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THE BOUNDARY MORERA THEOREM FOR DOMAIN $\tau^+(n-1)$

G. KHUDAYBERGANOV, J.Sh. ABDULLAYEV

Abstract. In this work, we will continue to construct an analysis in the future tube and move on to study the Lie ball. The Lie ball can be realized as a future tube. These realizations will be the subject of our research. These methods turn out to be convenient for computing the Bergman, Cauchy-Szegö and Poisson kernels in this domain.

In the theory of functions Morera's theorems have been studied by many mathematicians. In the complex plane, the functions with one-dimensional holomorphic extension property are trivial but Morera's boundary theorems are not available. Therefore, the results of the work are essential in the multidimensional case. In this article, we proved the boundary Morera theorem for the domain $\tau^+(n-1)$. An analog of Morera's theorem is given, in which integration is carried out along the boundaries of analytic disks. For this purpose, we use the automorphisms $\tau^+(n-1)$ and the invariant Poisson kernel in the domain $\tau^+(n-1)$. Moreover, an analogue of Stout's theorem on functions with the one-dimensional holomorphic continuation property is obtained for the domain $\tau^+(n-1)$. In addition, generalizations of Tumanov's theorem is obtained for a smooth function from the given class of CR manifolds.

Keywords: Classical domains, Lie ball, realization, future tube, Shilov boundary, Poisson kernel, holomorphic continuation, Morera's theorem, analytic disk, Hardy spaces.

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1. Introduction and preliminary results

The issue on holomorphic continuation of functions defined on the entire boundary into the domain was studied quite well. The usual (non-boundary) Morera theorems in domains of the space $\mathbb{C}^n$ are well known (see, for example, [5], [8])

A high great interest of specialists is attracted by the multidimensional boundary analogues of Morera's theorem (see [11], [14]–[16], [18], [23], [24], [29]). One way or another, they dealt with an entire boundary of the domain and studied a holomorphic continuation of the function f from the boundary ∂D of a domain D in $\mathbb{C}^n$ provided the integrals of f over the boundaries of the analytic disks lying on ∂D vanish.

The first result related to this topic was obtained by M.L. Agranovskii and R.E. Val'skii in [19], who studied functions with the one-dimensional property of holomorphic continuation in a ball. The proof was based on the properties of the automorphism group of a ball. In [12], E.L. Stout extended the Agranovskii and Val'skii theorem to an arbitrary bounded domains

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with a smooth boundary via using the complex Radon transform. A.E. Tumanov in [17] showed that each smooth function with the one-dimensional property of holomorphic continuation along analytic disks on generating minimal manifolds is a CR function. An alternative proof of Stout's theorem was given by A.M. Kytmanov and S.G. Myslivets (see [9]), who applied the Bochner – Martinelli integral. The idea of using integral representations (Bochner – Martinelli, Cauchy – Fantappie, logarithmic residue) turned out to be useful in studying functions with the one-dimensional holomorphic continuation property along complex curves, see [24], [25].

In papers [30], [37], a boundary version of Morera's theorem was considered for classical domains. Its starting point was the result of Nagel and Rudin [11], which stated that if the function f is continuous on the boundary of the ball in $\mathbb{C}^n$ and

$$\int_0^{2\pi} f\left(\varphi\left(e^{i\theta}, 0, \ldots, 0\right)\right) e^{i\theta} d\theta = 0$$

for all (holomorphic) automorphisms of the φ ball, then the function f can be continued holomorphically into the ball. An alternative proof of the theorem of Nagel and Rudin was given by S. Kosbergenov in [31]. It allowed one to generalize this statement to the case of classical domains.

In papers [32]–[34], a boundary version of Morera's theorem was studied for matrix balls, as well as for matrix balls associated with classical domains of the second and third types. This statement is generalized for a matrix ball by replacing the domain boundary by a Shilov boundary (skeleton).

In all previous results, the domain D was assumed to be bounded. In [35], [36], there was studied a subspace of the Schwartz space on a closed convex unbounded set in $\mathbb{R}^n$ was formed by the function admitting holomorphic continuation in $\mathbb{C}^n$. In ([20]–[22], [27], [28], [38]), the following case of unbounded domains of D were considered: a domain with a boundary being the Poincare spheré or the Heisenberg group, the matrix upper half-plane, and also Siegel domains of the second kind. Morera's theorem with the Heisenberg group was also considered by M. Agranovskii, K. Berenstein, and D. Chang in [20]. In this work, the integration was made over spheres of maximal dimension. We give an analogue of Morera's theorem, in which the integration is made along the boundaries of analytic disks.

In [28], [38], the Bochner –Martinelli integral was used in the proof of this theorem. The domain D was biholomorphically mapped onto the ball, the intersections of this ball with complex lines were considered, the inverse disks were the inverse images of these intersections in the domain D. B. Kurbanov gave an alternative proof of this theorem in [21] and in [22] considered Morera's theorem for the matrix upper half-plane.

In this paper, we consider the boundary Morera theorem for a future tube $\tau^+(n-1)$ and we use automorphisms $\tau^+(n-1)$ and the invariant Poisson kernel in the domain $\tau^+(n-1)$.

2. Realizations of Lie ball and automorphisms $\tau^+(n-1)$

We consider an n-dimensional complex space $\mathbb{C}^n$, with ordered set $z = (z_1, z_2, \ldots, z_n)$ of n complex numbers. For $z, w \in \mathbb{C}^n$ we let

$$\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n), \qquad \langle z, w\rangle = z_1 w_1 + \ldots + z_n w_n,$$
$$|z| = \sqrt{\langle z, \bar{z}\rangle} = \sqrt{|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2}.$$

The domain $\Re_{IV}^n$ (Lie ball) consists of n-dimensional complex vectors z satisfying certain conditions:

$$\Re_{IV}^n = \left\{ z \in \mathbb{C}^n \, : \, |\langle z, z\rangle|^2 - 2|z|^2 + 1 > 0, \ |\langle z, z\rangle| < 1 \right\}. \tag{2.1}$$

This classical domain is of the fourth type according to the classification by E. Cartan, see [1], or the Lie ball, see [3]. The Lie ball $\Re_{IV}^n$ is a complete circular convex bounded domain.

The Shilov's boundary (skeleton) $\Gamma_{\Re_{IV}^n}$ of the domain $\Re_{IV}^n$ is defined as follows:

$$\Gamma_{\Re_{IV}^n} = \{ z \in \mathbb{C}^n : \ |\langle z, z\rangle| = 1, \ |z| = 1 \}. \tag{2.2}$$

An unbounded domain of the form

$$\tau^+(n) = \left\{ w \in \mathbb{C}^{n+1} : \ (\operatorname{Im} w_{n+1})^2 > (\operatorname{Im} w_1)^2 + \ldots + (\operatorname{Im} w_n)^2, \ \operatorname{Im} w_{n+1} > 0 \right\}$$

is called future tube in $\mathbb{C}^{n+1}$.

