Big[T_1 + A_{1,1}\,x_1 + B_1\,\big(\tfrac{1}{2}\,x_1^2 + E^4 + \cdots + E^{m-2} + \mathrm{O}_x(m-1)\big)\Big] \\
& \cdot \Big[x_1 + E^4_{x_1} + \cdots + E^{m-1}_{x_1} + F^m_{x_1} + \mathrm{O}_x(m)\Big] \\
& + \big[T_2 + \mathrm{O}_x(1)\big]\,\big[F^m_{x_2} + \mathrm{O}_x(m)\big] \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot \\
& + \big[T_n + \mathrm{O}_x(1)\big]\,\big[F^m_{x_n} + \mathrm{O}_x(m)\big].
\end{aligned}$$

In the first two lines, all appearing monomials $x_1^{\tau_1}x_2^{\tau_2}\cdots x_n^{\tau_n}$ with $\tau_1+\tau_2+\cdots+\tau_n \leqslant m-1$ are such that $\tau_1 \geqslant 1$ except the ones in $T_1\,F^m_{x_1}$.

Therefore, when applying the coefficients-taking operators $[x_2^{\tau_2}\cdots x_n^{\tau_n}](\bullet)$ for all $(\tau_2,\ldots,\tau_n) \in \mathbb{N}^{n-1}$ with $\tau_2+\cdots+\tau_n = m-1$, it remains only

$$\begin{aligned}
0 \,=\, \quad & T_1\,\Big(\big[x_2^{\tau_2}\cdots x_n^{\tau_n}\big]\big(F^m_{x_1}\big)\Big) \\
& + T_2\,\Big(\big[x_2^{\tau_2}\cdots x_n^{\tau_n}\big]\big(F^m_{x_2}\big)\Big) \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \\
& + T_n\,\Big(\big[x_2^{\tau_2}\cdots x_n^{\tau_n}\big]\big(F^m_{x_n}\big)\Big),
\end{aligned}$$

whence:

$$0 \,=\, \big[x_2^{\tau_2}\cdots x_n^{\tau_n}\big]\big(F^m_{x_2}\big), \qquad \ldots\ldots, \qquad \big[x_2^{\tau_2}\cdots x_n^{\tau_n}\big]\big(F^m_{x_n}\big),$$

which is equivalent to

$$0 \,\equiv\, F^m_{x_2}, \qquad \ldots\ldots, \qquad 0 \,\equiv\, F^m_{x_n}.$$

Thus, $F^m(x)$ is independent of $x_2, \ldots, x_n$, it can be denoted by $E^m(x_1)$, and by the induction in $m \to \infty$ we complete the proof of the proposition:

$$u \,=\, \frac{1}{2}\,x_1^2 + \sum_{m=4}^{\infty} E^m(x_1) \,=:\, F(x_1).$$

□

9. Interlude: expansion of F in homogeneous (in)dependent monomials

We expand:

$$F(x) = \sum_{\sigma_1,\dots,\sigma_n \geqslant 0} x_1^{\sigma_1} \cdots x_n^{\sigma_n}\, F_{\sigma_1,\dots,\sigma_n},$$

with $F_\sigma := \frac{1}{\sigma!} F_{x^\sigma}(0)$. In accordance with Terminology 4.1, we introduce

Terminology 9.1. *A monomial $x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n}\, F_{\sigma_1,\sigma_2,\dots,\sigma_n}$ will be said:*

- *independent if $\sigma_2 + \cdots + \sigma_n \leqslant 1$;*
- *border-dependent if $\sigma_2 + \cdots + \sigma_n = 2$;*
- *body-dependent if $\sigma_2 + \cdots + \sigma_n \geqslant 3$.*

Then F decomposes into 3 parts:

$$F = \sum_{\sigma_2+\cdots+\sigma_n \leqslant 1} x^\sigma F_\sigma + \sum_{\sigma_2+\cdots+\sigma_n = 2} x^\sigma F_\sigma + \sum_{\sigma_2+\cdots+\sigma_n \geqslant 3} x^\sigma F_\sigma.$$

If we want to emphasize independent monomials only, the dependent monomials can be gathered as a plain remainder:

$$F = \sum_{\substack{\sigma_1,\sigma_2,\dots,\sigma_n \geqslant 0 \\ \sigma_2+\cdots+\sigma_n \leqslant 1}} x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n}\, F_{\sigma_1,\sigma_2,\dots,\sigma_n} + \mathrm{O}_{x_2,\dots,x_n}(2).$$

If we want to show also border-dependent monomials, the body-dependent monomials can be gathered as a plain remainder:

$$F = \sum_{\substack{\sigma_1,\sigma_2,\dots,\sigma_n \geqslant 0 \\ \sigma_2+\cdots+\sigma_n \leqslant 1}} x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n}\, F_{\sigma_1,\sigma_2,\dots,\sigma_n} + \sum_{\substack{\sigma_1,\sigma_2,\dots,\sigma_n \geqslant 0 \\ \sigma_2+\cdots+\sigma_n = 2}} x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n}\, F_{\sigma_1,\sigma_2,\dots,\sigma_n} + \mathrm{O}_{x_2,\dots,x_n}(3).$$

The decomposition of $F(x)$ as a sum of homogeneous terms will be constantly used:

$$F(x) = \sum_{m=2}^{\infty} \sum_{\sigma_1+\sigma_2+\cdots+\sigma_n = m} x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n}\, F_{\sigma_1,\sigma_2,\dots,\sigma_n} =: \sum_{m=2}^{\infty} F^m(x).$$

Then each $F^m(x)$ can be subjected to similar decompositions:

$$\begin{aligned} F^m(x) &= \sum_{\substack{\sigma_1+\sigma_2+\cdots+\sigma_n = m \\ \sigma_2+\cdots+\sigma_n \leqslant 1}} x^\sigma F_\sigma + \sum_{\substack{\sigma_1+\sigma_2+\cdots+\sigma_n = m \\ \sigma_2+\cdots+\sigma_n = 2}} x^\sigma F_\sigma + \sum_{\substack{\sigma_1+\sigma_2+\cdots+\sigma_n = m \\ \sigma_2+\cdots+\sigma_n \geqslant 3}} x^\sigma F_\sigma \\ &= \sum_{\substack{\sigma_1+\sigma_2+\cdots+\sigma_n = m \\ \sigma_2+\cdots+\sigma_n \leqslant 1}} x^\sigma F_\sigma + \sum_{\substack{\sigma_1+\sigma_2+\cdots+\sigma_n = m \\ \sigma_2+\cdots+\sigma_n = 2}} x^\sigma F_\sigma + \mathrm{O}_{x_2,\dots,x_n}(3) \\ &= \sum_{\substack{\sigma_1+\sigma_2+\cdots+\sigma_n = m \\ \sigma_2+\cdots+\sigma_n \leqslant 1}} x^\sigma F_\sigma + \mathrm{O}_{x_2,\dots,x_n}(2). \end{aligned}$$

Finally, we recall the abbreviation:

$$x' := (x_2, \dots, x_n).$$

10. The nondegenerate order 3 case and starting induction

Disregarding the product case of Section 8, we assume that $(\varphi_2, \dots, \varphi_n) \neq (0, \dots, 0)$. A natural affine coordinate change

$$u = \frac{1}{2} x_1^2 + \frac{1}{2} x_1^2 \big(\underbrace{\varphi_2 x_2 + \cdots + \varphi_n x_n}_{=: \text{ new } x_2} \big) + \mathrm{O}_x(4),$$

enables to normalize:

$$u = \frac{1}{2}\,x_1^2 + \frac{1}{2}\,x_1^2\,x_2 + \mathrm{O}_x(4).$$

Now we provide a general inductive reasoning. For some integer ν with $3 \leqslant \nu \leqslant n$, we assume that, modulo dependent monomials which, by Section 9, can be all gathered as a remainder $\mathrm{O}_{x'}(2)$, the hypersurface equation is of the form:

$$u = \frac{x_1^2}{2} + \frac{x_1^2x_2}{2} + \cdots + \frac{x_1^{\nu-1}x_{\nu-1}}{(\nu-1)!} + \mathrm{O}_{x'}(2) + \mathrm{O}_x(\nu+1).$$

Then let appear all independent monomials of order $\nu+1$:

$$\begin{aligned} u &= \frac{x_1^2}{2} + \frac{x_1^2x_2}{2} + \cdots + \frac{x_1^{\nu-1}x_{\nu-1}}{(\nu-1)!} \\ &\quad + \varphi_1\,\frac{x_1^\nu x_1}{\nu!} + \cdots + \varphi_{\nu-1}\,\frac{x_1^\nu x_{\nu-1}}{\nu!} + \varphi_\nu\,\frac{x_1^\nu x_\nu}{\nu!} + \cdots + \varphi_n\,\frac{x_1^\nu x_n}{\nu!} + \mathrm{O}_{x'}(2) + \mathrm{O}_x(\nu+2). \end{aligned}$$

Again, if $(\varphi_\nu, \dots, \varphi_n) \neq (0, \dots, 0)$, a natural affine coordinate change

$$\begin{aligned} u &= \frac{x_1^2}{2} + \frac{x_1^2x_2}{2} + \cdots + \frac{x_1^{\nu-1}x_{\nu-1}}{(\nu-1)!} \\ &\quad + \frac{x_1^\nu}{\nu!}\Big(\underbrace{\varphi_1\,x_1 + \cdots + \varphi_{\nu-1}\,x_{\nu-1} + \varphi_\nu\,x_\nu + \cdots + \varphi_n\,x_n}_{=:\ \text{new}\ x_\nu}\Big) + \mathrm{O}_{x'}(2) + \mathrm{O}_x(\nu+2), \end{aligned}$$

leads to

$$u = \frac{x_1^2}{2} + \frac{x_1^2x_2}{2} + \cdots + \frac{x_1^{\nu-1}x_{\nu-1}}{(\nu-1)!} + \frac{x_1^\nu x_\nu}{\nu!} + \mathrm{O}_{x'}(2) + \mathrm{O}_x(\nu+2),$$

and then inductively to

$$u = \frac{x_1^2}{2} + \frac{x_1^2x_2}{2} + \cdots + \frac{x_1^\nu x_\nu}{\nu!} + \cdots + \frac{x_1^n x_n}{n!} + \mathrm{O}_{x'}(2) + \mathrm{O}_x(n+2),$$

These are the hypersurfaces we mainly want to study: they involve all variables $x_1, x_2, \dots, x_n$.

But before going further, we must study the situation where $\varphi_\nu = \cdots = \varphi_n = 0$:

$$\begin{aligned} u &= \frac{x_1^2}{2} + \frac{x_1^2x_2}{2} + \cdots + \frac{x_1^{\nu-1}x_{\nu-1}}{(\nu-1)!} \\ &\quad + \varphi_1\,\frac{x_1^\nu x_1}{\nu!} + \cdots + \varphi_{\nu-1}\,\frac{x_1^\nu x_{\nu-1}}{\nu!} + 0 + \cdots + 0 + \mathrm{O}_{x'}(2) + \mathrm{O}_x(\nu+2). \end{aligned}$$

This degenerate branch will again lead to a product situation.

Since the arguments in the next Section 11 will again involve application of an affine vector field L to the equation $0 = -u + F(x)$, and since L is a first-order derivation, we need to know the border-dependent monomials as well. Recall that, by Section 9, body-dependent monomials that are not border-dependent can be gathered as $\mathrm{O}_{x'}(3)$.

To organize properly the thought, we consider simultaneously the two cases $\mathbf{0} \cdot \frac{x_1^\nu x_\nu}{\nu!}$ and $\mathbf{1} \cdot \frac{x_1^\nu x_\nu}{\nu!}$, by setting up an

Induction Hypothesis 10.1. *For some integer ν with $3 \leqslant \nu \leqslant n$, the hypersurface equation reads as*

$$\begin{aligned} u = \frac{x_1^2}{2} + \frac{x_1^2x_2}{2} + \sum_{m=3}^{\nu-1}\Big(\frac{x_1^m x_m}{m!} + x_1^{m-1}\sum_{\substack{i,j\geqslant 2\\ i+j=m+1}} \tfrac{1}{2}\,\frac{x_ix_j}{(i-1)!\,(j-1)!} + R^{m+1}(x_1, x_2, \dots, x_{m-1})\Big) \\ + \{\begin{smallmatrix}\mathbf{0}\\ \mathbf{1}\end{smallmatrix}\}\,\frac{x_1^\nu x_\nu}{\nu!} + x_1^{\nu-1}\sum_{i,j\geqslant 2} \tfrac{1}{2}\,x_ix_j\,\Lambda_{i,j}^\nu + R^{\nu+1}(x_1, x_2, \dots, x_\nu, \dots, x_n) + \mathrm{O}_x(\nu+2), \end{aligned}$$

where R^{m+1} is homogeneous of order $m+1$ in $(x_1, x_2, \dots, x_{m-1})$, and is of order $\geqslant 3$ in $(x_2, \dots, x_{m-1})$, where the $\Lambda^\nu_{i,j} = \Lambda^\nu_{j,i}$ are unknown constants, and where $R^{\nu+1}$ is homogeneous of order $\nu+1$ in $(x_1, x_2, \dots, x_\nu, \dots, x_n)$, and is of order $\geqslant 3$ in $(x_2, \dots, x_\nu, \dots, x_n)$.

We therefore assume that up to $m = \nu - 1$, the values of the border-dependent jets have been found, as they appear within the large parentheses. For $\nu = 3$, the formula holds true with empty $\sum_{m=3}^{\nu-1}$. To complete the induction on ν, we need to show that $R^{\nu+1} = R^{\nu+1}(x_1, \dots, x_{\nu-1})$ is independent of $x_\nu, x_{\nu+1}, \dots, x_n$, and we should determine the values of $\Lambda^\nu_{i,j}$. In the nondegenerate branch $\mathbf{1} \cdot \frac{x_1^\nu x_\nu}{\nu!}$ that we will treat later, we will have to show that $\Lambda^\nu_{i,j} = 0$ whenever $i + j \neq \nu + 1$, and that $\Lambda^\nu_{i,j} = \frac{1}{(i-1)!(j-1)!}$ when $i + j = \nu + 1$.

Abbreviate the homogeneous terms of $F(x)$ normalized up to order $\leqslant \nu$, namely up to $m = \nu - 1$ in the sum, as follows:

$$u = N^2(x_1) + N^3(x_1, x_2) + \cdots + N^\nu(x_1, x_2, \dots, x_{\nu-1}) + \mathrm{O}_x(\nu+1),$$

where, for $3 \leqslant m \leqslant \nu - 1$:

$$N^{m+1} := \frac{x_1^m x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j \geqslant 2 \\ i+j=m+1}} \tfrac{1}{2} \frac{x_i x_j}{(i-1)!\,(j-1)!} + R^{m+1}(x_1, x_2, \dots, x_{m-1}),$$

and abbreviate also the full dependent remainder homogeneous of order $\nu + 1$ after $\left\{ \begin{smallmatrix} \mathbf{0} \\ \mathbf{1} \end{smallmatrix} \right\} \frac{x_1^\nu x_\nu}{\nu!}$ as:

$$S^{\nu+1}\big(x_1, x_2, \dots, x_\nu, \dots, x_n\big) := x_1^{\nu-1} \sum_{i,j \geqslant 2} \tfrac{1}{2}\, x_i x_j\, \Lambda^\nu_{i,j} + R^{\nu+1}\big(x_1, x_2, \dots, x_\nu, \dots, x_n\big),$$

which is of order $\geqslant 2$ in $(x_2, \dots, x_\nu, \dots, x_n)$, so that:

$$u = F(x) = N^2 + \cdots + N^\nu + \left\{ \begin{smallmatrix} \mathbf{0} \\ \mathbf{1} \end{smallmatrix} \right\} \frac{x_1^\nu x_\nu}{\nu!} + S^{\nu+1} + \mathrm{O}_x(\nu+2).$$

Lemma 10.1. *This function $S^{\nu+1}$ is independent of $x_\nu, x_{\nu+1}, \dots, x_n$.*

Proof. For any two indices $\nu \leqslant k, \ell \leqslant n$, we must have by our constant Hessian rank 1 hypothesis:

$$\begin{aligned}
0 &\equiv F_{x_1x_1}(x) \cdot F_{x_kx_\ell}(x) - F_{x_1x_k}(x) \cdot F_{x_1x_\ell}(x) \\
&\equiv \big(1 + \mathrm{O}_x(1)\big) \cdot \Big(S^{\nu+1}_{x_kx_\ell} + \mathrm{O}_x(\nu)\Big) \\
&\quad - \Big(\left\{ \begin{smallmatrix} \mathbf{0} \\ \mathbf{1} \end{smallmatrix} \right\} \tfrac{x_1^{\nu-1}}{(\nu-1)!} \tfrac{\partial x_\nu}{\partial x_k} + S^{\nu+1}_{x_1x_k} + \mathrm{O}_x(\nu)\Big) \cdot \Big(\left\{ \begin{smallmatrix} \mathbf{0} \\ \mathbf{1} \end{smallmatrix} \right\} \tfrac{x_1^{\nu-1}}{(\nu-1)!} \tfrac{\partial x_\nu}{\partial x_\ell} + S^{\nu+1}_{x_1x_\ell} + \mathrm{O}_x(\nu)\Big) \\
&\equiv S^{\nu+1}_{x_kx_\ell} + \mathrm{O}_x(\nu) - \mathrm{O}_x(2\nu - 2),
\end{aligned}$$

since $2\nu - 2 \geqslant \nu$ as $\nu \geqslant 3$, and this yields:

$$0 \equiv S^{\nu+1}_{x_kx_\ell},$$

Since $S^{\nu+1}$ is of order $\geqslant 2$ in $(x_2, \dots, x_n)$, this completes the proof. □

11. The product case $H^n \cong H^{\nu-1} \times \mathbb{R}^{n-\nu}$

Here we treat the degenerate branch $\mathbf{0} \cdot \frac{x_1^\nu x_\nu}{\nu!}$. We have

$$u = F(x) = N^2(x_1) + \cdots + N^\nu(x_1, \dots, x_{\nu-1}) + \mathbf{0} + S^{\nu+1}(x_1, \dots, x_{\nu-1}) + \mathrm{O}_x(\nu+2).$$

Notice that the term $S^{\nu+1}$ (homogeneous of order $\nu + 1$) depends only on $x_1, \dots, x_{\nu-1}$, as does the preceding one N^ν.

Lemma 11.1. *If such hypersurface is affinely homogeneous, then $F = F(x_1, \dots, x_{\nu-1})$ is independent of $x_\nu, \dots, x_n$.*

Proof. Let homogeneous terms of order $\nu+2$ appear:

$$u = F = N^2 + \cdots + N^\nu + S^{\nu+1} + F^{\nu+2}(x_1,\dots,x_{\nu-1},x_\nu,\dots,x_n) + \mathrm{O}(\nu+3).$$

We claim that $F^{\nu+2}$ is independent of $x_\nu,\dots,x_n$.

Indeed, using L from (8.1), we write and examine tangency equations up to order $\nu+1$:

$$\begin{aligned}
0 \equiv & - T_1\, x_1 - 2\, A_{1,1} \Big(N^2 + \cdots + N^\nu + S^{\nu+1} + \mathrm{O}_x(\nu+2)\Big) \\
& + \Big[T_1 + A_{1,1}x_1 + B_1\big(N^2 + \cdots + N^\nu + \mathrm{O}_x(\nu+1)\big)\Big] \\
& \cdot \Big[N^2_{x_1} + \cdots + N^\nu_{x_1} + S^{\nu+1}_{x_1} + F^{\nu+2}_{x_1} + \mathrm{O}_x(\nu+2)\Big] \\
& + \Big[T_2 + A_{2,1}x_1 + \cdots + A_{2,\nu}x_\nu + \cdots + A_{2,n}x_n + B_2\big(N^2 + \cdots + N^{\nu-1} + \mathrm{O}_x(\nu)\big)\Big] \\
& \cdot \Big[N^3_{x_2} + \cdots + N^\nu_{x_2} + S^{\nu+1}_{x_2} + F^{\nu+2}_{x_2} + \mathrm{O}_x(\nu+2)\Big] \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \Big[T_{\nu-1} + A_{\nu-1,1}x_1 + \cdots + A_{\nu-1,\nu}x_\nu + \cdots + A_{\nu-1,n}x_n + B_{\nu-1}\big(N^2 + \mathrm{O}_x(3)\big)\Big] \\
& \cdot \Big[N^\nu_{x_{\nu-1}} + S^{\nu+1}_{x_{\nu-1}} + F^{\nu+2}_{x_{\nu-1}} + \mathrm{O}_x(\nu+2)\Big] \\
& + \Big[T_\nu + \mathrm{O}_x(1)\Big] \cdot \Big[F^{\nu+2}_{x_\nu} + \mathrm{O}_x(\nu+2)\Big] \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \\
& + \Big[T_n + \mathrm{O}_x(1)\Big] \cdot \Big[F^{\nu+2}_{x_n} + \mathrm{O}_x(\nu+2)\Big].
\end{aligned}$$

For all $(\tau_\nu,\dots,\tau_n) \in \mathbb{N}^{n-\nu+1}$ with $\tau_\nu + \cdots + \tau_n = \nu+1$, we apply the coefficients-picking operators $[x_\nu^{\tau_\nu} \cdots x_n^{\tau_n}](\bullet)$ to this equation,. Since N^2, N^3, ..., N^ν, $S^{\nu+1}$ depend only on $(x_1,\dots,x_{\nu-1})$, we obtain:

$$\begin{aligned}
0 \equiv \;\; & T_1 \Big([x_\nu^{\tau_\nu} \cdots x_n^{\tau_n}]\big(F^{\nu+2}_{x_1}\big)\Big) \\
& + \cdots\cdots\cdots\cdots \\
& + T_{\nu-1} \Big([x_\nu^{\tau_\nu} \cdots x_n^{\tau_n}]\big(F^{\nu+2}_{x_{\nu-1}}\big)\Big) \\
& + T_\nu \Big([x_\nu^{\tau_\nu} \cdots x_n^{\tau_n}]\big(F^{\nu+2}_{x_\nu}\big)\Big) \\
& + \cdots\cdots\cdots\cdots \\
& + T_n \Big([x_\nu^{\tau_\nu} \cdots x_n^{\tau_n}]\big(F^{\nu+2}_{x_n}\big)\Big),
\end{aligned}$$

whence

$$0 = \big[x_\nu^{\tau_\nu} \cdots x_n^{\tau_n}\big]\big(F^{\nu+2}_{x_\nu}\big), \qquad \dots\dots, \qquad \big[x_\nu^{\tau_\nu} \cdots x_n^{\tau_n}\big]\big(F^{\nu+2}_{x_n}\big),$$

which is equivalent to

$$0 \equiv F^{\nu+2}_{x_\nu}, \qquad \dots\dots, \qquad 0 \equiv F^{\nu+2}_{x_n}.$$

Thus, $F^{\nu+2}$ is independent of $x_\nu,\dots,x_n$, and can be denoted $E^{\nu+2}(x_1,\dots,x_{\nu-1})$.

Next, let homogeneous terms of order $\nu+3$ appear:

$$u = F = N^2 + \cdots + N^\nu + S^{\nu+1} + E^{\nu+2} + F^{\nu+3}(x_1,\dots,x_{\nu-1},x_\nu,\dots,x_n) + \mathrm{O}(\nu+4).$$

By writing the tangency equation up to order $\nu+2$, we realize similarly that $F^{\nu+3}$ is independent of $x_\nu,\dots,x_n$. Proceeding by induction, we conclude that

$$u = F = N^2 + \cdots + N^\nu + S^{\nu+1} + \sum_{m=\nu+2}^{\infty} E^m(x_1,\dots,x_{\nu-1}).$$

□

We now disregard this degenerate case.

12. Nondegenerate case $\mathbf{1} \cdot \frac{x_1^\nu x_\nu}{\nu!}$

At last, we can start to treat the most interesting branch $\mathbf{1} \cdot \frac{x_1^\nu x_\nu}{\nu!}$. Thus, as already stated by the Induction Hypothesis 10.1, we start from

$$u = \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \sum_{m=3}^{\nu-1} \Big(\frac{x_1^m x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j \geqslant 2 \\ i+j=m+1}} \frac{1}{2} \frac{x_i x_j}{(i-1)!\,(j-1)!} + \mathrm{O}_{x_2,\dots,x_{m-1}}(3) \Big)$$
$$+ \frac{x_1^\nu x_\nu}{\nu!} + x_1^{\nu-1} \sum_{2 \leqslant i,j \leqslant \nu-1} \tfrac{1}{2}\, x_i x_j\, \Lambda_{i,j}^\nu + \mathrm{O}_{x_2,\dots,x_{\nu-1}}(3) + \mathrm{O}_x(\nu+2),$$

where we already know from Lemma 10.1 that $\Lambda_{i,j}^\nu = 0$ when $i \geqslant \nu$ or $j \geqslant \nu$, and that the body-dependent remainder $R^{\nu+1}$ is independent of $x_\nu, x_{\nu+1}, \dots, x_n$, hence is of order $\geqslant 3$ in $x_2, \dots, x_{\nu-1}$.

Here is the statement we mentioned.

Lemma 12.1. *We have* $\Lambda_{i,j}^\nu = 0$ *whenever* $i + j \neq \nu + 1$, *and* $\Lambda_{i,j}^\nu = \frac{1}{(i-1)!\,(j-1)!}$ *when* $i + j = \nu + 1$.

Proof. First,

$$F_{x_1 x_1} = 1 + \mathrm{O}_{x_2,\dots,x_{\nu-1}}(1) + \mathrm{O}_x(\nu).$$

Second, for any two indices $2 \leqslant i, j \leqslant \nu - 1$:

$$F_{x_1 x_i} = \frac{x_1^{i-1}}{(i-1)!} + \mathrm{O}_{x_2,\dots,x_{\nu-1}}(1) + \mathrm{O}_x(\nu),$$
$$F_{x_1 x_j} = \frac{x_1^{j-1}}{(j-1)!} + \mathrm{O}_{x_2,\dots,x_{\nu-1}}(1) + \mathrm{O}_x(\nu).$$

Here, we can also write $\mathrm{O}_{x'}(1)$ instead of the more precise (but not useful) $\mathrm{O}_{x_2,\dots,x_{\nu-1}}(1)$.

Third, to compute $F_{x_i x_j}$, consider two subcases:

- when $4 \leqslant i + j \leqslant \nu$:

$$F_{x_i x_j} = x_1^{i+j-2} \tfrac{1}{(i-1)!\,(j-1)!} + x_1^{\nu-1} \Lambda_{i,j}^\nu + \mathrm{O}_{x'}(1) + \mathrm{O}_x(\nu);$$

- when $\nu + 1 \leqslant i + j \leqslant 2\nu - 2$:

$$F_{x_i x_j} = 0 + x_1^{\nu-1} \Lambda_{i,j}^\nu + \mathrm{O}_{x'}(1) + \mathrm{O}_x(\nu).$$

The vanishing of the Hessian yields:

$$\begin{aligned} 0 &\equiv F_{x_1 x_1} \cdot F_{x_i x_j} - F_{x_1 x_i} \cdot F_{x_1 x_j} \\ &\equiv \Big(1 + \mathrm{O}_{x'}(1) + \mathrm{O}_x(\nu)\Big) \Big(\left\{ \begin{smallmatrix} \mathbf{0} \\ \mathbf{1} \end{smallmatrix} \right\} \tfrac{1}{(i-1)!(j-1)!}\, x_1^{i+j-2} + x_1^{\nu-1} \Lambda_{i,j}^\nu + \mathrm{O}_{x'}(1) + \mathrm{O}_x(\nu) \Big) \\ &\quad - \Big(\tfrac{x_1^{i-1}}{(i-1)!} + \mathrm{O}_{x'}(1) + \mathrm{O}_x(\nu) \Big) \Big(\tfrac{x_1^{j-1}}{(j-1)!} + \mathrm{O}_{x'}(1) + \mathrm{O}_x(\nu) \Big), \end{aligned}$$

and it can be expanded into various cases:

- As $4 \leqslant i + j \leqslant \nu$, it gives $\Lambda_{i,j}^\nu = 0$.
- For $i + j = \nu + 1$, it gives $\Lambda_{i,j}^\nu = \frac{1}{(i-1)!(j-1)!}$.
- As $\nu + 2 \leqslant i + j \leqslant 2\nu - 2$, it gives $\Lambda_{i,j}^\nu = 0$. □

This completes the induction from $\nu - 1$ to ν, while $3 \leqslant \nu \leqslant n$, cf. Induction Hypothesis 10.1.

13. Summary and beyond

We started from arbitrary hypersurface $H^n \subset \mathbb{R}^{n+1}$ whose Hessian has constant rank 1 and we showed, generally, that one can let appear monomials $\frac{x_1^2}{2}$, $\frac{x_1^2 x_2}{2!}$, $\frac{x_1^3 x_3}{3!}$, $\frac{x_1^4 x_4}{4!}$, $\frac{x_1^5 x_5}{5!}$, ..., until the process stops, and we proved the following theorem.

Theorem 13.1. *Let $H^n \subset \mathbb{R}^{n+1}$ be a local affinely homogeneous hypersurface having constant Hessian rank 1. Then there exists an integer $1 \leqslant n_H \leqslant n$ and affine coordinates $(x_1, \dots, x_n)$, in which*

$$H^n \;=\; H^{n_H} \times \mathbb{R}^{n-n_H-1}_{x_{n_H+1},\dots,x_n}$$

is a product of an affinely homogeneous hypersurface $H^{n_H} \subset \mathbb{R}^{n_H+1}$ times a 'dumb' $\mathbb{R}^{n-n_H-1}$, and is graphed as:

$$u = \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \sum_{m=3}^{n_H} \left(\frac{x_1^m x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j \geqslant 2 \\ i+j=m+1}} \frac{1}{2} \frac{x_i x_j}{(i-1)!(j-1)!} + \mathrm{O}_{x_2,\dots,x_{m-1}}(3) \right) + \sum_{m=n_H+2}^{\infty} E^m(x_1, \dots, x_{n_H}),$$

with graphing function $F = F(x_1, \dots, x_{n_H})$ independent of $x_{n_H+1}, \dots, x_n$. □

Disregarding the product cases (branches) $n_H = 1, \dots, n_H = n-1$ which lead to similar considerations in lower dimensions, we will from now on study the class of *nondegenerate* hypersurfaces, those involving *all* variables $(x_1, \dots, x_n)$:

$$u \;=\; \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \sum_{m=3}^{n} \Big(\frac{x_1^m x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j \geqslant 2 \\ i+j=m+1}} \frac{1}{2} \frac{x_i x_j}{(i-1)!(j-1)!} + \mathrm{O}_{x_2,\dots,x_{m-1}}(3) \Big) + \mathrm{O}_x(n+2).$$

The next (substantial) task is to examine further the remainder $\mathrm{O}_x(n+2)$.

