ON DISCRETIZATION OF DARBOUX INTEGRABLE SYSTEMS ADMITTING SECOND-ORDER INTEGRALS

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Abstract. We consider a discretization problem for hyperbolic Darboux integrable systems. In particular, we discretize continuous systems admitting x- and y-integrals of the first and second order. Such continuous systems were classified by Zhyber and Kostrigina. In the present paper, continuous systems are discretized with respect to one of continuous variables and the resulting semi-discrete system is required to be also Darboux integrable.

To obtain such a discretization, we take x- or y-integrals of a given continuous system and look for a semi-discrete systems admitting the chosen integrals as n-integrals. This method was proposed by Habibullin. For all considered systems and corresponding sets of integrals we were able to find such semi-discrete systems. In general, the obtained semi-discrete systems are given in terms of solutions of some first order quasilinear differential systems. For all such first order quasilinear differential systems we find implicit solutions. New examples of semi-discrete Darboux integrable systems are obtained. Also for each of considered continuous systems we determine a corresponding semi-discrete system that gives the original system in the continuum limit.

Keywords: Darboux integrability, discretization.

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1. Introduction

In the present paper we study the problem of the discretization of integrable equations so that the integrability property is preserved. In particular, we consider hyperbolic systems

$$p^i_{xy} = f^i(x, y, p, p_x, p_y) \qquad i = 1, \dots, m\,, \tag{1.1}$$

where $p = (p^1, \dots, p^m)$, $p_x = (p^1_x, \dots, p^m_x)$ and $p_y = (p^1_y, \dots, p^m_y)$.

For such hyperbolic systems it is convenient to use Darboux integrability [1]. The above system is said to be integrable if it admits m functionally independent non-trivial x-integrals and m functionally independent non-trivial y-integrals. A function $I(x, y, p, p_y, p_{yy}, \dots)$ is called an x-integral of the system (1.1) if

$$D_x I(x, y, p, p_y, p_{yy}, \dots) = 0 \qquad \text{for all solutions of (1.1)}, \tag{1.2}$$

where D_x is the total derivative with respect to x. One can define y-integrals in a similar way. Darboux integrable systems are extensively studied, see [2]-[11] and a review paper [12].

The extension of the notion of Darboux integrability to discrete and semi-discrete Darboux integrable systems was developed by Habibullin and Pekcan [13], see also [14]. In recent years there is an interest in studying such systems, see [15]-[25]. A semi-discrete system

$$q^i_{1x} = f^i(x, n, q, q_x, q_1) \qquad i = 1, \dots, m\,, \tag{1.3}$$

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where $q = (q^1, \dots, q^m)$, $q_x = (q_x^1, \dots, q_x^m)$ and $q_1 = (q^1(x, n+1), \dots, q^m(x, n+1))$, is called Darboux integrable if it admits m functionally independent non-trivial x-integrals and m functionally independent non-trivial n-integrals. A function $J(x, n, q, q_x, q_{xx}, \dots)$ is called an n-integral of the system (1.3) if

$$DJ(x, n, q, q_x, q_{xx}, \dots) = J(x, n, q, q_x, q_{xx}, \dots) \qquad \text{for all solutions of (1.3)}, \tag{1.4}$$

where D is the shift operator, that is $Dq = q_1$. Note that $Dq_k = q_{k+1}$, $k = 1, 2, 3, \dots$. The x-integrals $I(x, n, q, q_1, q_2, \dots)$ for system (1.3) are defined in the same way as for continuous systems.

A hypothesis states that any continuous Darboux integrable system can be discretized with respect to one of the independent variables such that the resulting semi-discrete system is Darboux integrable and admits the set of x- or y-integrals of the original system as $n-$integrals [26]. The results of our work support the above conjecture. We complete the discretization of continuous Darboux integrable equations derived by Zhiber and Kostrigina in [8]. In their paper, Zhiber and Kostrigina considered the classification problem for continuous Darboux integrable systems with two integrals of the first order and two integrals of the second order. They found all such systems together with their x- and y-integrals. Following [8], we have two types of systems. The first system is