The boundary $\partial\tau^+(n)$ of the domain $\tau^+(n)$ consists of

$$\partial\tau^+(n) = \left\{ w \in \mathbb{C}^{n+1} : (\operatorname{Im} w_{n+1})^2 = (\operatorname{Im} w_1)^2 + \ldots + (\operatorname{Im} w_n)^2, \operatorname{Im} w_{n+1} \geqslant 0 \right\}$$

and skeleton

$$\Gamma_{\tau^+(n)} = \left\{ w \in \mathbb{C}^{n+1} : \ \operatorname{Im} w_1 = \ldots = \operatorname{Im} w_n = \operatorname{Im} w_{n+1} = 0 \right\} = \mathbb{R}^{n+1},$$

on which the boundary degenerates.

The following statement is true

Theorem 2.1. *The map $\Phi : \mathbb{C}_z^n \to \mathbb{C}_w^n$ defined by identities*

$$w_k = \frac{-2iz_k}{\sum\limits_{j=1}^{n-1} z_j^2 + (z_n - i)^2}, \qquad k = 1, \ldots, n-1, \qquad w_n = \frac{2(z_n - i)}{\sum\limits_{j=1}^{n-1} z_j^2 + (z_n - i)^2} - i, \tag{2.3}$$

biholomorphically maps the domain $\Re_{IV}^n$ onto $\tau^+(n-1)$, while $\Gamma_{\Re_{IV}^n}$ is mapped onto $\Gamma_{\tau^+(n-1)}$.

Remark 2.1. *We call the transformation (2.3) a generalized Cayley transform. In* [4]*, there was considered the issue on the existence of a biholomorphic map of the domain $\tau^+(n)$ into the classical domain of the fourth type. In* [7]*, the embedding of the future tube into the classical domain $\Re_{IV}^{n+1}$ was given. The goal of Theorem 2.1 is to find an explicit mapping establishing a biholomorphic equivalence of the domains $\Re_{IV}^n$ and $\tau^+(n-1)$; this is useful for calculating a Jacobian in other important concepts.*

Proof. It is known that the group G of analytic automorphisms of domains of the fourth type is described as follows [2], [4]. Consider a group of linear transformations with real coefficients of $(n+2)$ variables that keep the quadratic form on its place

$$-u_1^2 - \ldots - u_n^2 + v_1^2 + v_2^2 = A'HA \tag{2.4}$$

where

$$A = \begin{pmatrix} u_1 \\ \vdots \\ u_n \\ v_1 \\ v_2 \end{pmatrix}, \qquad H = \begin{pmatrix} -I^{(n)} & 0 \\ 0 & I^{(2)} \end{pmatrix}$$

and $I^{(n)}$ is the unit matrix of size n. More precisely, we consider a group of linear transformations of the matrix G satisfying the equalities $G = \overline{G}$ and $G'HG = H$. Note that the number of real parameters of this group is equal to $\frac{(n+1)(n+2)}{2}$.

The set of points of $(n+2)$ dimensional complex space satisfying the conditions

$$A'HA = -u_1^2 - \ldots - u_n^2 + v_1^2 + v_2^2 = 0 \tag{2.5}$$

and

$$A'H\overline{A} = -|u_1|^2 - \ldots - |u_n|^2 + |v_1|^2 + |v_2|^2 > 0 \tag{2.6}$$

transforms into itself under one-to-one transformations $A \to GA$.

Note that v_1 and v_2 are both nonzero and do not belong to the same line passing through the origin since otherwise we would have

$$|v_1|^2 + |v_2|^2 = \left|v_1^2 + v_2^2\right|^2 \qquad \text{and} \qquad |u_1|^2 + \ldots + |u_n|^2 < \left|u_1^2 + \ldots + u_n^2\right|. \tag{2.7}$$

Therefore, the imaginary part of the ratio $\frac{v_1}{v_2}$ is nonzero. It can be shown that our point set consists of two components, each containing points with a certain sign of the imaginary part $\frac{v_1}{v_2}$. It is also obvious that the transformations of our group either preserves the sign of the imaginary part $\frac{v_1}{v_2}$ for all points of the considered set or make it opposite for all points.

Thus, linear transformations preserving the sign of the imaginary part $\frac{v_1}{v_2}$, form a subgroup of index two. We consider this subgroup and the set of points defined by the conditions:

$$A'HA = 0, \qquad A'H\overline{A} > 0, \qquad \operatorname{Im}\frac{v_1}{v_2} > 0. \tag{2.8}$$

Passing to inhomogeneous coordinates, we transfer this set to a bounded domain in n dimensional complex space. Dividing the first relation in (2.8) by $(v_1 + iv_2)^2$, we obtain

$$\left(\frac{u_1}{v_1 + iv_2}\right)^2 + \ldots + \left(\frac{u_n}{v_1 + iv_2}\right)^2 = \frac{v_1 - iv_2}{v_1 + iv_2}. \tag{2.9}$$

Dividing the second relation in (2.8) by $|v_1 + iv_2|^2$ and keeping in mind that

$$\frac{|v_1|^2 + |v_2|^2}{|v_1 + iv_2|^2} = \frac{|v_1 - iv_2|^2 + |v_1 + iv_2|^2}{2\,|v_1 + iv_2|^2},$$

we get

$$\left|\frac{u_1}{v_1 + iv_2}\right|^2 + \ldots + \left|\frac{u_n}{v_1 + iv_2}\right|^2 < \frac{1}{2}\left(1 + \left|\frac{v_1 - iv_2}{v_1 + iv_2}\right|^2\right). \tag{2.10}$$

The third condition (2.8) means that

$$\left|\frac{v_1 - iv_2}{v_1 + iv_2}\right| < 1.$$

Hence, we let

$$z_k = \frac{u_k}{v_1 + iv_2}, \qquad k = 1, \ldots, n,$$

and then the set defined in (2.10) becomes a bounded domain

$$|z_1|^2 + \ldots + |z_n|^2 < \frac{1}{2}\left(1 + \left|z_1^2 + \ldots + z_n^2\right|\right) < 1. \tag{2.11}$$

and our group of linear transformations becomes a $\frac{(n+1)(n+2)}{2}$-parametric group of linear fractional transformations of this domain onto itself. It can be proved that this group of mappings is transitive in the considered domain. Finally, if we put $G = H$, then we get the map $z \to -z$, so that (2.11) is the symmetric domain.