We gather all remainders $\mathrm{O}_{x_2,\dots,x_{m-1}}(3)$ as $\mathrm{O}_{x'}(3)$, where $x' := (x_2, \dots, x_n)$, and we let appear the independent homogeneous monomials of order $n+2$, namely,

$$F_{n+2,0,\dots,0}\, \frac{x_1^{n+1} x_1}{(n+2)!} + F_{n+1,1,\dots,0}\, \frac{x_1^{n+1} x_2}{(n+1)!} + \cdots + F_{n+1,0,\dots,1}\, \frac{x_1^{n+1} x_n}{(n+1)!}.$$

A reasoning similar to that in the proof of Proposition 12.1 shows that border-dependent monomials of homogeneous order $n+2$ have the same form, hence the equation of the hypersurface reads as

$$\begin{aligned} u = {} & \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \sum_{m=3}^{n} \Big(\frac{x_1^m x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j \geqslant 2 \\ i+j=m+1}} \frac{1}{2} \frac{x_i x_j}{(i-1)!(j-1)!} \Big) \\ & + F_{n+2,0,\dots,0}\, \frac{x_1^{n+1} x_1}{(n+2)!} + F_{n+1,1,\dots,0}\, \frac{x_1^{n+1} x_2}{(n+1)!} + \cdots + F_{n+1,0,\dots,1}\, \frac{x_1^{n+1} x_n}{(n+1)!} \\ & + x_1^n \sum_{\substack{i,j \geqslant 2 \\ i+j=n+2}} \tfrac{1}{2} \frac{x_i x_j}{(i-1)!(j-1)!} + \mathrm{O}_{x'}(3) + \mathrm{O}_x(n+3). \end{aligned}$$

Question 13.1. *How to normalize these order $n+2$ independent coefficients $F_{n+2,0,\dots,0}$, $F_{n+1,1,\dots,0}$, ..., $F_{n+1,0,\dots,1}$?*

14. Normalization of order $n+2$ terms

Consider therefore *another* similar (nondegenerate) hypersurface, in coordinates $(y_1, \dots, y_n, v)$:

$$\begin{aligned} v &= \frac{y_1^2}{2} + \frac{y_1^2 y_2}{2} + \sum_{m=3}^{n} \bigg(\frac{y_1^m y_m}{m!} + y_1^{m-1} \sum_{\substack{i,j \geqslant 2 \\ i+j=m+1}} \tfrac{1}{2} \frac{y_i y_j}{(i-1)!(j-1)!} \bigg) \\ &+ G_{n+2,0,\dots,0} \frac{y_1^{n+1} y_1}{(n+2)!} + G_{n+1,1,\dots,0} \frac{y_1^{n+1} y_2}{(n+1)!} + \cdots + G_{n+1,0,\dots,1} \frac{y_1^{n+1} y_n}{(n+1)!} \\ &+ y_1^n \sum_{\substack{i,j \geqslant 2 \\ i+j=n+2}} \tfrac{1}{2} \frac{y_i y_j}{(i-1)!(j-1)!} + \mathrm{O}_{y'}(3) + \mathrm{O}_y(n+3). \end{aligned}$$

From Section 5, we know what is the stability subgroup at order 2.

Step 1. Determine subgroup reduction stabilizing terms of order less than or equal to $n+1$:

$$\overset{G^2_{\mathrm{stab}}}{\begin{bmatrix} a_{1,1} & \mathbf{0} & \cdots & \mathbf{0} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & b_n \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{2}}} \rightsquigarrow \overset{G^{n+1}_{\mathrm{stab}}}{\begin{bmatrix} ? & \mathbf{0} & \cdots & \mathbf{0} & ? \\ ? & ? & \cdots & ? & ? \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ? & ? & \cdots & ? & ? \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & a_{1,1}^{n+1} \end{bmatrix}^{\mathbf{2}}}.$$

Step 2. Determine how this subgroup acts on order $n+2$ terms, and normalize those coefficients among $G_{n+2,1,\dots,0}$, $G_{n+1,1,\dots,0}$, $\dots$, $G_{n+1,0,\dots,1}$ which can be normalized.

This computational task being nontrivial, let us start in low dimensions. The calculations of the next Section will not be detailed, and the remainders $\mathrm{O}_x(*)$ will not be written.

15. Stabilizing order $n+1$ terms in dimensions $n = 2, 3, 4, 5, 6$

- In dimension $n = 2$:

$$u = \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + F_{4,0} \frac{x_1^3 x_1}{24} + F_{3,1} \frac{x_1^3 x_2}{6},$$

the stability group is

$$G^3_{\mathrm{stab}}: \qquad \begin{bmatrix} a_{1,1} & \mathbf{0} & -a_{1,1} a_{2,1} \\ a_{2,1} & 1 & b_2 \\ \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{3}},$$

and its action gives:

$$\begin{aligned} 0 &\overset{40}{=} -\frac{1}{24} a_{1,1}^2 F_{4,0} + \frac{1}{24} a_{1,1}^4 G_{4,0} + \frac{1}{6} a_{1,1}^3 a_{2,1} G_{3,1} + \frac{1}{8} a_{1,1}^2 a_{2,1}^2 + \frac{1}{4} a_{1,1}^2 \boxed{b_2}, \\ 0 &\overset{31}{=} -\frac{1}{6} a_{1,1}^2 F_{3,1} + \frac{1}{6} a_{1,1}^3 G_{3,1}. \end{aligned}$$

The free group parameter b_2 can be used to normalize $G_{4,0} := 0$.

- In dimension $n = 3$:

$$\begin{aligned} u &= \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \frac{x_1^3 x_3}{6} + \frac{x_1^2 x_2^2}{2} \\ &+ F_{5,0,0} \frac{x_1^4 x_1}{120} + F_{4,1,0} \frac{x_1^4 x_2}{24} + F_{4,0,1} \frac{x_1^4 x_3}{24} + \frac{x_1^3 x_2 x_3}{2}, \end{aligned}$$

the stability group is

$$G^4_{\rm stab}\colon \qquad \begin{bmatrix} a_{1,1} & \mathbf{0} & \mathbf{0} & -a_{1,1}a_{2,1} \\ a_{2,1} & 1 & \mathbf{0} & -\frac{1}{2}a_{2,1}^2 - \frac{2}{3}a_{1,1}a_{3,1} \\ a_{3,1} & \mathbf{0} & \frac{1}{a_{1,1}} & b_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{4}},$$

and its action gives

$$\begin{aligned} 0 &\overset{500}{=} -\frac{1}{120}\,a_{1,1}^2\,F_{5,0,0} + \frac{1}{120}\,a_{1,1}^5\,G_{5,0,0} + \frac{1}{24}\,a_{1,1}^4\,a_{2,1}\,G_{4,1,0} + \frac{1}{12}\,a_{1,1}^3\,a_{2,1}\,a_{3,1} \\ &\quad + \frac{1}{24}\,a_{1,1}^4\,a_{3,1}\,G_{4,0,1} + \frac{1}{12}\,a_{1,1}^3\,\boxed{b_3} \\ 0 &\overset{410}{=} -\frac{1}{24}\,a_{1,1}^2\,F_{4,1,0} + \frac{1}{24}\,a_{1,1}^4\,G_{4,1,0}, \\ 0 &\overset{401}{=} -\frac{1}{24}\,a_{1,1}^2\,F_{4,0,1} + \frac{1}{24}\,a_{1,1}^3\,G_{4,0,1} + \frac{1}{12}\,a_{1,1}^2\,\boxed{a_{2,1}}. \end{aligned}$$

The free group parameter $a_{2,1}$ can be used to normalize $G_{4,0,1} := 0$, and the free group parameter b_3 can be used to normalize $G_{5,0,0} := 0$.

- In dimension $n = 4$:

$$\begin{aligned} u &= \frac{x_1^2}{2} + \frac{x_1^2x_2}{2} + \frac{x_1^3x_3}{6} + \frac{x_1^2x_2^2}{2} + \frac{x_1^4x_4}{24} + \frac{x_1^2x_2^3}{2} + \frac{x_1^3x_2x_3}{2} \\ &\quad + F_{6,0,0,0}\,\frac{x_1^5x_1}{720} + F_{5,1,0,0}\,\frac{x_1^5x_2}{120} + F_{5,0,1,0}\,\frac{x_1^5x_3}{120} \\ &\quad + F_{5,0,0,1}\,\frac{x_1^5x_4}{120} + \frac{x_1^2x_2^4}{2} + x_1^3x_2^2x_3 + \frac{x_1^4x_2x_4}{6} + \frac{x_1^4x_3^2}{8}, \end{aligned}$$

the stability group is:

$$G^5_{\rm stab}\colon \qquad \begin{bmatrix} a_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -a_{1,1}a_{2,1} \\ a_{2,1} & 1 & \mathbf{0} & \mathbf{0} & -\frac{1}{2}a_{2,1}^2 - \frac{2}{3}a_{1,1}a_{3,1} \\ a_{3,1} & \mathbf{0} & \frac{1}{a_{1,1}} & \mathbf{0} & -\frac{1}{2}a_{1,1}a_{4,1} - a_{2,1}a_{3,1} \\ a_{4,1} & \mathbf{0} & \frac{-2a_{2,1}}{a_{1,1}^2} & \frac{1}{a_{1,1}^2} & b_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{1,1}^2 \end{bmatrix}^{\mathbf{5}},$$

and its action gives

$$\begin{aligned} 0 &\overset{6000}{=} -\frac{1}{720}\,a_{1,1}^2\,F_{6,0,0,0} + \frac{1}{720}\,a_{1,1}^6\,G_{6,0,0,0} + \frac{1}{72}\,a_{1,1}^4\,a_{3,1}^2 + \frac{1}{48}\,a_{1,1}^4\,a_{2,1}\,a_{4,1} \\ &\quad + \frac{1}{120}\,G_{5,1,0,0}\,a_{1,1}^5\,a_{2,1} + \frac{1}{120}\,G_{5,0,1,0}\,a_{1,1}^5\,a_{3,1} + \frac{1}{120}\,G_{5,0,0,1}\,a_{1,1}^5\,a_{4,1} + \frac{1}{48}\,a_{1,1}^4\,\boxed{b_4} \\ 0 &\overset{5100}{=} -\frac{1}{120}\,a_{1,1}^2\,F_{5,1,0,0} + \frac{1}{120}\,a_{1,1}^5\,G_{5,1,0,0}, \\ 0 &\overset{5010}{=} -\frac{1}{120}\,a_{1,1}^2\,F_{5,0,1,0} + \frac{1}{120}\,a_{1,1}^4\,G_{5,0,1,0} - \frac{1}{24}\,a_{1,1}^2\,a_{2,1}^2 - \frac{1}{60}\,a_{1,1}^3\,a_{2,1} + \frac{1}{36}\,a_{1,1}^3\,\boxed{a_{3,1}}, \\ 0 &\overset{5001}{=} -\frac{1}{120}\,a_{1,1}^2\,F_{5,0,0,1} + \frac{1}{120}\,a_{1,1}^3\,G_{5,0,0,1} + \frac{1}{24}\,a_{1,1}^2\,\boxed{a_{2,1}}. \end{aligned}$$

The free group parameters $a_{2,1}$, $a_{3,1}$, b_4 can be used to normalize $G_{5,0,0,1} := 0$, $G_{5,0,1,0} := 0$, $G_{6,0,0,0} := 0$.

• In dimension $n = 5$:

$$
\begin{aligned}
u = & \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \frac{x_1^3 x_3}{6} + \frac{x_1^2 x_2^2}{2} + \frac{x_1^4 x_4}{24} + \frac{x_1^2 x_2^3}{2} + \frac{x_1^3 x_2 x_3}{2} \\
& + \frac{x_1^5 x_5}{120} + \frac{x_1^2 x_2^4}{2} + x_1^3 x_2^2 x_3 + \frac{x_1^4 x_2 x_4}{6} + \frac{x_1^4 x_3^2}{8} \\
& + F_{7,0,0,0,0} \frac{x_1^6 x_1}{5040} + F_{6,1,0,0,0} \frac{x_1^6 x_2}{720} + F_{6,0,1,0,0} \frac{x_1^6 x_3}{720} + F_{6,0,0,1,0} \frac{x_1^6 x_4}{720} + F_{6,0,0,0,1} \frac{x_1^6 x_5}{720} \\
& + \frac{1}{2} x_1^2 x_2^5 + \frac{5}{3} x_1^3 x_2^3 x_3 + \frac{5}{8} x_1^4 x_2 x_3^2 + \frac{5}{12} x_1^4 x_2^2 x_4 + \frac{1}{12} x_1^5 x_3 x_4 + \frac{1}{24} x_1^5 x_2 x_5,
\end{aligned}
$$

the stability group is:

$$
G^6_{\text{stab}}: \quad \begin{bmatrix}
a_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -a_{1,1} a_{2,1} \\
a_{2,1} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2} a_{2,1}^2 - \frac{2}{3} a_{1,1} a_{3,1} \\
a_{3,1} & \mathbf{0} & \frac{1}{a_{1,1}} & \mathbf{0} & \mathbf{0} & -a_{2,1} a_{3,1} - \frac{1}{2} a_{1,1} a_{4,1} \\
a_{4,1} & \mathbf{0} & -\frac{2 a_{2,1}}{a_{1,1}^2} & \frac{1}{a_{1,1}^2} & \mathbf{0} & -a_{2,1} a_{4,1} - \frac{2}{3} a_{3,1}^2 - \frac{2}{5} a_{1,1} a_{5,1} \\
a_{5,1} & \mathbf{0} & \frac{5 a_{2,1}^2}{a_{1,1}^3} - \frac{10}{3} \frac{a_{3,1}}{a_{1,1}^2} & -\frac{5 a_{2,1}}{a_{1,1}^3} & \frac{1}{a_{1,1}^3} & b_5 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{1,1}^2
\end{bmatrix}^{\mathbf{6}},
$$

and its action gives:

$$
\begin{aligned}
0 \overset{70000}{=} & - \frac{1}{5040} a_{1,1}^2 F_{7,0,0,0,0} + \frac{1}{5040} a_{1,1}^7 G_{7,0,0,0,0} + \frac{1}{144} a_{1,1}^5 a_{3,1} a_{4,1} + \frac{1}{240} a_{1,1}^5 a_{2,1} a_{5,1} \\
& + \frac{1}{720} G_{6,1,0,0,0} a_{1,1}^6 a_{2,1} + \frac{1}{720} G_{6,0,1,0,0} a_{1,1}^6 a_{3,1} + \frac{1}{720} G_{6,0,0,1,0} a_{1,1}^6 a_{4,1} \\
& + \frac{1}{720} G_{6,0,0,0,1} a_{1,1}^6 a_{5,1} + \frac{1}{240} a_{1,1}^5 \boxed{b_5} \\
0 \overset{61000}{=} & - \frac{1}{720} a_{1,1}^2 F_{6,1,0,0,0} + \frac{1}{720} a_{1,1}^6 G_{6,1,0,0,0}, \\
0 \overset{60100}{=} & - \frac{1}{720} a_{1,1}^2 F_{6,0,1,0,0} + \frac{1}{720} a_{1,1}^5 G_{6,0,1,0,0} - \frac{1}{36} a_{1,1}^3 a_{2,1} a_{3,1} + \frac{1}{48} a_{1,1}^2 a_{2,1}^3 + \frac{1}{144} a_{1,1}^3 a_{2,1}^2 G_{6,0,0,0,1} \\
& - \frac{1}{216} a_{1,1}^4 a_{3,1} G_{6,0,0,0,1} - \frac{1}{360} a_{1,1}^4 a_{2,1} G_{6,0,0,1,0} + \frac{1}{144} a_{1,1}^4 \boxed{a_{4,1}}, \\
0 \overset{60010}{=} & - \frac{1}{720} a_{1,1}^2 F_{6,0,0,1,0} + \frac{1}{720} a_{1,1}^4 G_{6,0,0,1,0} - \frac{1}{32} a_{1,1}^2 a_{2,1}^2 - \frac{1}{144} a_{1,1}^3 a_{2,1} G_{6,0,0,0,1} + \frac{1}{72} a_{1,1}^3 \boxed{a_{3,1}}, \\
0 \overset{60001}{=} & - \frac{1}{720} a_{1,1}^2 F_{6,0,0,0,1} + \frac{1}{720} a_{1,1}^3 G_{6,0,0,0,1} + \frac{1}{80} a_{1,1}^2 \boxed{a_{2,1}}.
\end{aligned}
$$

The free group parameters $a_{2,1}$, $a_{3,1}$, $a_{4,1}$, b_4 can be used to normalize $G_{6,0,0,0,1} := 0$, $G_{6,0,0,1,0} := 0$, $G_{6,0,1,0,0} := 0$, $G_{7,0,0,0,0} := 0$.

• In dimension $n = 6$:

$$
\begin{aligned}
u = & \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \frac{x_1^3 x_3}{6} + \frac{x_1^2 x_2^2}{2} + \frac{x_1^4 x_4}{24} + \frac{x_1^2 x_2^3}{2} + \frac{x_1^3 x_2 x_3}{2} \\
& + \frac{x_1^5 x_5}{120} + \frac{x_1^2 x_2^4}{2} + x_1^3 x_2^2 x_3 + \frac{x_1^4 x_2 x_4}{6} + \frac{x_1^4 x_3^2}{8} \\
& + \frac{x_1^6 x_6}{720} + \frac{x_1^2 x_2^5}{2} + \frac{5}{3} x_1^3 x_2^3 x_3 + \frac{5}{8} x_1^4 x_2 x_3^2 + \frac{5}{12} x_1^4 x_2^2 x_4 + \frac{1}{12} x_1^5 x_3 x_4 + \frac{1}{24} x_1^5 x_2 x_5 \\
& + F_{8,0,0,0,0,0} \frac{x_1^7 x_1}{40320} + F_{7,1,0,0,0,0} \frac{x_1^7 x_2}{5040} + F_{7,0,1,0,0,0} \frac{x_1^7 x_3}{5040} + F_{7,0,0,1,0,0} \frac{x_1^7 x_4}{5040}
\end{aligned}
$$

$$+ F_{7,0,0,0,1,0}\,\frac{x_1^7x_5}{5040} + F_{7,0,0,0,0,1}\,\frac{x_1^7x_6}{5040} + \frac{1}{8}\,x_1^5x_3^3 + \frac{1}{72}\,x_1^6x_4^2 + \frac{5}{2}\,x_1^3x_2^4x_3 + \frac{15}{8}\,x_1^4x_2^2x_3^2$$
$$+ \frac{5}{6}\,x_1^4x_2^3x_4 + \frac{1}{8}\,x_1^5x_2^2x_5 + \frac{1}{48}\,x_1^6x_3x_5 + \frac{1}{2}\,x_1^5x_2x_3x_4 + \frac{1}{2}\,x_1^2x_2^6 + \frac{1}{120}\,x_1^6x_2x_6,$$

the stability group is:

$$\begin{bmatrix}
a_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -a_{1,1}a_{2,1} \\
a_{2,1} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2}a_{2,1}^2 - \frac{2}{3}a_{1,1}a_{3,1} \\
a_{3,1} & \mathbf{0} & \frac{1}{a_{1,1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -a_{2,1}a_{3,1} - \frac{1}{2}a_{1,1}a_{4,1} \\
a_{4,1} & \mathbf{0} & -\frac{2a_{2,1}}{a_{1,1}^2} & \frac{1}{a_{1,1}^2} & \mathbf{0} & \mathbf{0} & -a_{2,1}a_{4,1} - \frac{2}{3}a_{3,1}^2 - \frac{2}{5}a_{1,1}a_{5,1} \\
a_{5,1} & \mathbf{0} & \frac{5a_{2,1}^2}{a_{1,1}^3} - \frac{10}{3}\frac{a_{3,1}}{a_{1,1}^2} & -\frac{5a_{2,1}}{a_{1,1}^3} & \frac{1}{a_{1,1}^3} & \mathbf{0} & -a_{2,1}a_{5,1} - \frac{1}{3}a_{1,1}a_{6,1} - \frac{5}{3}a_{3,1}a_{4,1} \\
a_{6,1} & \mathbf{0} & 20\frac{a_{2,1}a_{3,1}}{a_{1,1}^3} - 15\frac{a_{2,1}^3}{a_{1,1}^4} - 5\frac{a_{4,1}}{a_{2,1}^2} & \frac{45}{2}\frac{a_{2,1}^2}{a_{1,1}^4} - 10\frac{a_{3,1}}{a_{1,1}^3} & -9\frac{a_{2,1}}{a_{1,1}^4} & \frac{1}{a_{1,1}^4} & b_6 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{1,1}^2
\end{bmatrix}^{\mathbf{7}},$$

and its action gives:

$$\begin{aligned}
0 \overset{800000}{=} &- \frac{1}{40320}\,a_{1,1}^2\,F_{8,0,0,0,0,0,0} + \frac{1}{40320}\,a_{1,1}^7\,G_{8,0,0,0,0,0} + \\
&+ \frac{1}{5040}\,G_{7,1,0,0,0,0}\,a_{1,1}^7\,a_{2,1} + \frac{1}{5040}\,G_{7,0,1,0,0,0}\,a_{1,1}^7\,a_{3,1} + \frac{1}{5040}\,G_{7,0,0,1,0,0}\,a_{1,1}^7\,a_{4,1} \\
&+ \frac{1}{5040}\,G_{7,0,0,0,1,0}\,a_{1,1}^7\,a_{5,1} + \frac{1}{5040}\,G_{7,0,0,0,0,1}\,a_{1,1}^7\,a_{6,1} \\
&+ \frac{1}{1152}\,a_{1,1}^6\,a_{4,1}^2 + \frac{1}{1440}\,a_{1,1}^6\,a_{2,1}\,a_{6,1} + \frac{1}{720}\,a_{1,1}^6\,a_{3,1}\,a_{5,1} + \frac{1}{1440}\,a_{1,1}^6\,\boxed{b_6} \\
0 \overset{71000}{=} &- \frac{1}{5040}\,a_{1,1}^2\,F_{7,1,0,0,0,0} + \frac{1}{5040}\,a_{1,1}^7\,G_{7,1,0,0,0,0}, \\
0 \overset{701000}{=} &- \frac{1}{5040}\,a_{1,1}^2\,F_{7,0,1,0,0,0} + \frac{1}{5040}\,a_{1,1}^6\,G_{7,0,1,0,0,0} - \frac{1}{144}\,a_{1,1}^4\,a_{2,1}\,a_{4,1} + \frac{1}{48}\,a_{1,1}^3\,a_{2,1}^2\,a_{3,1} \\
&- \frac{1}{96}\,a_{1,1}^2\,a_{2,1}^4 - \frac{1}{216}\,a_{1,1}^4\,a_{3,1}^2 - \frac{1}{2520}\,a_{1,1}^5\,a_{2,1}\,G_{7,0,0,1,0,0} \\
&+ \frac{1}{252}\,a_{1,1}^4\,a_{2,1}\,a_{3,1}\,G_{7,0,0,0,0,1} - \frac{1}{336}\,a_{1,1}^3\,a_{2,1}^3\,G_{7,0,0,0,0,1} \\
&- \frac{1}{1008}\,a_{1,1}^4\,a_{4,1}\,G_{7,0,0,0,0,1} - \frac{1}{1008}\,a_{1,1}^4\,a_{2,1}^2\,G_{7,0,0,0,1,0} \\
&- \frac{1}{1512}\,a_{1,1}^5\,a_{3,1}\,G_{7,0,0,0,1,0} + \frac{1}{720}\,a_{1,1}^5\,\boxed{a_{5,1}}, \\
0 \overset{700100}{=} &- \frac{1}{5040}\,a_{1,1}^2\,F_{7,0,0,1,0,0} + \frac{1}{5040}\,a_{1,1}^5\,G_{7,0,0,1,0,0} + \frac{1}{224}\,a_{1,1}^3\,a_{2,1}^2\,G_{7,0,0,0,0,1} \\
&- \frac{1}{504}\,a_{1,1}^4\,a_{3,1}\,G_{7,0,0,0,0,1} + \frac{1}{48}\,a_{1,1}^2\,a_{2,1}^3 - \frac{1}{48}\,a_{1,1}^3\,a_{2,1}\,a_{3,1} \\
&- \frac{1}{1008}\,a_{1,1}^4\,a_{2,1}\,G_{7,0,0,0,1,0} + \frac{1}{288}\,a_{1,1}^4\,\boxed{a_{4,1}}, \\
0 \overset{700010}{=} &- \frac{1}{5040}\,a_{1,1}^2\,F_{7,0,0,0,1,0} + \frac{1}{5040}\,a_{1,1}^4\,G_{7,0,0,0,1,0} \\
&- \frac{1}{560}\,a_{1,1}^3\,a_{2,1}\,G_{7,0,0,0,0,1} - \frac{1}{80}\,a_{1,1}^2\,a_{2,1}^2 + \frac{1}{240}\,a_{1,1}^3\,\boxed{a_{3,1}}, \\
0 \overset{700001}{=} &- \frac{1}{5040}\,a_{1,1}^2\,F_{7,0,0,0,0,1} + \frac{1}{5040}\,a_{1,1}^3\,G_{7,0,0,0,0,1} + \frac{1}{360}\,a_{1,1}^2\,\boxed{a_{2,1}}.
\end{aligned}$$

The free group parameters $a_{2,1}$, $a_{3,1}$, $a_{4,1}$, $a_{5,1}$, b_4 can be used to normalize $G_{7,0,0,0,0,1} := 0$, $G_{7,0,0,0,1,0} := 0$, $G_{7,0,0,1,0,0} := 0$, $G_{7,0,1,0,0,0} := 0$, $G_{8,0,0,0,0,0} := 0$.

Instead of attempting to *dominate* the combinatorics of such formulas in any dimension $n \geqslant 2$, we shall *infinitesimalize* the determination of the stability group at order $n+1$, and also, we shall *infinitesimalize* its action on coefficients of order $n+2$.

16. Tangency at order 2 in dimension n

Before starting, in any dimension $n \geqslant 2$ and in continuation with Section 7, let us examine the tangency of L up to order 2 to the hypersurface $u = \frac{1}{2}\,x_1^2 + \mathrm{O}_x(3)$. Thus, in (7.2), we let $T_1 := 0, \dots, T_n := 0$:

$$
\begin{aligned}
L = & \Big(A_{1,1}\,x_1 + \cdots + A_{1,n}\,x_n + B_1\,u\Big)\,\frac{\partial}{\partial x_1} \\
& + \Big(A_{2,1}\,x_1 + \cdots + A_{2,n}\,x_n + B_2\,u\Big)\,\frac{\partial}{\partial x_2} \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \Big(A_{n,1}\,x_1 + \cdots + A_{n,n}\,x_n + B_n\,u\Big)\,\frac{\partial}{\partial x_n} \\
& + \Big(\qquad\qquad\qquad\qquad\qquad D\,u\Big)\,\frac{\partial}{\partial u}.
\end{aligned}
$$

Lemma 16.1. *Tangency $\pi^2\big(L(-u+F)\big|_{u=F}\big)$ up to order 2 holds if and only if the coefficients matrix of L reads:*

$$
\begin{bmatrix}
A_{1,1} & \mathbf{0} & \cdots & \mathbf{0} & B_1 \\
A_{2,1} & A_{2,2} & \cdots & A_{2,n} & B_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{n,1} & A_{n,2} & \cdots & A_{n,n} & B_n \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 2A_{1,1}
\end{bmatrix}^{\mathbf{2}}.
$$

Proof. We write the graph as $u = \frac{1}{2}\,x_1^2 + \mathrm{O}_x(3)$, and compute modulo $\mathrm{O}_x(3)$:

$$
\begin{aligned}
0 \equiv & -D\,\big(\frac{1}{2}\,x_1^2 + \mathrm{O}_x(3)\big) \\
& + \Big(A_{1,1}\,x_1 + A_{1,2}\,x_2 + \cdots + A_{1,n}\,x_n + B_1\,\mathrm{O}_x(2)\Big)\,\big(x_1 + \mathrm{O}_x(2)\big) \\
& + \Big(A_{2,1}\,x_1 + A_{2,2}\,x_2 + \cdots + A_{2,n}\,x_n + B_2\,\mathrm{O}_x(2)\Big)\,\big(0 + \mathrm{O}_x(2)\big) \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \Big(A_{n,1}\,x_1 + A_{n,2}\,x_2 + \cdots + A_{n,n}\,x_n + B_n\,\mathrm{O}_x(2)\Big)\,\big(0 + \mathrm{O}_x(2)\big).
\end{aligned}
$$

The coefficients of x_1^2, of x_1x_2, ..., of x_1x_n must vanish, which concludes:

$$
D = 2\,A_{1,1}, \qquad A_{1,2} = \mathbf{0}, \qquad \dots\dots, \qquad A_{1,n} = \mathbf{0}.
$$

□

17. Tangency in dimensions $n = 2, 3, 4, 5, 6$

Before proceeding to treating the crucial order $n+1$ in general dimension $n \geqslant 2$, let us show what formulae exist in low dimensions.