$$\begin{cases} u_{xy} = \dfrac{u_x u_y}{u+v} + \left(\dfrac{1}{u+v} + \dfrac{\alpha}{u+\alpha^2 v}\right) u_x v_y, \\ v_{xy} = \dfrac{\alpha^2 v_x v_y}{u+\alpha^2 v} + \left(\dfrac{1}{\alpha(u+v)} + \dfrac{1}{u+\alpha^2 v}\right) u_x v_y, \end{cases} \tag{1.5}$$

with α being a nonzero constant. We mention that in the case $\alpha = 1$, system (1.5) was discretized in [26]. For $\alpha \neq 1$, it possesses y-integrals

$$I_1 = \left(1 + \frac{1}{\alpha}\right) v \left(\frac{u_x}{u+v}\right)^{1-\alpha} - v_x \left(\frac{u_x}{u+v}\right)^{-\alpha}, \tag{1.6}$$

$$J_1 = \frac{u_{xx}}{u_x} - \frac{(\alpha+1)u_x + \alpha v_x}{\alpha(u+v)}, \tag{1.7}$$

and the x-integrals have the same form in u, u_y, u_{yy} and v, v_y, v_{yy} variables.

The second system reads as

$$\begin{cases} u_{xy} = \dfrac{v u_x u_y}{uv+d} + \left(\dfrac{1}{uv+d} + \dfrac{1}{\alpha(uv+c)}\right) u u_x v_y, \\ v_{xy} = \dfrac{u v_x v_y}{uv+c} + \left(\dfrac{\alpha}{uv+d} + \dfrac{1}{uv+c}\right) v u_x v_y, \end{cases} \tag{1.8}$$

where α, c and d are nonzero constants. For $\alpha = -1$ it possesses y-integrals

$$I_2 = \frac{(d-c)v^2 u_x^2}{2(uv+d)^2} - \frac{c u_x v_x}{uv+d} \tag{1.9}$$

and

$$J_2 = \frac{u_{xx}}{u_x} + \frac{(d-c)v u_x - c u v_x}{c(uv+d)}, \tag{1.10}$$

where c and d are non-zero constants and the x-integrals have the same form in u, u_y, u_{yy} and v, v_y, v_{yy} variables.

For $\alpha \neq -1$ it possesses y-integrals

$$I_3 = \frac{u_x^{\beta} v_x}{(uv+d)^{\beta}} + \frac{\beta v^2 u_x^{\beta+1}}{(uv+d)^{\beta+1}} \tag{1.11}$$

and

$$J_3 = -\frac{u_{xx}}{u_x} + \frac{2vu_x + uv_x}{uv + d}, \tag{1.12}$$

where d and $\beta = -\alpha \neq 1$ are nonzero constants, and the x-integrals have the same form in u, u_y , u_{yy} and v, v_y, v_{yy} variables.

In order to discretize the systems (1.5) and (1.8), we employ a method introduced by Habibullin et. all [20], see also [24]-[26]. According to this approach, one takes the x- or y-integrals of a system and looks for a semi-discrete system admitting such integrals as n-integrals. In general, one gets a set of semi-discrete systems admitting these n-integrals. For all sets of the y-integrals of systems (1.5) and (1.8) we obtain corresponding semi-discrete systems. Note that initially we allow the parameters α, c and d in integrals (1.6), (1.7) and (1.9)-(1.12) to depend on n. It turns out that only d may depend non-trivially on n in one case. In all cases we are able to choose a semi-discrete system that gives the original system in the continuum limit. Also in examples, where we can write a semi-discrete system explicitly, we show that the system is Darboux integrable.

The following theorems are formulated for a hyperbolic type semi-discrete system

$$\begin{cases} u_{1x} = f(x, n, u, v, u_x, v_x, u_1, v_1), \\ v_{1x} = g(x, n, u, v, u_x, v_x, u_1, v_1), \end{cases} \tag{1.13}$$

where variables u, v depend on a continuous variable $x \in \mathbb{R}$ and a discrete variable $n \in \mathbb{Z}$.