The inequality (2.11) can be written as

$$\Re_{IV}^n = \left\{z \in \mathbb{C}^n \,:\, |\langle z, z\rangle|^2 - 2|z|^2 + 1 > 0,\ |\langle z, z\rangle| < 1\right\}.$$

We let

$$u_k = (v_2 + u_n)\, w_k, \qquad k = 1, \ldots, n-1, \qquad v_1 = (v_2 + u_n)\, w_n,$$

then,

$$\frac{-2iz_k}{\sum\limits_{k=1}^{n-1} z_k^2 + (z_n - i)^2} = \frac{\frac{-2iu_k}{v_1+iv_2}}{\sum\limits_{k=1}^{n-1}\left(\frac{u_k}{v_1+iv_2}\right)^2 + \left(\frac{u_n}{v_1+iv_2}\right)^2 - \frac{2iu_n}{v_1+iv_2} - 1}$$

$$= \frac{\frac{-2iu_k}{v_1+iv_2}}{\frac{\sum\limits_{k=1}^{n} u_k^2}{(v_1+iv_2)^2} - \frac{2iu_n}{(v_1+iv_2)} - 1}$$

$$= \frac{-2iu_k}{v_1 - iv_2 - 2iu_n - v_1 - iv_2}$$

$$= \frac{u_k}{v_2 + u_n} = w_k, \qquad k = 1, \ldots, (n-1),$$

and

$$\frac{2\,(z_n - i)}{\sum\limits_{k=1}^{n-1} z_k^2 + (z_n - i)^2} - i = \frac{2\left(\frac{u_n}{v_1+iv_2} - i\right)}{\sum\limits_{k=1}^{n-1}\left(\frac{u_k}{v_1+iv_2}\right)^2 + \left(\frac{u_n}{v_1+iv_2}\right)^2 - \frac{2iu_n}{v_1+iv_2} - 1} - i$$

$$= \frac{2\left(\frac{u_n}{v_1+iv_2} - i\right)}{\frac{\sum\limits_{k=1}^{n} u_k^2}{(v_1+iv_2)^2} - \frac{2iu_n}{(v_1+iv_2)} - 1} - i$$

$$= \frac{2u_n - 2iv_1 + 2v_2}{v_1 - iv_2 - 2iu_n - v_1 - iv_2} - i$$

$$=? \frac{v_1}{v_2 + u_n} = w_n,$$

i.e.,

$$w_k = \frac{-2iz_k}{\sum\limits_{k=1}^{n-1} z_k^2 + (z_n - i)^2}, \qquad k = 1, \ldots, n-1, \qquad w_n = \frac{2\,(z_n - i)}{\sum\limits_{k=1}^{n-1} z_k^2 + (z_n - i)^2} - i.$$

On the other hand,

$$(w_1 - \bar{w}_1)^2 + \ldots + (w_{n-1} - \bar{w}_{n-1})^2 - (w_n - \bar{w}_n)^2 = \frac{-|u_1|^2 - \ldots - |u_n|^2 + |v_1|^2 + |v_2|^2}{|v_2 + u_n|^2}.$$

Then, according inequality (2.6), relation (2.1) becomes

$$(\operatorname{Im} w_n)^2 - (\operatorname{Im} w_1)^2 - \ldots - (\operatorname{Im} w_{n-1})^2 > 0. \tag{2.12}$$

Domain (2.12) consists of two connected pieces which analytically equivalent each to other. In one piece the inequality $\operatorname{Im} w_n > 0$ holds, while in the other the opposite inequality $\operatorname{Im} w_n < 0$ is satisfied. The domain $\tau^+(n-1)$ coincides with one of them. This means that Φ maps $\Re_{IV}^n$ onto $\tau^+(n-1)$. The map Φ is holomorphic because $\sum\limits_{k=1}^{n-1} z_k^2 + (z_n - i)^2 \neq 0$.

Now we are going to find $z_1, \ldots, z_n$. We have the following identities:

$$\frac{-2iw_k}{\sum\limits_{k=1}^{n-1} w_k^2 - (w_n + i)^2} = \frac{-2iw_k}{w_1^2 + \ldots + w_{n-1}^2 - w_n^2 - 2iw_n + 1} = \frac{-2i \cdot \frac{u_k}{v_2+u_n}}{\frac{v_2-u_n}{v_2+u_n} - 2i\frac{v_1}{v_2+u_n} + 1}$$

$$
\begin{aligned}
&= \frac{-2iu_k}{v_2 - u_n - 2iv_1 + v_2 + u_n} = \frac{-2iu_k}{2v_2 - 2iv_1} \\
&= \frac{u_k}{v_1 + iv_2} = z_k, \quad k = 1, \ldots, (n-1),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{-2\left(w_n + i\right)}{\sum\limits_{k=1}^{n-1} w_k^2 - \left(w_n + i\right)^2} &= \frac{-2 \cdot \left(\frac{v_1}{v_2+u_n} + i\right)}{\frac{v_2-u_n}{v_2+u_n} - 2i\frac{v_1}{v_2+u_n} + 1} = \frac{-2v_1 - 2iv_2 - 2iu_n}{v_2 - u_n - 2iv_1 + v_2 + u_n} \\
&= \frac{-2v_1 - 2iu_2 - 2iu_n}{2v_2 - 2iv_1} = \frac{u_n - iv_1 + v_2}{v_1 + iv_2} \\
&= \frac{u_n}{v_1 + iv_2} - i = z_n - i.
\end{aligned}
$$

Then by (2.3) we can find the inverse map $\Psi = \Phi^{-1} : \mathbb{C}_w^n \to \mathbb{C}_z^n$, which is defined as

$$
z_k = \frac{-2iw_k}{\sum\limits_{k=1}^{n-1} w_k^2 - \left(w_n + i\right)^2}, \quad k = 1, \ldots, (n-1), \qquad z_n = i - \frac{2\left(w_n + i\right)}{\sum\limits_{k=1}^{n-1} w_k^2 - \left(w_n + i\right)^2}. \tag{2.13}
$$

By the inequality

$$
w_1^2 + \ldots + w_{n-1}^2 - w_n^2 - 2iw_n + 1 \neq 0,
$$

the map Φ^{-1} is holomorphic in $\tau^+(n-1)$. In the same way we prove that Φ^{-1} maps holomorphically $\tau^+(n-1)$ into $\Re_{IV}^n$. Thus, Φ maps biholomorphically the domain $\Re_{IV}^n$ onto $\tau^+(n-1)$.