• In dimension $n=2$, with

$$
\begin{aligned}
u &= \frac{x_1^2}{2}+\frac{x_1^2x_2}{2},\\
L &= \big(A_{1,1}\,x_1+A_{1,2}\,x_2+B_1\,u\big)\,\partial_{x_1}+\big(A_{2,1}\,x_1+A_{2,2}\,x_2+B_2\,u\big)\,\partial_{x_2}+\big(C_1\,x_1+C_2\,x_2+D\,u\big)\,\partial_u,
\end{aligned}
$$

the tangency equation in orders not exceeding 3

$$
\begin{aligned}
0 \equiv & -C_1\,x_1-C_2\,x_2-D\,\bigg(\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}\bigg)\\
&+\bigg(A_{1,1}\,x_1+A_{1,2}\,x_2+B_1\,\frac{x_1^2}{2}\bigg)\,\big(x_1+x_1x_2\big)\\
&+\big(A_{2,1}\,x_1+A_{2,2}\,x_2\big)\,\big(\tfrac{x_1^2}{2}\big),
\end{aligned}
$$

gives at order 1 as we know

$$
C_1 := 0, \qquad C_2 := 0,
$$

then at order 2 as we know

$$
D := 2\,A_{1,1}, \qquad A_{1,2} := 0,
$$

and it remains

$$
\begin{aligned}
0 &\equiv -2\,A_{1,1}\,\big(\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}\big)+\bigg(A_{1,1}\,x_1+B_1\,\frac{x_1^2}{2}\bigg)\,\big(x_1+x_1x_2\big)+\big(A_{2,1}\,x_1+A_{2,2}\,x_2\big)\,\frac{x_1^2}{2}\\
&\equiv x_1^3\,\bigg[\frac{1}{2}\,B_1+\frac{1}{2}\,A_{2,1}\bigg]+x_1^2x_2\,\bigg[-A_{1,1}+A_{1,1}+\frac{1}{2}\,A_{2,2}\bigg],
\end{aligned}
$$

which gives at order 3:

$$
B_1 := -A_{2,1}, \qquad A_{2,2} := 0.
$$

The reductions of the coefficients matrix of L read:

$$
\begin{bmatrix} A_{1,1} & A_{1,2} & B_1\\ A_{2,1} & A_{2,2} & B_2\\ C_1 & C_2 & D\end{bmatrix}^0 \rightsquigarrow \begin{bmatrix} A_{1,1} & A_{1,2} & B_1\\ A_{2,1} & A_{2,2} & B_2\\ \mathbf{0} & \mathbf{0} & D\end{bmatrix}^1 \rightsquigarrow \begin{bmatrix} A_{1,1} & \mathbf{0} & B_1\\ A_{2,1} & A_{2,2} & B_2\\ \mathbf{0} & \mathbf{0} & 2A_{1,1}\end{bmatrix}^2 \rightsquigarrow \begin{bmatrix} A_{1,1} & \mathbf{0} & -A_{2,1}\\ A_{2,1} & \mathbf{0} & B_2\\ \mathbf{0} & \mathbf{0} & D\end{bmatrix}^3.
$$

• In dimension $n=3$, with

$$
\begin{aligned}
u &= \frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\frac{x_1^3x_3}{6}+\frac{x_1^2x_2^2}{2},\\
L &= \big(A_{1,1}\,x_1+A_{1,2}\,x_2+A_{1,3}\,x_3+B_1\,u\big)\,\partial_{x_1}+\big(A_{2,1}\,x_1+A_{2,2}\,x_2+A_{2,3}\,x_3+B_2\,u\big)\,\partial_{x_2}\\
&\quad+\big(A_{3,1}\,x_1+A_{3,2}\,x_2+A_{3,3}\,x_3+B_3\,u\big)\,\partial_{x_3}+\big(C_1\,x_1+C_2\,x_2+C_3\,x_3+D\,u\big)\,\partial_u,
\end{aligned}
$$

the tangency equation up to order 4, the expansion of which can be done, is

$$
\begin{aligned}
0 \equiv & -C_1\,x_1-C_2\,x_2-C_3\,x_3-D\,\bigg(\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\frac{x_1^3x_3}{6}\bigg)\\
&+\bigg(A_{1,1}x_1+A_{1,2}x_2+A_{1,3}x_3+B_1\,\bigg[\frac{x_1^2}{2}+\tfrac{x_1^2x_2}{2}\bigg]\bigg)\,\bigg(x_1+x_1x_2+\frac{x_1^2x_3}{2}\bigg)\\
&+\bigg(A_{2,1}x_1+A_{2,2}x_2+A_{2,3}x_3+B_2\,\bigg[\frac{x_1^2}{2}\bigg]\bigg)\,\bigg(\frac{x_1^2}{2}+x_1^2x_2\bigg)\\
&+\big(A_{3,1}x_1+A_{3,2}x_2+A_{3,3}x_3\big)\,\frac{x_1^3}{6}.
\end{aligned}
$$

Starting from order 2 thanks to Lemma 16.1, the matrix reductions read

$$\begin{bmatrix} A_{1,1} & \mathbf{0} & \mathbf{0} & B_1 \\ A_{2,1} & A_{2,2} & A_{2,3} & B_2 \\ A_{3,1} & A_{3,2} & A_{3,3} & B_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2A_{1,1} \end{bmatrix}^2 \rightsquigarrow \begin{bmatrix} A_{1,1} & \mathbf{0} & \mathbf{0} & -A_{2,1} \\ A_{2,1} & \mathbf{0} & \mathbf{0} & B_2 \\ A_{3,1} & A_{3,2} & A_{3,3} & B_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2A_{1,1} \end{bmatrix}^3$$

$$\rightsquigarrow \begin{bmatrix} A_{1,1} & \mathbf{0} & \mathbf{0} & -A_{2,1} \\ A_{2,1} & \mathbf{0} & \mathbf{0} & -\frac{2}{3}A_{3,1} \\ A_{3,1} & \mathbf{0} & -A_{1,1} & B_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2A_{1,1} \end{bmatrix}^4.$$

- In dimension $n = 4$, with

$$\begin{aligned} u &= \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \frac{x_1^3 x_3}{6} + \frac{x_1^2 x_2^2}{2} + \frac{x_1^4 x_4}{24} + \frac{x_1^2 x_2^3}{2} + \frac{x_1^3 x_2 x_3}{2}, \\ L &= \big(A_{1,1}x_1 + A_{1,2}x_2 + A_{1,3}x_3 + A_{1,4}x_4 + B_1 u\big)\partial_{x_1} \\ &\quad + \big(A_{2,1}x_1 + A_{2,2}x_2 + A_{2,3}x_3 + A_{2,4}x_4 + B_2 u\big)\partial_{x_2} \\ &\quad + \big(A_{3,1}x_1 + A_{3,2}x_2 + A_{3,3}x_3 + A_{3,4}x_4 + B_3 u\big)\partial_{x_3} \\ &\quad + \big(A_{4,1}x_1 + A_{4,2}x_2 + A_{4,3}x_3 + A_{4,4}x_4 + B_4 u\big)\partial_{x_4} \\ &\quad + \big(C_1 x_1 + C_2 x_2 + C_3 x_3 + C_4 x_4 + D u\big)\partial_u, \end{aligned}$$

one can confirm that up to order 4, the same equations appear as in dimension $n = 3$, and hence we can replace the coefficients of L from what has been obtained just above, so that the tangency equation up to order 5 becomes

$$\begin{aligned} 0 \equiv &- 2\,A_{1,1}\left(\frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \frac{x_1^3 x_3}{6} + \frac{x_1^4 x_4}{24}\right) \\ &+ \left(A_{1,1}\,x_1 - A_{2,1}\left[\frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \frac{x_1^3 x_3}{6}\right]\right)\left(x_1 + x_1 x_2 + \frac{x_1^2 x_3}{2} + \frac{x_1^3 x_4}{6}\right) \\ &+ \left(A_{2,1}\,x_1 - \frac{2}{3}\,A_{3,1}\left[\frac{x_1^2}{2} + \frac{x_1^2 x_2}{2}\right]\right)\left(\frac{x_1^2}{2} + x_1^2 x_2 + \frac{x_1^3 x_3}{2}\right) \\ &+ \left(A_{3,1}\,x_1 - A_{1,1}\,x_3 + B_3\left[\frac{x_1^2}{2}\right]\right)\left(\frac{x_1^3}{6} + \frac{x_1^3 x_2}{2}\right) \\ &+ \big(A_{4,1}\,x_1 + A_{4,2}\,x_2 + A_{4,3}\,x_3 + A_{4,4}\,x_4\big)\left(\frac{x_1^4}{24}\right). \end{aligned}$$

By expanding this equation, one can see that the matrix reduction at order 5 reads:

$$\begin{bmatrix} A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{2}A_{2,1} \\ A_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{3}A_{3,1} \\ A_{3,1} & \mathbf{0} & -A_{1,1} & \mathbf{0} & B_3 \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & B_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2A_{1,1} \end{bmatrix}^4 \rightsquigarrow \begin{bmatrix} A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{2}A_{2,1} \\ A_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{3}A_{3,1} \\ A_{3,1} & \mathbf{0} & -A_{1,1} & \mathbf{0} & -\frac{2}{4}A_{4,1} \\ A_{4,1} & \mathbf{0} & -2A_{2,1} & -2A_{1,1} & B_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2A_{1,1} \end{bmatrix}^5.$$

- In dimension $n = 5$, with:

$$\begin{aligned} u &= \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \frac{x_1^3 x_3}{6} + \frac{x_1^2 x_2^2}{2} + \frac{x_1^4 x_4}{24} + \frac{x_1^2 x_2^3}{2} + \frac{x_1^3 x_2 x_3}{2} \\ &\quad + \frac{x_1^5 x_5}{120} + \frac{x_1^2 x_2^4}{2} + x_1^3 x_2^2 x_3 + \frac{x_1^4 x_2 x_4}{6} + \frac{x_1^4 x_3^2}{8}, \end{aligned}$$

the tangency equation up to order 5 is

$$
\begin{aligned}
0 \equiv & -2\,A_{1,1}\left(\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\frac{x_1^3x_3}{6}+\frac{x_1^4x_4}{24}+\frac{x_1^5x_5}{120}\right)\\
& +\left(A_{1,1}\,x_1-A_{2,1}\left[\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\frac{x_1^3x_3}{6}+\frac{x_1^4x_4}{24}\right]\right)\left(x_1+x_1x_2+\frac{x_1^2x_3}{2}+\frac{x_1^3x_4}{6}+\frac{x_1^4x_5}{24}\right)\\
& +\left(A_{2,1}\,x_1-\frac{2}{3}\,A_{3,1}\left[\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\frac{x_1^3x_3}{6}\right]\right)\left(\frac{x_1^2}{2}+x_1^2x_2+\frac{x_1^3x_3}{2}+\frac{x_1^4x_4}{6}\right)\\
& +\left(A_{3,1}\,x_1-A_{1,1}\,x_3-\frac{2}{4}\,A_{4,1}\left[\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}\right]\right)\left(\frac{x_1^3}{6}+\frac{x_1^3x_2}{2}+\frac{x_1^4x_3}{4}\right)\\
& +\left(A_{4,1}\,x_1-2\,A_{2,1}\,x_3-2\,A_{1,1}\,x_4+B_4\left[\frac{x_1^2}{2}\right]\right)\left(\frac{x_1^4}{24}+\frac{x_1^4x_2}{6}\right)\\
& +\left(A_{5,1}\,x_1+A_{5,2}\,x_2+A_{5,3}\,x_3+A_{5,4}\,x_4+A_{5,5}\,x_5\right)\frac{x_1^5}{120},
\end{aligned}
$$

and the matrix reduction is

$$
\begin{bmatrix}
A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{2}A_{2,1}\\
A_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{3}A_{3,1}\\
A_{3,1} & \mathbf{0} & -A_{1,1} & \mathbf{0} & \mathbf{0} & -\frac{2}{4}A_{4,1}\\
A_{4,1} & \mathbf{0} & -2A_{2,1} & -2A_{1,1} & \mathbf{0} & B_4\\
A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & B_5\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2A_{1,1}
\end{bmatrix}^5
$$

$$
\rightsquigarrow
\begin{bmatrix}
A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{2}A_{2,1}\\
A_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{3}A_{3,1}\\
A_{3,1} & \mathbf{0} & -A_{1,1} & \mathbf{0} & \mathbf{0} & -\frac{2}{4}A_{4,1}\\
A_{4,1} & \mathbf{0} & -2A_{2,1} & -2A_{1,1} & \mathbf{0} & -\frac{2}{5}A_{5,1}\\
A_{5,1} & \mathbf{0} & -\frac{10}{3}A_{3,1} & -5A_{2,1} & -3A_{1,1} & B_5\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2A_{1,1}
\end{bmatrix}^6.
$$

- In dimension $n=6$, with:

$$
\begin{aligned}
u =& \frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\frac{x_1^3x_3}{6}+\frac{x_1^2x_2^2}{2}+\frac{x_1^4x_4}{24}+\frac{x_1^2x_2^3}{2}+\frac{x_1^3x_2x_3}{2}+\frac{x_1^5x_5}{120}+\frac{x_1^2x_2^4}{2}+x_1^3x_2^2x_3+\frac{x_1^4x_2x_4}{6}\\
&+\frac{x_1^4x_3^2}{8}+\frac{x_1^6x_6}{720}+\frac{x_1^2x_2^5}{2}+\frac{5}{3}x_1^3x_2^3x_3+\frac{5}{8}x_1^4x_2x_3^2+\frac{5}{12}x_1^4x_2^2x_4+\frac{1}{12}x_1^5x_3x_4+\frac{1}{24}x_1^5x_2x_5,
\end{aligned}
$$

the tangency equation up to order 6 is

$$
\begin{aligned}
0 \equiv & -2\,A_{1,1}\left(\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\frac{x_1^3x_3}{6}+\frac{x_1^4x_4}{24}+\frac{x_1^5x_5}{120}+\frac{x_1^6x_6}{720}\right)\\
& +\left(A_{1,1}\,x_1-A_{2,1}\left[\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\frac{x_1^3x_3}{6}+\frac{x_1^4x_4}{24}+\frac{x_1^5x_5}{120}\right]\right)\\
& \left(x_1+x_1x_2+\frac{x_1^2x_3}{2}+\frac{x_1^3x_4}{6}+\frac{x_1^4x_5}{24}+\frac{x_1^5x_6}{120}\right)\\
& +\left(A_{2,1}\,x_1-\frac{2}{3}\,A_{3,1}\left[\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\frac{x_1^3x_3}{6}+\frac{x_1^4x_4}{24}\right]\right)\left(\frac{x_1^2}{2}+x_1^2x_2+\frac{x_1^3x_3}{2}+\frac{x_1^4x_4}{6}+\frac{x_1^5x_5}{24}\right)
\end{aligned}
$$

$$+\left(A_{3,1}x_1 - A_{1,1}x_3 - \frac{2}{4}\,A_{4,1}\left[\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\frac{x_1^3x_3}{6}\right]\right)\left(\frac{x_1^3}{6}+\frac{x_1^3x_2}{2}+\frac{x_1^4x_3}{4}+\frac{x_1^5x_4}{12}\right)$$
$$+\left(A_{4,1}\,x_1 - 2\,A_{2,1}\,x_3 - 2\,A_{1,1}\,x_4 - \frac{2}{5}\,A_{5,1}\left[\frac{x_1^2}{2}+\frac{x_1^2x_2}{2}\right]\right)\left(\frac{x_1^4}{24}+\frac{x_1^4x_2}{6}+\frac{x_1^5x_3}{12}\right)$$
$$+\left(A_{5,1}\,x_1 - \frac{10}{3}\,A_{3,1}\,x_3 - 5\,A_{2,1}\,x_4 - 3\,A_{1,1}\,x_5 + B_5\left[\frac{x_1^2}{2}\right]\right)\left(\frac{x_1^5}{120}+\frac{x_1^5x_2}{24}\right)$$
$$+\left(A_{6,1}\,x_1 + A_{6,2}\,x_2 + A_{6,3}\,x_3 + A_{6,4}\,x_4 + A_{6,5}\,x_5 + A_{6,6}\,x_6\right)\left(\frac{x_1^6}{720}\right),$$

and the matrix reduction is

$$\begin{bmatrix}
A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{2}A_{2,1}\\
A_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{3}A_{3,1}\\
A_{3,1} & \mathbf{0} & -A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{4}A_{4,1}\\
A_{4,1} & \mathbf{0} & -2A_{2,1} & -2A_{1,1} & \mathbf{0} & \mathbf{0} & -\frac{2}{5}A_{5,1}\\
A_{5,1} & \mathbf{0} & -\frac{10}{3}A_{3,1} & -5A_{2,1} & -3A_{1,1} & \mathbf{0} & B_5\\
A_{6,1} & A_{6,2} & A_{6,3} & A_{6,4} & A_{6,5} & A_{6,6} & B_6\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2A_{1,1}
\end{bmatrix}^6$$
$$\rightsquigarrow\begin{bmatrix}
A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{2}A_{2,1}\\
A_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{3}A_{3,1}\\
A_{3,1} & \mathbf{0} & -A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{4}A_{4,1}\\
A_{4,1} & \mathbf{0} & -2A_{2,1} & -2A_{1,1} & \mathbf{0} & \mathbf{0} & -\frac{2}{5}A_{5,1}\\
A_{5,1} & \mathbf{0} & -\frac{10}{3}A_{3,1} & -5A_{2,1} & -3A_{1,1} & \mathbf{0} & -\frac{2}{6}A_{6,1}\\
A_{6,1} & \mathbf{0} & -5A_{4,1} & -10A_{3,1} & -9A_{2,1} & -4A_{1,1} & B_6\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2A_{1,1}
\end{bmatrix}^7.$$

18. Projections $\pi^m(\bullet)$ and $\pi^m_{\mathrm{ind}}(\bullet)$

For each integer $m \geqslant 1$ and each power series vanishing at the origin

$$E(x) \,=\, E(x_1,\dots,x_n) \,=\, \sum_{\substack{\sigma_1,\dots,\sigma_n\geqslant 0\\ \sigma_1+\cdots+\sigma_n\geqslant 1}} x_1^{\sigma_1}\cdots x_n^{\sigma_n}\,E_{\sigma_1,\dots,\sigma_n},$$

we define:

$$\pi^m\left(E(x)\right) := \sum_{\sigma_1+\cdots+\sigma_n\leqslant m} x_1^{\sigma_1}\cdots x_n^{\sigma_n}\,E_{\sigma_1,\dots,\sigma_n}.$$

Several times later, the following elementary fact will be employed.

Observation 18.1. *Given two integers $k_1, k_2 \in \mathbb{N}$ and two power series $H_1(x) \in \mathrm{O}_x(k_1)$ and $H_2(x) \in \mathrm{O}_x(k_2)$, namely,*

$$H_1(x) \,=\, \sum_{\sigma_1+\cdots+\sigma_n\geqslant k_1} H_{1,\sigma_1,\dots,\sigma_n}\,x_1^{\sigma_1}\cdots x_n^{\sigma_n}, \qquad H_2(x) \,=\, \sum_{\sigma_1+\cdots+\sigma_n\geqslant k_2} H_{2,\sigma_1,\dots,\sigma_n}\,x_1^{\sigma_1}\cdots x_n^{\sigma_n},$$

for each (homogeneous) order $m \geqslant 0$, we have

$$\pi^m\left(H_1\cdot H_2\right) \,=\, \pi^m\Big(\pi^{m-k_2}(H_1)\cdot\pi^{m-k_1}(H_2)\Big),$$

the right-hand side being understood as 0 when $m < \min\left(k_1,k_2\right)$.

Since our hypersurfaces $H^n \subset \mathbb{R}^{n+1}$ have constant Hessian rank 1, independent monomials are of interest. Accordingly, we define

$$\pi_{\rm ind}\,(E(x)) \,:=\, \sum_{i=0}^{\infty} \Big(E_{i+1,0,\dots,0}\, x_1^{i+1} + E_{i,1,\dots,0}\, x_1^i x_2 + \cdots + E_{i,0,\dots,1}\, x_1^i x_n.\Big),$$

and also

$$\pi_{\rm ind}^m \,:=\, \pi^m \circ \pi_{\rm ind} \,=\, \pi_{\rm ind} \circ \pi^m,$$

that is,

$$\pi_{\rm ind}^m\,(E(x)) \,:=\, \sum_{0\leqslant i\leqslant m-1} \Big(E_{i+1,0,\dots,0}\, x_1^{i+1} + E_{i,1,\dots,0}\, x_1^i x_2 + \cdots + E_{i,0,\dots,1}\, x_1^i x_n.\Big).$$

Observation 18.2. *Given two integers $k_1, k_2 \in \mathbb{N}$ and two power series $H_1(x) \in {\rm O}_x(k_1)$ and $H_2(x) \in {\rm O}_x(k_2)$, for each (homogeneous) order $m \geqslant 0$, we have*

$$\pi_{\rm ind}^m\,(H_1 \cdot H_2) \,=\, \pi_{\rm ind}^m\Big(\pi_{\rm ind}^{m-k_2}(H_1)\cdot \pi_{\rm ind}^{m-k_1}(H_2)\Big),$$

the right-hand side being understood as 0 when $m < \min\,(k_1, k_2)$.

We shall need a notation to select *homogeneous* monomials of order exactly equal to some fixed integer $m \geqslant 1$ (notice that the index m is lower-case now):

$$\pi_m\,(E(x)) \,:=\, \sum_{\sigma_1+\cdots+\sigma_n=m} x_1^{\sigma_1}\cdots x_n^{\sigma_n}\, E_{\sigma_1,\dots,\sigma_n},$$

and also to select the *independent* homogeneous ones:

$$\pi_m^{\rm ind}\,(E(x)) \,:=\, E_{m,0,\dots,0}\, x_1^m + E_{m-1,1,\dots,0}\, x_1^{m-1} x_2 + \cdots + E_{m-1,0,\dots,1}\, x_1^{m-1} x_n.$$

19. Tangency at order $n+1$ in dimension n

Now we begin the induction in dimension. In each dimension $n-1$, in coordinates $(x_1, \dots, x_{n-1}, u)$, we know that the hypersurface equation up to order $n-1+1$ reads as

$$\begin{aligned}u &= \frac{x_1^2}{2!} + \frac{x_1^2 x_2}{2!} + \sum_{m=3}^{n-2}\Big(\frac{x_1^m x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j\geqslant 2\\ i+j=m+1}} \frac{1}{2}\,\frac{x_i x_j}{(i-1)!(j-1)!}\Big)\\ &\quad + \frac{x_1^{n-1} x_{n-1}}{(n-1)!} + x_1^{n-2} \sum_{\substack{i,j\geqslant 2\\ i+j=n}} \frac{1}{2}\,\frac{x_i x_j}{(i-1)!(j-1)!} + {\rm O}_{x_2,\dots,x_{n-1}}(2) + {\rm O}_{x_1,x_2,\dots,x_{n-1}}(n+1),\end{aligned}$$

with explicit independent and border-dependent monomials.

A linear (vanishing at the origin) affine vector field reads as

$$L \,=\, X_1\,\frac{\partial}{\partial x_1} + \cdots + X_{n-1}\,\frac{\partial}{\partial x_{n-1}} + U\,\frac{\partial}{\partial u},$$

with:

$$\begin{aligned}X_1 &\,=\, A_{1,1}\, x_1 + \cdots + A_{1,n-1}\, x_{n-1} + B_1\, u,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ X_{n-1} &\,=\, A_{n-1,1}\, x_1 + \cdots + A_{n-1,n-1}\, x_{n-1} + B_{n-1}\, u,\\ U &\,=\, C_1\, x_1 + \cdots + C_{n-1}\, x_{n-1} + D\, u.\end{aligned}$$

We consider tangency of L to such a hypersurface $u = F(x_1, \dots, x_{n-1})$ up to order $n-1$, namely, the condition

$$0 \,=\, \pi^{n-1}\Big(L\,(-u+F)\,\big|_{u=F}\Big),$$

which is equivalent to

$$0 \,=\, \pi_{\rm ind}^{n-1}\Big(L\,(-u+F)\,\big|_{u=F}\Big).$$

We start arguing from Lemma 16.1 at order 2 considered in dimension $n-1$.

Induction Hypothesis 19.1. *The vector space of fields L which are tangent to order not exceeding $n-1$ is of dimension $n-1+1$ parametrized by $A_{1,1}, \dots, A_{n-1,1}, B_{n-1}$, with the other constants defined by the formulae*

$$\underset{(1\leqslant i\leqslant n-1)}{A_{i,2} \,=\, 0}, \qquad \underset{(3\leqslant j\leqslant n-1)}{A_{1,j} \,=\, A_{2,j} \,=\, 0}, \qquad \underset{(3\leqslant j\leqslant i\leqslant n-1)}{A_{i,j} \,=\, \tfrac{-(j-2)}{i-j+1}\tbinom{i}{j}\,A_{i-j+1,1}}, \qquad \underset{(3\leqslant j\leqslant n-2)}{B_j \,=\, -\frac{2}{j+1}\,A_{j+1,1}},$$

or equivalently, the matrix of the coefficients of L is

$$\begin{bmatrix} A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\frac{2}{2}A_{2,1} \\ A_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\frac{2}{3}A_{3,1} \\ A_{3,1} & \mathbf{0} & -A_{1,1} & \mathbf{0} & \cdots & \mathbf{0} & -\frac{2}{4}A_{4,1} \\ A_{4,1} & \mathbf{0} & -2A_{2,1} & -A_{1,1} & \cdots & \mathbf{0} & -\frac{2}{5}A_{4,1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n-2,1} & \mathbf{0} & \frac{-1}{n-4}\binom{n-2}{3}A_{n-4,1} & \frac{-2}{n-5}\binom{n-2}{4}A_{n-5,1} & \cdots & \mathbf{0} & -\frac{2}{n-1}A_{n-1,1} \\ A_{n-1,1} & \mathbf{0} & \frac{-1}{n-3}\binom{n-1}{3}A_{n-3,1} & \frac{-2}{n-4}\binom{n-1}{4}A_{n-4,1} & \cdots & -(n-3)A_{1,1} & B_{n-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 2\,A_{1,1} \end{bmatrix}.$$

The goal is to show that similar expressions hold in dimension n. Thus, in coordinates $(x_1,\dots,x_{n-1},x_n,u)$, we consider a hypersurface, the equation of which up to order $n+1$ reads as

$$\begin{aligned} u \,=\, &\frac{x_1^2}{2!}+\frac{x_1^2x_2}{2!}+\sum_{m=3}^{n-2}\bigg(\frac{x_1^mx_m}{m!}+x_1^{m-1}\sum_{\substack{i,j\geqslant 2\\ i+j=m+1}}\frac{1}{2}\,\frac{x_ix_j}{(i-1)!(j-1)!}\bigg)+\frac{x_1^{n-1}x_{n-1}}{(n-1)!} \\ &+x_1^{n-2}\sum_{\substack{i,j\geqslant 2\\ i+j=n}}\frac{1}{2}\,\frac{x_ix_j}{(i-1)!(j-1)!}+\frac{x_1^nx_n}{n!}+x_1^{n-1}\sum_{\substack{i,j\geqslant 2\\ i+j=n+1}}\frac{1}{2}\,\frac{x_ix_j}{(i-1)!(j-1)!} \\ &+\mathrm{O}_{x_2,\dots,x_{n-1},x_n}(2)+\mathrm{O}_{x_1,x_2,\dots,x_{n-1},x_n}(n+2), \end{aligned}$$

and we consider a linear vector field

$$L \,=\, X_1\,\frac{\partial}{\partial x_1}+\cdots+X_{n-1}\,\frac{\partial}{\partial x_{n-1}}+X_n\,\frac{\partial}{\partial x_n}+U\,\frac{\partial}{\partial u},$$

with:

$$\begin{aligned} X_1 \,&=\, A_{1,1}\,x_1+\cdots+A_{1,n-1}\,x_{n-1}+B_1\,u, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ X_{n-1} \,&=\, A_{n-1,1}\,x_1+\cdots+A_{n-1,n}\,x_n+B_{n-1}\,u, \\ X_n \,&=\, A_{n,1}\,x_1+\cdots+A_{n,n}\,x_n+B_n\,u, \\ U \,&=\, C_1\,x_1+\cdots+C_n\,x_n+D\,u. \end{aligned}$$

Similarly, we consider tangency of L to such hypersurface $u=F(x_1,\dots,x_{n-1},x_n)$ up to order n, namely, the condition

$$0 \,=\, \pi^n\Big(L\,(-u+F)\,\big|_{u=F}\Big),$$

which is equivalent to

$$0 \,=\, \pi_{\rm ind}^n\Big(L\,(-u+F)\,\big|_{u=F}\Big). \tag{19.1}$$

Letting $x_n := 0$, taking only $\pi_{\mathsf{ind}}^{n-1}(\bullet)$, applying the Induction Hypothesis 19.1, we see that the matrix of coefficients of L involves known elements in dimension $n-1$:

$$\begin{bmatrix}
A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & A_{1,n} & -\frac{2}{2}A_{2,1} \\
A_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & A_{2,n} & -\frac{2}{3}A_{3,1} \\
A_{3,1} & \mathbf{0} & -A_{1,1} & \mathbf{0} & \cdots & \mathbf{0} & A_{3,n} & -\frac{2}{4}A_{4,1} \\
A_{4,1} & \mathbf{0} & -2A_{2,1} & -2A_{1,1} & \cdots & \mathbf{0} & A_{4,n} & -\frac{2}{5}A_{4,1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
A_{n-2,1} & \mathbf{0} & \frac{-1}{n-4}\binom{n-2}{3}A_{n-4,1} & \frac{-2}{n-5}\binom{n-2}{4}A_{n-5,1} & \cdots & \mathbf{0} & A_{n-2,n} & -\frac{2}{n-1}A_{n-1,1} \\
A_{n-1,1} & \mathbf{0} & \frac{-1}{n-3}\binom{n-1}{3}A_{n-3,1} & \frac{-2}{n-4}\binom{n-1}{4}A_{n-4,1} & \cdots & -(n-3)A_{1,1} & A_{n-1,n} & B_{n-1} \\
A_{n,1} & A_{n,2} & A_{n,3} & A_{n,4} & \cdots & A_{n-1,n} & A_{n,n} & B_n \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_n & 2\,A_{1,1}
\end{bmatrix}.$$

Without letting $x_n := 0$, by examining $\pi_{\mathsf{ind}}^{n-1}(\bullet)$, one can see that

$$A_{1,n} \,=\, \cdots \,=\, A_{n-1,n} \,=\, \mathbf{0}, \qquad C_n \,=\, \mathbf{0},$$

and the detailed verification of these vanishings is implicitly contained in the proof of the next Lemma 19.1. Hence, the matrix is

$$\begin{bmatrix}
A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\frac{2}{2}A_{2,1} \\
A_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\frac{2}{3}A_{3,1} \\
A_{3,1} & \mathbf{0} & -A_{1,1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\frac{2}{4}A_{4,1} \\
A_{4,1} & \mathbf{0} & -2A_{2,1} & -2A_{1,1} & \cdots & \mathbf{0} & \mathbf{0} & -\frac{2}{5}A_{4,1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
A_{n-2,1} & \mathbf{0} & \frac{-1}{n-4}\binom{n-2}{3}A_{n-4,1} & \frac{-2}{n-5}\binom{n-2}{4}A_{n-5,1} & \cdots & \mathbf{0} & \mathbf{0} & -\frac{2}{n-1}A_{n-1,1} \\
A_{n-1,1} & \mathbf{0} & \frac{-1}{n-3}\binom{n-1}{3}A_{n-3,1} & \frac{-2}{n-4}\binom{n-1}{4}A_{n-4,1} & \cdots & -(n-3)A_{1,1} & \mathbf{0} & B_{n-1} \\
A_{n,1} & A_{n,2} & A_{n,3} & A_{n,4} & \cdots & A_{n-1,n} & A_{n,n} & B_n \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & 2\,A_{1,1}
\end{bmatrix},$$

and in order to complete the induction, we need to determine the values of $A_{n,1}$, $A_{n,2}$, $A_{n,3}$, $A_{n,4}$, $\dots$, $A_{n-1,n}$, $A_{n,n}$, B_{n-1}, as follows.

Lemma 19.1. *The tangency condition* (19.1) *at order* $n+1$ *forces the announced values*

$$B_{n-1} \,=\, -\frac{2}{n}\,A_{n,1}, \qquad A_{n,2} \,=\, \mathbf{0}, \qquad \underset{(3\leqslant k\leqslant n)}{A_{n,k} \,=\, -\,\frac{k-2}{n-k+1}\binom{n}{k}\,A_{n-k+1,1}}.$$

Proof. We abbreviate the hypersurface equation (19.1) including independent and border-dependent monomials up to order $n+1$ as follows:

$$u \,=\, \overline{F}^{n+1}(x) + \mathrm{O}_{x'}(2) + \mathrm{O}_x(n+2),$$

and start to examine the tangency equation:

$$\begin{aligned}
0 &\equiv \pi_{\rm ind}^{n+1}\Big(L\,(-\,u+F)\,\big|_{u=F}\Big)\\
&\equiv \pi_{\rm ind}^{n+1}\bigg(-\,U(x,u)+\sum_{i=1}^{n}X_i(x,u)\cdot F_{x_i}(x)\Big|_{u=F(x)}\bigg)\\
&\equiv \pi_{\rm ind}^{n+1}\bigg(-\,U\Big(x,\overline{F}^{n+2}(x)+{\rm O}_{x'}(2)+{\rm O}_x(n+2)\Big)\\
&\quad+\sum_{i=1}^{n}X_i\Big(x,\overline{F}^{n+2}(x)+{\rm O}_{x'}(2)+{\rm O}_x(n+2)\Big)\cdot\Big(\overline{F}_{x_i}^{n+1}+{\rm O}_{x'}(1)+{\rm O}_x(n+1)\Big)\bigg)\\
&\equiv \pi_{\rm ind}^{n+1}\bigg(-\,U\,\Big(x,\overline{F}^{n+2}(x)\Big)+\sum_{i=1}^{n}X_i\,\Big(x,\overline{F}^{n+2}(x)\Big)\cdot\overline{F}_{x_i}^{n+1}(x)\bigg),
\end{aligned}\tag{19.2}$$

where we use the identity

$$0=\pi_{\mathsf{ind}}^{n+1}\left({\rm O}_{x'}(1)\right)=\pi^{n+1}\left({\rm O}_x(n+2)\right)$$

as well as the fact that $X_i\left(x,\overline{F}^{n+1}(x)\right)$ are all ${\rm O}_x(1)$, hence multiplied by the last remainder ${\rm O}_x(n+1)$ they become ${\rm O}_x(n+2)$.