Theorem 1.1. *Let $\alpha \neq 1$. System* (1.13) *admits n-integrals* (1.6) *and* (1.7) *if and only if it is of the form*

$$\begin{cases} u_{1x} = \dfrac{u_1 + v_1}{u + v} \mathcal{D}_1^{\alpha^{-1}} u_x, \\ v_{1x} = \dfrac{\alpha + 1}{\alpha} \dfrac{v_1 \mathcal{D}_1^{\alpha^{-1}} - v\mathcal{D}_1}{u + v} u_x + \mathcal{D}_1 v_x. \end{cases} \tag{1.14}$$

The function $\mathcal{D}_1$ is equal to 1 *or given implicitly by $H(n, K_1, L_1) = 0$, where, for each n, the symbol H denotes an arbitrary smooth function and*

$$K_1 = \frac{\alpha v_1 \mathcal{D}_1^{\alpha^{-1}} - \alpha v \mathcal{D}_1^{1+\alpha^{-1}} + (1 - \mathcal{D}_1^{\alpha^{-1}}) u_1}{(\mathcal{D}_1^{\alpha^{-1}} - 1)^{\alpha+1}}, \tag{1.15}$$

$$L_1 = \frac{(u_1 - \mathcal{D}_1^{1+\alpha^{-1}} u) e^{\mathcal{D}_1^{\alpha^{-1}}}}{\mathcal{D}_1 (\mathcal{D}_1^{\alpha^{-1}} - 1)^a} + \frac{(-1)^\alpha e^{\mathcal{D}_1^{\alpha^{-1}}} (\alpha v_1 \mathcal{D}_1^{\alpha^{-1}} - \alpha v \mathcal{D}_1^{1+\alpha^{-1}} + (1 - \mathcal{D}_1^{\alpha^{-1}}) u_1)}{(\mathcal{D}_1^{\alpha^{-1}} - 1)^{\alpha+1}}. \tag{1.16}$$

Let us construct some examples.

Example 1.1. *In the case $\mathcal{D}_1 = 1$ system* (1.14) *becomes*

$$\begin{cases} u_{1x} = \dfrac{u_1 + v_1}{u + v} u_x, \\ v_{1x} = \left(1 + \dfrac{1}{\alpha}\right) \dfrac{v_1 - v}{u + v} u_x + v_x. \end{cases} \tag{1.17}$$

This system is Darboux integrable. Indeed, it possesses two independent non-trivial n-integrals (1.6), (1.7) *and two independent non-trivial x-integrals*

$$\mathcal{F}_1 = \frac{v - v_1}{v_1 - v_2} \qquad \text{and} \qquad \mathcal{F}_2 = \frac{u_1 - u}{(v_1 - v)^{\frac{\alpha}{1+\alpha}}} - \alpha (v_1 - v)^{\frac{1}{1+\alpha}}. \tag{1.18}$$

The x-integrals can be found by considering the x-ring corresponding to the system.

Example 1.2. *Letting $K_1 = 0$ and $\alpha = -1$, we get*

$$\mathcal{D}_1 = \frac{u_1 + v_1}{u_1 + v}.$$

Using (1.14), *we find the system*

$$\begin{cases} u_{1x} = \dfrac{u_1 + v}{u + v} u_x, \\ v_{1x} = \dfrac{u_1 + v_1}{u_1 + v} v_x. \end{cases} \tag{1.19}$$

This system is Darboux integrable. It possesses two independent non-trivial n-integrals (1.6), (1.7) *and two independent non-trivial x-integrals*

$$\mathcal{F}_1 = \frac{(u_2 + v_1)(v - v_1)}{(u_1 + v_1)(v_1 - v_2)} \qquad \text{and} \qquad \mathcal{F}_2 = \frac{(u - u_1)(u_1 + v_1)}{(-u_1 + u_2)(u_1 + v)}. \tag{1.20}$$