Now for the points $z_1, \ldots, z_n \in \Gamma_{\Re_{IV}^n}$ we calculate:

$$
\langle z, z \rangle = \frac{-W - 4iw_n}{W}
$$

and

$$
|z|^2 = \frac{4\sum\limits_{k=1}^{n-1} |w_k|^2 + \left|iW - 2\left(w_n + i\right)\right|^2}{|W|^2}, \tag{2.14}
$$

where $W = \sum\limits_{k=1}^{n-1} w_k^2 - \left(w_n + i\right)^2$. In addition, the complex numbers $w_1, w_2, \ldots, w_n$ satisfy the conditions

$$
|\langle z, z \rangle| = 1, \qquad |z| = 1
$$

if and only if

$$
\operatorname{Im} w_1 = \ldots = \operatorname{Im} w_n = 0.
$$

This means that the map Φ biholomorphically converts $\Gamma_{\Re_{IV}^n}$ to $\Gamma_\tau^+(n)$.

Thus, Φ maps biholomorphically the domain $\Re_{IV}^n$ onto $\tau^+(n-1)$, and it is clear that it transforms $\Gamma_{\Re_{IV}^n}$ into $\Gamma_\tau^+(n)$. $\square$

It is known that each biholomorphic map $\Phi : D \to G$ establishes a group isomorphism of $\operatorname{Aut} D$ and $\operatorname{Aut} G$ by the formula

$$
\widetilde{\Phi} : \varphi \to \Phi \circ \varphi \circ \Phi^{-1}, \qquad \varphi \in \operatorname{Aut} D, \tag{2.15}
$$

i.e., the isomorphism of the groups $\operatorname{Aut} D$ and $\operatorname{Aut} G$ is necessary for the holomorphic equivalence of the domains D and G (but not enough), see [8].

Thus, using Theorem 2.1 and the formula (2.15), we obtain a transitive group of holomorphic automorphisms of the domain $\tau^+(n-1)$. We recall that the group automorphisms of a domain are called transitive if there exists an automorphism transferring each point a of a domain to any other point of this domain.

We know that [3] the automorphism Φ_a of a Lie ball $\Re_{IV}^n$ transfering the point $a = (a_1, a_2, \ldots, a_n)$ into $0 = (0, \ldots, 0)$ can be found explicitly:

$$\Phi_a(z) = \left(\left(\left(\frac{1}{2}(\langle z, z\rangle + 1), \frac{i}{2}(\langle z, z\rangle - 1) \right) A' - zX'_0 A' \right) \begin{pmatrix} 1 \\ i \end{pmatrix} \right)^{-1}$$
$$\cdot \left(zQ' - \left(\frac{1}{2}(\langle z, z\rangle + 1), \frac{i}{2}(\langle z, z\rangle - 1) \right) X_0 Q' \right),$$

where

$$A = \frac{1}{2}(1 + |\langle a, a\rangle|^2 - 2|a|^2)^{-\frac{1}{2}} \begin{pmatrix} -i(\langle a, a\rangle - \overline{\langle a, a\rangle}) & \langle a, a\rangle + \overline{\langle a, a\rangle} - 2 \\ \langle a, a\rangle + \overline{\langle a, a\rangle} + 2 & i(\langle a, a\rangle - \overline{\langle a, a\rangle}) \end{pmatrix},$$

$$X_0 = \frac{1}{1 - |\langle a, a\rangle|^2} \begin{pmatrix} a + \bar{a} - (\overline{\langle a, a\rangle} \cdot a + \langle a, a\rangle \cdot \bar{a})) \\ i(a - \bar{a}) + i(\overline{\langle a, a\rangle} \cdot a - \langle a, a\rangle \cdot \bar{a}) \end{pmatrix},$$

and the nondegenerate matrix Q is chosen as

$$Q(I^{(n)} - X_0' X_0)Q' = I^{(n)}. \tag{2.16}$$

Using the transformation (2.3) and the automorphism Φ_a of the Lie ball $\Re_{IV}^n$ transferring the point a to $(0, 0, \ldots, 0)$, we define the following map:

$$\Psi_b = \Phi \circ \Phi_a \circ \Phi^{-1}, \text{where } b = \Phi(a). \tag{2.17}$$

It is clear that Ψ_b is an automorphism of the domain $\tau^+(n-1)$ transferring the point $b \in \tau^+(n-1)$ into the point $\mathrm{i} = (0, 0, \ldots, i)$.

In this way, using map (2.3), we can find the group of automorphisms of the domain $\tau^+(n-1)$.

Under the map Φ, the generalized unitary transformation of the Lie ball $\Re_{IV}^n$ transferring the point $(0, 0, \ldots, 0)$ onto itself becomes the transformation of the domain $\tau^+(n-1)$ preserving the point $\mathrm{i} = (0, 0, \ldots, i)$.

3. Boundary Morera theorem for the future tube

Let Φ be the Cayley transform (2.3) that maps the points of the Lie ball $\Re_{IV}^n$ to the points in the future tube $\tau^+(n-1)$. We consider the following embedding of the circle $\Delta = \{|t| < 1\}$ into the future tube $\tau^+(n-1)$:

$$\left\{ \zeta_t \in \mathbb{C}^n : \zeta_t = \Phi\left(t\Phi^{-1}(\lambda)\right), t \in \Delta \right\}, \tag{3.1}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma_{\tau^+(n-1)}$. If Ψ is an automorphism of the domain $\tau^+(n-1)$, then this automorphism maps set (3.1) into some analytic disk with boundaries on $\Gamma_{\tau^+(n-1)}$.

We denote $T = \{t \in \mathbb{C} : |t| = 1\}$ and we define the Hardy space $H^p(\tau^+(n-1))$, $0 < p < +\infty$ on the domain $\tau^+(n-1)$ as a space of holomorphic functions F in $\tau^+(n-1)$ with a finite norm

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\int\limits_{\Gamma_{\tau^+(n-1)}} |f(rz)|^p d\eta \right)^{\frac{1}{p}},$$

where η is the Lebesgue measure on the skeleton of $\Gamma_{\tau^+(n-1)}$. The corresponding spaces $H^\infty(\tau^+(n-1))$ are defined as functions with a finite norm defined by

$$\|f\|_{H^\infty} = \sup_{z \in \tau^+(n-1)} |f(z)|.$$

Theorem 3.1. *Let f be a continuous bounded function on $\Gamma_{\tau^+(n-1)}$. If the function f satisfies the condition*

$$\int\limits_T f(\Psi(\zeta_t))\,dt = 0, \tag{3.2}$$

for all automorphisms $\Psi \in \mathrm{Aut}\,(\tau^+(n-1))$ and for a fixed $\lambda^0 \in \Gamma_{\tau^+(n-1)}$, then the function f can be continued holomorphically on $\tau^+(n-1)$ to a function F in the class $H^\infty(\tau^+(n-1))$ and F is continuous up to $\Gamma_{\tau^+(n-1)}$.