In two steps, we want to apply Observation 18.2 to the products $X_i\cdot\overline{F}_{x_i}^{n+2}$ above for $i=1,\dots,n$. First, each X_i is an ${\rm O}_x(1)$ and hence,

$$\pi_{\rm ind}^{n+1}\left(X_i\cdot\overline{F}_{x_i}^{n+1}\right)\;=\;\pi_{\rm ind}^{n+1}\Big(\pi_{\rm ind}^{n+1}(X_i)\cdot\pi_{\rm ind}^{n}\left(\overline{F}_{x_i}^{n+1}\right)\Big).$$

Second, the vanishing orders at 0 of $\overline{F}_{x_i}^{n+2}$ increase, as one can confirm by differentiating 19.1:

$$\begin{aligned}
\pi_{\rm ind}^{n}\left(\overline{F}_{x_1}^{n+1}\right)&=\frac{x_1^1}{1!}\bigg|+\frac{x_1^1x_2}{2!}+\cdots+\frac{x_1^{n-1}x_n}{(n-1)!},\\
\pi_{\rm ind}^{n}\left(\overline{F}_{x_2}^{n+1}\right)&=\frac{x_1^2}{2!}\bigg|+\frac{x_1^2x_2}{1!\,1!}+\cdots+\frac{x_1^{n-1}x_{n-1}}{1!\,(n-2)!},\\
\pi_{\rm ind}^{n}\left(\overline{F}_{x_3}^{n+1}\right)&=\frac{x_1^3}{3!}\bigg|+\frac{x_1^3x_2}{2!\,1!}+\cdots+\frac{x_1^{n-1}x_{n-2}}{2!\,(n-3)!},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\pi_{\rm ind}^{n}\left(\overline{F}_{x_{n-3}}^{n+1}\right)&=\frac{x_1^{n-3}}{(n-3)!}\bigg|+\frac{x_1^{n-3}x_2}{(n-4)!\,1!}+\frac{x_1^{n-2}x_3}{(n-4)!\,2!}+\frac{x_1^{n-1}x_4}{(n-4)!\,3!},\\
\pi_{\rm ind}^{n}\left(\overline{F}_{x_{n-2}}^{n+1}\right)&=\frac{x_1^{n-2}}{(n-2)!}\bigg|+\frac{x_1^{n-2}x_2}{(n-3)!\,1!}+\frac{x_1^{n-1}x_3}{(n-3)!\,2!}\\
\pi_{\rm ind}^{n}\left(\overline{F}_{x_{n-1}}^{n+1}\right)&=\frac{x_1^{n-1}}{(n-1)!}\bigg|+\frac{x_1^{n-1}x_2}{(n-2)!\,1!}\\
\pi_{\rm ind}^{n}\left(\overline{F}_{x_n}^{n+1}\right)&=\frac{x_1^{n}}{n!};
\end{aligned}$$

here for later use, the vertical bar separates the pure monomials x_1^* from the monomials $x_1^*x_2,\dots,x_1^*x_n$. Therefore, applying again Observation 18.2, it suffice to know the independent parts of X_1, X_2, X_3, $\dots$, X_{n-3}, X_{n-2}, X_{n-1}, X_n up to decreasing orders:

$$\begin{aligned}
\pi_{\rm ind}^{n+1}\left(X_1\cdot\overline{F}_{x_1}^{n+1}\right)&=\pi_{\rm ind}^{n+1}\Big(\pi_{\rm ind}^{n}(X_1)\cdot\pi_{\rm ind}^{n}\left(\overline{F}_{x_1}^{n+1}\right)\Big),\\
\pi_{\rm ind}^{n+1}\left(X_2\cdot\overline{F}_{x_2}^{n+1}\right)&=\pi_{\rm ind}^{n+1}\Big(\pi_{\rm ind}^{n-1}(X_2)\cdot\pi_{\rm ind}^{n}\left(\overline{F}_{x_2}^{n+1}\right)\Big),
\end{aligned}$$

$$
\begin{aligned}
\pi_{\text{ind}}^{n+1}\Big(X_3\cdot\overline{F}_{x_3}^{n+1}\Big) &= \pi_{\text{ind}}^{n+1}\Big(\pi_{\text{ind}}^{n-2}(X_3)\cdot\pi_{\text{ind}}^{n}\Big(\overline{F}_{x_3}^{n+1}\Big)\Big),\\
&\cdots\cdots\cdots\\
\pi_{\text{ind}}^{n+1}\Big(X_{n-3}\cdot\overline{F}_{x_{n-3}}^{n+1}\Big) &= \pi_{\text{ind}}^{n+1}\Big(\pi_{\text{ind}}^{4}(X_{n-3})\cdot\pi_{\text{ind}}^{n}\Big(\overline{F}_{x_{n-3}}^{n+1}\Big)\Big),\\
\pi_{\text{ind}}^{n+1}\Big(X_{n-2}\cdot\overline{F}_{x_{n-2}}^{n+1}\Big) &= \pi_{\text{ind}}^{n+1}\Big(\pi_{\text{ind}}^{3}(X_{n-2})\cdot\pi_{\text{ind}}^{n}\Big(\overline{F}_{x_{n-2}}^{n+1}\Big)\Big),\\
\pi_{\text{ind}}^{n+1}\Big(X_{n-1}\cdot\overline{F}_{x_{n-1}}^{n+1}\Big) &= \pi_{\text{ind}}^{n+1}\Big(\pi_{\text{ind}}^{2}(X_{n-1})\cdot\pi_{\text{ind}}^{n}\Big(\overline{F}_{x_{n-1}}^{n+1}\Big)\Big),\\
\pi_{\text{ind}}^{n+1}\Big(X_{n}\cdot\overline{F}_{x_{n}}^{n+1}\Big) &= \pi_{\text{ind}}^{n+1}\Big(\pi_{\text{ind}}^{1}(X_{n})\cdot\pi^{n}\Big(\overline{F}_{x_{n}}^{n+1}\Big)\Big).
\end{aligned}
$$

We recall that in all the X_i, before taking $\pi_{\text{ind}}^{n-i+1}(X_i)$ as above, we have to replace u by $\overline{F}^{n+1}(x)$ and for $3\leqslant i\leqslant n-2$ this gives

$$
X_i = X_i\Big(x,\overline{F}^{n+1}(x)\Big) = A_{i,1}x_1 - \sum_{3\leqslant j\leqslant i}\frac{j-2}{i-j+1}\tbinom{i}{j}A_{i-j+1}\,x_j - \frac{2}{i+1}A_{i+1,1}\,\overline{F}^{n+2}(x),
$$

with similar formulae for $i=1,2,n-1$. Fortunately, for all $3\leqslant j\leqslant i$, the values of the coefficients of the x_j will be unimportant, hence we shall abbreviate:

$$
\pi_{\text{ind}}^{n-i+1}(X_i) = A_{i,1}\,x_1 - *\,x_3 - \cdots - *\,x_i - \frac{2}{i+1}A_{i+1,1}\left[\frac{x_1^2}{2!}\bigg| + \frac{x_1^2x_2}{2!} + \cdots + \frac{x_1^{n-i}x_{n-i}}{(n-i)!}\right].
$$

In X_n, there is also no contribution of $B_n\,u\equiv B_n\,\overline{F}^{n+1}$ since $\overline{F}^{n+1}=\mathrm{O}_x(2)$:

$$
\begin{aligned}
\pi_{\text{ind}}^{1}(X_n) &= \pi_{\text{ind}}^{1}\Big(A_{n,1}\,x_1+\cdots+A_{n,n}\,x_n+B_n\,\mathrm{O}_x(2)\Big)\\
&= A_{n,1}\,x_1+\cdots+A_{n,n}\,x_n.
\end{aligned}
$$

Finally, we can write in length all the terms of the tangency equation (19.2), as follows without mentionining $\pi_{\text{ind}}^{n+1}(\bullet)$:

$$
\begin{aligned}
0 \equiv &-2\,A_{1,1}\left(\frac{x_1^2}{2!}+\frac{x_1^2x_2}{2!}+\cdots+\frac{x_1^nx_n}{n!}\right)\\
&+\left(A_{1,1}x_1-\frac{2}{2}A_{2,1}\left[\frac{x_1^2}{2!}\bigg|+\frac{x_1^2x_2}{2!}+\cdots+\frac{x_1^{n-1}x_{n-1}}{(n-1)!}\right]\right)\\
&\cdot\left(\frac{x_1^1}{1!}\bigg|+\frac{x_1^2x_2}{1!}+\cdots+\frac{x_1^{n-2}x_{n-1}}{(n-2)!}+\frac{x_1^{n-1}x_n}{(n-1)!}\right)\\
&+\left(A_{2,1}x_1-\frac{2}{3}A_{3,1}\left[\frac{x_1^2}{2!}\bigg|+\frac{x_1^2x_2}{2!}+\cdots+\frac{x_1^{n-2}x_{n-2}}{(n-2)!}\right]\right)\\
&\cdot\left(\frac{x_1^2}{2!}\bigg|+\frac{x_1^2x_2}{1!\,1!}+\cdots+\frac{x_1^{n-2}x_{n-2}}{1!\,(n-3)!}+\frac{x_1^{n-1}x_{n-1}}{1!\,(n-2)!}\right)\\
&+\left(A_{3,1}x_1-*x_3-\frac{2}{4}A_{4,1}\left[\frac{x_1^2}{2!}\bigg|+\frac{x_1^2x_2}{2!}+\cdots+\frac{x_1^{n-3}x_{n-3}}{(n-3)!}\right]\right)\\
&\cdot\left(\frac{x_1^3}{3!}\bigg|+\frac{x_1^3x_2}{2!\,1!}+\cdots+\frac{x_1^{n-2}x_{n-3}}{2!\,(n-4)!}+\frac{x_1^{n-1}x_{n-2}}{2!\,(n-3)!}\right)\\
&+\cdots\cdots\cdots\\
&+\left(A_{i,1}x_1-*x_3-\cdots-*x_i-\frac{2}{i+1}A_{i+1,1}\left[\frac{x_1^2}{2!}\bigg|+\frac{x_1^2x_2}{2!}+\cdots+\frac{x_1^{n-i}x_{n-i}}{(n-i)!}\right]\right)
\end{aligned}
$$

$$\cdot\left(\frac{x_1^i}{i!}\bigg| + \frac{x_1^i x_2}{(i-1)!1!} + \cdots + \frac{x_1^{n-2}x_{n-i}}{(i-1)!(n-i-1)!} + \frac{x_1^{n-1}x_{n-i+1}}{(i-1)!(n-i)!}\right)$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\left(A_{n-3,1}x_1 - *x_3 - \cdots - *x_{n-3} - \frac{2}{n-2}A_{n-2,1}\left[\frac{x_1^2}{2!}\bigg| + \frac{x_1^2x_2}{2!} + \frac{x_1^3x_3}{3!}\right]\right)$$

$$\cdot\left(\frac{x_1^{n-3}}{(n-3)!}\bigg| + \frac{x_1^{n-3}x_2}{(n-4)!1!} + \frac{x_1^{n-2}x_3}{(n-4)!2!} + \frac{x_1^{n-1}x_4}{(n-4)!3!}\right)$$

$$+\left(A_{n-2,1}x_1 - *x_3 - \cdots - *x_{n-2} - \frac{2}{n-1}A_{n-1,1}\left[\frac{x_1^2}{2!}\bigg| + \frac{x_1^2x_2}{2!}\right]\right)$$

$$\cdot\left(\frac{x_1^{n-2}}{(n-2)!}\bigg| + \frac{x_1^{n-2}x_2}{(n-3)!1!} + \frac{x_1^{n-1}x_3}{(n-3)!2!}\right)$$

$$+\left(A_{n-1,1}x_1 - *x_3 - \cdots - *x_{n-1} + B_{n-1}\left[\frac{x_1^2}{2!}\right]\right)\left(\frac{x_1^{n-1}}{(n-1)!}\bigg| + \frac{x_1^{n-1}x_2}{(n-2)!1!}\right)$$

$$+\left(A_{n,1}x_1 + \cdots + A_{n,k}x_k + \cdots + A_{n,n}x_n\right)\left(\frac{x_1^n}{n!}\right).$$

We know by the induction assumption that $\pi_{\rm ind}^n(\bullet)$ applied to this gives zero. Thus, when applying $\pi_{\rm ind}^{n+1}(\bullet)$, it suffices to collect all independent monomials of order $n+1$, namely $x_1^n x_1$, $x_1^n x_2, \dots, x_1^n x_n$, and to determine the coefficients of these monomials, which should all vanish.

Let us explain how to determine the coefficient of a general 'intermediate' monomial $x_1^k x_k$ with $2 \leqslant k \leqslant n-1$:

- the line $i = n-k$ contributes *two* terms;
- the line $i = n-k+1$ contributes *one* term;
- the line $i = k$ contributes *one* term;

and that is all. Therefore, we obtain:

$$x_1^k x_k\left(-\frac{2}{n-k+1}A_{n-k+1,1}\left[\frac{1}{2}\frac{1}{(n-k-1)!(k-1)!} + \frac{1}{k!\,(n-k)!}\right]\right.$$
$$\left. + A_{n-k+1,1}\frac{1}{(n-k)!(k-1)!} + \frac{1}{n!}A_{n,k}\right).$$

The coefficients of $x_1^n x_1$ and of $x_1^n x_2$ can be determined directly from what is written as well as the coefficients of $x_1^n x_{n-1}$ and of $x_1^n x_n$:

$$0 \equiv x_1^{n+1}\left(\frac{1}{2!}\frac{1}{(n-1)!}B_{n-1} + \frac{1}{n!}A_{n,1}\right)$$
$$+ x_1^n x_2\left(\left[-\frac{2}{n-1}\frac{1}{2!}\frac{1}{(n-3)!1!} - \frac{2}{n-1}\frac{1}{2!(n-2)!} + \frac{1}{(n-2)!1!}\right]A_{n-1,1} + \frac{1}{n!}A_{n,2}\right)$$
$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ x_1^k x_k\left(\left[-\frac{2}{n-k+1}\frac{1}{2!}\frac{1}{(n-k+1)!(k-1)!} - \frac{2}{n-k+1}\frac{1}{k!(n-k)!}\right.\right.$$
$$\left.\left. + \frac{1}{(n-k)!(k-1)!}\right]A_{n-k+1,1} + \frac{1}{n!}A_{n,k}\right)$$
$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ x_1^{n-1}x_{n-1}\left(\left[-\frac{2}{2}\frac{1}{2!}\frac{1}{0!(n-2)!} - \frac{2}{2}\frac{1}{(n-1)!1!} + \frac{1}{1!(n-2)!}\right]A_{2,1} + \frac{1}{n!}A_{n,n-1}\right)$$

$$+\, x_1^n x_n \left(\left[-\,2\,\frac{1}{n!} + \frac{1}{(n-1)!} \right] + \frac{1}{n!}\,A_{n,n} \right)$$

So, we obtain the announced values for B_{n-1}, $A_{n,2}$, $\dots$, $A_{n,k}$, $\dots$, $A_{n,n-1}$, $A_{n,n}$. □

In conclusion, the induction on the dimension n is complete.

20. Infinitesimal action at order $n+2$

Beyond dimension $n \leqslant 6$ (Section 15), we have no explicit formula for the subgroup of $\mathrm{GL}(n+1,\mathbb{R})$, which stabilizes the normalization (19.1) up to order $n+1$ and which would be valid for each $n \geqslant 2$. We will therefore proceed in an infinitesimal manner. This will be less expensive, computationally speaking.

Thanks to Section 19, we can take a vector field tangent to (19.1) up to order $\leqslant n+1$:

$$\begin{aligned}
L \,=\, & \left(A_{1,1}\,x_1 - \frac{2}{1+1}\,A_{2,1}\,u \right) \partial_{x_1} + \left(A_{2,1}\,x_1 - \frac{2}{2+1}\,A_{3,1}\,u \right) \partial_{x_2} \\
& + \left(A_{3,1}\,x_1 + 0 + A_{3,1}\,x_3 - \frac{2}{3+1}\,A_{4,1}\,u \right) \partial_{x_3} \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \left(A_{n-1,1}x_1 + 0 + A_{n-1,1}\,x_3 + \cdots + A_{n-1,n-1}x_{n-1} - \frac{2}{n}\,A_{n,1}\,u \right) \partial_{x_{n-1}} \\
& + \left(A_{n,1}x_1 + 0 + A_{n,3}\,x_3 + \cdots + A_{n,n-1}x_{n-1} + A_{n,n}x_n + B_n\,u \right) \partial_{x_n} \\
& + \left(2\,A_{1,1}\,u \right) \partial_u,
\end{aligned} \tag{20.1}$$

where, for all $3 \leqslant j \leqslant i \leqslant n$,

$$A_{i,j} \,=\, -\,\frac{(j-2)}{i-j+1}\binom{i}{j}. \tag{20.2}$$

With small $\varepsilon \approx 0$, this L has an approximate flow:

$$\begin{aligned}
y_1 \,&=\, x_1 + \varepsilon \left(A_{1,1}\,x_1 - \frac{2}{1+1}\,A_{2,1}\,u \right) + \mathrm{O}(\varepsilon^2), \\
y_2 \,&=\, x_2 + \varepsilon \left(A_{2,1}\,x_1 - \frac{2}{2+1}\,A_{3,1}\,u \right) + \mathrm{O}(\varepsilon^2), \\
y_3 \,&=\, x_3 + \varepsilon \left(A_{3,1}\,x_1 + 0 + A_{3,1}\,x_3 - \frac{2}{3+1}\,A_{4,1}\,u \right) + \mathrm{O}(\varepsilon^2), \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y_{n-1} \,&=\, x_{n-1} + \varepsilon \left(A_{n-1,1}x_1 + 0 + A_{n-1,1}\,x_3 + \cdots + A_{n-1,n-1}x_{n-1} - \frac{2}{n}\,A_{n,1}\,u \right) \\
& \qquad + \mathrm{O}(\varepsilon^2), \\
y_n \,&=\, x_n + \varepsilon \left(A_{n,1}x_1 + 0 + A_{n,3}\,x_3 + \cdots + A_{n,n-1}x_{n-1} + A_{n,n}x_n + B_n\,u \right) + \mathrm{O}(\varepsilon^2), \\
v \,&=\, u + \varepsilon \left(2\,A_{1,1}\,u \right) + \mathrm{O}(\varepsilon^2).
\end{aligned} \tag{20.3}$$

For $1 \leqslant i \leqslant n-1$, we can write the intermediate lines as follows:

$$y_i \,=\, x_i + \varepsilon \Big(A_{i,i}\,x_1 + 0 + \sum_{3\leqslant j\leqslant i} A_{i,j}\,x_j + 0 + \cdots + 0 - \frac{2}{i+1}\,A_{i+1,1}\,u \Big) + \mathrm{O}(\varepsilon^2). \tag{20.4}$$

With its independent monomials of order $n+2$, the hypersurface equation on the left is

$$\begin{aligned} u &= \frac{x_1^2}{2} + \frac{x_1^2 x_2}{2} + \sum_{m=3}^{n} \Big(\frac{x_1^m x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j\geqslant 2\\ i+j=m+1}} \frac{1}{2}\,\frac{x_i x_j}{(i-1)!(j-1)!} \Big) \\ &+ F_{n+2,0,\dots,0}\,\frac{x_1^{n+1}x_1}{(n+2)!} + F_{n+1,1,\dots,0}\,\frac{x_1^{n+1}x_2}{(n+1)!} + \cdots + F_{n+1,0,\dots,1}\,\frac{x_1^{n+1}x_n}{(n+1)!} \\ &+ x_1^n \sum_{\substack{i,j\geqslant 2\\ i+j=n+2}} \frac{1}{2}\,\frac{x_i x_j}{(i-1)!(j-1)!} + \mathrm{O}_{x'}(3) + \mathrm{O}_x(n+3). \end{aligned} \tag{20.5}$$

Similarly, the hypersurface equation on the right is

$$\begin{aligned} v &= \frac{y_1^2}{2} + \frac{y_1^2 y_2}{2} + \sum_{m=3}^{n} \Big(\frac{y_1^m y_m}{m!} + y_1^{m-1} \sum_{\substack{i,j\geqslant 2\\ i+j=m+1}} \frac{1}{2}\,\frac{y_i y_j}{(i-1)!(j-1)!} \Big) \\ &+ G_{n+2,0,\dots,0}\,\frac{y_1^{n+1}y_1}{(n+2)!} + G_{n+1,1,\dots,0}\,\frac{y_1^{n+1}y_2}{(n+1)!} + \cdots + G_{n+1,0,\dots,1}\,\frac{y_1^{n+1}y_n}{(n+1)!} \\ &+ y_1^n \sum_{\substack{i,j\geqslant 2\\ i+j=n+2}} \frac{1}{2}\,\frac{y_i y_j}{(i-1)!(j-1)!} + \mathrm{O}_{y'}(3) + \mathrm{O}_y(n+3). \end{aligned} \tag{20.6}$$

Now we assume that $(y_1, \dots, y_n, v)$ are replaced from (20.3) in $0 = -v + G(y)$, that u is replaced by $F(x_1, \dots, x_n)$ from (20.5):

$$0 \equiv E(x_1, \dots, x_n, \varepsilon) = E(x, \varepsilon).$$

We consider the terms of orders not exceeding $n+1$:

$$\pi^{n+1}\Bigg(\sum_{\sigma_1,\dots,\sigma_n} E_{\sigma_1,\dots,\sigma_n}(\varepsilon)\, x_1^{\sigma_1} \cdots x_n^{\sigma_n} \Bigg) = \sum_{\sigma_1+\cdots+\sigma_n\leqslant n+1} E_{\sigma_1,\dots,\sigma_n}(\varepsilon)\, x_1^{\sigma_1} \cdots x_n^{\sigma_n}.$$

Lemma 20.1. *The fact that L is tangent up to order $n+1$ implies*

$$0 \equiv \pi^{n+1}\left(-v + G(y)\right) = \pi^{n+1}\left(E(x,\varepsilon)\right).$$

We look at order $n+2$ terms:

$$\begin{aligned} 0 &\equiv \pi^{n+2}\left(E(x,\varepsilon)\right) = \pi_{n+2}\left(E(x,\varepsilon)\right) \\ &=: E_{n+2,0,\dots,0}(\varepsilon)\, x_1^{n+1}x_1 + E_{n+1,1,\dots,0}(\varepsilon)\, x_1^{n+1}x_2 + \cdots + E_{n+1,0,\dots,1}(\varepsilon)\, x_1^{n+1}x_n. \end{aligned}$$

For $\varepsilon = 0$, the map is the identity onr, hence without computation we know that

$$0 \equiv \sum_{\nu_1+\cdots+\nu_n=n+2} \left(-\frac{F_{\nu_1,\dots,\nu_n}}{\nu_1!\cdots\nu_n!} + \frac{G_{\nu_1,\dots,\nu_n}}{\nu_1!\cdots\nu_n!} + \varepsilon\, T_{\nu_1,\dots,\nu_n} + \mathrm{O}(\varepsilon^2) \right) + \mathrm{O}_{x_1,\dots,x_n}(n+3),$$

and our key goal is to determine the terms $T_{\nu_1,\dots,\nu_n}$ that are of order 1 in ε, more precisely, to compute the independent ones:

$$T_{n+2,0,\dots,0}, \qquad T_{n+1,1,\dots,0}, \qquad \dots, \qquad T_{n+1,0,\dots,1}. \tag{20.7}$$

Equivalently, we can write the fundamental equation by specifying the dependent monomials as the remainder $\mathrm{O}_{x_2,\dots,x_n}(2)$:

$$
\begin{aligned}
0 &\equiv E(x,\varepsilon)\\
&\equiv x_1^{n+1}x_1\bigg(-\frac{F_{n+2,0,\dots,0}}{(n+2)!}+\frac{G_{n+2,0,\dots,0}}{(n+2)!}+\varepsilon\,T_{n+2,0,\dots,0}+\mathrm{O}(\varepsilon^2)\bigg)\\
&\quad+x_1^{n+1}x_2\bigg(-\frac{F_{n+1,1,\dots,0}}{(n+1)!}+\frac{G_{n+1,1,\dots,0}}{(n+1)!}+\varepsilon\,T_{n+1,1,\dots,0}+\mathrm{O}(\varepsilon^2)\bigg)\\
&\quad+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad+x_1^{n+1}x_n\bigg(-\frac{F_{n+1,0,\dots,1}}{(n+1)!}+\frac{G_{n+1,0,\dots,1}}{(n+1)!}+\varepsilon\,T_{n+1,0,\dots,1}+\mathrm{O}(\varepsilon^2)\bigg)\\
&\quad+\mathrm{O}_{x_2,\dots,x_n}(2)+\mathrm{O}_{x_1,x_2,\dots,x_n}(n+3).
\end{aligned}
\tag{20.8}
$$

We introduce an operator

$$
\Pi_{n+2}^{\mathrm{ind}}(\bullet)\;:=\;\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\Big(\pi_{n+2}^{\mathrm{ind}}(\bullet)\Big)\;=\;\pi_{n+2}^{\mathrm{ind}}\Big(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}(\bullet)\Big),
$$

which selects what we want to compute:

$$
\Pi_{n+2}^{\mathrm{ind}}\left(E(x,\varepsilon)\right)\;=\;\frac{x_1^{n+1}x_1}{(n+2)!}\,T_{n+2,0,\dots,0}+\frac{x_1^{n+1}x_2}{(n+1)!}\,T_{n+1,1,\dots,0}+\cdots+\frac{x_1^{n+1}x_n}{(n+1)!}\,T_{n+1,0,\dots,1}.
$$

Also, for studying the remainders, we shall need to consider all independent monomials of order not exceeding $n+2$:

$$
\Pi_{\mathrm{ind}}^{n+2}(\bullet)\;:=\;\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\Big(\pi_{\mathrm{ind}}^{n+2}(\bullet)\Big)\;=\;\pi_{\mathrm{ind}}^{n+2}\Big(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}(\bullet)\Big).
$$

Thus, we should apply $\Pi_{n+2}(\bullet)$ to all (numerous) terms of (20.6). We start by the remainders.

Lemma 20.2. *The identity*

$$
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(\mathrm{O}_{y_2,\dots,y_n}(3)\right)\;=\;\mathrm{O}_{x_2,\dots,x_n}(2)
$$

holds.

Proof. Take any monomial $y_2^{\nu_2}\cdots y_n^{\nu_n}$ with $\nu_2+\cdots+\nu_n\geqslant 3$, and abbreviate (20.3) as:

$$
y_i\;=\;x_i+\varepsilon\,R_i(x)+\mathrm{O}(\varepsilon^2),\qquad 2\leqslant\,i\,\leqslant\,n,
$$

whence,

$$
\begin{aligned}
&\big(x_2+\varepsilon\,R_2+\mathrm{O}(\varepsilon^2)\big)^{\nu_2}\cdots\big(x_n+\varepsilon\,R_n+\mathrm{O}(\varepsilon^2)\big)^{\nu_n}\\
&\qquad=\;\Big(x_2^{\nu_2}+\nu_2\,x_2^{\nu_2-1}\,\varepsilon\,R_2+\mathrm{O}(\varepsilon^2)\Big)\cdots\Big(x_n^{\nu_n}+\nu_n\,x_n^{\nu_n-1}\,\varepsilon\,R_n+\mathrm{O}(\varepsilon^2)\Big)\\
&\qquad=\;x_2^{\nu_2}\cdots x_n^{\nu_n}+\varepsilon\,\Big[\nu_2\,x_2^{\nu_2-1}R_2x_3^{\nu_3}\cdots x_n^{\nu_n}+\cdots+x_2^{\nu_2}\cdots x_{n-1}^{\nu_{n-1}}\nu_nx_n^{\nu_n-1}R_n\Big]+\mathrm{O}(\varepsilon^2),
\end{aligned}
$$

and here $\nu_2-1+\nu_3+\cdots+\nu_n\geqslant 2,\,\dots,\,\nu_2+\cdots+\nu_{n-1}+\nu_n-1\geqslant 2$ as well. □

Thus, the remainder $\mathrm{O}_{y_2,\dots,y_n}(3)$ in (20.6) has contribution equal to 0 in $\Pi_{\mathrm{ind}}^{n+2}(E)$.

Next, still in (20.6), we consider border-dependent monomials from the sum $\sum\limits_{m=3}^{n}$, but only for $m\leqslant n-1$ at first.

Lemma 20.3. *For every* $3\leqslant m\leqslant n-1$ *and for all* $i,j\geqslant 2$ *with* $i+j=m+1$*, we have*

$$
0\;=\;\Pi_{n+2}^{\mathrm{ind}}\left(y_1^{m-1}y_iy_j\right).
$$

Proof. To simplify (20.4) after replacement of u by F, we abbreviate

$$\alpha_i := -\frac{2}{i+1}\,\frac{A_{i+1}}{(n+1-m)!},$$

and use $\cdots$ to denote monomials of order not exceeding $n-m+1$:

$$\begin{aligned} y_i &= x_i + \varepsilon\bigg\{A_{i,1} + \sum_{3\leqslant j\leqslant i} A_{i,j}\,x_j - \frac{2}{i+1}\,A_{i+1,1}\,\bigg[\cdots + \frac{x_1^{n+1-m}x_{n+1-m}}{(n+1-m)!} + \mathrm{O}_x(n-m+3)\bigg]\bigg\} \\ &\quad + \mathrm{O}(\varepsilon^2) \\ &= x_i + \varepsilon\Big\{\cdots + \alpha_i\,x_1^{n+1-m}\,x_{n+1-m} + \mathrm{O}_x(n-m+3)\Big\} + \mathrm{O}(\varepsilon^2). \end{aligned}$$

We write a product

$$\begin{aligned} y_1^{m-1}y_iy_j &= \Big(x_1 + \varepsilon\Big\{\cdots + \alpha_1\,x_1^{n+1-m}\,x_{n+1-m} + \mathrm{O}_x(n-m+3)\Big\} + \mathrm{O}(\varepsilon^2)\Big)^{m-1} \\ &\quad \cdot\Big(x_i + \varepsilon\Big\{\cdots + \alpha_i\,x_1^{n+1-m}\,x_{n+1-m} + \mathrm{O}_x(n-m+3)\Big\} + \mathrm{O}(\varepsilon^2)\Big)^1 \\ &\quad \cdot\Big(x_j + \varepsilon\Big\{\cdots + \alpha_j\,x_1^{n+1-m}\,x_{n+1-m} + \mathrm{O}_x(n-m+3)\Big\} + \mathrm{O}(\varepsilon^2)\Big)^1, \end{aligned}$$

and we expand:

$$\begin{aligned} \pi_{n+2}\big(y_1^{m-1}y_iy_j\big) &= \underline{\pi_{n+2}\big(x_1^{m-1}x_ix_j\big)}_{\circ} + \varepsilon\Big\{x_1^{m-2}\tbinom{m-1}{1}\,\alpha_1\,x_1^{n+1-m}\,x_{n+1-m}\,x_i\,x_j \\ &\quad + x_1^{m-1}\,\alpha_i\,x_1^{n+1-m}\,x_{n+1-m}\,x_j \\ &\quad + x_1^{m-1}\,x_i\,\alpha_j\,x_1^{n+1-m}\,x_{n+1-m}\Big\} + \mathrm{O}(\varepsilon^2). \end{aligned}$$

To complete the proof, we observe that x_ix_j are dependent monomials, and since $n-m+1\geqslant 2$, observe that the two monomials $x_{n+1-m}\,x_j$ and $x_{n+1-m}\,x_i$ are also dependent. □

It therefore remains to compute

$$\begin{aligned} \Pi_{n+2}^{\mathrm{ind}}\bigg(&-v + \frac{y_1^2}{2!} + \frac{y_1^2y_2}{2!} + \frac{y_1^3y_3}{3!} + \cdots + \frac{y_1^{n-1}y_{n-1}}{(n-1)!} + \frac{y_1^ny_n}{n!} + y_1^{n-1}\sum_{\substack{i,j\geqslant 2\\ i+j=n+1}}\frac{1}{2}\,\frac{y_iy_j}{(i-1)!(j-1)!} \\ &+ G_{n+2,0,\dots,0}\,\frac{y_1^{n+1}y_1}{(n+2)!} + G_{n+1,1,\dots,0}\,\frac{y_1^{n+1}y_2}{(n+1)!} + \cdots + G_{n+1,0,\dots,1}\,\frac{y_1^{n+1}y_n}{(n+1)!} \\ &+ y_1^n\sum_{\substack{i,j\geqslant 2\\ i+j=n+2}}\frac{1}{2}\,\frac{y_iy_j}{(i-1)!(j-1)!}\bigg). \end{aligned}$$

Since $\Pi_{n+2}^{\mathrm{ind}}(\bullet)$ is linear, we can proceed termwise.