Example 1.3. *Choosing $K_1 = 0$ and $\alpha = -\dfrac{1}{2}$, we get*

$$\mathcal{D}_1 = \frac{4u_1 + 2v_1}{v + \sqrt{v^2 + 16u_1^2 + 8u_1 v_1}}.$$

By (1.14), *we obtain the system*

$$\begin{cases} u_{1x} = \dfrac{u_1 + v_1}{u + v} \left(\dfrac{v + \sqrt{v^2 + 16u_1^2 + 8u_1 v_1}}{4u_1 + 2v_1} \right)^2 u_x, \\ v_{1x} = -\left(\dfrac{v_1}{u + v} \left(\dfrac{v + \sqrt{v^2 + 16u_1^2 + 8u_1 v_1}}{4u_1 + 2v_1} \right)^2 \right. \\ \qquad \left. - \dfrac{v(4u_1 + 2v_1)}{(u + v)(v + \sqrt{v^2 + 16u_1^2 + 8u_1 v_1})} \right) u_x \\ \qquad + \dfrac{4u_1 + 2v_1}{v + \sqrt{v^2 + 16u_1^2 + 8u_1 v_1}} v_x. \end{cases} \tag{1.21}$$

This system possesses two independent non-trivial n-integrals (1.6) *and* (1.7).

Example 1.4. *Considering $L_1 = 0$ and $\alpha = -1/2$, we get*

$$\mathcal{D}_1 = \frac{v_1 + \sqrt{v_1^2 + 16u^2 + 8uv}}{2v + 4u}.$$

By (1.14) *we then arrive at the system*

$$\begin{cases} u_{1x} = \dfrac{u_1 + v_1}{u + v} \left(\dfrac{2v + 4u}{v_1 + \sqrt{v_1^2 + 16u^2 + 8uv}} \right)^2 u_x, \\ v_{1x} = -\left(\dfrac{v_1}{u + v} \left(\dfrac{2v + 4u}{v_1 + \sqrt{v_1^2 + 16u^2 + 8uv}} \right)^2 - \dfrac{v(v_1 + \sqrt{v_1^2 + 16u^2 + 8uv})}{(u + v)(2v + 4u)} \right) u_x \\ \qquad + \dfrac{v_1 + \sqrt{v_1^2 + 16u^2 + 8uv}}{2v + 4u} v_x. \end{cases} \tag{1.22}$$

This system possesses two independent non-trivial n-integrals (1.6) *and* (1.7).

Remark 1.1. *In both previous examples let us consider the corresponding x-rings. Let*

$$X = D_x, \qquad Y_1 = \frac{\partial}{\partial u_x}, \qquad Y_2 = \frac{\partial}{\partial v_x},$$
$$E_1 = [Y_1, X], \qquad E_2 = [Y_2, X], \qquad E_3 = [E_1, E_2].$$

We observe that

$$X = u_x E_1 + v_x E_2 + Y_1 + Y_2.$$

The following multiplication table

$[E_i, E_j]$	E_1	E_2	E_3
E_1	0	E_3	$-2(u+v)^{-1}E_3$
E_2	$-E_3$	0	$-2(u+v)^{-1}E_3$
E_3	$2(u+v)^{-1}E_3$	$2(u+v)^{-1}E_3$	0

shows that the x-rings are finite-dimensional. Therefore, systems (1.21) and (1.22) are Darboux integrable.

Remark 1.2. *We consider the function $\mathcal{D}_1^{\alpha^{-1}}$ defined implicitly by*

$$H(K_1, L_1) = K_1 = 0,$$

that is, by

$$\alpha v_1 \mathcal{D}_1^{\alpha^{-1}} - \alpha v \mathcal{D}_1^{1+\alpha^{-1}} + (1 - \mathcal{D}_1^{\alpha^{-1}}) u_1 = 0,$$

and expand it into a series of the form

$$\mathcal{D}_1^{\alpha^{-1}}(u_1, v, v_1) = \sum_{n=0}^{\infty} a_n (v_1 - v)^n,$$

where the coefficients a_n depend on variables u_1 and v only. This yields

$$\mathcal{D}_1^{\alpha^{-1}}(u_1, v, v_1) = 1 + \frac{\alpha}{u_1 + \alpha^2 v}(v_1 - v) + \sum_{n=2}^{\infty} a_n (v_1 - v)^n$$

and

$$\mathcal{D}_1(u_1, v, v_1) = 1 + \frac{\alpha^2}{u_1 + \alpha^2 v}(v_1 - v) + \sum_{n=2}^{\infty} a_n (v_1 - v)^n.$$

Letting $u_1 = u + \varepsilon u_y$, $v_1 = v + \varepsilon v_y$ and passing to the limit as $\varepsilon \to 0$, one can see that system (1.14) becomes (1.5).