Proof. Without loss of generality, we can assume that condition (3.2) is satisfied for arbitrary $\lambda \in \Gamma_{\tau^+(n-1)}$. In condition (3.2), we substitute an automorphism

$$\Psi = \Phi \circ \Phi_a^{-1} \circ \Phi^{-1}$$

instead of Ψ and we get:

$$\int\limits_T f(\Phi \circ \Phi_a^{-1} \circ \Phi^{-1}(\zeta_t))dt = 0. \tag{3.3}$$

Since $\zeta_t = \Phi(t\Phi^{-1}(\lambda))$ and denoting $\Theta = \Phi^{-1}(\lambda)$, we rewrite condition (3.3) in the form

$$\int\limits_T f(\Phi \circ \Phi_a^{-1}(\Theta t))dt = 0. \tag{3.4}$$

Let us consider in more detail the Shilov boundary (skeleton) $\Gamma_{\Re^n_{IV}}$ for the domain $\Re^n_{IV}$. Put $z = x + iy$, where x and y are real vectors. The intersection of $\Gamma_{\Re^n_{IV}}$ with the complex sphere $S^n = \{z \in \mathbb{C}^n : \langle z, z\rangle = 1\}$ reads as

$$\left\{|x|^2 = |y|^2 + 1,\ \langle x, y\rangle = 0,\ |x|^2 + |y|^2 = 1\right\},$$

hence $y = 0$ and S^n intersects $\Gamma_{\Re^n_{IV}}$ along the n-dimensional real sphere $\left\{x \in \mathbb{R}^n : |x|^2 = 1\right\}$. Thats why the skeleton $\Gamma_{\Re^n_{IV}}$ can be written as

$$\Gamma_{\Re^n_{IV}} = \left\{z = e^{i\varphi}x :\ |x|^2 = 1,\ 0 \leqslant \varphi \leqslant 2\pi\right\}.$$

It means that the skeleton $\Gamma_{\Re^n_{IV}}$ has real dimension n. Then we can consider the following parameterization of the core of the Lie ball:

$$\xi = \Theta t,\ t = e^{i\varphi},\ 0 \leqslant \varphi \leqslant 2\pi,\ \Theta \in S^n,$$

where S^n is the real unit sphere and the normalized Lebesgue measure of the skeleton of the Lie ball $\Gamma_{\Re^n_{IV}}$ can be represented as

$$d\sigma = \frac{d\varphi}{2\pi} \wedge d\sigma_1(\Theta) = \frac{1}{2\pi i}\frac{dt}{t} \wedge d\sigma_1(\Theta),$$

where $t = e^{i\varphi}$ and $d\sigma_1$ is a positive measure on S^n. Using this representation, we multiply the identity in condition (3.4) by $d\sigma_1(\Theta)$ and integrate over the skeleton of $\Gamma_{\Re^n_{IV}}$:

$$\int\limits_{\Gamma_{\Re^n_{IV}}} f(\Phi \circ \Phi_a^{-1}(\xi))\xi_k d\sigma(\xi) = 0, \tag{3.5}$$

where ξ_k, $k = 1, 2, \ldots$, are the coordinates of the vector $\xi \in \Gamma_{\Re^n_{IV}}$. After changing the variables $\omega = \Phi_a^{-1}(\xi)$, identity (3.5) casts into the form:

$$\int\limits_{\Gamma_{\Re^n_{IV}}} f(\Phi(\omega))\Phi_a^k(\omega)d\sigma(\Phi_a(\omega)) = 0, \tag{3.6}$$

where Φ_a^k is the kth component of the automorphism Φ_a. By virtue of [13], we have that

$$d\sigma\left(\Phi_a\left(\omega\right)\right) = P_{\Re_{IV}^n}\left(\omega, a\right) d\sigma\left(\omega\right),$$

here $P_{\Re_{IV}^n}(\omega, a)$ is the Poisson kernel in $\Re_{IV}^n$. Hence,

$$\int\limits_{\Gamma_{\Re_{IV}^n}} f(\Phi(\omega))\Phi_a^k(\omega) P_{\Re_{IV}^n}(\omega, a) d\sigma(\omega) = 0, \tag{3.7}$$

for all points $a \in \Re_{IV}^n$ and for all k.

Now in this integral, using the mapping (2.3), we make the replacement $u = \Phi\left(\omega\right)$:

$$\Phi_a^k\left(\omega\right) = \Phi_a^k \circ \Phi^{-1}\left(u\right) = \Psi_b^k\left(u\right),$$

where $\Psi_b^k\left(u\right)$ is the kth component of the map $\Psi_b = \Phi_a \circ \Phi^{-1}$. Due to [30] we can write the automorphism Φ_a in the following form:

$$\Phi_a(\omega) = \frac{(1 + |(a,a)|^2 - 2|a|^2)^{\frac{1}{2}}}{1 + (\omega,\omega)\overline{(a,a)} - 2(\omega,\bar{a})} \left(\omega - a + \frac{(a,a) - (\omega,\omega)}{1 - |(a,a)|^2}(\bar{a} - a \cdot \overline{(a,a)})\right) Q'.$$

Since the nondegenerate matrix Q depends only on a, the condition (3.7) also holds for the map

$$\Phi_a\left(\omega\right) = \frac{1}{1 + (\omega,\omega)\overline{(a,a)} - 2\left(\omega,\overline{a}\right)} \left(\omega - a + \frac{(a,a) - (\omega,\omega)}{1 - |(a,a)|^2}\left(\overline{a} - a\overline{(a,a)}\right)\right).$$