We compute the first term

$$\begin{aligned} \Pi_{n+2}^{\mathrm{ind}}(-v) &= \pi_{n+2}^{\mathrm{ind}}\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(-u - \varepsilon\,2\,A_{1,1}\,u + \mathrm{O}(\varepsilon^2)\Big)\bigg) \\ &= \pi_{n+2}^{\mathrm{ind}}\bigg(-2\,A_{1,1}\,\Big[\frac{x_1^2}{2!} + \frac{x_1^2x_2}{2!} + \cdots + \frac{x_1^nx_n}{n!} + F_{n+2,0,\dots,0}\,\frac{x_1^{n+1}x_1}{(n+2)!} \\ &\qquad + F_{n+1,1,\dots,0}\,\frac{x_1^{n+1}x_2}{(n+1)!} + \cdots + F_{n+1,0,\dots,1}\,\frac{x_1^{n+1}x_n}{(n+1)!} + \mathrm{O}_{x_1,\dots,x_n}(n+3)\Big]\bigg), \end{aligned}$$

and we obtain:

$$\begin{aligned}\Pi^{\text{ind}}_{n+2}(-v) &= -\frac{2}{(n+2)!}\,F_{n+2,0,\dots,0}\,A_{1,1}\,x_1^{n+2} - \frac{2}{(n+1)!}\,F_{n+1,1,\dots,0}\,A_{1,1}\,x_1^{n+1}x_2 \\ &\quad - \cdots - \frac{2}{(n+1)!}\,F_{n+1,0,\dots,1}\,A_{1,1}\,x_1^{n+1}x_n.\end{aligned}\tag{20.9}$$

Next, for $\kappa \in \mathbb{N}$, abbreviating

$$\mathcal{R}^\kappa := \mathrm{O}_{x_2,\dots,x_n}(2) + \mathrm{O}_{x_1,x_2,\dots,x_n}(\kappa),$$

we compute the second term:

$$\begin{aligned}\Pi^{\text{ind}}_{n+2}\left(\frac{y_1^2}{2}\right) &= \Pi^{\text{ind}}_{n+2}\bigg(\frac{1}{2}\Big(x_1+\varepsilon\Big\{A_{1,1}x_1 - \frac{2}{2}A_{2,1}\Big[\frac{x_1^2}{2} \\ &\qquad + \cdots + \frac{x_1^{n-1}x_{n-1}}{(n-1)!} + \frac{x_1^n x_n}{n!} + \mathrm{O}_{x'}(2) + \mathrm{O}_x(n+2)\Big]\Big\} + \mathrm{O}(\varepsilon^2)\Big)^2\bigg) \\ &= \Pi^{\text{ind}}_{n+2}\bigg(\frac{1}{2}\Big(x_1+\varepsilon\Big\{\cdots - \frac{2}{2}A_{2,1}\,\frac{x_1^n x_n}{n!} + \mathcal{R}^{n+2}\Big\} + \mathrm{O}(\varepsilon^2)\Big)^2\bigg),\end{aligned}$$

expand the square, select the coefficient of ε^1, select the (single) independent monomial, and we obtain:

$$\Pi^{\text{ind}}_{n+2}\left(\frac{y_1^2}{2}\right) = \frac{1}{2}\,2\,x_1\,\left(-\frac{2}{2}A_{2,1}\right)\frac{x_1^n x_n}{n!} = -\frac{1}{n!}\,A_{2,1}\,x_1^{n+1}x_n.\tag{20.10}$$

We treat the third term:

$$\begin{aligned}\Pi^{\text{ind}}_{n+2}\left(\frac{y_1^2 y_2}{2}\right) &= \Pi^{\text{ind}}_{n+2}\bigg(\frac{1}{2!}\Big(x_1+\varepsilon\Big[\cdots - \frac{2}{2}\,A_{2,1}\,\frac{x_1^{n-1}x_{n-1}}{(n-1)!} + \mathcal{R}^{n+1}\Big] + \mathrm{O}(\varepsilon^2)\Big)^2 \\ &\qquad \cdot\Big(x_2+\varepsilon\Big[\cdots - \frac{2}{3}\,A_{3,1}\,\frac{x_1^{n-1}x_{n-1}}{(n-1)!} + \mathcal{R}^{n+1}\Big] + \mathrm{O}(\varepsilon^2)\Big)^1\bigg)\end{aligned}$$

that is,

$$\Pi^{\text{ind}}_{n+2}\left(\frac{y_1^2 y_2}{2}\right) = \frac{1}{2}\,x_1^2\,\left(-\frac{2}{3}\,A_{3,1}\right)\frac{x_1^{n-1}x_{n-1}}{(n-1)!} = -\frac{2}{3}\,\frac{1}{2!\,(n-1)!}\,A_{3,1}\,x_1^{n+1}x_{n+1}.\tag{20.11}$$

Similarly,

$$\begin{aligned}\Pi^{\text{ind}}_{n+2}\left(\frac{y_1^3 y_3}{3!}\right) &= \Pi^{\text{ind}}_{n+2}\bigg(\frac{1}{3!}\Big(x_1+\varepsilon\Big[\cdots - \frac{2}{2}\,A_{2,1}\,\frac{x_1^{n-2}x_{n-2}}{(n-2)!} + \mathcal{R}^{n}\Big] + \mathrm{O}(\varepsilon^2)\Big)^3 \\ &\qquad \cdot\Big(x_3+\varepsilon\Big[\cdots - \frac{2}{4}\,A_{4,1}\,\frac{x_1^{n-2}x_{n-2}}{(n-2)!} + \mathcal{R}^{n}\Big] + \mathrm{O}(\varepsilon^2)\Big)^1\bigg)\end{aligned}$$

that is,

$$\Pi^{\text{ind}}_{n+2}\left(\frac{y_1^3 y_3}{3!}\right) = \frac{1}{3!}\,x_1^3\,\left(-\frac{2}{4}\,A_{4,1}\right)\frac{x_1^{n-2}x_{n-2}}{(n-2)!} = -\frac{2}{4}\,\frac{1}{3!\,(n-2)!}\,A_{4,1}\,x_1^{n+1}x_{n-2}.$$

Now we consider general m with $3 \leqslant m \leqslant n-1$

$$\begin{aligned}\Pi^{\text{ind}}_{n+2}\left(\frac{y_1^m y_m}{m!}\right) &= \Pi^{\text{ind}}_{n+2}\bigg(\frac{1}{m!}\Big(x_1+\varepsilon\Big[\cdots - \frac{2}{2}\,A_{2,1}\,\frac{x_1^{n-m+1}x_{n-m+1}}{(n-m+1)!} + \mathcal{R}^{n-m+3}\Big] + \mathrm{O}(\varepsilon^2)\Big)^m \\ &\qquad \cdot\Big(x_m+\varepsilon\Big[\cdots - \frac{2}{m+1}\,A_{m+1,1}\,\frac{x_1^{n-m+1}x_{n-m+1}}{(n-m+1)!} + \mathcal{R}^{n-m+3}\Big] + \mathrm{O}(\varepsilon^2)\Big)^1\bigg),\end{aligned}$$

that is,

$$\Pi_{n+2}^{\rm ind}\left(\frac{y_1^m y_m}{m!}\right) = \frac{1}{m!}\,x_1^m\left(-\frac{2}{m+1}\,A_{m+1,1}\right)\frac{x_1^{n-m+1}x_{n-m+1}}{(n-m+1)!} = -\,\frac{2}{m+1}\,\frac{1}{m!(n-m+1)!}\,A_{m+1,1}\,x_1^{n+1}x_{n-m+1}. \tag{20.12}$$

For $m=n$, the result is different, two monomials are obtained:

$$\begin{aligned}\Pi_{n+2}^{\rm ind}\left(\frac{1}{n!}\,y_1^n y_n\right) &= \Pi_{n+2}^{\rm ind}\bigg(\frac{1}{n!}\left(x_1+\varepsilon\left[\cdots-\frac{2}{2}\,A_{2,1}\,\frac{x_1^2}{2!}+\mathcal{R}^3\right]+{\rm O}(\varepsilon^2)\right)^n \\ &\quad\cdot\left(x_n+\varepsilon\left[\cdots+B_n\,\frac{x_1^2}{2}+\mathcal{R}^3\right]+{\rm O}(\varepsilon^2)\right)^1\bigg)^n \\ &= \frac{1}{n!}\,x_1^n\,B_n\,\frac{x_1^2}{2!}+\frac{1}{n!}\tbinom{n}{1}\,x_1^{n-1}\left(-\frac{2}{2}\,A_{2,1}\right)\frac{x_1^2}{2!}\,x_n,\end{aligned}$$

that is,

$$\Pi_{n+2}^{\rm ind}\left(\frac{1}{n!}\,y_1^n y_n\right) = \frac{1}{n!\,2!}\,B_n\,x_1^{n+2}-\frac{1}{(n-1)!\,2!}\,A_{2,1}\,x_1^{n+1}\,x_n. \tag{20.13}$$

Next, for all $i,j\geqslant 2$ satisfying $i+j=n+1$, whence $i,j\leqslant n-1$:

$$\begin{aligned}\Pi_{n+2}^{\rm ind}\left(y_1^{n-1}\frac{1}{2}\,\frac{y_iy_j}{(i-1)!(j-1)!}\right) &= \frac{1}{2}\,\frac{1}{(i-1)!(j-1)!} \\ &\quad\cdot\Pi_{n+2}^{\rm ind}\bigg(\left(x_1+\varepsilon\left[\cdots-\frac{2}{2}\,A_{2,1}\,\frac{x_1^2}{2}+\mathcal{R}^3\right]+{\rm O}(\varepsilon^2)\right)^{n-1} \\ &\quad\cdot\left(x_i+\varepsilon\left[\cdots-\frac{2}{i+1}\,A_{i+1,1}\,\frac{x_1^2}{2}+\mathcal{R}^3\right]+{\rm O}(\varepsilon^2)\right)^1 \\ &\quad\cdot\left(x_j+\varepsilon\left[\cdots-\frac{2}{j+1}\,A_{j+1,1}\,\frac{x_1^2}{2}+\mathcal{R}^3\right]+{\rm O}(\varepsilon^2)\right)^1\bigg) \\ &= \frac{1}{2}\,\frac{1}{(i-1)!(j-1)!}\,x_1^{n-1}\left(-\frac{2}{i+1}\,A_{i+1,1}\right)\frac{x_1^2}{2}\,x_j \\ &\quad+\frac{1}{2}\,\frac{1}{(i-1)!(j-1)!}\,x_1^{n-1}\left(-\frac{2}{j+1}\,A_{j+1,1}\right)\frac{x_1^2}{2}\,x_i,\end{aligned}$$

that is,

$$\begin{aligned}\Pi_{n+2}^{\rm ind}\left(y_1^{n-1}\frac{1}{2}\,\frac{y_iy_j}{(i-1)!(j-1)!}\right) &= -\,\frac{1}{2(i+1)}\,\frac{1}{(i-1)!(j-1)!}\,A_{i+1,1}\,x_1^{n+1}x_j \\ &\quad-\frac{1}{2(j+1)}\,\frac{1}{(i-1)!(j-1)!}\,A_{j+1,1}\,x_1^{n+1}x_i.\end{aligned}$$

In fact, we need to find a sum:

$$\begin{aligned}\Pi_{n+2}^{\rm ind}\bigg(y_1^{n-1}\sum_{\substack{i,j\geqslant 2\\ i+j=n+1}}\frac{1}{2}\,\frac{y_iy_j}{(i-1)!(j-1)!}\bigg) &= -\,\frac{1}{2}\sum_{j=2}^{n-1}\frac{1}{n-j+2}\,\frac{1}{(n-j)!(j-1)!}\,A_{n-j+2,1}\,x_1^{n+1}x_j \\ &\quad-\frac{1}{2}\sum_{i=2}^{n-1}\frac{1}{n-i+2}\,\frac{1}{(i-1)!(n-i)!}\,A_{n-i+2,1}\,x_1^{n+1}x_i\end{aligned}$$

and by exchanging $j \longleftrightarrow i$ in the first sum, the two sums are *equal*:

$$\Pi^{\text{ind}}_{n+2}\Big(y_1^{n-1}\sum_{\substack{i,j\geqslant 2\\ i+j=n+1}}\frac{1}{2}\,\frac{y_iy_j}{(i-1)!(j-1)!}\Big) \;=\; -\sum_{i=2}^{n-1}\frac{1}{n-i+2}\,\frac{1}{(i-1)!(n-i)!}\,A_{n-i+2,1}\,x_1^{n+1}x_i. \tag{20.14}$$

The next term is

$$\begin{aligned}\Pi^{\text{ind}}_{n+2}\bigg(G_{n+2,0,\dots,0}\,\frac{y_1^{n+2}}{(n+2)!}\bigg) &= \frac{G_{n+2,0,\dots,0}}{(n+2)!}\,\Pi^{\text{ind}}_{n+2}\bigg(\Big(x_1+\varepsilon\,\Big[A_{1,1}\,x_1+\mathcal{R}^2\Big]+\mathrm{O}(\varepsilon^2)\Big)^{n+2}\bigg)\\ &= \frac{G_{n+2,0,\dots,0}}{(n+2)!}\,\tbinom{n+2}{1}\,x_1^{n+1}\,A_{1,1}\,x_1\end{aligned}$$

has value

$$\Pi^{\text{ind}}_{n+2}\Big(G_{n+2,0,\dots,0}\,\frac{y_1^{n+2}}{(n+2)!}\Big) \;=\; \frac{G_{n+2,0,\dots,0}}{(n+1)!}\,A_{1,1}\,x_1^{n+2}. \tag{20.15}$$

Then

$$\begin{aligned}\Pi^{\text{ind}}_{n+2}\Big(G_{n+1,1,\dots,0}\,\frac{y_1^{n+1}y_2}{(n+1)!}\Big) &= \frac{G_{n+1,1,\dots,0}}{(n+1)!}\,\Pi^{\text{ind}}_{n+2}\bigg(\Big(x_1+\varepsilon\,\Big[A_{1,1}\,x_1+\mathcal{R}^2\Big]+\mathrm{O}(\varepsilon^2)\Big)^{n+1}\\ &\qquad\cdot\Big(x_2+\varepsilon\,\Big[A_{2,1}\,x_1+\mathcal{R}^2\Big]+\mathrm{O}(\varepsilon^2)\Big)^{1}\bigg)\\ &= \frac{G_{n+1,1,\dots,0}}{(n+1)!}\,x_1^{n+1}\,A_{2,1}\,x_1+\frac{G_{n+1,1,\dots,0}}{(n+1)!}\,x_1^{n}\,\tbinom{n+1}{1}\,A_{1,1}\,x_1x_2,\end{aligned}$$

that is,

$$\Pi^{\text{ind}}_{n+2}\Big(G_{n+1,1,\dots,0}\,\frac{y_1^{n+1}y_2}{(n+1)!}\Big) \;=\; \frac{G_{n+1,1,\dots,0}}{(n+1)!}\,A_{2,1}\,x_1^{n+2}+\frac{G_{n+1,1,\dots,0}}{n!}\,A_{1,1}\,x_1^{n+1}x_2. \tag{20.16}$$

Next term is

$$\begin{aligned}\Pi^{\text{ind}}_{n+2}\Big(G_{n+1,0,1,\dots,0}\,\frac{y_1^{n+1}y_3}{(n+1)!}\Big) &= \frac{G_{n+1,0,1,\dots,0}}{(n+1)!}\,\Pi^{\text{ind}}_{n+2}\bigg(\Big(x_1+\varepsilon\,\Big[A_{1,1}\,x_1+\mathcal{R}^2\Big]+\mathrm{O}(\varepsilon^2)\Big)^{n+1}\\ &\qquad\cdot\Big(x_3+\varepsilon\,\Big[A_{3,1}\,x_1-A_{1,1}\,x_3+\mathcal{R}^2\Big]+\mathrm{O}(\varepsilon^2)\Big)^{1}\bigg)\\ &= \frac{G_{n+1,0,1,\dots,0}}{(n+1)!}\,\Big\{x_1^{n+1}\,A_{3,1}\,x_1-x_1^{n+1}\,A_{1,1}\,x_3+x_1^{n}\,\tbinom{n+1}{1}\,A_{1,1}\,x_1x_3\Big\},\end{aligned}$$

that is,

$$\Pi^{\text{ind}}_{n+2}\Big(G_{n+1,0,1,\dots,0}\,\frac{y_1^{n+1}y_3}{(n+1)!}\Big) \;=\; \frac{G_{n+1,0,1,\dots,0}}{(n+1)!}\,\Big\{A_{3,1}\,x_1^{n+2}+\frac{n}{(n+1)!}\,A_{1,1}\,x_1^{n+1}\,x_3\Big\}. \tag{20.17}$$

Then for general k with $3\leqslant k\leqslant n$, we compute in $0\cdots1\cdots0$, the 1 is at position $k-1$:

$$\begin{aligned}\Pi^{\text{ind}}_{n+2}\Big(G_{n+1,0\cdots1\cdots1}\,\frac{y_1^{n+1}y_k}{(n+1)!}\Big) &= \frac{G_{n+1,0\cdots1\cdots0}}{(n+1)!}\,\Pi^{\text{ind}}_{n+2}\bigg(\Big(x_1+\varepsilon\,\Big[A_{1,1}\,x_1+\mathcal{R}^2\Big]+\mathrm{O}(\varepsilon^2)\Big)^{n+1}\\ &\qquad\cdot\Big(x_k+\varepsilon\,\Big[A_{k,1}\,x_1+\sum_{3\leqslant j\leqslant k}A_{k,j}\,x_j+\mathcal{R}^2\Big]+\mathrm{O}(\varepsilon^2)\Big)^{1}\bigg)\\ &= \frac{G_{n+1,0\cdots1\cdots0}}{(n+1)!}\,\Big\{x_1^{n+1}\,\Big(A_{k,1}\,x_1+\sum_{3\leqslant j\leqslant k}A_{k,j}\,x_j\Big)+x_1^{n}\,\tbinom{n+1}{1}\,A_{1,1}\,x_1x_k\Big\}\\ &= \frac{G_{n+1,0\cdots1\cdots0}}{(n+1)!}\,\Big\{A_{k,1}\,x_1^{n+2}+\sum_{3\leqslant j\leqslant k-1}A_{k,j}\,x_1^{n+1}x_j\end{aligned}$$

$$+ \left(A_{k,k} + (n+1)\,A_{1,1}\right)\, x_1^{n+1} x_k \Big\}.$$

Now we replace the $A_{k,j}$ by their values from (20.2),

$$A_{k,k} \;=\; -\,(k-2)\,A_{1,1},$$

and we obtain:

$$\begin{aligned}
\Pi_{n+2}^{\mathrm{ind}}\Big(G_{n+1,0\cdots 1\cdots 0}\,\frac{y_1^{n+1}y_k}{(n+1)!}\Big) &= \frac{1}{(n+1)!}\,G_{n+1,0\cdots1\cdots0}\,A_{k,1}\,x_1^{n+2} \\
&\quad - \sum_{3\leqslant j\leqslant k-1} \frac{1}{(n+1)!}\,\frac{j-2}{k-j+1}\,\binom{k}{j}\,G_{n+1,0\cdots1\cdots0}\,A_{k-j+1,1}\,x_1^{n+1}x_j \\
&\quad + \frac{n-k+3}{(n+1)!}\,G_{n+1,0\cdots1\cdots0}\,A_{1,1}\,x_1^{n+1}x_k.
\end{aligned} \tag{20.18}$$

Finally, for all $i,j\geqslant 2$ with $i+j=n+2$ we have

$$\begin{aligned}
\Pi_{n+2}^{\mathrm{ind}}\Big(y_1^n\,\frac{1}{2}\,\frac{y_iy_j}{(i-1)!(j-1)!}\Big) &= \frac{1}{2}\,\frac{1}{(i-1)!(j-1)!}\,\Pi_{n+2}^{\mathrm{ind}}\bigg(\Big(x_1+\varepsilon\Big[A_{1,1}\,x_1+\mathcal{R}^2\Big]+\mathrm{O}(\varepsilon^2)\Big)^n \\
&\quad\cdot\Big(x_i+\varepsilon\Big[A_{i,1}\,x_1+\sum_{3\leqslant k\leqslant i}A_{i,k}\,x_k+\mathcal{R}^2\Big]+\mathrm{O}(\varepsilon^2)\Big)^1 \\
&\quad\cdot\Big(x_j+\varepsilon\Big[A_{j,1}\,x_1+\sum_{3\leqslant l\leqslant i}A_{j,l}\,x_l+\mathcal{R}^2\Big]+\mathrm{O}(\varepsilon^2)\Big)^1\bigg),
\end{aligned}$$

that is,

$$\Pi_{n+2}^{\mathrm{ind}}\Big(y_1^n\,\frac{1}{2}\,\frac{y_iy_j}{(i-1)!(j-1)!}\Big) \;=\; \frac{1}{2}\,\frac{1}{(i-1)!(j-1)!}\,A_{i,1}\,x_1^{n+1}x_j + \frac{1}{2}\,\frac{1}{(i-1)!(j-1)!}\,A_{j,1}\,x_1^{n+1}x_i.$$

We find a sum:

$$\begin{aligned}
\Pi_{n+2}^{\mathrm{ind}}\Big(y_1^n\sum_{\substack{i,j\geqslant2\\ i+j=n+2}}\frac{1}{2}\,\frac{y_iy_j}{(i-1)!(j-1)!}\Big) &= -\,\frac{1}{2}\sum_{j=2}^{n}\frac{1}{(n-j+1)!(j-1)!}\,A_{n-j+2,1}\,x_1^{n+1}x_j \\
&\quad -\,\frac{1}{2}\sum_{i=2}^{n}\frac{1}{(i-1)!(n-i+1)!}\,A_{n-i+2,1}\,x_1^{n+1}x_i,
\end{aligned}$$

and by exchanging $j\longleftrightarrow i$ in the first sum, the two sums are *equal*:

$$\Pi_{n+2}^{\mathrm{ind}}\left(y_1^n\sum_{\substack{i,j\geqslant2\\ i+j=n+2}}\frac{1}{2}\,\frac{y_iy_j}{(i-1)!(j-1)!}\right) \;=\; -\sum_{i=2}^{n}\frac{1}{(i-1)!(n-i+1)!}\,A_{n-i+2,1}\,x_1^{n+1}x_i. \tag{20.19}$$

We sum all terms (20.9), (20.10), (20.11), (20.12), (20.13), (20.14), (20.15), (20.16), (20.17), (20.18), (20.19):

$$\begin{aligned}
\Pi_{n+2}\left(-v+G(y)\right) \;=\; &-\,\frac{2}{(n+2)!}\,F_{n+2,0,\dots,0}\,A_{1,1}\,x_1^{n+2} - \frac{2}{(n+1)!}\,F_{n+1,1,\dots,0}\,A_{1,1}\,x_1^{n+1}x_2 \\
&-\cdots-\frac{2}{(n+1)!}\,F_{n+1,0,\dots,1}\,A_{1,1}\,x_1^{n+1}x_n - \frac{1}{n!}\,A_{2,1}\,x_1^{n+1}x_n \\
&-\,\frac{1}{3}\,\frac{1}{(n-1)!}\,A_{3,1}\,x_1^{n+1}x_{n-1}-\cdots \\
&-\,\frac{2}{m+1}\,\frac{1}{m!(n-m+1)!}\,A_{m+1,1}\,x_1^{n+1}x_{n-m+1}-\cdots
\end{aligned}$$

$$
\begin{aligned}
&-\frac{2}{n}\,\frac{1}{(n-1)!2!}\,A_{n,1}\,x_1^{n+1}x_2+\frac{1}{n!2}\,B_n\,x_1^{n+2}-\frac{1}{(n-1)!2}\,A_{2,1}\,x_1^{n+1}x_n\\
&-\sum_{i=2}^{n-1}\frac{1}{n-i+2}\,\frac{1}{(i-1)!(n-i)!}\,A_{n+2-i,1}\,x_1^{n+1}x_i\\
&+\frac{1}{(n+1)!}\,G_{n+2,0,\dots,0}\,A_{1,1}\,x_1^{n+2}+\frac{1}{(n+1)!}\,G_{n+1,1,\dots,0}\,A_{2,1}\,x_1^{n+2}\\
&+\frac{1}{n!}\,G_{n+1,1,0,\dots,0}\,A_{1,1}\,x_1^{n}x_2\\
&+\sum_{3\leqslant k\leqslant n}\bigg\{\frac{1}{(n+1)!}\,G_{n+1,0\cdots1\cdots0}\,A_{k,1}\,x_1^{n+2}\\
&\qquad-\sum_{3\leqslant j\leqslant k-1}\frac{1}{(n+1)!}\,\frac{j-2}{k-j+1}\,\binom{k}{j}\,G_{n+1,0\cdots1\cdots0}\,A_{k-j+1,1}\,x_1^{n+1}x_j\\
&\qquad+\frac{n-k+3}{(n+1)!}\,G_{n+1,0\cdots1\cdots0}\,A_{1,1}\,x_1^{n+1}k_x\bigg\}\\
&+\sum_{i=2}^{n}\frac{1}{(i-1)!(n-i+1)!}\,A_{n-i+2,1}\,x_1^{n+1}x_i,
\end{aligned}
$$

and collect the coefficients of the independent monomials:

$$
\begin{aligned}
0\equiv x_1^{n+2}\bigg\{&-\frac{2}{(n+2)!}\,F_{n+2,0,\dots,0}\,A_{1,1}+\frac{1}{(n+1)!}\,G_{n+2,0,\dots,0}\,A_{1,1}\\
&+\sum_{2\leqslant k\leqslant n}\frac{1}{(n+1)!}\,G_{n+1,0\cdots1\cdots}\,A_{k,1}+\frac{1}{n!2}\,B_n\bigg\}\\
+x_1^{n+1}x_2\bigg\{&-\frac{2}{(n+1)!}\,F_{n+1,1,\dots,0}\,A_{1,1}+\frac{1}{n!}\,G_{n+1,1,\dots,0}\,A_{1,1}\\
&\underbrace{-\frac{2}{n}\,\frac{1}{(n-1)!2!}\,A_{n,1}-\frac{1}{n}\,\frac{1}{1!(n-2)!}\,A_{n,1}+\frac{1}{1!}\,\frac{1}{(n-1)!}\,A_{n,1}}_{=A_{n,1}\frac{1}{n!2!}\left[-2-(n-1)2+n\cdot 2\right]=A_{n,1}\frac{1}{n!2!}\left[-2+2\right]=0}\bigg\}\\
+x_1^{n+1}x_3\bigg\{&-\frac{2}{(n+1)!}\,F_{n+1,01\cdots0}\,A_{1,1}+\frac{(n-3+3)}{(n+1)!}\,G_{n+1,01\cdots0}\,A_{1,1}+*\,A_{2,1}+\cdots+*\,A_{n-2,1}\\
&+\underbrace{A_{n-1,1}\Big[-\frac{2}{n-1}\,\frac{1}{(n-2)!3!}-\frac{1}{n-1}\,\frac{1}{2!(n-3)!}+\frac{1}{2!(n-2)!}\Big]}_{=A_{n-1,1}\frac{1}{(n-1)!3!}\left[-2-(n-2)3+(n-1)3\right]=A_{n-1,1}\frac{1}{(n-1)!3!}\left[-2+3\right]}\bigg\}\\
+x_1^{n+1}x_4\bigg\{&-\frac{2}{(n+1)!}\,F_{n+1,001\cdots0}\,A_{1,1}+\frac{n-4+3}{(n+1)!}\,G_{n+1,001\cdots0}\,A_{1,1}+*\,A_{2,1}+\cdots+*\,A_{n-3,1}\\
&+\underbrace{A_{n-2,1}\Big[-\frac{2}{n-2}\,\frac{1}{(n-3)!4!}-\frac{1}{n-2}\,\frac{1}{3!(n-4)!}+\frac{1}{3!(n-3)!}\Big]}_{=A_{n-2,1}\frac{1}{(n-2)!4!}\left[-2-(n-3)4+(n-2)4\right]=A_{n-2,1}\frac{1}{(n-2)!4!}\left[-2+4\right]}\bigg\}\\
+x_1^{n+1}x_5\bigg\{&-\frac{2}{(n+1)!}\,F_{n+1,0001\cdots0}\,A_{1,1}+\frac{n-5+3}{(n+1)!}\,G_{n+1,0001\cdots0}\,A_{1,1}+*\,A_{2,1}+\cdots+*\,A_{n-4,1}
\end{aligned}
$$

$$+\underbrace{A_{n-3,1}\big[-\frac{2}{n-3}\,\frac{1}{(n-4)!5!}-\frac{1}{n-3}\,\frac{1}{4!(n-5)!}+\frac{1}{4!(n-4)!}\big]}_{=\,A_{n-3,1}\,\frac{1}{(n-3)!5!}\,\big[-2-(n-4)5+(n-3)5\big]\,=\,A_{n-3,1}\,\frac{1}{(n-3)!5!}\,\big[-2+5\big]}\Bigg\}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\,x_1^{n+1}x_k\Bigg\{-\frac{2}{(n+1)!}\,F_{n+1,0\cdots1\cdots0}\,A_{1,1}+\frac{n-k+3}{(n+1)!}\,G_{n+1,0\cdots1\cdots0}\,A_{1,1}+*\,A_{2,1}+\cdots+*\,A_{n-k+1,1}$$

$$+\,A_{n-k+2,1}\,\frac{1}{(n-k+2)!k!}\,\big[-2+k\big]\Bigg\}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\,x_1^{n+1}x_{n-1}\Bigg\{-\frac{2}{(n+1)!}\,F_{n+1,0\cdots10}\,A_{1,1}+\frac{4}{(n+1)!}\,G_{n+1,0\cdots10}\,A_{1,1}+*\,A_{2,1}$$

$$+\,A_{3,1}\,\frac{1}{3!(n-1)!}\,\big[-2+n-1\big]\Bigg\}$$

$$+\,x_1^{n+1}x_n\Bigg\{-\frac{2}{(n+1)!}\,F_{n+1,0\cdots01}\,A_{1,1}+\frac{3}{(n+1)!}\,G_{n+1,0\cdots01}\,A_{1,1}$$

$$+\underbrace{A_{2,1}\big[-\frac{1}{n!}-\frac{1}{(n-1)!2!}-\emptyset+\frac{1}{(n-1)!1!}\big]}_{=\,A_{2,1}\,\frac{1}{2!n!}\,\big[-2-n+2n\big]\,=\,A_{2,1}\,\frac{1}{2!n!}\,\big[n-2\big]}\Bigg\}.$$

With this expression, we have therefore computed the quantities $T_{n+2,0,\dots,0}$, $T_{n+1,1,\dots,0}$, $T_{n+1,0,1,\dots,0}$, $\dots$, $T_{n+1,0,\dots,1}$ introduced in (20.7).

21. Normalizations of order $(n+2)$ monomials

Coming back to (20.8), thanks to the above expressions of $T_{n+2,0,\dots,0}$, $T_{n+1,1,\dots,0}$, $T_{n+1,0,1,\dots,0}$, $\dots$, $T_{n+1,0,\dots,1}$, the coefficients at x_1^{n+2}, $x_1^{n+1}x_2$, $x_1^{n+1}x_3$, $\dots$, $x_1^{n+1}x_n$ should vanish and hence, for the first three of them we obtain:

$$\begin{aligned}
0 \equiv & -\frac{1}{(n+2)!}\,F_{n+2,0,\dots,0}+\frac{1}{(n+2)!}\,G_{n+2,0,\dots,0}\\
& +\varepsilon\Bigg\{-\frac{2}{(n+2)!}\,F_{n+2,0,\dots,0}\,A_{1,1}+\frac{1}{(n+1)!}\,G_{n+2,0,\dots,0}\,A_{1,1}\\
& +\sum_{2\leqslant k\leqslant n}\frac{1}{(n+1)!}\,G_{n+1,0\cdots1\cdots0}\,A_{k,1}+\frac{1}{n!\,2}\,B_n\Bigg\}+\mathrm{O}(\varepsilon^2),\\
0 \equiv & -\frac{1}{(n+1)!}\,F_{n+1,1,\dots,0}+\frac{1}{(n+1)!}\,G_{n+1,1,\dots,0},\\
0 \equiv & -\frac{1}{(n+1)!}\,F_{n+1,0,1,\dots,0}+\frac{1}{(n+1)!}\,G_{n+1,0,1,\dots,0}\\
& +\varepsilon\Bigg\{-\frac{2}{(n+1)!}\,F_{n+1,0,1,\dots,0}\,A_{1,1}+\frac{n-3+3}{(n+1)!}\,G_{n+1,0,1,\dots,0}\,A_{1,1}+*\,A_{2,1}+\cdots
\end{aligned}$$

$$+ * A_{n-2,1} + \frac{-2+3}{(n-1)!3!}\, A_{n-1,1} \Big\} + \mathrm{O}(\varepsilon^2),$$

and for general k with $3 \leqslant k \leqslant n-1$ we get

$$\begin{aligned} 0 \equiv & - \frac{1}{(n+1)!}\, F_{n+1,0\cdots 1\cdots 0} + \frac{1}{(n+1)!}\, G_{n+1,0\cdots 1\cdots 0} \\ & + \varepsilon \Big\{ - \frac{2}{(n+1)!}\, F_{n+1,0\cdots 1\cdots 0}\, A_{1,1} + \frac{n-k+3}{(n+1)!}\, G_{n+1,0\cdots 1\cdots 0}\, A_{1,1} \\ & + * A_{2,1} + \cdots + * A_{n-k+1,1} + \frac{-2+k}{(n-k+2)!k!}\, A_{n-k+2,1} \Big\} + \mathrm{O}(\varepsilon^2), \end{aligned}$$

and for the last two we find

$$\begin{aligned} 0 \equiv & - \frac{1}{(n+1)!}\, F_{n+1,0,\dots,1,0} + \frac{1}{(n+1)!}\, G_{n+1,0,\dots,1,0} \\ & + \varepsilon \Big\{ - \frac{2}{(n+1)!}\, F_{n+1,0,\dots,1,0}\, A_{1,1} + \frac{4}{(n+1)!}\, G_{n+1,0,\dots,1,0}\, A_{1,1} + * A_{2,1} + \frac{-2+n-1}{3!\,(n-1)!} \Big\} \\ & + \mathrm{O}(\varepsilon^2), \\ 0 \equiv & - \frac{1}{(n+1)!}\, F_{n+1,0,\dots,0,1} + \frac{1}{(n+1)!}\, G_{n+1,0,\dots,0,1} \\ & + \varepsilon \Big\{ - \frac{2}{(n+1)!}\, F_{n+1,0,\dots,0,1}\, A_{1,1} + \frac{3}{(n+1)!}\, G_{n+1,0,\dots,0,1}\, A_{1,1} + \frac{-2+n}{2!\,n!}\, A_{2,1} \Big\} + \mathrm{O}(\varepsilon^2). \end{aligned}$$

We are going to solve the $G_\bullet$ in terms of the $F_\bullet$. For some $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$, we consider an equation with constant α, β, Λ_ν:

$$0 \equiv -F_\nu + G_\nu + \varepsilon \big\{ \alpha\, F_\nu + \beta\, G_\nu + \Lambda_\nu \big\} + \mathrm{O}(\varepsilon^2).$$

To determine the term S_ν with

$$G_\nu \overset{?}{=} F_\nu + \varepsilon\, S_\nu + \mathrm{O}(\varepsilon^2),$$

we replace and identify

$$\begin{aligned} 0 & \equiv \underline{-F_\nu + F_\nu}_{\circ} + \varepsilon\, S_\nu + \mathrm{O}(\varepsilon^2) + \varepsilon \Big\{ \alpha\, F_\nu + \beta\, \big(G_\nu + \varepsilon\, S_\nu + \mathrm{O}(\varepsilon^2) \big) + \Lambda_\nu \Big\} + \mathrm{O}(\varepsilon^2) \\ & \equiv \varepsilon \big\{ S_\nu + (\alpha+\beta)\, G_\nu + \Lambda_\nu \big\} + \mathrm{O}(\varepsilon^2), \end{aligned}$$

and hence,

$$S_\nu := -(\alpha+\beta)\, G_\nu - \Lambda_\nu.$$

Formulae of the same kind exist for a linear system involving several F_ν, G_ν as above.