Theorem 1.2. *System* (1.13) *admits n-integrals* (1.9) *and* (1.10) *if and only if it is of the form*

$$\begin{cases} u_{1x} = \dfrac{v(u_1 v_1 + d)\mathcal{D}_2}{v_1(uv + d)} u_x, \\ v_{1x} = \dfrac{(d - c) v v_1 (\mathcal{D}_2^2 - 1)}{2c(uv + d)\mathcal{D}_2} u_x + \dfrac{v_1}{v \mathcal{D}_2} v_x. \end{cases} \tag{1.23}$$

The function $\mathcal{D}_2$ is defined implicitly by $H(n, K_2, L_2) = 0$, where, for each n, the symbol H denotes an arbitrary smooth function and

$$K_2 = \frac{v_1(\mathcal{D}_2 - 1)M^{-\frac{2d}{c+d}}}{v\mathcal{D}_2}\left(-2cdu_1v_1 + uv\mathcal{D}_2M\right), \qquad L_2 = \frac{v\mathcal{D}_2M^{\frac{2d}{c+d}}}{v_1}, \tag{1.24}$$

where

$$M = 2cd + \frac{(c+d)(\mathcal{D}_2 - 1)u_1v_1}{\mathcal{D}_2}.$$

Example 1.5. *Let $K_2 = 0$, then we get*

$$\mathcal{D}_2 = \frac{(2cd + (c+d)uv)u_1v_1}{(2cd + (c+d)u_1v_1)uv}.$$

Using (1.23), *we obtain the system*

$$\begin{cases} u_{1x} = \dfrac{u_1(u_1v_1 + d)(2cd + (c+d)uv)}{u(uv+d)(2cd + (c+d)u_1v_1)}u_x, \\ v_{1x} = \dfrac{(d-c)}{2c(uv+d)}\left(\dfrac{u_1v_1^2(2cd + (c+d)uv)}{u(2cd + (c+d)u_1v_1)} - \dfrac{uv^2(2cd + (c+d)u_1v_1)}{u_1(2cd + (c+d)uv)}\right)u_x \\ \qquad + \dfrac{u(2cd + (c+d)u_1v_1)}{u_1(2cd + (c+d)uv)}v_x. \end{cases} \tag{1.25}$$

This system possesses two independent non-trivial n-integrals (1.9) *and* (1.10). *One can confirm that this system possesses also the following two n-integrals*

$$I_2^* = \frac{(2cd + (c+d)uv)u_x}{u(uv+d)}, \qquad J_2^* = \frac{(c-d)uv^2u_x}{2c(uv+d)(2cd + (c+d)uv)} + \frac{uv_x}{2cd + (c+d)uv}.$$

Considering the corresponding x-ring we can also find the x-integrals given by

$$\mathcal{F}_1 = \frac{u_1}{u}\left(\frac{2cd + (c+d)uv}{2cd + (c+d)u_1v_1}\right)^{\frac{c-d}{c+d}}, \qquad \mathcal{F}_2 = \frac{u_1v_1 - uv}{u_2v_2 - uv}.$$

Example 1.6. *Let $K_2 = 0$, then*

$$\mathcal{D}_2 = \frac{(c+d)u_1v_1}{2cd + (c+d)u_1v_1}.$$

Using (1.23) *we get the system*

$$\begin{cases} u_{1x} = \dfrac{(c+d)u_1v(u_1v_1 + d)}{(uv+d)(2cd + (c+d)u_1v_1)}u_x, \\ v_{1x} = \dfrac{(d-c)v}{2c(uv+d)}\left(\dfrac{(c+d)u_1v_1^2}{2cd + (c+d)u_1v_1} - \dfrac{2cd + (c+d)u_1v_1}{(c+d)u_1}\right)u_x \\ \qquad + \dfrac{2cd + (c+d)u_1v_1}{(c+d)u_1v}v_x. \end{cases}$$