Then, the relation (2.13) for $\Psi_b^k\left(u\right)$ is of the form:

$$\Psi_b^k(u) = \frac{U\overline{B}}{2U + 2\overline{B} + 8u_n\overline{b_n} - 8\sum\limits_{k=1}^{n-1} u_k\bar{b}_k + i\left(4u_n - 4\bar{b}_n\right) + 2}$$
$$\cdot \left(\frac{-2iu_k}{U} - \frac{-2ib_k}{B} + \frac{\left(\left(4iu_n + U\right)B - \left(4ib_n + B\right)U\right)\overline{B}}{U\left(4i\overline{b_n}B - 4ib_n\overline{B} - 16|b_n|^2\right)}\left(\frac{2i\bar{b}_k}{\overline{B}} + \frac{2ib_k}{B} \cdot \frac{4i\bar{b}_n + \overline{B}}{\overline{B}}\right)\right),$$

where $k = 1, \ldots, (n-1)$, and

$$\Psi_b^n(u) = \frac{U\overline{B}}{2U + 2\overline{B} + 8u_n\overline{b_n} - 8\sum\limits_{k=1}^{n-1} u_k\bar{b}_k + i\left(4u_n - 4\bar{b}_n\right) + 2}$$
$$\cdot \left(-\frac{2\left(u_n + i\right)}{U} + \frac{2\left(b_n + i\right)}{B} + \frac{\left(\left(4iu_n + U\right)B - \left(4ib_n + B\right)U\right)\overline{B}}{U\left(4i\overline{b_n}B - 4ib_n\overline{B} - 16|b_n|^2\right)}\right.$$
$$\left. \cdot \left(-i - \frac{2\left(\bar{b}_n - i\right)}{\overline{B}} + \left(i - \frac{2\left(b_n + i\right)}{B}\right) \cdot \frac{4i\bar{b}_n + \overline{B}}{\overline{B}}\right)\right),$$

where

$$B = \sum_{k=1}^{n-1} b_k^2 - (b_n + i)^2, \qquad U = \sum_{k=1}^{n-1} u_k^2 - (u_n + i)^2.$$

By the Lemma 3.4 in [13] we have:

$$P_{\Re_{IV}^n}\left(\Phi^{-1}\left(u\right), \, \Phi^{-1}\left(b\right)\right) d\sigma\left(\Phi^{-1}\left(u\right)\right) = P_{\tau^+(n-1)}\left(u, \, b\right) d\eta\left(u\right),$$

here the Poisson kernel $P_{\tau^+(n-1)}\left(u, \, b\right)$ for the domain $\tau^+\left(n-1\right)$ has the following form [6]:

$$P_{\tau^+(n-1)}\left(u, b\right) = \frac{\left[(y^2)\right]^{\frac{n}{2}}}{\left|(x + iy - u)^2\right|^n},$$

where

$$y^2 = y_n^2 - (y')^2 = y_n^2 - \left(y_1^2 + y_2^2 + \ldots + y_{n-1}^2\right), \qquad x + iy = b \in \tau^+(n-1), \qquad u \in \Gamma_{\tau^+(n-1)}.$$

Then, in condition (3.7), the integration set becomes $\Gamma_{\tau^+(n-1)}$ and we obtain the condition

$$\int\limits_{\Gamma_{\tau^+(n-1)}} f(u)\Psi_b^k(u)P_{\tau^+(n-1)}(u,b)d\eta(u) = 0, \tag{3.8}$$

for all points $b \in \tau^+(n-1)$ and for all $k = 1, \ldots, n$.

We introduce the following notation:

$$F(b) = \int\limits_{\Gamma_{\tau^+(n-1)}} f(u)P_{\tau^+(n-1)}(u,b)d\eta(u).$$

According Proposition 2.5 in [13], the Poisson integral $F(b)$ is well-defined and it is a real analytic function in $\tau^+(n-1)$ continuous up to $\Gamma_{\tau^+(n-1)}$ and equal to f on $\Gamma_{\tau^+(n-1)}$. We need to show that F is a holomorphic function once f satisfies the assumptions of Theorem 3.1. In order to do this, we consider a differential operator of Euler type of the following form:

$$\tilde{\partial} = \sum_{k=1}^{n-1} 2i\overline{b_k}\frac{\partial}{\partial \overline{b_k}} - \left(i\overline{B} + 2(\overline{b_n} - i)\right)\frac{\partial}{\partial \overline{b_n}}.$$

We denote

$$\Delta(y) := y_n^2 - (y')^2 = y_n^2 - \left(y_2^2 + y_3^2 + \ldots + y_{n-1}^2\right),$$

then the Poisson kernel $P_{\tau^+(n-1)}(u,\, b)$ can be written as

$$P_{\tau^+(n-1)}(u,\, b) = \frac{\Delta^{\frac{n}{2}}\left(i\left(\bar{b} - b\right)\right)}{|\Delta(b-u)|^n}, \quad b \in \tau^+(n-1), \quad u \in \Gamma_{\tau^+(n-1)}.$$

Then the partial derivatives $P_{\tau^+(n-1)}(u,\, b)$ cast into the form:

$$\begin{aligned}
\frac{\partial P_{\tau^+(n-1)}(u,b)}{\partial \overline{b_k}} =& \frac{1}{\Delta^{\frac{n}{2}}(b-u)}\frac{\partial}{\partial \overline{b_k}}\left(\frac{\Delta^{\frac{n}{2}}\left(i\left(\bar{b}-b\right)\right)}{\Delta^{\frac{n}{2}}\left(\bar{b}-u\right)}\right) \\
=& \frac{1}{\Delta^{\frac{n}{2}}(b-u)}\left(\frac{n\Delta^{\frac{n}{2}-1}\left(i\left(\bar{b}-b\right)\right)2\left(\bar{b}_k-b_k\right)}{2\Delta^{\frac{n}{2}}\left(\bar{b}-u\right)} + \frac{n}{2}\frac{\Delta^{\frac{n}{2}}\left(i\left(\bar{b}-b\right)\right)2\left(\bar{b}_k-u_k\right)}{\Delta^{\frac{n}{2}+1}\left(\bar{b}-u\right)}\right) \\
=& n\left(\frac{\left(\bar{b}_k-b_k\right)}{\Delta\left(i\left(\bar{b}-b\right)\right)} + \frac{\left(\bar{b}_k-u_k\right)}{\Delta\left(\bar{b}-u\right)}\right)\frac{\Delta^{\frac{n}{2}}\left(i\left(\bar{b}-b\right)\right)}{|\Delta^n(b-u)|} \\
=& n\left(\frac{\left(\bar{b}_k-b_k\right)}{\Delta\left(i\left(\bar{b}-b\right)\right)} + \frac{\left(\bar{b}_k-u_k\right)}{\Delta\left(\bar{b}-u\right)}\right)P_{\tau^+(n-1)}(u,b) \\
=& n\left(\frac{\left(\bar{b}_k-b_k\right)}{\left(\sum\limits_{k=1}^{n-1}\left(\bar{b}_k-b_k\right)^2-\left(\bar{b}_n-b_n\right)^2\right)^{\frac{n}{2}}}\right. \\
&\left. + \frac{\left(\bar{b}_k-u_k\right)}{\left|\left(b_n-u_n\right)^2-\sum\limits_{k=1}^{n-1}\left(b_k-u_k\right)^2\right|^n}\right)P_{\tau^+(n-1)}(u,b),
\end{aligned}$$