We therefore obtain:

$$\begin{aligned} G_{n+2,0,\dots,0} & = F_{n+2,0,\dots,0} - \varepsilon \Big\{ n\, F_{n+2,0,\dots,0}\, A_{1,1} + \sum_{2\leqslant k\leqslant n} (n+2)\, F_{n+1,0\cdots 1\cdots 0}\, A_{k,1} + \tbinom{n+2}{2}\, B_n \Big\} + \mathrm{O}(\varepsilon^2), \\ G_{n+1,1,\dots,0} & = F_{n+1,1,\dots,0} - \varepsilon \Big\{ (n-1)\, F_{n+1,1,\dots,0}\, A_{1,1} \Big\} + \mathrm{O}(\varepsilon^2) \\ G_{n+1,0,1,,\dots,0} & = F_{n+1,0,1,\dots,0} - \varepsilon \Big\{ (n-2)\, F_{n+1,0,1,\dots,0}\, A_{1,1} + * A_{2,1} + \cdots + * A_{n-2,1} + \frac{-2+3}{(n-1)!3!}\, (n+1)!\, A_{n-1,1} \end{aligned}$$

for general k with $3 \leqslant k \leqslant n-1$:

$$G_{n+1,0\cdots 1\cdots 0} = F_{n+1,0\cdots 1\cdots 0} - \varepsilon \bigg\{ (n-k+1)\, F_{n+1,0\cdots 1\cdots 0}\, A_{1,1} + *\, A_{2,1} + \cdots + *\, A_{n-k+1,1} + \frac{-2+k}{(n-k+2)!k!}\,(n+1)!\, A_{n-k+2,1} \bigg\} + \mathrm{O}(\varepsilon^2),$$

and

$$G_{n+1,0,\dots,1,0} = F_{n+1,0,\dots,1,0} - \varepsilon \bigg\{ 2\, F_{n+1,0,\dots,1,0}\, A_{1,1} + *\, A_{2,1} + \frac{-2+n-1}{3!(n-1)!}\,(n+1)!\, A_{3,1} \bigg\} + \mathrm{O}(\varepsilon^2),$$

$$G_{n+1,0,\dots,0,1} = F_{n+1,0,\dots,0,1} - \varepsilon \bigg\{ 1\, F_{n+1,0,\dots,0,1}\, A_{1,1} + \frac{-2+n}{2!n!}\,(n+1)!\, A_{2,1} \bigg\} + \mathrm{O}(\varepsilon^2).$$

It is more natural to begin with the last equation.

Lemma 21.1. *It is possible to on the right, and then on the left:*

$$\begin{aligned} &G_{n+1,0,\dots,0,1} := 0, && F_{n+1,0,\dots,0,1} := 0,\\ &G_{n+1,0,\dots,1,0} := 0, && F_{n+1,0,\dots,0,1} := 0,\\ &\cdots\cdots\cdots && \cdots\cdots\cdots\\ &G_{n+1,0,1,\dots,0} := 0, && F_{n+1,0,1,\dots,0} := 0,\\ &G_{n+2,0,0,\dots,0} := 0, && F_{n+2,0,0,\dots,0} := 0, \end{aligned}$$

while the coefficient of $\frac{1}{(n+1)!}\, x_1^{n+1} x_2$ *is a relative invariant:*

$$G_{n+1,10\cdots 0} \propto F_{n+1,10\cdots 0}.$$

Proof. Indeed, the variable $A_{2,1}$ is free to make the first normalization, then the variable $A_{3,1}$ is free to make the second one, and so on until B_n is free to make the last normalization. We also observe that the system is triangular. Two other views of these normalizations will be given in the next two Section 23 and 22. □

22. Alternative more direct normalizations at order $(n+2)$

As we know from Section 19, the matrix of coefficients of a vector field L tangent up to order $n+1$ is

$$\begin{bmatrix} A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ A_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ A_{3,1} & \mathbf{0} & -A_{1,1} & \mathbf{0} & \cdots \\ A_{4,1} & \mathbf{0} & -2A_{2,1} & -2A_{1,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ A_{n-2,1} & \mathbf{0} & \frac{-1}{n-4}\binom{n-2}{3} A_{n-4,1} & \frac{-2}{n-5}\binom{n-2}{4} A_{n-5,1} & \cdots \\ A_{n-1,1} & \mathbf{0} & \frac{-1}{n-3}\binom{n-1}{3} A_{n-3,1} & \frac{-2}{n-4}\binom{n-1}{4} A_{n-4,1} & \cdots \\ A_{n,1} & \mathbf{0} & \frac{-1}{n-2}\binom{n}{3} A_{n-2,1} & \frac{-2}{n-3}\binom{n}{4} A_{n-3,1} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \end{bmatrix}$$

$$
\left.\begin{array}{cccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{2}A_{2,1} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{3}A_{3,1} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{4}A_{4,1} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{2}{5}A_{4,1} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{-(n-4)}{1}\binom{n-2}{n-2}A_{1,1} & \mathbf{0} & \mathbf{0} & -\frac{2}{n-1}A_{n-1,1} \\
\frac{-(n-4)}{2}\binom{n-1}{n-2}A_{2,1} & \frac{-(n-3)}{1}\binom{n-1}{n-1}A_{1,1} & \mathbf{0} & -\frac{2}{n}A_{n,1} \\
\frac{-(n-4)}{3}\binom{n}{n-2}A_{3,1} & \frac{-(n-3)}{2}\binom{n}{n-1}A_{2,1} & \frac{-(n-2)}{1}\binom{n}{n}A_{1,1} & B_n \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & 2\,A_{1,1}
\end{array}\right],
$$

With these coefficients,

$$
0 \equiv \pi_{\mathrm{ind}}^{n+1}\Big(L\left(-u+F(x)\right)\big|_{u=F(x)}\Big).
$$

Then we apply the derivation $L(\bullet)$ to the hypersurface equation (20.5) written up to order $n+2$, and we are going to compute $\pi_{n+2}^{\mathrm{ind}}\Big(L\left(-u+F(x)\right)\big|_{u=F(x)}\Big)$, namely we are going to find $\pi_{n+2}^{\mathrm{ind}}(\bullet)$ of

$$
\begin{aligned}
0 \equiv {} & -\,2\,A_{1,1}\bigg(\frac{x_1^2}{2!}+\frac{x_1^2x_2}{2!}+\cdots+\frac{x_1^nx_n}{n!}+F_{n+2,0,\dots,0}\frac{x_1^{n+1}x_1}{(n+2)!} \\
& \qquad\qquad +F_{n+1,1,\dots,0}\frac{x_1^{n+1}x_2}{(n+1)!}+\cdots+F_{n+1,0,\dots,1}\frac{x_1^{n+1}x_n}{(n+1)!}\bigg) \\
& +\bigg(A_{1,1}x_1-\frac{2}{2}A_{2,1}\Big[\frac{x_1^2}{2!}\Big|+\frac{x_1^2x_2}{2!}+\cdots+\frac{x_1^nx_n}{n!}\Big]\bigg) \\
& \cdot\bigg(\frac{x_1^1}{1!}\Big|+\frac{x_1^2x_2}{1!}+\cdots+\frac{x_1^{n-1}x_n}{(n-1)!}+F_{n+2,0,\dots,0}\frac{x_1^{n+1}}{(n+1)!} \\
& \quad +F_{n+1,1,\dots,0}\frac{x_1^nx_2}{n!}+\cdots+F_{n+1,0,\dots,1}\frac{x_1^nx_n}{n!}\bigg) \\
& +\bigg(A_{2,1}x_1-\frac{2}{3}A_{3,1}\Big[\frac{x_1^2}{2!}\Big|+\frac{x_1^2x_2}{2!}+\cdots+\frac{x_1^{n-1}x_{n-1}}{(n-1)!}\Big]\bigg) \\
& \cdot\bigg(\frac{x_1^2}{2!}\Big|+\frac{x_1^2x_2}{1!\,1!}+\cdots+\frac{x_1^{n-1}x_{n-1}}{1!\,(n-2)!}+F_{n+1,1,\dots,0}\frac{x_1^{n+1}}{(n+1)!}+\frac{x_1^nx_n}{1!(n-1)!}\bigg) \\
& +\bigg(A_{3,1}x_1-*A_{1,1}x_3-\frac{2}{4}A_{4,1}\Big[\frac{x_1^2}{2!}\Big|+\frac{x_1^2x_2}{2!}+\cdots+\frac{x_1^{n-2}x_{n-2}}{(n-2)!}\Big]\bigg) \\
& \cdot\bigg(\frac{x_1^3}{3!}\Big|+\frac{x_1^3x_2}{2!\,1!}+\cdots+\frac{x_1^{n-1}x_{n-2}}{2!\,(n-3)!}+F_{n+1,0,1,\dots,0}\frac{x_1^{n+1}}{(n+1)!}+\frac{x_1^nx_{n-1}}{2!(n-2)!}\bigg) \\
& +\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& +\bigg(A_{i,1}x_1-*A_{i-2,1}x_3-\cdots-*A_{2,1}x_{i-1}-(i-2)A_{1,1}x_i-\frac{2}{i+1}A_{i+1,1}\Big[\frac{x_1^2}{2!}\Big| \\
& \quad +\frac{x_1^2x_2}{2!}+\cdots+\frac{x_1^{n-i+1}x_{n-i+1}}{(n-i+1)!}\Big]\bigg)
\end{aligned}
$$

$$\cdot\left(\frac{x_1^i}{i!}\Big| + \frac{x_1^i x_2}{(i-1)!1!} + \cdots + \frac{x_1^{n-1}x_{n-i+1}}{(i-1)!(n-i)!} + F_{n+1,0\cdots1\cdots0}\frac{x_1^{n+1}}{(n+1)!} + \frac{x_1^n x_{n-i+2}}{(i-1)!(n-i+1)!}\right)$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\bigg(A_{n-3,1}x_1 - *A_{n-5,1}x_3 - \cdots - *A_{2,1}x_{n-4} - (n-5)A_{1,1}x_{n-3}$$

$$\qquad - \frac{2}{n-2}A_{n-2,1}\left[\frac{x_1^2}{2!}\Big| + \frac{x_1^2x_2}{2!} + \frac{x_1^3x_3}{3!} + \frac{x_1^4x_4}{4!}\right]\bigg)$$

$$\cdot\left(\frac{x_1^{n-3}}{(n-3)!}\Big| + \frac{x_1^{n-3}x_2}{(n-4)!1!} + \frac{x_1^{n-2}x_3}{(n-4)!2!} + \frac{x_1^{n-1}x_4}{(n-4)!3!} + F_{n+1,0,\dots,1,0,0,0}\frac{x_1^{n+1}}{(n+1)!} + \frac{x_1^n x_5}{(n-4)!4!}\right)$$

$$+\bigg(A_{n-2,1}x_1 - *A_{n-4,1}x_3 - \cdots - *A_{2,1}x_{n-3} - (n-4)A_{1,1}x_{n-2}$$

$$\qquad - \frac{2}{n-1}A_{n-1,1}\left[\frac{x_1^2}{2!}\Big| + \frac{x_1^2x_2}{2!} + \frac{x_1^3x_3}{3!}\right]\bigg)$$

$$\cdot\left(\frac{x_1^{n-2}}{(n-2)!}\Big| + \frac{x_1^{n-2}x_2}{(n-3)!1!} + \frac{x_1^{n-1}x_3}{(n-3)!2!} + F_{n+1,0,\dots,1,0,0}\frac{x_1^{n+1}}{(n+1)!} + \frac{x_1^n x_4}{(n-3)!3!}\right)$$

$$+\left(A_{n-1,1}x_1 - *A_{n-3,1}x_3 - \cdots - *A_{2,1}x_{n-2} - (n-3)A_{1,1}x_{n-1} - \frac{2}{n}A_{n,1}\left[\frac{x_1^2}{2!}\Big| + \frac{x_1^2x_2}{2!}\right]\right)$$

$$\cdot\left(\frac{x_1^{n-1}}{(n-1)!}\Big| + \frac{x_1^{n-1}x_2}{(n-2)!1!} + F_{n+1,0,\dots,1,0}\frac{x_1^{n+1}}{(n+1)!} + \frac{x_1^n x_3}{(n-2)!2!}\right)$$

$$+\left(A_{n,1}x_1 - *A_{n-2,1}x_3 - \cdots - *A_{2,1}x_{n-1} - (n-2)A_{1,1}x_n + B_n\left[\frac{x_1^2}{2!}\right]\right)$$

$$\cdot\left(\frac{x_1^n}{n!}\Big| + \frac{x_1^n x_2}{(n-1)!1!} + F_{n+1,0,\dots,0,1}\frac{x_1^{n+1}}{(n+1)!} + \frac{x_1^n x_2}{(n-1)!2!}\right).$$

By careful inspection of the products and after relevant simplifications, we find exactly the same factors (up to sign) of ε^1 as in the equations preceding Lemma 21.1:

$$0 \equiv \frac{x_1^{n+2}}{(n+2)!}\left\{n\,F_{n+2,0,\dots,0}\,A_{1,1} + \sum_{2\leqslant k\leqslant n}(n+2)\,F_{n+1,0\cdots1\cdots0}\,A_{k,1} + \tbinom{n+2}{2}\,B_n\right\}$$

$$\qquad + \frac{x_1^{n+1}x_2}{(n+1)!}\Big\{(n-1)\,F_{n+1,1,\dots,0}\,A_{1,1}\Big\}$$

$$+ \frac{x_1^{n+1}x_3}{(n+1)!}\left\{(n-2)\,F_{n+1,0,1,\dots,0}\,A_{1,1} + *\,A_{2,1} + \cdots + *\,A_{n-2,1} + \frac{-2+3}{(n-1)!3!}\,(n+1)!\,A_{n-1,1}\right\}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{x_1^{n+1}x_k}{(n+1)!}\bigg\{(n-k+1)\,F_{n+1,0\cdots1\cdots0}\,A_{1,1} + *\,A_{2,1} + \cdots + *\,A_{n-k+1,1}$$

$$\qquad + \frac{-2+k}{(n-k+2)!k!}\,(n+1)!\,A_{n-k+2,1}\bigg\}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{x_1^{n+1}x_{n-1}}{(n+1)!}\bigg\{2\,F_{n+1,0,\dots,1,0}\,A_{1,1}+*\,A_{2,1}+\frac{-2+n-1}{3!(n-1)!}\,(n+1)!\,A_{3,1}\bigg\}$$

$$+\frac{x_1^{n+1}x_{n-1}}{(n+1)!}\bigg\{1\,F_{n+1,0,\dots,0,1}\,A_{1,1}+\frac{-2+n}{2!n!}\,(n+1)!\,A_{2,1}\bigg\}.$$

In conclusion, this computation is a *shortcut* of the longer computation done previously with infinitesimal ε.

Examples 22.1. *In dimension* $n=2$*:*

$$\frac{x_1^4}{4!}\Big\{2\,F_{4,0}\,A_{1,1}+4\,F_{3,1}\,A_{2,1}+6\,B_2\Big\},$$
$$+\frac{x_1^3x_2}{3!}\Big\{F_{3,1}\,A_{1,1}\Big\}.$$

In dimension $n=3$*:*

$$\frac{x_1^5}{5!}\Big\{3\,F_{5,0,0}\,A_{1,1}+5\,F_{4,1,0}\,A_{2,1}+5\,F_{4,0,1}\,A_{3,1}+10\,B_3\Big\},$$
$$+\frac{x_1^4x_2}{4!}\Big\{2\,F_{4,1,0}\,A_{1,1}\Big\},$$
$$+\frac{x_1^4x_3}{4!}\Big\{F_{4,0,1}\,A_{1,1}+2\,A_{2,1}\Big\}.$$

In dimension $n=4$*:*

$$\frac{x_1^6}{6!}\Big\{4\,F_{6,0,0,0}\,A_{1,1}+6\,F_{5,1,0,0}\,A_{2,1}+6\,F_{5,0,1,0}\,A_{3,1}+6\,F_{5,0,0,1}\,A_{4,1}+15\,B_4\Big\},$$
$$+\frac{x_1^5x_2}{5!}\Big\{3\,F_{5,1,0,0}\,A_{1,1}\Big\},$$
$$+\frac{x_1^5x_3}{5!}\Big\{2\,F_{5,0,1,0}\,A_{1,1}-2\,F_{5,0,0,1}\,A_{2,1}+\frac{10}{3}\,A_{3,1}\Big\},$$
$$+\frac{x_1^5x_4}{5!}\Big\{1\,F_{5,0,0,1}\,A_{1,1}+5\,A_{2,1}\Big\}.$$

In dimension $n=5$*:*

$$\frac{x_1^7}{7!}\Big\{5\,F_{7,0,0,0,0}\,A_{1,1}+7\,F_{6,1,0,0,0}\,A_{2,1}+7\,F_{6,0,1,0,0}\,A_{3,1}+7\,F_{6,0,0,1,0}\,A_{4,1}+7\,F_{6,0,0,0,1}\,A_{5,1}+21\,B_5\Big\},$$
$$+\frac{x_1^6x_2}{6!}\Big\{4\,F_{6,1,0,0,0}\,A_{1,1}\Big\},$$
$$+\frac{x_1^6x_3}{6!}\Big\{3\,F_{6,0,1,0,0}\,A_{1,1}-2\,F_{6,0,0,1,0}\,A_{2,1}-\frac{10}{3}\,F_{6,0,0,0,1}\,A_{3,1}+5\,A_{4,1}\Big\},$$
$$+\frac{x_1^6x_4}{6!}\Big\{2\,F_{6,0,0,1,0}\,A_{1,1}-5\,F_{6,0,0,0,1}\,A_{2,1}+10\,A_{3,1}\Big\},$$
$$+\frac{x_1^6x_4}{6!}\Big\{1\,F_{6,0,0,0,1}\,A_{1,1}+9\,A_{2,1}\Big\}.$$

In dimension $n=6$*:*

$$\frac{x_1^8}{8!}\Big\{6\,F_{8,0,0,0,0,0}\,A_{1,1}+8\,F_{7,1,0,0,0,0}\,A_{2,1}+8\,F_{7,0,1,0,0,0}\,A_{3,1}+8\,F_{7,0,0,1,0,0}\,A_{4,1}$$
$$+8\,F_{7,0,0,0,1,0}\,A_{5,1}+8\,F_{7,0,0,0,0,1}\,A_{6,1}+28\,B_6\Big\},$$

$$
\begin{aligned}
&+\frac{x_1^7x_2}{7!}\Big\{5\,F_{7,1,0,0,0,0}\,A_{1,1}\Big\},\\
&+\frac{x_1^7x_3}{7!}\Big\{4\,F_{7,0,1,0,0,0}\,A_{1,1}-2\,F_{7,0,0,1,0,0}\,A_{2,1}-\frac{10}{3}\,F_{7,0,0,0,1,0}\,A_{3,1}-5\,F_{7,0,0,0,0,1}\,A_{4,1}+7\,A_{5,1}\Big\},\\
&+\frac{x_1^7x_4}{7!}\bigg\{3\,F_{7,0,0,1,0,0}\,A_{1,1}-5\,F_{7,0,0,0,1,0}\,A_{2,1}-10\,F_{7,0,0,0,0,1}\,A_{3,1}+\frac{35}{2}\,A_{4,1}\bigg\},\\
&+\frac{x_1^7x_5}{7!}\Big\{2\,F_{7,0,0,0,1,0}\,A_{1,1}-9\,F_{7,0,0,0,0,1}\,A_{2,1}+21\,A_{3,1}\Big\},\\
&+\frac{x_1^7x_6}{7!}\Big\{1\,F_{7,0,0,0,0,1}\,A_{1,1}+14\,A_{2,1}\Big\}.
\end{aligned}
$$

23. Normalizations at order $(n+2)$ via jet prolongations

As before, we treat $u = u(x_1, \dots, x_n)$ as a function of $(x_1, \dots, x_n)$. The letter F will not be used. Then to each partial derivative $u_{x_1^{\nu_1}\cdots x_n^{\nu_n}}(x_1, \dots, x_n)$, we can associate an independent coordinate (variable), denoted similarly $u_{x_1^{\nu_1}\cdots x_n^{\nu_n}}$, or sometimes more simply $u_{\nu_1,\dots,\nu_n}$. Background appears e.g. in [11].

For each integer $\kappa \geqslant 0$, we introduce a *jet space of order* κ, namely, $\mathbb{R}^{n+\binom{n+\kappa}{\kappa}}$ equipped with the coordinates

$$
\Big(x_1, \dots, x_n, u, \, \big(u_{x_1^{\nu_1}\cdots x_n^{\nu_n}}\big)_{1\leqslant \nu_1+\cdots+\nu_n\leqslant\kappa}\Big).
$$

We shall employ an abbreviation:

$$
u^{(\kappa)} := \big(u_{x_1^{\nu_1}\cdots x_n^{\nu_n}}\big)_{1\leqslant \nu_1+\cdots+\nu_n\leqslant\kappa}.
$$

For $i = 1, \dots, n$ we also introduce also *total differentiation operators*

$$
\mathsf{D}_{x_i} := \frac{\partial}{\partial x_i} + u_{x_i}\frac{\partial}{\partial u} + \sum_{m=1}^{\infty}\sum_{\nu_1+\cdots+\nu_n=m} u_{x_ix_1^{\nu_1}\cdots x_n^{\nu_n}}\frac{\partial}{\partial u_{x_1^{\nu_1}\cdots x_n^{\nu_n}}},
$$

which commute one with another.

Given a general vector field in the (x,u)-space

$$
L = \sum_{i=1}^{n} X_i(x,u)\frac{\partial}{\partial x_i} + U(x,u)\frac{\partial}{\partial u},
$$

its extension [11] to the infinite jet space

$$
L^{(\infty)} = L + \sum_{\nu_1+\cdots+\nu_n\geqslant 1} U_{\nu_1,\dots,\nu_n}\frac{\partial}{\partial u_{x_1^{\nu_1}\cdots x_n^{\nu_n}}},
$$

expresses how the (differentiated) flow of L acts on higher order jets, and its coefficients $U_{\nu_1,\dots,\nu_n}$ are uniquely determined by the formulae

$$
U_{\nu_1,\dots,\nu_n} := \mathsf{D}_{x_1^{\nu_1}}\cdots\mathsf{D}_{x_n^{\nu_n}}\Big(U - \sum_{1\leqslant i\leqslant n} X_i\cdot u_{x_i}\Big) + \sum_{1\leqslant i\leqslant n} X^i\cdot u_{x_ix_1^{\nu_1}\cdots x_n^{\nu_n}}.
$$

It is known that they depend on jet coordinates of order not exceeding $\nu_1+\cdots+\nu_n$:

$$
U_{\nu_1,\dots,\nu_n} = U_{\nu_1,\dots,\nu_n}\Big(x, u, \, u^{(\nu_1+\cdots+\nu_n)}\Big).
$$

According to Section 19 (see also the matrix in the beginning of Section 22), the general vector field, which stabilizes the normalizations up order $n+1$, i.e., which is tangent up to order $n+1$ to (19.1), reads as

$$\begin{aligned}
L =& \bigg(A_{1,1}x_1 - \frac{2}{2}A_{2,1}u\bigg)\frac{\partial}{\partial x_1} + \bigg(A_{2,1}x_1 - \frac{2}{3}A_{3,1}u\bigg)\frac{\partial}{\partial x_2} + \bigg(A_{3,1}x_1 - A_{1,1}x_3 - \frac{2}{4}A_{4,1}u\bigg)\frac{\partial}{\partial x_3} \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ \bigg(A_{n-1,1}x_1 - \frac{1}{n-3}\tbinom{n-1}{3}A_{n-3,1}x_3 - \cdots - \frac{n-3}{1}\tbinom{n-1}{n-1}A_{1,1}x_{n-1} - \frac{2}{n}A_{n,1}u\bigg)\frac{\partial}{\partial x_{n-1}} \\
&+ \bigg(A_{n,1}x_1 - \frac{1}{n-2}\tbinom{n}{3}A_{n-2,1}x_3 - \cdots - \frac{n-3}{2}\tbinom{n}{n-1}A_{2,1}x_{n-1} - \frac{n-2}{1}\tbinom{n}{n}A_{1,1}x_n + B_n u\bigg)\frac{\partial}{\partial x_n} \\
&+ 2A_{1,1}u\frac{\partial}{\partial u}.
\end{aligned}$$

Then we extend it to the jet space of order $n+2$:

$$\begin{aligned}
L^{(n+2)} =& L + \sum_{\nu_1+\cdots+\nu_n=1} U_{\nu_1,\ldots,\nu_n}\big(x, u, u^{(1)}\big) + \cdots + \sum_{\nu_1+\cdots+\nu_n=n+1} U_{\nu_1,\ldots,\nu_n}\big(x, u, u^{(n+1)}\big)\frac{\partial}{\partial u_{\nu_1,\ldots,\nu_n}} \\
&+ \sum_{\nu_1+\cdots+\nu_n=n+2} U_{\nu_1,\ldots,\nu_n}\big(x, u, u^{(n+2)}\big)\frac{\partial}{\partial u_{\nu_1,\ldots,\nu_n}}.
\end{aligned}$$

The following (admitted) statement can be established in a general theoretical context.

Lemma 23.1. *If L is tangent to* (20.5) *up to order $n+1$, then at the origin $(x,u)=(0,0)$, it holds:*

$$0 = U_{\nu_1,\ldots,\nu_n}\big(0,0,u^{(\nu_1+\cdots+\nu_n)}\big) \qquad \textit{for all} \qquad \nu_1+\cdots+\nu_n \leqslant n+1.$$

Furthermore, at order equal to $n+2$, by considering only independent jets, still at the origin, it can be shown in a general theoretical context that one recovers, up to sign and a change of notation, the expressions appearing in Section 22.

Lemma 23.2. *If L is tangent to* (20.5) *up to order $n+1$, then at the origin $(x,u)=(0,0)$, the identities hold:*

$$\begin{aligned}
U_{n+2,0,\ldots,0}\big(0,0,u^{(n+2)}\big) &= -n\,u_{n+2,0,\ldots,0}\,A_{1,1} - \sum_{2\leqslant k\leqslant n}(n+2)\,u_{n+1,0\cdots1\cdots0}\,A_{k,1} - \tbinom{n+2}{2}B_n, \\
U_{n+1,1,\ldots,0}\big(0,0,u^{(n+2)}\big) &= -(n-1)\,u_{n+1,1,\ldots,0}\,A_{1,1}, \\
U_{n+1,0,1,\ldots,0}\big(0,0,u^{(n+2)}\big) &= -(n-2)\,u_{n+1,0,1,\ldots,0}\,A_{1,1} - *\,A_{2,1} - \cdots - *\,A_{n-2,1} \\
&\quad - \frac{-2+3}{(n-1)!3!}(n+1)!\,A_{n-1,1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
U_{n+1,0\cdots1\cdots0}\big(0,0,u^{(n+2)}\big) &= -(n-k+1)\,u_{n+1,0\cdots1\cdots0}\,A_{1,1} - *\,A_{2,1} - \cdots - *\,A_{n-k+1,1} \\
&\quad - \frac{-2+k}{(n-k+2)!k!}(n+1)!\,A_{n-k+2,1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
U_{n+1,0,\ldots,1,0}\big(0,0,u^{(n+2)}\big) &= -2\,F_{n+1,0,\ldots,1,0}\,A_{1,1} - *\,A_{2,1} - \frac{-2+n-1}{3!(n-1)!}(n+1)!\,A_{3,1} \\
U_{n+1,0,\ldots,0,1}\big(0,0,u^{(n+2)}\big) &= -1\,F_{n+1,0,\ldots,0,1}\,A_{1,1} - \frac{-2+n}{2!n!}(n+1)!\,A_{2,1}.
\end{aligned}$$

Example 23.1. *For instance, as $n=3$, the vector field from Section 17 reads as*

$$\begin{aligned} L \,=\, & \left(A_{1,1}\,x_1 - \frac{2}{2}A_{2,1}\,u\right)\frac{\partial}{\partial x_1} + \left(A_{2,1}\,x_1 - \frac{2}{3}A_{3,1}\,u\right)\frac{\partial}{\partial x_2} \\ & + \left(A_{3,1}\,x_1 - A_{1,1}\,x_3 - \frac{2}{4}A_{4,1}\,u\right)\frac{\partial}{\partial x_3} + 2\,A_{1,1}\,u\frac{\partial}{\partial u}, \end{aligned}$$

has continuation of order 5 *above the origin* $(x,u)=(0,0)$ *given by*

$$\begin{aligned} L^{(5)} \,=\, & -\,(3\,u_{5,0,0}\,A_{1,1} + 5\,u_{4,1,0}\,A_{2,1} + 5\,u_{4,0,1}\,A_{3,1} + 10\,B_3)\,\frac{\partial}{\partial u_{5,0,0}} \\ & -\,2\,u_{4,1,0}\,A_{1,1}\frac{\partial}{\partial u_{4,1,0}} - (u_{4,0,1}\,A_{1,1} + 2\,A_{2,1})\,\frac{\partial}{\partial u_{4,0,1}}. \end{aligned}$$

Hence, for $n=3$, in the space $\mathbb{R}^3 \in (u_{5,0,0}, u_{4,1,0}, u_{4,0,1})$ of pure $(n+2)$-jets, we have a linear space of vector fields parametrized by four free coefficients $A_{1,1}$, $A_{2,1}$, $A_{3,1}$, B_3. This is in fact the Lie algebra of the action of the stability group in orders $\leqslant n+1$ acting on pure $n+2$ order monomials.

Without computing the flows of these vector fields (which would amount to recover the formulae of Section 15), we can realize that the two normalizations

$$F_{4,0,1} \,:=\, 0, \qquad F_{5,0,0} \,:=\, 0,$$

are possible.

- *By taking $A_{2,1} := 1$ and the others zero, we have*

$$L^{(5)} \,=\, -\,5\,u_{4,1,0}\,\frac{\partial}{\partial u_{5,0,0}} - 0\,\frac{\partial}{\partial u_{4,1,0}} - 2\,\frac{\partial}{\partial u_{4,0,1}},$$

and the flow of the constant vector field along the $u_{4,0,1}$-axis clearly crosses the axis $\{u_{4,0,1}=0\}$.