This system possesses two independent non-trivial n-integrals (1.9) *and* (1.10) *and two independent x-integrals*

$$\mathcal{F}_1 = \frac{1}{c+d}\left(\frac{2cd + (c+d)u_1v_1}{vu_1}\right)^{\frac{c+d}{2d}} + \frac{u}{u_1}\left(\frac{2cd + (c+d)u_1v_1}{vu_1}\right)^{\frac{c-d}{2d}}$$

and

$$\mathcal{F}_2 = \frac{v_1u_1^{\frac{d-c}{2d}}\left(2cd + (c+d)u_1v_1\right)^{\frac{c+d}{2d}}}{v^{\frac{c+d}{2d}}\left(2cd + (c+d)(u_1v_1 + u_2v_2)\right)}.$$

Remark 1.3. *We consider a function $\mathcal{D}_2$ defined implicitly by*

$$H(K_2, L_2) = L_2 - (2cd)^{2d/(c+d)} = 0$$

and expand it into a series of the form

$$\mathcal{D}_2(u_1, v, v_1) = \sum_{n=0}^{\infty} a_n (v_1 - v)^n,$$

where coefficients a_n depend on variables u_1 and v only. This gives:

$$\mathcal{D}_2(u_1, v, v_1) = 1 + \frac{c}{v(uv+c)}(v_1 - v) + \sum_{n=2}^{\infty} a_n (v_1 - v)^n.$$

By letting $u_1 = u + \varepsilon u_y$, $v_1 = v + \varepsilon v_y$ and passing to the limit as $\varepsilon \to 0$ one can see that system (1.23) becomes (1.8) with $\alpha = -1$.

Theorem 1.3. *System* (1.13) *admits n-integrals* (1.11) *and* (1.12) *if and only if it is of the form*

$$\begin{cases} u_{1x} = \dfrac{u_1 v_1 + d_1}{\mathcal{D}_3(uv+d)} u_x, \\ v_{1x} = \left(\dfrac{-\beta v_1^2}{\mathcal{D}_3(uv+d)} + \dfrac{\beta v^2 \mathcal{D}_3^\beta}{uv+d} \right) u_x + \mathcal{D}_3^\beta v_x. \end{cases} \tag{1.26}$$

The function $\mathcal{D}_3$ is given implicitly by $H(n, K_3, L_3) = 0$, where, for each n, the symbol H denotes an arbitrary smooth function and

$$K_3 = \frac{(v_1 - v\mathcal{D}_3^\beta)^{\beta^{-1}}(d_1 u - d u_1 \mathcal{D}_3)}{\mathcal{D}_3}, \tag{1.27}$$

$$L_3 = (v_1 - v\mathcal{D}_3^\beta)^{(1-\beta)\beta^{-1}} \Big(d_1 \mathcal{D}_3^{\beta-1} - d_1 + (\beta - 1) u_1 (v_1 - v\mathcal{D}_3^\beta) \Big). \tag{1.28}$$

Here $d_1 = Dd$ and D is the shift operator.

Example 1.7. *Considering $K_3 = 0$, we find*

$$\mathcal{D}_3 = \frac{v_1^{1/\beta}}{v^{1/\beta}}.$$

Using (1.26), *we get the system*

$$\begin{cases} u_{1x} = \dfrac{(u_1 v_1 + d_1) v^{1/\beta}}{(uv+d) v_1^{1/\beta}} u_x, \\ v_{1x} = \left(-\dfrac{\beta v_1^2 v^{1/\beta}}{v_1^{1/\beta}(uv+d)} + \dfrac{\beta v^2 v_1}{v(uv+d)} \right) u_x + \dfrac{v_1}{v} v_x. \end{cases} \tag{1.29}$$

This system possesses two independent non-trivial n-integrals (1.11) *and* (1.12) *and two independent x-integrals*

$$\mathcal{F}_1 = \left(1 - \left(\frac{v_1}{v}\right)^{\frac{1-\beta}{\beta}} \right) \left(-d_1 u + d u_1 \left(\frac{v_1}{v}\right)^{\frac{1}{\beta}} \right)^{\beta - 1}$$

and

$$\mathcal{F}_2 = \frac{v^{\frac{1-\beta}{\beta}} - v_2^{\frac{1-\beta}{\beta}}}{v^{\frac{1-\beta}{\beta}} - v_1^{\frac{1-\beta}{\beta}}}.$$