for all $k = 1, 2, \ldots, n-1$, and

$$\begin{aligned}
\frac{\partial P_{\tau^+(n-1)}(u,b)}{\partial \overline{b_n}} =& \frac{1}{\Delta^{\frac{n}{2}}(b-u)} \frac{\partial}{\partial \overline{b_n}} \left(\frac{\Delta^{\frac{n}{2}}\left(i\left(\bar{b}-b\right)\right)}{\Delta^{\frac{n}{2}}\left(\bar{b}-u\right)} \right) \\
=& \frac{1}{\Delta^{\frac{n}{2}}(b-u)} \left(-\frac{n\Delta^{\frac{n}{2}-1}\left(i\left(\bar{b}-b\right)\right)\left(\bar{b}_n-b_n\right)}{\Delta^{\frac{n}{2}}\left(\bar{b}-u\right)} - \frac{n\Delta^{\frac{n}{2}}\left(i\left(\bar{b}-b\right)\right)\left(\bar{b}_n-u_n\right)}{\Delta^{\frac{n}{2}+1}\left(\bar{b}-u\right)} \right) \\
=& -n \left(\frac{\left(\bar{b}_n-u_n\right)}{\Delta\left(\bar{b}-u\right)} + \frac{\left(\bar{b}_n-b_n\right)}{\Delta\left(i\left(\bar{b}-b\right)\right)} \right) \frac{\Delta^{\frac{n}{2}}\left(i\left(\bar{b}-b\right)\right)}{\left|\Delta^n(b-u)\right|} \\
=& -n \left(\frac{\left(\bar{b}_n-b_n\right)}{\Delta\left(i\left(\bar{b}-b\right)\right)} + \frac{\left(\bar{b}_n-u_n\right)}{\Delta\left(\bar{b}-u\right)} \right) P_{\tau^+(n-1)}(u,b) \\
=& -n \left(\frac{\left(\bar{b}_n-b_n\right)}{\left(\sum\limits_{k=1}^{n-1}\left(\bar{b}_k-b_k\right)^2 - \left(\bar{b}_n-b_n\right)^2\right)^{\frac{n}{2}}} \right. \\
& \left. + \frac{\left(\bar{b}_n-u_n\right)}{\left|\left(b_n-u_n\right)^2 - \sum\limits_{k=1}^{n-1}\left(b_k-u_k\right)^2\right|^n} \right) P_{\tau^+(n-1)}(u,b),
\end{aligned}$$

Hence, we have the following relation:

$$\begin{aligned}
\tilde{\partial} P_{\tau^+(n-1)}(u,b) =& \sum_{k=1}^{n-1} 2i\overline{b_k} \frac{\partial P_{\tau^+(n-1)}(u,b)}{\partial \overline{b_k}} - \left(i\overline{B} + 2(\overline{b_n}-i)\right) \frac{\partial P_{\tau^+(n-1)}(u,b)}{\partial \overline{b_n}} \\
=& nP_{\tau^+(n-1)}(u,b) \left(\frac{4\sum\limits_{k=1}^{n-1}|b_k|^2 + 2i\overline{B}(b_n+i) - 2iB\left(\bar{b}_n-i\right) + 4|b_n+i|^2}{-4ib_n\overline{B} - \left(8\sum\limits_{k=1}^{n-1}|b_k|^2 + 2|iB-2(b_n+i)|^2\right)} \right. \\
& \left. - \frac{-4\sum\limits_{k=1}^{n-1}u_k\bar{b}_k + i\left((2u_n+i)\overline{B} - U\left(2\bar{b}_n-i\right)\right) + (2u_n+i)\left(2\bar{b}_n-i\right)}{2U + 2\overline{B} + 8u_n\overline{b_n} - 8\sum\limits_{k=1}^{n-1}u_k\bar{b}_k + i\left(4u_n-4\bar{b}_n\right) + 2} \right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{k=1}^{n-1}\Psi_b^k(u)\frac{2i\bar{b}_k}{\overline{B}} - \Psi_b^n(u)\left(\frac{2\left(\overline{b_n}-i\right)}{\overline{B}}+i\right) =& \frac{U\overline{B}}{2U + 2\overline{B} + 8u_n\overline{b_n} - 8\sum\limits_{k=1}^{n-1}u_k\bar{b}_k + i\left(4u_n-4\bar{b}_n\right) + 2} \\
\cdot \Bigg(-4\sum_{k=1}^{n-1}\frac{u_k\bar{b}_k}{U\overline{B}} + 1 + i\left(\frac{(2u_n+i)}{U} - \frac{\left(2\bar{b}_n-i\right)}{\overline{B}}\right) + \frac{(2u_n+i)}{U}\frac{\left(2\bar{b}_n-i\right)}{\overline{B}} \\
- \sum_{k=1}^{n-1}\left|\frac{-2ib_k}{B}\right|^2 - \left|i - \frac{2(b_n+i)}{B}\right|^2 + \frac{\left[(4iu_n+U)B - (4ib_n+B)U\right]\overline{B}}{U\left(4i\overline{b_n}B - 4ib_n\overline{B} - 16|b_n|^2\right)} \\
\left(\frac{-4i\overline{b_n}+\overline{B}}{\overline{B}} \left(1 - \sum_{k=1}^{n-1}\left|\frac{-2ib_k}{B}\right|^2 + \left|i - \frac{2(b_n+i)}{B}\right|^2 \right) \right) \Bigg)
\end{aligned}$$

$$= \left(\frac{4\sum\limits_{k=1}^{n-1} |b_k|^2 + 2i\overline{B}\,(b_n+i) - 2iB\left(\overline{b}_n - i\right) + 4|b_n+i|^2}{-4ib_n\overline{B} - \left(8\sum\limits_{k=1}^{n-1}|b_k|^2 + 2|iB - 2\,(b_n+i)|^2\right)} - \frac{-4\sum\limits_{k=1}^{n-1} u_k\overline{b}_k + i\left((2u_n+i)\,\overline{B} - U\left(2\overline{b}_n - i\right)\right) + (2u_n+i)\left(2\overline{b}_n - i\right)}{2U + 2\overline{B} + 8u_n\overline{b_n} - 8\sum\limits_{k=1}^{n-1} u_k\overline{b}_k + i\left(4u_n - 4\overline{b}_n\right) + 2} \right) B.$$