- *Assuming $u_{4,0,1}=0$, by taking $B_3 := 1$ and the others zero, we have*

$$L^{(5)} \,=\, -\,10\,\frac{\partial}{\partial u_{5,0,0}} - 0\,\frac{\partial}{\partial u_{4,1,0}} - 0\,\frac{\partial}{\partial u_{4,0,1}},$$

a vector field that stabilizes $\{u_{4,0,1}=0\}$ and whose flow crosses the axis $\{u_{5,0,0}=0\}$.

24. Normalizations of order $(n+3)$ monomials

Thanks to Proposition 21.1, several order $n+2$ independent monomials can be normalized to zero:

$$F_{n+2,0,\dots,0} \,:=\, 0, \qquad F_{n+1,0,1,\dots,0} \,:=\, 0, \qquad \dots\dots \qquad F_{n+1,0,\dots,1} \,:=\, 0, \tag{24.1}$$

hence if we come back to (20.5) and let appear the next order $n+3$ monomials, disregarding as before the body-dependent ones, we get the equation

$$\begin{aligned} u \,=\, & \frac{x_1^2}{2} + \frac{x_1^2x_2}{2} + \sum_{m=3}^{n}\Big(\frac{x_1^m x_m}{m!} + x_1^{m-1}\sum_{\substack{i,j\geqslant 2\\ i+j=m+1}}\frac{1}{2}\,\frac{x_ix_j}{(i-1)!(j-1)!}\Big) \\ & + 0 + F_{n+1,1,\dots,0}\,\frac{x_1^{n+1}x_2}{(n+1)!} + 0 + \cdots + 0 + x_1^n\sum_{\substack{i,j\geqslant 2\\ i+j=n+2}}\frac{1}{2}\,\frac{x_ix_j}{(i-1)!(j-1)!} \\ & + F_{n+3,0,\dots,0}\,\frac{x_1^{n+2}x_1}{(n+3)!} + F_{n+2,1,\dots,0}\,\frac{x_1^{n+2}x_2}{(n+2)!} + F_{n+2,0,1,\dots,0}\,\frac{x_1^{n+2}x_3}{(n+2)!} + \cdots + F_{n+2,0,\dots,1}\,\frac{x_1^{n+2}x_n}{(n+2)!} \end{aligned}$$

$$+ F_{n+1,1,\dots,0}\,\frac{x_1^{n+1}x_2x_2}{n!} + x_1^{n+1}\sum_{\substack{i,j\geqslant 2\\ i+j=n+3}}\frac{1}{2}\,\frac{x_ix_j}{(i-1)!(j-1)!} + \mathrm{O}_{x'}(3) + \mathrm{O}_x(n+4),$$

where the border-dependent monomials written in the last line involve a supplementary monomial $F_{n+1,1,\dots,0}\,\frac{x_1^n x_2 x_2}{n!}$; this can be confirmed by reasoning as in Proposition 12.1.

We come back to the expression of a general affine vector field L tangent up to order $n+1$. Taking into consideration current normalization (24.1), by looking at the equation in the end of Section 22, we see that the tangency of L up to order $n+2$ requires

$$A_{2,1} := 0, \qquad A_{3,1} := 0, \qquad \dots\dots, \qquad A_{n-1,1} := 0, \qquad B_n := 0,$$

hence coming back to (20.1), we get the following vector field:

$$\begin{aligned} L &= A_{1,1}\,x_1,\partial_{x_1} + 0\,\partial_{x_2} - A_{1,1}\,x_3\,\partial_{x_3} + \dots + + \Big(-(n-3)A_{1,1}x_{n-1} - \frac{2}{n}\,A_{n,1}\,u\Big)\,\partial_{x_{n-1}} \\ &\quad + (A_{n,1}x_1 - (n-2)A_{1,1}x_n)\,\partial_{x_n} + 2\,A_{1,1}\,u\,\partial_u, \end{aligned}$$

the flow of it, according to Lemma 20.1, at the next order stabilizes monomials of order not exceeding $n+2$.

In order to see the action on order $n+3$ monomials, we apply $\pi_{n+3}^{\mathrm{ind}}(\bullet)$ to the tangency equation, and not writing terms which do not contribute, we find:

$$\begin{aligned} 0 \equiv &- 2\,A_{1,1}\Big(\dots + F_{n+3,0,\dots,0}\frac{x_1^{n+2}x_1}{(n+3)!} + F_{n+2,1,\dots,0}\frac{x_1^{n+2}x_2}{(n+2)!} \\ &\qquad + F_{n+2,0,1,\dots,0}\frac{x_1^{n+2}x_3}{(n+2)!} + \dots + F_{n+2,0,\dots,1}\frac{x_1^{n+2}x_n}{(n+2)!}\Big) \\ &+ A_{1,1}x_1\Big(\dots + F_{n+3,0,\dots,0}\frac{x_1^{n+2}}{(n+2)!} + F_{n+2,1,\dots,0}\frac{x_1^{n+1}x_2}{(n+1)!} \\ &\qquad + F_{n+2,0,1,\dots,0}\frac{x_1^{n+1}x_3}{(n+1)!} + \dots + F_{n+2,0,\dots,1}\frac{x_1^{n+1}x_n}{(n+2)!}\Big) \\ &+ 0 - A_{1,1}x_3\Big(\dots + F_{n+2,0,1,\dots,0}\frac{x_1^{n+2}}{(n+2)!} + x_1^{n+1}\,\frac{x_n}{2!(n-1)!}\Big) \\ &- 2A_{1,1}x_4\Big(\dots + F_{n+2,0,0,1,\dots,0}\frac{x_1^{n+2}}{(n+2)!} + x_1^{n+1}\,\frac{x_{n-1}}{3!(n-2)!}\Big) \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &- (n-4)A_{1,1}x_{n-2}\Big(\dots + F_{n+2,0,\dots,1,0,0}\frac{x_1^{n+2}}{(n+2)!} + x_1^{n+1}\frac{x_5}{(n-3)!4!}\Big) \\ &+ \Big(-(n-3)A_{1,1}x_{n-1} - \frac{2}{n}A_{n,1}\big[\frac{x_1^2}{2!}\Big| + \frac{x_1^2x_2}{2!} + \frac{x_1^3x_3}{3!}\big]\Big) \\ &\cdot\Big(\frac{x_1^{n-1}}{(n-1)!}\Big| + \frac{x_1^{n-1}x_2}{1!(n-2)!} + \frac{x_1^nx_3}{2!(n-2)!} + F_{n+2,0,\dots,1,0}\frac{x_1^{n+2}}{(n+2)!} + \frac{x_1^{n+1}x_4}{3!(n-2)!}\Big) \\ &+ \big(A_{n,1}x_1 - (n-2)A_{1,1}x_n\big)\Big(\frac{x_1^n}{n!}\Big| + \frac{x_1^nx_2}{1!(n-1)!} + F_{n+2,0,\dots,1}\frac{x_1^{n+2}}{(n+2)!} + \frac{x_1^{n+1}x_3}{2!(n-1)!}\Big). \end{aligned}$$

We then collect all the independent monomials of order $n+3$:

$$\begin{aligned} 0 \equiv &x_1^{n+3}\Big\{-\frac{2}{(n+3)!}F_{n+3,0,\dots,0}A_{1,1} + \frac{1}{(n+2)!}F_{n+3,0,\dots,0}A_{1,1} + \frac{1}{(n+2)!}F_{n+2,0,\dots,1}A_{n,1}\Big\} \\ &+ x_1^{n+2}x_2\Big\{\frac{-2}{(n+2)!}F_{n+2,1,\dots,0}A_{1,1} + \frac{1}{(n+1)!}F_{n+2,1,\dots,0}A_{1,1}\Big\} \end{aligned}$$

$$
\begin{aligned}
&+x_1^{n+2}x_3\bigg\{\frac{-2}{(n+2)!}F_{n+2,0,1,\dots,0}A_{1,1}+\frac{1}{(n+1)!}F_{n+2,0,1,\dots,0}A_{1,1}\\
&\qquad -\frac{1}{(n+2)!}F_{n+2,0,1,\dots,0}A_{1,1}-\frac{2}{n}\frac{1}{3!(n-1)!}A_{n,1}\\
&\qquad -\frac{2}{n}\frac{1}{2!}\frac{1}{(n-2)!2!}A_{n,1}+\frac{1}{2!(n-1)!}A_{n,1}\bigg\}\\
&+x_1^{n+2}x_4\bigg\{\frac{-2}{(n+2)!}F_{n+2,0,0,1,\dots,0}A_{1,1}+\frac{1}{(n+1)!}F_{n+2,0,0,1,\dots,0}A_{1,1}-\frac{2}{(n+2)!}F_{n+2,0,0,1,\dots,0}A_{1,1}\bigg\}\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+x_1^{n+2}x_{n-2}\bigg\{\frac{-2}{(n+2)!}F_{n+2,0,\dots,1,0,0}A_{1,1}+\frac{1}{(n+1)!}F_{n+2,0,\dots,1,0,0}A_{1,1}\\
&\qquad -\frac{n-4}{(n+2)!}F_{n+2,0,\dots,1,0,0}A_{1,1}\bigg\}\\
&+x_1^{n+2}x_{n-1}\bigg\{\frac{-2}{(n+2)!}F_{n+2,0,\dots,1,0}A_{1,1}+\frac{1}{(n+1)!}F_{n+2,0,\dots,1,0}A_{1,1}-\frac{n-3}{(n+2)!}F_{n+2,0,\dots,1,0}A_{1,1}\bigg\}\\
&+x_1^{n+2}x_n\bigg\{\frac{-2}{(n+2)!}F_{n+2,0,\dots,1}A_{1,1}+\frac{1}{(n+1)!}F_{n+2,0,\dots,1}A_{1,1}-\frac{n-2}{(n+2)!}F_{n+2,0,\dots,1}A_{1,1}\bigg\}.
\end{aligned}
$$

After simplifications, this becomes

$$
\begin{aligned}
0 \equiv\, & x_1^{n+3}\bigg\{\frac{n+1}{(n+3)!}F_{n+3,0,\dots,0}A_{1,1}+\frac{1}{(n+2)!}F_{n+2,0,\dots,1}A_{n,1}\bigg\}\\
&+x_1^{n+2}x_2\bigg\{\frac{n}{(n+2)!}F_{n+2,1,\dots,0}A_{1,1}\bigg\}\\
&+x_1^{n+2}x_3\bigg\{\frac{n-1}{(n+2)!}F_{n+2,0,1,\dots,0}A_{1,1}+\frac{1}{3!n!}A_{n,1}\bigg\}\\
&+x_1^{n+2}x_4\bigg\{\frac{n-2}{(n+2)!}F_{n+2,0,0,1,\dots,0}A_{1,1}\bigg\}\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+x_1^{n+2}x_{n-2}\bigg\{\frac{4}{(n+2)!}F_{n+2,0,\dots,1,0,0}A_{1,1}\bigg\}\\
&+x_1^{n+2}x_{n-1}\bigg\{\frac{3}{(n+2)!}F_{n+2,0,\dots,1,0}A_{1,1}\bigg\}\\
&+x_1^{n+2}x_n\bigg\{\frac{2}{(n+2)!}F_{n+2,0,\dots,1}A_{1,1}\bigg\}.
\end{aligned}
$$

Hence, we can normalize as

$$F_{n+2,0,1,\dots,0} := 0,$$

while all the other independent coefficients of order $n+3$ are relative invariants.

Finally, to stabilize the obtained order $(n+3)$ normalizations,

$$
\begin{aligned}
u =\, & \frac{x_1^2}{2}+\frac{x_1^2x_2}{2}+\sum_{m=3}^{n}\Bigg(\frac{x_1^mx_m}{m!}+x_1^{m-1}\sum_{\substack{i,j\geqslant 2\\ i+j=m+1}}\frac{1}{2}\frac{x_ix_j}{(i-1)!(j-1)!}\Bigg)+0\\
&+F_{n+1,1,\dots,0}\frac{x_1^{n+1}x_2}{(n+1)!}+0+\cdots+0+x_1^n\sum_{\substack{i,j\geqslant 2\\ i+j=n+2}}\frac{1}{2}\frac{x_ix_j}{(i-1)!(j-1)!}
\end{aligned}
$$

$$+ F_{n+3,0,\dots,0}\,\frac{x_1^{n+2}x_1}{(n+3)!} + F_{n+2,1,\dots,0}\,\frac{x_1^{n+2}x_2}{(n+2)!} + 0 + F_{n+2,0,0,1,\dots,0}\,\frac{x_1^{n+2}x_4}{(n+2)!} + \cdots$$
$$+ F_{n+2,0,\dots,1}\,\frac{x_1^{n+2}x_n}{(n+2)!} + F_{n+1,1,\dots,0}\,\frac{x_1^{n+1}x_2x_2}{n!} + x_1^{n+1}\sum_{\substack{i,j\geqslant 2\\ i+j=n+3}}\frac{1}{2}\,\frac{x_ix_j}{(i-1)!(j-1)!}$$
$$+ \mathrm{O}_{x'}(3) + \mathrm{O}_x(n+4),$$

we deduce that

$$A_{n,1} := 0.$$

The isotropy Lie algebra is 1-dimensional and is represented by the matrix:

$$\begin{bmatrix} A_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -A_{1,1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -2A_{1,1} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -(n-3)A_{1,1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -(n-2)A_{1,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & 2\,A_{1,1} \end{bmatrix}.$$

25. Orders $(n+4)$ and $(n+3)$

The corresponding matrix Lie group

$$\begin{bmatrix} a_{1,1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a_{1,1}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a_{1,1}^2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{a_{1,1}^{n-3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{a_{1,1}^{n-2}} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{1,1}^2 \end{bmatrix},$$

consists of the plain dilations

$$y_1 = a_{1,1}\,x_1, \qquad y_2 = 0, \qquad y_3 = \frac{1}{a_{1,1}}\,x_3, \qquad \dots, \qquad y_n = \frac{1}{a_{1,1}^{n-2}}\,x_n, \qquad v = a_{1,1}^2\,u.$$

Lemma 25.1. *All power series coefficients $F_{\sigma_1,\dots,\sigma_n}$ are relative invariants.*

Proof. Possible remaining maps fixing the origin which send a hypersurface normalized as above

$$u = \sum_{\sigma_1,\dots,\sigma_n} x_1^{\sigma_1}\cdots x_n^{\sigma_n}\,F_{\sigma_1,\dots,\sigma_n}$$

to a similarly normalized hypersurface

$$v = \sum_{\sigma_1,\dots,\sigma_n} y_1^{\sigma_1}\cdots y_n^{\sigma_n}\,G_{\sigma_1,\dots,\sigma_n},$$

are only the dilations

$$y_1 = \boldsymbol{a}\,x_1, \qquad y_2 = 0, \qquad y_3 = \frac{1}{\boldsymbol{a}}\,x_3, \qquad \dots, \qquad y_n = \frac{1}{\boldsymbol{a}^{n-2}}\,x_n, \qquad v = \boldsymbol{a}^2\,u.$$

where we have abbreviated $a_{1,1} =: \boldsymbol{a}$. After replacement

$$\boldsymbol{a}^2\, u = \sum_{\sigma_1,\dots,\sigma_n} (\boldsymbol{a}\, x_1)^{\sigma_1}\, (x_2)^{\sigma_2} \left(\frac{1}{\boldsymbol{a}}\, x_3\right)^{\sigma_3} \cdots \left(\frac{1}{\boldsymbol{a}^{n-2}}\, x_n\right)^{\sigma_n} G_{\sigma_1,\dots,\sigma_n},$$

an identification gives:

$$F_{\sigma_1,\dots,\sigma_n} = \boldsymbol{a}^{-2}\, \boldsymbol{a}^{\sigma_1}\, \boldsymbol{a}^{-\sigma_3} \cdots \boldsymbol{a}^{-(n-2)\sigma_n}\, G_{\sigma_1,\dots,\sigma_n}.$$

Thus, each pair of power series coefficients are nonzero multiple one of the other. Thus, they vanish (or do not vanish) simultaneously. □

Consequently, at the next two orders $n+4$ and $n+5$ we shall not perform any further normalization since this would create some branching. But we must determine the independent and border-dependent monomials of homogeneous degrees $n+4$ and $n+5$.

Proceeding as in Lemma 12.1 to take account the assumption that the Hessian matrix is of constant rank 1, we obtain the following explicit expressions. We skip presenting the details of computations.

Theorem 25.1. *In each dimension $n \geqslant 2$, every local hypersurface $H^n \subset \mathbb{R}^{n+1}$ having constant Hessian rank 1 which is not affinely equivalent to a product of $\mathbb{R}^m$ ($1 \leqslant m \leqslant n$) with a hypersurface $H^{n-m} \subset \mathbb{R}^{n-m+1}$ can be affinely normalized as*

$$\begin{aligned}
u = {} & \frac{x_1^2}{2} + \frac{x_1^2\, x_2}{2} + \sum_{m=3}^{n} \left(\frac{x_1^m\, x_m}{m!} + x_1^{m-1} \sum_{\substack{i,j\geqslant 2\\ i+j=m+1}} \frac{1}{2}\, \frac{x_i\, x_j}{(i-1)!(j-1)!} \right) + F_{n+1,10\cdots 0}\, \frac{x_1^{n+1} x_2}{(n+1)!} \\
& + x_1^n \sum_{\substack{i,j\geqslant 2\\ i+j=n+2}} \frac{1}{2}\, \frac{x_i\, x_j}{(i-1)!(j-1)!} + F_{n+3,0\cdots 0}\, \frac{x_1^{n+3}}{(n+3)!} + F_{n+2,10\cdots 0}\, \frac{x_1^{n+2}\, x_2}{(n+2)!} \\
& + F_{n+2,0010\cdots 0}\, \frac{x_1^{n+2}\, x_4}{(n+2)!} + \cdots + F_{n+2,0\cdots 01}\, \frac{x_1^{n+2}\, x_n}{(n+2)!} + F_{n+1,10\cdots 0}\, \frac{x_1^{n+1}\, x_2\, x_2}{n!} \\
& + x_1^{n+1} \sum_{\substack{i,j\geqslant 2\\ i+j=n+3}} \frac{1}{2}\, \frac{x_i\, x_j}{(i-1)!(j-1)!} + F_{n+4,0\cdots 0}\, \frac{x_1^{n+4}}{(n+4)!} + F_{n+3,10\cdots 0}\, \frac{x_1^{n+3}\, x_2}{(n+3)!} \\
& + F_{n+3,010\cdots 0}\, \frac{x_1^{n+3}\, x_3}{(n+3)!} + F_{n+3,0010\cdots 0}\, \frac{x_1^{n+3}\, x_4}{(n+3)!} + \cdots + F_{n+3,0\cdots 01}\, \frac{x_1^{n+3}\, x_n}{(n+3)!} \\
& + x_1^{n+2} \left[\frac{F_{n+2,10\cdots 0}}{(n+1)!}\, x_2\, x_2 + \frac{F_{n+1,10\cdots 0}}{2!\, n!}\, x_2\, x_3 + \frac{F_{n+2,0010\cdots 0}}{(n+1)!}\, x_2\, x_4 + \cdots + \frac{F_{n+2,0\cdots 01}}{(n+1)!}\, x_2\, x_n \right. \\
& \qquad \left. + \sum_{\substack{i,j\geqslant 2\\ i+j=n+4}} \frac{1}{2}\, \frac{x_i\, x_j}{(i-1)!(j-1)!} \right] \\
& + F_{n+5,0\cdots 0}\, \frac{x_1^{n+5}}{(n+5)!} + F_{n+4,10\cdots 0}\, \frac{x_1^{n+4}\, x_2}{(n+4)!} + F_{n+4,010\cdots 0}\, \frac{x_1^{n+4}\, x_3}{(n+4)!} \\
& + F_{n+4,0010\cdots 0}\, \frac{x_1^{n+4}\, x_4}{(n+4)!} + \cdots + F_{n+4,0\cdots 01}\, \frac{x_1^{n+4}\, x_n}{(n+4)!} \\
& + x_1^{n+3} \left[\left(\frac{F_{n+3,10\cdots 0}}{(n+2)!} - \frac{F_{n+3,0\cdots 0}}{2!\,(n+1)!} \right) x_2 x_2 + \left(\frac{F_{n+3,010\cdots 0}}{(n+2)!} + \frac{F_{n+2,10\cdots 0}}{2!\,(n+1)!} \right) x_2 x_3 \right. \\
& \qquad + \left(\frac{F_{n+3,0010\cdots 0}}{(n+2)!} + \frac{F_{n+1,10\cdots 0}}{3!\, n!} \right) x_2 x_4 + \frac{F_{n+3,00010\cdots 0}}{(n+2)!}\, x_2 x_5 + \cdots + \frac{F_{n+3,0\cdots 01}}{(n+2)!}\, x_2 x_n
\end{aligned}$$

$$+\frac{F_{n+2,0010\cdots 0}}{2!\,(n+2)!}\,x_3x_4+\cdots+\frac{F_{n+2,0\cdots 01}}{2!\,(n+2)!}\,x_3x_n+\sum_{\substack{i,j\geqslant 2\\ i+j=n+5}}\frac{1}{2}\,\frac{x_i\,x_j}{(i-1)!(j-1)!}\Bigg]$$

$$+\mathrm{O}_{x_2,\dots,x_n}(3)+\mathrm{O}_{x_1,x_2,\dots,x_n}(n+6).$$

This explicit expression of the graphing function $F(x_1,\dots,x_n)$ is our new starting point.

26. Summary of proof of main Theorem 1.3

We take a general affine vector field which does not necessarily vanish at the origin:

$$\begin{aligned}L=&\Big(T_0+C_1\,x_1+C_2\,x_2+\cdots+C_{n-1}\,x_{n-1}+C_n\,x_n\Big)\,\frac{\partial}{\partial u}\\&+\Big(T_1+A_{1,1}\,x_1+A_{1,2}\,x_2+\cdots+A_{1,n-1}\,x_{n-1}+A_{1,n}\,x_n\Big)\,\frac{\partial}{\partial x_1}\\&+\Big(T_2+A_{2,1}\,x_1+A_{2,2}\,x_2+\cdots+A_{2,n-1}\,x_{n-1}+A_{2,n}\,x_n\Big)\,\frac{\partial}{\partial x_2}\\&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&+\Big(T_{n-1}+A_{n-1,1}\,x_1+A_{n-1,2}\,x_2+\cdots+A_{n-1,n-1}\,x_{n-1}+A_{n-1,n}\,x_n\Big)\,\frac{\partial}{\partial x_{n-1}}\\&+\Big(T_n+A_{n,1}\,x_1+A_{n,2}\,x_2+\cdots+A_{n,n-1}\,x_{n-1}+A_{n,n}\,x_n\Big)\,\frac{\partial}{\partial x_n}.\end{aligned}$$

We recall that if L is tangent to the hypersurface $H=\big\{u=F(x_1,\dots,x_n)\big\}$, then $T_0=0$, since $u=F=\mathrm{O}_x(2)$. Here the parameters $T_1,\dots,T_n$ are tightly related to the *infinitesimal transitivity* of the action since the value of L at the origin is

$$L\big|_0=T_1\,\frac{\partial}{\partial x_1}+\cdots+T_n\,\frac{\partial}{\partial x_n},$$

since homogeneity requires

$$T_0H=\mathrm{Span}_{\mathbb{R}}\,\big\{L\big|_0\colon\,L|_H\text{ tangent to }H\big\},$$

and since, again due to $F=\mathrm{O}_x(2)$,

$$T_0H=\mathrm{Span}\,\bigg\{\frac{\partial}{\partial x_1},\,\dots,\,\frac{\partial}{\partial x_n}\bigg\}.$$

Observation 26.1. *For (infinitesimal) affine homogeneity to hold, the parameters* $T_1,\dots,T_n$ *should remain absolutely free in all computations.*

Such general affine vector field L is an infinitesimal affine symmetry of our hypersurface $H=\big\{u=F(x_1,\dots,x_n)\big\}$ graphed as in Theorem 25.1 if and only if $L|_H$ is tangent to H and if and only if the following power series identity holds in $\mathbb{R}\{x_1,\dots,x_n\}$:

$$0\equiv L\,(-u+F)\,\Big|_{u=F}.$$

We shall in fact 'only' study independent monomials of order not exceeding $n+4$ in this fundamental equation, namely, we shall examine $\pi_{\mathrm{ind}}^{n+4}\big(L(-u+F)\big|_{u=F}\big)$. Recalling that

$$F_{x_1}=\mathrm{O}_x(1),\qquad F_{x_2}=\mathrm{O}_x(2),\qquad\cdots\qquad F_{x_{n-1}}=\mathrm{O}_x(n-1),\qquad F_{x_n}=\mathrm{O}_x(n),$$

we can therefore begin with writing

$$0\equiv\Lambda_0+\Lambda_1+\cdots+\Lambda_{n-1}+\Lambda_n,$$

where

$$\Lambda_0:=-C_1\,x_1-C_2\,x_2-\cdots-C_{n-1}\,x_{n-1}-C_n\,x_n-D\,\big[\pi_{\mathrm{ind}}^{n+4}(F)\big],$$

$$\Lambda_1 := \Big(T_1 + A_{1,1}\,x_1 + A_{1,2}\,x_2 + \cdots + A_{1,n-1}\,x_{n-1} + A_{1,n}\,x_n + B_1\,\big[\pi_{\rm ind}^{n+3}(F)\big]\Big)\,F_{x_1},$$
$$\Lambda_2 := \Big(T_2 + A_{2,1}\,x_1 + A_{2,2}\,x_2 + \cdots + A_{2,n-1}\,x_{n-1} + A_{2,n}\,x_n + B_2\,\big[\pi_{\rm ind}^{n+2}(F)\big]\Big)\,F_{x_2},$$
$$\Lambda_3 := \Big(T_3 + A_{3,1}\,x_1 + A_{3,2}\,x_2 + \cdots + A_{3,n-1}\,x_{n-1} + A_{3,n}\,x_n + B_3\,\big[\pi_{\rm ind}^{n+1}(F)\big]\Big)\,F_{x_3},$$
$$\Lambda_4 := \Big(T_4 + A_{4,1}\,x_1 + A_{4,2}\,x_2 + \cdots + A_{4,n-1}\,x_{n-1} + A_{4,n}\,x_n + B_4\,\big[\pi_{\rm ind}^{n}(F)\big]\Big)\,F_{x_4},$$
$$\Lambda_5 := \Big(T_5 + A_{5,1}\,x_1 + A_{5,2}\,x_2 + \cdots + A_{5,n-1}\,x_{n-1} + A_{5,n}\,x_n + B_5\,\big[\pi_{\rm ind}^{n-1}(F)\big]\Big)\,F_{x_5},$$
$$\Lambda_6 := \Big(T_6 + A_{6,1}\,x_1 + A_{6,2}\,x_2 + \cdots + A_{6,n-1}\,x_{n-1} + A_{6,n}\,x_n + B_6\,\big[\pi_{\rm ind}^{n-2}(F)\big]\Big)\,F_{x_6},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Lambda_{n-1} := \Big(T_{n-1} + A_{n-1,1}x_1 + A_{n-1,2}x_2 + \cdots + A_{n-1,n-1}x_{n-1} + A_{n-1,n}x_n + B_{n-1}\big[\pi_{\rm ind}^{5}(F)\big]\Big)F_{x_{n-1}},$$
$$\Lambda_n := \Big(T_n + A_{n,1}\,x_1 + A_{n,2}\,x_2 + \cdots + A_{n,n-1}\,x_{n-1} + A_{n,n}\,x_n + B_n\,\big[\pi_{\rm ind}^{4}(F)\big]\Big)\,F_{x_n}.$$

In the next Section 27, we shall compute some of the coefficients of the monomials in this large equation, namely, we shall find

$$E_{[\sigma_1,\dots,\sigma_n]} \,:=\, \big[x^{\sigma_1}\cdots x_n^{\sigma_n}\big]\,\Big(L\,(-\,u+F)\,\Big|_{u=F}\Big), \qquad \sigma_1+\cdots+\sigma_n \,\leqslant\, n+4,$$

which are linear in $C_\bullet$, D, $T_\bullet$, $A_{\bullet,\bullet}$, $B_\bullet$, and which should vanish for L to be tangent to H:

$$E_{[\sigma_1,\dots,\sigma_n]} \,=\, 0.$$

In particular, we shall find

$$\mathbf{I} \,:=\, E_{[n+2,0,\dots,0,1]} \,=\, 0,$$
$$\mathbf{II} \,:=\, E_{[n+3,0,\dots,0,1]} \,=\, 0.$$

But before proceeding to the (non-straightforward) computations, let us summarize the key reason why affinely homogeneous models do *not* exist in dimension $n \geqslant 5$. We use $*$ to denote any unspecified real number whose value does not matter.

Lemma 26.1. *For a hypersurface $\{u = F(x)\}$ normalized as in Theorem 25.1, after taking into consideration some of the other equations $E_{[\sigma_1,\dots,\sigma_n]} = 0$, these two specific equations* **I**, **II** *become*

$$0 \overset{\mathbf{I}}{=} *\,T_1 + *\,T_2 - \frac{1}{12\,(n-3)\,n!}\,T_4 + \frac{2}{(n+2)!}\,F_{n+2,0\cdots 01}\,A_{1,1},$$
$$0 \overset{\mathbf{II}}{=} *\,T_1 + *\,T_2 + *\,T_3 - \frac{1}{30\,(n-4)\,n!}\,T_5 + \frac{3}{(n+3)!}\,F_{n+3,0\cdots 01}\,A_{1,1}.$$

Admitting temporarily this fact, we can easily complete our main *non-existence* result.

Proof of Theorem 1.3. If the power series coefficient $F_{n+2,0\cdots 01} = 0$ would be zero, then the first equation:

$$0 \overset{\mathbf{I}}{=} *\,T_1 + *\,T_2 - \frac{1}{12\,(n-3)\,n!}\,T_4,$$

would consist of a *nontrivial* linear dependence relation between $T_1, \dots, T_n$, contradicting infinitesimal transitivity. Hence, $F_{n+2,0\cdots 01} \neq 0$. But then we can find the isotropy parameter by equation **I**:

$$A_{1,1} \,=\, *\,T_1 + *\,T_2 + *\,T_4,$$

that we replace in **II**, getting, whatever the value of $F_{n+3,0\cdots 01}$ is

$$0 \,=\, *\,T_1 + *\,T_2 + *\,T_3 + *\,T_4 - \frac{1}{30\,(n-4)\,n!}\,T_5.$$

But such an equation is also *always* a *nontrivial* linear dependence relation between $T_1, \dots, T_n$ contradicting again infinitesimal transitivity. □

Observe that $n \geqslant 5$ was used in this argumentation.

27. Tangency equations at orders $\leqslant n+4$

It remains to prove Lemma 26.1.