One can confirm that this system possesses also the following two n-integrals

$$I_3^* = \frac{v^{1/\beta}u_x}{uv+d}, \qquad J_3^* = \frac{v_x}{v} + \frac{\beta v u_x}{uv+d}.$$

Example 1.8. *Let $K_3 = 0$, we find*

$$\mathcal{D}_3 = \frac{d_1 u}{du_1}.$$

By (1.26) *we get the system*

$$\begin{cases} u_{1x} = \dfrac{(u_1v_1+d_1)du_1}{(uv+d)d_1u}u_x, \\ v_{1x} = \left(-\dfrac{\beta dv_1^2u_1}{d_1u(uv+d)} + \dfrac{\beta d_1^\beta v^2u^\beta}{d^\beta u_1^\beta(uv+d)}\right)u_x + \dfrac{d_1^\beta u^\beta}{d^\beta u_1^\beta}v_x. \end{cases} \tag{1.30}$$

This system possesses two independent non-trivial n-integrals (1.11) *and* (1.12) *and two independent x-integrals*

$$\mathcal{F}_1 = \frac{d_1^\beta u^\beta v - d^\beta u_1^\beta v_1}{d_2^\beta u_1^\beta v_1 - d_1^\beta u_2^\beta v_2}$$

and

$$\mathcal{F}_2 = \frac{(d_1^\beta u^\beta v - d^\beta u_1^\beta v_1)(dd_1^\beta u^\beta u - d_1 d^\beta u_1^\beta u + (1-\beta)uu_1)}{dd_1uu_1}.$$

We confirm that this system possesses also the following two n-integrals

$$I_3^{**} = \frac{du_x}{u(uv+d)}, \qquad J_3^{**} = \frac{u^\beta v_x}{d^\beta} + \frac{\beta v^2u^\beta u_x}{d^\beta(uv+d)}.$$

Example 1.9. *Considering $L_3 = 0$ with $\beta = 2$, we get $\mathcal{D}_3 = \dfrac{d_1+R}{2u_1v}$, where*

$$R = \sqrt{d_1^2 + 4u_1v(u_1v_1 - d_1)}.$$

Then by (1.26) *we arrive at the system*

$$\begin{cases} u_{1x} = \dfrac{(u_1v_1+d_1)(d_1-R)}{2(uv+d)(d_1-u_1v_1)}u_x, \\ v_{1x} = \left(\dfrac{v_1^2(R-d_1)}{d_1-u_1v_1} + \dfrac{d_1^2+2u_1v(u_1v_1-d_1)+d_1R}{u_1^2}\right)\dfrac{u_x}{uv+d} \\ \qquad + \dfrac{d_1^2+2u_1v(u_1v_1-d_1)+d_1R}{2u_1^2v^2}v_x. \end{cases} \tag{1.31}$$

This system possesses two independent non-trivial n-integrals (1.11) *and* (1.12).

Example 1.10. *Considering $L_3 = 0$ with $\beta = 1/2$, we see that*

$$\mathcal{D}_3^{1/2} = \frac{2d_1 + u_1v_1 + R}{2u_1v},$$

where

$$R = \sqrt{(2d_1+u_1v_1)^2 - 8d_1u_1v}.$$

Employing (1.26), *we get the system*

$$\begin{cases} u_{1x} = \dfrac{(u_1v_1+d_1)(2d_1+u_1v_1-R)^2}{16d_1^2(uv+d)}u_x, \\ v_{1x} = \left(\dfrac{-v_1^2(2d_1+u_1v_1-R)^2}{32d_1^2} + \dfrac{v(2d_1+u_1v_1+R)}{4u_1}\right)\dfrac{u_x}{uv+d} + \dfrac{2d_1+u_1v_1+R}{2u_1v}v_x. \end{cases} \tag{1.32}$$

This system possesses two independent non-trivial n-integrals (1.11) *and* (1.12).