Then

$$\begin{aligned}\tilde{\partial} F =& \sum_{k=1}^{n-1} 2i\overline{b_k} \int\limits_{\Gamma_{\tau^+(n-1)}} f(u)\frac{\partial P_{\tau^+(n-1)}(u,b)}{\partial \overline{b_k}} d\eta(u)\\
&- \left(i\overline{B} + 2(\overline{b_n} - i)\right) \int\limits_{\Gamma_{\tau^+(n-1)}} f(u)\frac{\partial P_{\tau^+(n-1)}(u,b)}{\partial \overline{b_n}} d\eta(u)\\
=& \int\limits_{\Gamma_{\tau^+(n-1)}} f(u)\left(\sum_{k=1}^{n-1} 2i\overline{b_k}\frac{\partial P_{\tau^+(n-1)}(u,b)}{\partial \overline{b_k}} - \left(i\overline{B} + 2(\overline{b_n} - i)\right)\frac{\partial P_{\tau^+(n-1)}(u,b)}{\partial \overline{b_n}}\right) d\eta(u)\\
=& n\int\limits_{\Gamma_{\tau^+(n-1)}} f(u)\left(\frac{4\sum\limits_{k=1}^{n-1} |b_k|^2 + 2i\overline{B}\,(b_n+i) - 2iB\left(\overline{b}_n - i\right) + 4|b_n+i|^2}{-4ib_n\overline{B} - \left(8\sum\limits_{k=1}^{n-1}|b_k|^2 + 2|iB - 2\,(b_n+i)|^2\right)}\right.\\
&\left. - \frac{-4\sum\limits_{k=1}^{n-1} u_k\overline{b}_k + i\left((2u_n+i)\,\overline{B} - U\left(2\overline{b}_n - i\right)\right) + (2u_n+i)\left(2\overline{b}_n - i\right)}{2U + 2\overline{B} + 8u_n\overline{b_n} - 8\sum\limits_{k=1}^{n-1} u_k\overline{b}_k + i\left(4u_n - 4\overline{b}_n\right) + 2} \right)\\
&\cdot P_{\tau^+(n-1)}\,(u,b)\,d\eta(u)\\
=& \frac{n}{|B|^2}\left(\int\limits_{\Gamma_{\tau^+(n-1)}} f(u)\sum_{k=1}^{n-1}\Psi_b^k(u)2i\overline{b}_k P_{\tau^+(n-1)}(u,b)d\eta(u)\right.\\
&\left. - \int\limits_{\Gamma_{\tau^+(n-1)}} f(u)\Psi_b^n(u)\left(i\bar{B} + 2\left(\overline{b_n} - i\right)\right)P_{\tau^+(n-1)}(u,b)d\eta(u)\right).\end{aligned} \tag{3.9}$$

Now by (3.8) and (3.9) we obtain:

$$\tilde{\partial} F = 0 \tag{3.10}$$

for all points $b \in \tau^+(n-1)$. This yields that F is holomorphic in $\tau^+(n-1)$. Indeed, expanding the function into a Taylor series in powers of $(b - \mathrm{i})$, where $\mathrm{i} = (0,0,\dots,i) \in \tau^+(n-1)$, we get:

$$F(b) = \sum_{\substack{\|\alpha\|\geqslant 0\\ \|\beta\|\geqslant 0}} c_{\alpha,\beta}(b-\mathrm{i})^{\alpha}\overline{(b-\mathrm{i})^{\beta}},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ are multi-indexes and

$$b^\alpha = b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_n^{\alpha_n}, \qquad \|\alpha\| = \alpha_1 + \alpha_2 + \ldots + \alpha_n.$$

Then by (3.10) we obtain

$$\tilde{\partial} F = \sum_{\|\alpha\|, \|\beta\|} \|\beta\| \, c_{\alpha,\beta} (b - \mathrm{i})^\alpha \overline{(b - \mathrm{i})^\beta} = 0,$$

i.e., if $\|\beta\| > 0$, then all the coefficients $c_{\alpha,\beta}$ are equal to zero. Consequently, the function F is holomorphic in $\tau^+(n-1)$ and $F \in H^\infty(\tau^+(n-1))$. □

The proof of this theorem shows that a more general theorem is also true.

Theorem 3.2. *Let f be a continuous bounded function on $\Gamma_{\tau^+(n-1)}$. If condition (3.2) is satisfied for the function f for all automorphisms mapping the point $i = (0, 0, \ldots, i)$ into the points from some open set $V \subset \tau^+(n-1)$. Then f can be holomorphically continued in $\tau^+(n-1)$ to a function $F \in H^\infty(\tau^+(n-1))$ continuous up to $\Gamma_{\tau^+(n-1)}$.*

Let $\Delta_\Psi = \{\zeta : \zeta = \Psi(\xi_t)\}$ be an analytic disk, where ξ_t is defined as in (3.1) and Ψ is an automorphism of the domain $\tau^+(n-1)$.

Corollary 3.1. *If a continuous and bounded in $\Gamma_{\tau^+(n-1)}$ function f can be holomorphically continued into the analytic disks Δ_Ψ for all automorphisms Ψ mapping the point $i = (0, 0, \ldots, i)$ into the points from some open set $V \subset \tau^+(n-1)$, then f can be holomorphically continued into $\tau^+(n-1)$ to a function $F \in H^\infty(\tau^+(n-1))$ continuous up to $\Gamma_{\tau^+(n-1)}$.*

This Corollary 3.1 is an analogue Stout theorem [12] on functions with the one dimensional property of holomorphic continuation but in our case it is formulated for the domain $\tau^+(n-1)$.

The following corollary generalizes a similar Tumanov theorem [17] for a smooth function of a given class of CR manifolds.

Corollary 3.2. *If a continuous bounded function f defined on $\Gamma_{\tau^+(n-1)}$ can be holomorphically continued in t into each analytic disk Δ_Ψ lying in $\tau^+(n-1)$ with a boundary on $\Gamma_{\tau^+(n-1)}$, then f can be holomorphically continued into the domain $\tau^+(n-1)$ and, therefore, is a CR function on $\Gamma_{\tau^+(n-1)}$.*

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Gulmirza Khudayberganov,
National University of Uzbekistan named after M. Ulugbek,
Universitetskaya street, Vuzgorodok,
100174, Tashkent, Uzbekistan
E-mail: gkhudaiberg@mail.ru

Jonibek Shokirovich Abdullayev,
National University of Uzbekistan named after M. Ulugbek,
Universitetskaya street, Vuzgorodok, 100174
Tashkent, Uzbekistan
E-mail: jonibek-abdullayev@mail.ru