Proof of Lemma 26.1. We find Λ_0:

$$\begin{aligned}\Lambda_0 = & -C_1\,x_1 - \cdots - C_n\,x_n - D\bigg[\frac{x_1^2}{2!} + \frac{x_1^2x_2}{2!} + \cdots + \frac{x_1^{n-1}x_{n-1}}{(n-1)!} + \frac{x_1^n x_n}{n!} + F_{n+1,10\cdots0}\,\frac{x_1^{n+2}x_2}{(n+1)!} \\ & + F_{n+3,0\cdots0}\,\frac{x_1^{n+3}}{(n+3)!} + F_{n+2,10\cdots0}\,\frac{x_1^{n+2}x_2}{(n+2)!} + F_{n+2,0010\cdots0}\,\frac{x_1^{n+2}x_4}{(n+2)!} + \cdots + F_{n+2,0\cdots01}\,\frac{x_1^{n+2}x_n}{(n+2)!} \\ & + F_{n+4,0\cdots0}\,\frac{x_1^{n+4}}{(n+4)!} + F_{n+3,10\cdots0}\,\frac{x_1^{n+3}x_2}{(n+3)!} + F_{n+3,010\cdots0}\,\frac{x_1^{n+3}x_3}{(n+3)!} + F_{n+3,0010\cdots0}\,\frac{x_1^{n+4}x_4}{(n+3)!} \\ & + \cdots + F_{n+3,0\cdots01}\,\frac{x_1^{n+3}x_n}{(n+3)!}\bigg].\end{aligned}$$

Now we write $\Lambda_1, \dots, \Lambda_n$, which all involve products. We denote the products using the sign "$\cdot$". Here is Λ_1:

$$\begin{aligned}\Lambda_1 = \bigg(& T_1 + A_{1,1}\,x_1 + A_{1,2}\,x_2 + \cdots + A_{1,n-1}\,x_{n-1} + A_{1,n}\,x_n \\ & + B_1\bigg[\frac{x_1^2}{2!} + \frac{x_1^2x_2}{2!} + \cdots + \frac{x_1^n x_n}{n!} + F_{n+1,10\cdots0}\,\frac{x_1^{n+1}x_2}{(n+1)!} + F_{n+3,0\cdots0}\,\frac{x_1^{n+3}}{(n+3)!} \\ & + F_{n+2,10\cdots0}\,\frac{x_1^{n+2}x_2}{(n+2)!} + F_{n+2,0010\cdots0}\,\frac{x_1^{n+2}x_4}{(n+2)!} + \cdots + F_{n+2,0\cdots01}\,\frac{x_1^{n+2}x_n}{(n+2)!}\bigg]\bigg) \\ & \cdot\bigg(x_1 + x_1x_2 + \frac{x_1^2x_3}{2!} + \cdots + \frac{x_1^{n-1}x_n}{(n-1)!} + F_{n+1,10\cdots0}\,\frac{x_1^n x_2}{n!} + F_{n+3,0\cdots0}\,\frac{x_1^{n+2}}{(n+2)!} \\ & + F_{n+2,10\cdots0}\,\frac{x_1^{n+1}x_2}{(n+1)!} + F_{n+2,0010\cdots0}\,\frac{x_1^{n+1}x_4}{(n+1)!} + \cdots + F_{n+2,0\cdots01}\,\frac{x_1^{n+1}x_n}{(n+1)!} \\ & + F_{n+4,0\cdots0}\,\frac{x_1^{n+3}}{(n+3)!} + F_{n+3,10\cdots0}\,\frac{x_1^{n+2}x_2}{(n+2)!} + F_{n+3,010\cdots0}\,\frac{x_1^{n+2}x_3}{(n+2)!} + F_{n+3,0010\cdots0}\,\frac{x_1^{n+2}x_4}{(n+2)!} \\ & + \cdots + F_{n+3,0\cdots01}\,\frac{x_1^{n+2}x_n}{(n+2)!} + F_{n+5,0\cdots0}\,\frac{x_1^{n+4}}{(n+4)!} + F_{n+4,10\cdots0}\,\frac{x_1^{n+3}x_2}{(n+3)!} + F_{n+4,010\cdots0}\,\frac{x_1^{n+3}x_3}{(n+3)!} \\ & + F_{n+4,0010\cdots0}\,\frac{x_1^{n+3}x_4}{(n+3)!} + \cdots + F_{n+4,0\cdots01}\,\frac{x_1^{n+3}x_n}{(n+3)!}\bigg).\end{aligned}$$

Quantity Λ_2 reads as

$$\begin{aligned}\Lambda_2 = \bigg(& T_2 + A_{2,1}\,x_1 + A_{2,2}\,x_2 + \cdots + A_{2,n-1}\,x_{n-1} + A_{2,n}\,x_n \\ & + B_2\bigg[\frac{x_1^2}{2!} + \frac{x_1^2x_2}{2!} + \cdots + \frac{x_1^n x_n}{n!} + F_{n+1,10\cdots0}\,\frac{x_1^{n+1}x_2}{(n+1)!}\bigg]\bigg) \\ & \cdot\bigg(\frac{x_1^2}{2!} + \sum_{m=3}^{n}\frac{x_1^{m-1}x_{m-1}}{1!\,(m-2)!} + F_{n+1,10\cdots0}\,\frac{x_1^{n+1}}{(n+1)!} + \frac{x_1^n x_n}{(n-1)!} + F_{n+2,10\cdots0}\,\frac{x_1^{n+2}}{(n+2)!}\end{aligned}$$

$$+ F_{n+1,10\cdots 0}\,\frac{2}{n!}\,x_1^{n+1}x_2 + F_{n+3,10\cdots 0}\,\frac{x_1^{n+3}}{(n+3)!} + F_{n+2,10\cdots 0}\,\frac{2\,x_1^{n+2}x_2}{(n+1)!} + F_{n+1,10\cdots 0}\,\frac{x_1^{n+2}x_3}{2!\,n!}$$
$$+ F_{n+2,0010\cdots 0}\,\frac{x_1^{n+2}x_4}{(n+1)!} + \cdots + F_{n+2,0\cdots 01}\,\frac{x_1^{n+2}x_n}{(n+1)!} + F_{n+4,10\cdots 0}\,\frac{x_1^{n+4}}{(n+1)!}$$
$$+ \left(\frac{2\,F_{n+3,10\cdots 0}}{(n+2)!} - \frac{F_{n+3,0\cdots 0}}{(n+1)!}\right)x_1^{n+3}x_2 + \left(\frac{F_{n+3,010\cdots 0}}{(n+2)!} + \frac{F_{n+2,10\cdots 0}}{2!\,(n+1)!}\right)x_1^{n+3}x_3$$
$$+ \left(\frac{F_{n+3,0010\cdots 0}}{(n+2)!} + \frac{F_{n+1,10\cdots 0}}{3!\,n!}\right)x_1^{n+3}x_4\Bigg).$$

We find Λ_3:

$$\begin{aligned}\Lambda_3 = \Bigg(& T_3 + A_{3,1}\,x_1 + A_{3,2}\,x_2 + \cdots + A_{3,n-1}\,x_{n-1} + A_{3,n}\,x_n + B_3\left[\frac{x_1^2}{2!} + \frac{x_1^2x_2}{2!} + \cdots + \frac{x_1^nx_n}{n!}\right]\Bigg) \\ &\cdot\Bigg(\frac{x_1^3}{3!} + \sum_{m=4}^{n}\frac{x_1^{m-1}x_{m-2}}{2!\,(m-3!)} + \frac{x_1^nx_{n-1}}{2!\,(n-2)!} + \frac{x_1^{n+1}x_n}{2!\,(n-1)!} + F_{n+3,010\cdots 0}\,\frac{x_1^{n+3}}{(n+3)!} \\ &+ F_{n+1,10\cdots 0}\,\frac{x_1^{n+2}x_3}{2!\,n!} + F_{n+4,010\cdots 0}\,\frac{x_1^{n+4}}{(n+4)!} + \left(\frac{F_{n+3,010\cdots 0}}{(n+2)!} + \frac{F_{n+2,10\cdots 0}}{2!\,(n+1)!}\right)x_1^{n+3}x_2 \\ &+ F_{n+2,0010\cdots 0}\,\frac{x_1^{n+3}x_4}{2!\,(n+1)!} + \cdots + F_{n+2,0\cdots 01}\,\frac{x_1^{n+3}x_n}{2!\,(n+1)!}\Bigg).\end{aligned}$$

For Λ_4 we get:

$$\begin{aligned}\Lambda_4 = \Bigg(& T_4 + A_{4,1}\,x_1 + A_{4,2}\,x_2 + \cdots + A_{4,n-1}\,x_{n-1} + A_{4,n}\,x_n \\ &+ B_4\left[\frac{x_1^2}{2!} + \frac{x_1^2x_2}{2!} + \cdots + \frac{x_1^{n-1}x_{n-1}}{(n-1)!}\right]\Bigg) \\ &\cdot\Bigg(\frac{x_1^4}{4!} + \sum_{m=5}^{n}\frac{x_1^{m-1}x_{m-3}}{3!\,(m-4)!} + \frac{x_1^nx_{n-2}}{3!\,(n-3)!} + F_{n+2,0010\cdots 0}\,\frac{x_1^{n+2}}{(n+2)!} + \frac{x_1^{n+1}x_{n-1}}{3!\,(n-2)!} \\ &+ F_{n+3,0010\cdots 0}\,\frac{x_1^{n+3}}{(n+3)!} + F_{n+2,0010\cdots 0}\,\frac{x_1^{n+2}x_2}{(n+1)!} + \frac{x_1^{n+2}x_n}{3!\,(n-1)!} \\ &+ F_{n+4,0010\cdots 0}\,\frac{x_1^{n+4}}{(n+4)!} + \left(\frac{F_{n+3,0010\cdots 0}}{(n+2)!} + \frac{F_{n+1,10\cdots 0}}{3!\,n!}\right)x_1^{n+3}x_2 + F_{n+2,0010\cdots 0}\,\frac{x_1^{n+3}x_3}{2!\,(n+1)!}\Bigg).\end{aligned}$$

Here is Λ_5:

$$\begin{aligned}\Lambda_5 = \Bigg(& T_5 + A_{5,1}\,x_1 + A_{5,2}\,x_2 + \cdots + A_{5,n-1}\,x_{n-1} + A_{5,n}\,x_n \\ &+ B_5\left[\frac{x_1^2}{2!} + \frac{x_1^2x_2}{2!} + \cdots + \frac{x_1^{n-2}x_{n-2}}{(n-2)!}\right]\Bigg) \\ &\cdot\Bigg(\frac{x_1^5}{5!} + \sum_{m=6}^{n}\frac{x_1^{m-1}x_{m-4}}{4!\,(m-5)!} + \frac{x_1^nx_{n-3}}{4!\,(n-4)!} + F_{n+2,00010\cdots 0}\,\frac{x_1^{n+2}}{(n+2)!} + \frac{x_1^{n+1}x_{n-2}}{4!\,(n-3)!} \\ &+ F_{n+3,00010\cdots 0}\,\frac{x_1^{n+3}}{(n+3)!} + F_{n+2,00010\cdots 0}\,\frac{x_1^{n+2}x_2}{(n+1)!} + \frac{x_1^{n+2}x_{n-1}}{4!\,(n-2)!} \\ &+ F_{n+4,00010\cdots 0}\,\frac{x_1^{n+4}}{(n+4)!} + F_{n+3,00010\cdots 0}\,\frac{x_1^{n+3}x_2}{(n+2)!} + F_{n+2,00010\cdots 0}\,\frac{x_1^{n+3}x_3}{2!\,(n+1)!} + \frac{x_1^{n+3}x_n}{4!\,(n-1)!}\Bigg).\end{aligned}$$

We find Λ_6:

$$
\begin{aligned}
\Lambda_6 = \bigg(& T_6 + A_{6,1}\, x_1 + A_{6,2}\, x_2 + \cdots + A_{6,n-1}\, x_{n-1} + A_{6,n}\, x_n \\
& + B_6 \left[\frac{x_1^2}{2!} + \frac{x_1^2 x_2}{2!} + \cdots + \frac{x_1^{n-3} x_{n-3}}{(n-3)!} \right] \bigg) \\
& \cdot \bigg(\frac{x_1^6}{6!} + \sum_{m=7}^{n} \frac{x_1^{m-1} x_{m-5}}{5!\,(m-6)!} + \frac{x_1^n x_{n-4}}{5!\,(n-5)!} + F_{n+2,000010\cdots 0}\, \frac{x_1^{n+2}}{(n+2)!} + \frac{x_1^{n+1} x_{n-3}}{5!\,(n-4)!} \\
& + F_{n+3,000010\cdots 0}\, \frac{x_1^{n+3}}{(n+3)!} + F_{n+2,000010\cdots 0}\, \frac{x_1^{n+2} x_2}{(n+1)!} + \frac{x_1^{n+2} x_{n-2}}{5!\,(n-3)!} \\
& + F_{n+4,000010\cdots 0}\, \frac{x_1^{n+4}}{(n+4)!} + F_{n+3,000010\cdots 0}\, \frac{x_1^{n+3} x_2}{(n+2)!} + F_{n+2,000010\cdots 0}\, \frac{x_1^{n+3} x_3}{2!\,(n+1)!} + \frac{x_1^{n+3} x_{n-1}}{5!\,(n-2)!} \bigg).
\end{aligned}
$$

We calculate Λ_{n-1}:

$$
\begin{aligned}
\Lambda_{n-1} = \bigg(& T_{n-1} + A_{n-1,1}\, x_1 + A_{n-1,2}\, x_2 + \cdots + A_{n-1,n-1}\, x_{n-1} + A_{n-1,n}\, x_n \\
& + B_{n-1} \left[\frac{x_1^2}{2!} + \frac{x_1^2 x_2}{2!} + \frac{x_1^3 x_3}{3!} + \frac{x_1^4 x_4}{4!} \right] \bigg) \\
& \cdot \bigg(\frac{x_1^{n-1}}{(n-1)!} + \frac{x_1^{n-1} x_2}{(n-2)!\,1!} + \frac{x_1^n x_3}{(n-2)!\,2!} + F_{n+2,0\cdots 010}\, \frac{x_1^{n+2}}{(n+2)!} + \frac{x_1^{n+1} x_4}{(n-2)!\,3!} \\
& + F_{n+3,0\cdots 010}\, \frac{x_1^{n+3}}{(n+3)!} + F_{n+2,0\cdots 010}\, \frac{x_1^{n+2} x_2}{(n+1)!} + \frac{x_1^{n+2} x_5}{(n-2)!\,4!} \\
& + F_{n+4,0\cdots 010}\, \frac{x_1^{n+4}}{(n+4)!} + F_{n+3,0\cdots 010}\, \frac{x_1^{n+3} x_2}{(n+2)!} + F_{n+2,0\cdots 010}\, \frac{x_1^{n+3} x_3}{2!\,(n+1)!} + \frac{x_1^{n+3} x_6}{(n-2)!\,5!} \bigg).
\end{aligned}
$$

Finally, we find Λ_n:

$$
\begin{aligned}
\Lambda_n = \bigg(& T_n + A_{n,1}\, x_1 + A_{n,2}\, x_2 + \cdots + A_{n,n-1}\, x_{n-1} + A_{n,n}\, x_n + B_n \left[\frac{x_1^2}{2!} + \frac{x_1^2 x_2}{2!} + \frac{x_1^3 x_3}{3!} \right] \bigg) \\
& \cdot \bigg(\frac{x_1^n}{n!} + \frac{x_1^n x_2}{(n-1)!\,1!} + F_{n+2,0\cdots 01}\, \frac{x_1^{n+2}}{(n+2)!} + \frac{x_1^{n+1} x_3}{(n-1)!\,2!} + F_{n+3,0\cdots 01}\, \frac{x_1^{n+3}}{(n+3)!} \\
& + F_{n+2,0\cdots 01}\, \frac{x_1^{n+2} x_2}{(n+1)!} + \frac{x_1^{n+2} x_4}{(n-1)!\,3!} + F_{n+4,0\cdots 01}\, \frac{x_1^{n+4}}{(n+4)!} \\
& + F_{n+3,0\cdots 01}\, \frac{x_1^{n+3} x_2}{(n+2)!} + F_{n+2,0\cdots 01}\, \frac{x_1^{n+3} x_3}{2!\,(n+1)!} + \frac{x_1^{n+3} x_5}{(n-1)!\,4!} \bigg).
\end{aligned}
$$

By looking carefully at these products, we can determine the coefficients of some relevant monomials.

First, to get equation **I**, we extract the coefficient of the monomial $x_1^{n+2}x_n$. By inserting vertical bars, we indicate from which $\Lambda_{0,1,\dots,n}$ the written terms come from:

$$\begin{aligned}
0 \overset{\mathbf{I}}{=} \Big|^{\Lambda_0} - D\,\frac{F_{n+2,0\cdots01}}{(n+2)!}\;\Big|^{\Lambda_1} + T_1\,\frac{F_{n+3,0\cdots01}}{(n+2)!} + A_{1,1}\,\frac{F_{n+2,0\cdots01}}{(n+1)!} + A_{1,n}\,\frac{F_{n+3,0\cdots0}}{(n+2)!} \\
\Big|^{\Lambda_2} + T_2\,\frac{F_{n+2,0\cdots01}}{(n+1)!} + A_{2,n}\,\frac{F_{n+2,10\cdots0}}{(n+2)!} + B_2\,\frac{1}{2!\,(n-1)!} + B_2\,\frac{1}{2!\,n!} \\
\Big|^{\Lambda_3} + A_{3,1}\,\frac{1}{2!\,(n-1)!}\;\Big|^{\Lambda_4} + T_4\,\frac{1}{3!\,(n-1)!} + A_{4,n}\,\frac{F_{n+2,0010\cdots0}}{(n+2)!}\;\Big|^{\Lambda_5} + A_{5,n}\,\frac{F_{n+2,00010\cdots0}}{(n+2)!} \\
\Big|^{\Lambda_6} + A_{6,n}\,\frac{F_{n+2,000010\cdots0}}{(n+2)!}\;\Big|^{\Lambda_7} + \cdots\;\Big|^{\Lambda_{n-1}} + A_{n-1,n}\,\frac{F_{n+2,0\cdots010}}{(n+2)!}\;\Big|^{\Lambda_n} + A_{n,n}\,\frac{F_{n+2,0\cdots01}}{(n+2)!}.
\end{aligned}$$

Second, to get equation **II**, we extract the coefficient of the monomial $x_1^{n+3}x_n$:

$$\begin{aligned}
0 \overset{\mathbf{II}}{=} \Big|^{\Lambda_0} - D\,\frac{F_{n+3,0\cdots01}}{(n+3)!}\;\Big|^{\Lambda_1} + T_1\,\frac{F_{n+4,0\cdots01}}{(n+3)!} + A_{1,1}\,\frac{F_{n+3,0\cdots01}}{(n+2)!} + A_{1,n}\,\frac{F_{n+4,0\cdots0}}{(n+3)!} \\
+ B_1\,\frac{F_{n+2,0\cdots01}}{2!\,(n+1)!} + B_1\,\frac{F_{n+2,0\cdots01}}{(n+2)!}\;\Big|^{\Lambda_2} + A_{2,1}\,\frac{F_{n+2,0\cdots01}}{(n+1)!} + A_{2,n}\,\frac{F_{n+2,10\cdots0}}{(n+2)!}\;\Big|^{\Lambda_3} + T_3\,\frac{F_{n+2,0\cdots01}}{2!\,(n+1)!} \\
+ A_{3,n}\,\frac{F_{n+3,010\cdots0}}{(n+3)!} + B_3\,\frac{1}{2!\,(n-1)!} + B_3\,\frac{1}{3!\,n!}\;\Big|^{\Lambda_4} + A_{4,1}\,\frac{1}{3!\,(n-1)!} + A_{4,n}\,\frac{F_{n+3,0010\cdots0}}{(n+3)!}\;\Big|^{\Lambda_5} \\
+ T_5\,\frac{1}{4!\,(n-1)!} + A_{5,n}\,\frac{F_{n+3,00010\cdots0}}{(n+3)!}\;\Big|^{\Lambda_6} + A_{6,n}\,\frac{F_{n+3,000010\cdots0}}{(n+3)!}\;\Big|^{\Lambda_7} + \cdots\;\Big|^{\Lambda_{n-1}} \\
+ A_{n-1,n}\,\frac{F_{n+3,0\cdots010}}{(n+3)!}\;\Big|^{\Lambda_n} + A_{n,n}\,\frac{F_{n+3,0\cdots01}}{(n+3)!}.
\end{aligned}$$

We see that we need to determine the parameters:

$$\begin{array}{llll}
D, & A_{1,n}, & B_1, & A_{2,1}, \\
 & A_{2,n}, & B_2, & A_{3,1}, \\
 & A_{3,n}, & B_3, & A_{4,1}, \\
 & \cdots\cdot & & \\
 & A_{n-2,n}, & & \\
 & A_{n-1,n}, & & \\
 & A_{n,n}, & &
\end{array}$$

in terms of $T_1, \dots, T_n, A_{1,1}$.

By

$$E_{[2,0,\dots,0]}: \quad 0 = \frac{1}{2}\,T_2 + A_{1,1} - \frac{1}{2}\,D,$$

we determine

$$D = T_2 + 2\,A_{1,1}.$$

We then get:

$$
\begin{aligned}
E_{[1,0,\dots,0,1]}: \qquad & 0 = A_{1,n}, \\
E_{[2,0,\dots,0,1]}: \qquad & 0 = \frac{1}{2!}\,A_{2,n}, \\
E_{[3,0,\dots,0,1]}: \qquad & 0 = \frac{1}{3!}\,A_{3,n}, \\
\cdots\cdots\cdots \qquad & \cdots\cdots\cdots \\
E_{[n-2,0,\dots,0,1]}: \qquad & 0 = \frac{1}{(n-2)!}\,A_{n-2,n}, \\
E_{[n-1,0,\dots,0,1]}: \qquad & 0 = T_1\,\frac{1}{(n-1)!} + \frac{1}{(n-1)!}\,A_{n-1,n}, \\
E_{[n,0,\dots,0,1]}: \qquad & 0 = -\,D\,\frac{1}{n!} + A_{1,1}\,\frac{1}{(n-1)!} + T_2\,\frac{1}{(n-1)!} + \frac{1}{n!}\,A_{n,n}.
\end{aligned}
$$

Replacing the obtained values, we find:

$$
\begin{aligned}
0 \overset{\mathbf{I}}{=} & -\,(T_2 + 2\,A_{1,1})\,\frac{F_{n+2,0\cdots 01}}{(n+2)!} + T_1\,\frac{F_{n+3,0\cdots 01}}{(n+2)!} + A_{1,1}\,\frac{F_{n+2,0\cdots 01}}{(n+1)!} + T_2\,\frac{F_{n+2,0\cdots 01}}{(n+1)!} + B_2\,\frac{n+1}{2!\,n!} \\
& + A_{3,1}\,\frac{1}{2!\,(n-1)!} + T_4\,\frac{1}{3!\,(n-1)!} + A_{n-1,n}\,\frac{F_{n+2,0\cdots 010}}{(n+2)!} + A_{n,n}\,\frac{F_{n+2,0\cdots 01}}{(n+2)!},
\end{aligned}
$$

and:

$$
\begin{aligned}
0 \overset{\mathbf{II}}{=} & -\,(T_2 + 2\,A_{1,1})\,\frac{F_{n+3,0\cdots 01}}{(n+3)!} + T_1\,\frac{F_{n+4,0\cdots 01}}{(n+3)!} + A_{1,1}\,\frac{F_{n+3,0\cdots 01}}{(n+2)!} + B_1\,F_{n+2,0\cdots 01}\,\frac{n+4}{2!\,(n+2)!} \\
& + A_{2,1}\,\frac{F_{n+2,0\cdots 01}}{(n+1)!} + T_3\,\frac{F_{n+2,0\cdots 01}}{2!\,(n+1)!} + B_3\,\frac{3\,n+2}{12\cdot n!} + A_{4,1}\,\frac{1}{3!\,(n-1)!} + T_5\,\frac{1}{4!\,(n-1)!} \\
& + A_{n-1,n}\,\frac{F_{n+3,0\cdots 010}}{(n+3)!} + A_{n,n}\,\frac{F_{n+3,0\cdots 01}}{(n+3)!}.
\end{aligned}
$$

To find B_1, B_2, B_3, and $A_{2,1}$, $A_{3,1}$, $A_{4,1}$, we consider by patiently chasing in $\Lambda_0, \Lambda_1, \dots, \Lambda_n$, the three equations:

$$
\begin{aligned}
E_{[n+1,0,\dots,0,1]}: \quad 0 = & \left|^{\Lambda_1} T_1\,\frac{F_{n+2,0\cdots 01}}{(n+1)!} + B_1\,\frac{1}{2!\,(n-1)!} + B_1\,\frac{1}{n!}\right|^{\Lambda_2} + A_{2,1}\,\frac{1}{(n-1)!} \\
& + \underline{A_{2,n}}_{\circ}\,\frac{F_{n+1,10\cdots 0}}{(n+1)!}\,\bigg|^{\Lambda_3} + T_3\,\frac{1}{2!\,(n-1)!}, \\
E_{[n+1,0,\dots,0,1,0]}: \quad 0 = & \left|^{\Lambda_1} T_1\,\frac{F_{n+2,0\cdots 010}}{(n+1)!}\right|^{\Lambda_2} + \underline{A_{2,n-1}}_{\circ}\,\frac{F_{n+1,10\cdots 0}}{(n+1)!} \\
& + B_2\,\frac{1}{2!\,(n-2)!} + B_2\,\frac{1}{(n-1)!\,2!}\,\bigg|^{\Lambda_3} + A_{3,1}\,\frac{1}{2!\,(n-2)!}\,\bigg|^{\Lambda_4} + T_4\,\frac{1}{3!\,(n-2)!}, \\
E_{[n+1,0,\dots,0,1,0,0]}: \quad 0 = & \left|^{\Lambda_1} T_1\,\frac{F_{n+2,0\cdots 0100}}{(n+1)!}\right|^{\Lambda_2} + \underline{A_{2,n-2}}_{\circ}\,\frac{F_{n+1,10\cdots 0}}{(n+1)!}\,\bigg|^{\Lambda_3} + B_3\,\frac{1}{2!\,2!\,(n-3)!} \\
& + B_3\,\frac{1}{(n-2)!\,3!}\,\bigg|^{\Lambda_4} + A_{4,1}\,\frac{1}{3!\,(n-3)!}\,\bigg|^{\Lambda_5} + T_5\,\frac{1}{4!\,(n-3)!}.
\end{aligned}
$$

in which the three underlined parameters vanish thanks to:

$$\begin{aligned} E_{[2,0,\dots,0,1]}\colon && 0 &= A_{2,n}\,\frac{1}{2!}, \\ E_{[2,0,\dots,0,1,0]}\colon && 0 &= A_{2,n-1}\,\frac{1}{2!}, \\ E_{[2,0,\dots,0,1,0,0]}\colon && 0 &= A_{2,n-2}\,\frac{1}{2!}. \end{aligned}$$

After simplification, these three equations become

$$\begin{aligned} 0 &= T_1\,\frac{F_{n+2,0\cdots 01}}{(n+1)!} + T_3\,\frac{1}{2!\,(n-1)!} + A_{2,1}\,\frac{1}{(n-1)!} + B_1\,\frac{n+2}{2!\,n!}, \\ 0 &= T_1\,\frac{F_{n+2,0\cdots 010}}{(n+1)!} + T_4\,\frac{1}{3!\,(n-2)!} + A_{3,1}\,\frac{1}{2!\,(n-2)!} + B_2\,\frac{n}{2!\,(n-1)!}, \\ 0 &= T_1\,\frac{F_{n+2,0\cdots 0100}}{(n+1)!} + T_5\,\frac{1}{4!\,(n-3)!} + A_{4,1}\,\frac{1}{3!\,(n-3)!} + B_3\,\frac{3\,n-4}{2!\,3!\,(n-2)!}. \end{aligned}$$

The obtained $3+3$ equations are organized as 3 pairs and they allow us to find

$$\begin{aligned} B_1 &= T_1\,F_{n+2,0\cdots 01}\,\frac{2}{(n+1)(n-2)} + T_3\,\frac{n}{3\,(n-2)}, \\ A_{2,1} &= -\,T_1\,F_{n+2,0\cdots 01}\,\frac{2}{(n+1)(n-2)} - T_3\,\frac{2}{3}\,\frac{n-1}{n-2}, \end{aligned}$$

and

$$\begin{aligned} B_2 &= T_1\,F_{n+2,0\cdots 010}\,\frac{4}{(n-3)n(n+1)} + T_4\,\frac{1}{6}\,\frac{n-1}{n-3}, \\ A_{3,1} &= -\,T_1\,F_{n+2,0\cdots 010}\,\frac{6}{(n-3)n(n+1)} - T_4\,\frac{1}{2}\,\frac{n-2}{n-3}, \end{aligned}$$

as well as

$$\begin{aligned} B_3 &= T_1\,F_{n+2,0\cdots 0100}\,\frac{12}{n(n-4)(n^2-1)} + T_5\,\frac{1}{10}\,\frac{n-2}{n-4}, \\ A_{4,1} &= -\,T_1\,F_{n+2,0\cdots 0100}\,\frac{24}{n(n-4)(n^2-1)} - T_5\,\frac{2}{5}\,\frac{n-3}{n-4}. \end{aligned}$$

Replacing all these values, we obtain **I**:

$$\begin{aligned} 0 = \stackrel{\mathbf{I}}{=} {*T_1} + {*T_2} + T_4\left(\frac{1}{6}\,\frac{n-1}{n-3}\,\frac{n+1}{2!\,n!} - \frac{1}{2}\,\frac{n-2}{n-3}\,\frac{1}{2!\,(n-1)!} + \frac{1}{3!\,(n-1)!}\right) \\ + A_{1,1}\,F_{n+2,0\cdots 01}\left(-\,\frac{2}{(n+2)!} + \frac{1}{(n+1)!} - \frac{n-2}{(n+2)!}\right), \end{aligned}$$

which is exactly what Lemma 26.1 stated and we also obtain **II**:

$$\begin{aligned} 0 \stackrel{\mathbf{II}}{=} *\,T_1 + {*T_2} + {*T_3} + T_5\left(\frac{1}{10}\,\frac{n-2}{n-4}\,\frac{3\,n+2}{12\cdot n!} - \frac{2}{5}\,\frac{n-3}{n-4}\,\frac{1}{3!\,(n-1)!} + \frac{1}{4!\,(n-1)!}\right) \\ + A_{1,1}\,F_{n+3,0\cdots 01}\left(-\,\frac{2}{(n+3)!} + \frac{1}{(n+2)!} - \frac{n-2}{(n+3)!}\right), \end{aligned}$$

which is also exactly what Proposition 26.1 stated. □

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REFERENCES

1. S. Lie, J. Merker. *Theory of transformation groups I. General properties of continuous transformation groups. A contemporary approach and translation.* Springer-Verlag, Berlin, Heidelberg (2015).
2. B. Abdalla, F. Dillen, L. Vrancken. *Affine homogeneous surfaces in* $\mathbb{R}^3$ *with vanishing Pick invariant* // Abh. Math. Sem. Univ. Hamburg. **67**:1, 105–115 (1997).
3. B. Doubrov, B. Komrakov, M. Rabinovich. *Homogeneous surfaces in the three-dimensional affine geometry* // in "Geometry and topology of submanifolds", World Sci. Publ., River Edge, New Jersey, 168–178 (1996).
4. M. Eastwood, V. Ezhov. *On affine normal forms and a classification of homogeneous surfaces in affine three-space* // Geom. Dedicata. **77**:1, 11–69 (1999).
5. Ö. Arnaldsson, F. Valiquette. *Invariants of surfaces in three-dimensional affine geometry* // SIGMA. **17**, 033 (2021).
6. Z. Chen, J. Merker. *Affine homogeneous surfaces with Hessian rank 2 and algebras of differential invariants* // Preprint: `arXiv:2010.02873` (2010).
7. B. Doubrov, B. Komrakov. *Classification of homogeneous submanifolds in homogeneous spaces* // Lobachevskii J. Math. **3**, 19–38 (1999).
8. M. Wermann. *Homogene Hyperflächen im vierdimensionalen äqui-affinen Raum*, Dortmund Univ., Diss. (2001), Shaker Verlag, Aachen (2001).
9. M. Eastwood, V. Ezhov. *Homogeneous hypersurfaces with isotropy in affine four-space* // Proc. Steklov Inst. Math. **235**, 49–63 (2001).
10. M. Eastwood, V. Ezhov. *A classification of non-degenerate homogeneous equiaffine hypersurfaces in four complex dimensions* // Asian J. Math. **5**:4, 721–739 (2001).
11. Z. Chen, J. Merker. *On differential invariants of parabolic surfaces* // Dissertationes Mathematicæ **559**, 1–88 (2021).
12. W.G. Foo, J. Merker, T.A. Ta. *On convergent Poincaré-Moser reduction for Levi degenerate embedded* 5*-dimensional CR manifolds* // New York J. Math. **28**, 250–336 (2022).
13. W.G. Foo, J. Merker, P. Nurowski, T.A. Ta. *Homogeneous* $\mathfrak{C}_{2,1}$ *models* // Preprint: `arXiv:2104.09608` (2021).
14. J. Merker. *Affine rigidity without integration* // Preprint: `arXiv:1903.00889` (2021).
15. J. Merker. *Classification of Hessian rank 1 affinely homogeneous hypersurfaces* $H^n \subset \mathbb{R}^{n+1}$ *in dimensions* $n = 2, 3, 4$ // Preprint: `arXiv:2206.01449` (2022).

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