Remark 1.4. *In both previous examples the corresponding x-rings have the following multiplication table*

$[E_i, E_j]$	E_1	E_2	E_3
E_1	0	E_3	$\dfrac{-2v}{d+uv}E_3$
E_2	$-E_3$	0	$\dfrac{-2u}{d+uv}E_3$
E_3	$\dfrac{2v}{d+uv}E_3$	$\dfrac{2u}{d+uv}E_3$	0

where the fields X, Y_1, Y_2, E_1, E_2 and E_3 are introduced in the same way as in Remark 1. This shows that the x-rings are finite-dimensional and the corresponding systems (1.31) *and* (1.32) *are Darboux integrable.*

Remark 1.5. *We consider a function $\mathcal{D}_3$ given implicitly by $H(K_3, L_3) = L_3 = 0$ and expand it into a series of the form*

$$\mathcal{D}_3(u_1, v, v_1) = \sum_{n=0}^{\infty} a_n(v_1 - v)^n,$$

where coefficients a_n depend on variables u_1 and v only. Then

$$\mathcal{D}_3(u_1, v, v_1) = 1 + \frac{u_1}{\beta u_1 v - d_1}(v_1 - v) + \sum_{n=2}^{\infty} a_n(v_1 - v)^n.$$

By letting $u_1 = u + \varepsilon u_y$, $v_1 = v + \varepsilon v_y$ and passing to the limit as $\varepsilon \to 0$ one can see that system (1.26) becomes (1.8). We observe that $\beta = -\alpha$ and constants α, c, d satisfy the identity $d = \alpha c$.

2. Proof of Theorem 1.1

It follows from the identity $DJ_1 = J_1$ that

$$\frac{u_{1xx}}{u_{1x}} - \left(1 + \frac{1}{\alpha_1}\right)\frac{u_{1x}}{u_1+v_1} - \frac{v_{1x}}{u_1+v_1} = \frac{u_{xx}}{u_x} - \frac{(\alpha+1)u_x + \alpha v_x}{\alpha(u+v)},$$

that is

$$\begin{aligned} \frac{f_x + f_u u_x + f_v v_x + f_{u_1} f + f_{v_1} g + f_{u_x} u_{xx} + f_{v_x} v_{xx}}{f} - \left(1 + \frac{1}{\alpha_1}\right)\frac{f}{u_1+v_1} \\ - \frac{g}{u_1+v_1} = \frac{u_{xx}}{u_x} - \frac{(\alpha+1)u_x + \alpha v_x}{\alpha(u+v)}. \end{aligned} \tag{2.1}$$

By comparing the coefficients at v_{xx} and u_{xx}, we get

$$f_{v_x} = 0 \quad \text{and} \quad \frac{f_{u_x}}{f} = \frac{1}{u_x}.$$

Hence,

$$f(x, n, u, v, u_1, v_1, u_x, v_x) = A(x, n, u, v, u_1, v_1)u_x. \tag{2.2}$$

It follows from $DI_1 = I_1$ that

$$\left(1+\frac{1}{\alpha_1}\right) v_1 \left(\frac{Au_x}{u_1+v_1}\right)^{1-\alpha_1} - g\left(\frac{Au_x}{u_1+v_1}\right)^{-\alpha_1} = \left(1+\frac{1}{\alpha}\right) v \left(\frac{u_x}{u+v}\right)^{1-\alpha} - v_x\left(\frac{u_x}{u+v}\right)^{-\alpha}.$$

We first consider the case $\alpha_1 \neq \alpha$. We have:

$$g = Tu_x + Mu_x^{1+\alpha_1-\alpha} + Nv_x u_x^{\alpha_1-\alpha},$$

where

$$T = \left(1+\frac{1}{\alpha_1}\right)\frac{v_1 A}{u_1+v_1}, \quad M = -\left(1+\frac{1}{\alpha}\right)\frac{v(u+v)^{\alpha-1}A^{\alpha_1}}{(u_1+v_1)^{\alpha_1}}, \qquad N = \frac{(u+v)^{\alpha}A^{\alpha_1}}{(u_1+v_1)^{\alpha_1}}.$$

Substituting the expression for g and f into (2.1) and comparing the coefficients at u_x^0, u_x, v_x, $u_x^{1+\alpha_1-\alpha}$ and $v_x u_x^{\alpha_1-\alpha}$, we get

$$\frac{A_x}{A} = 0, \tag{2.3}$$

$$\frac{A_u}{A} + A_{u_1} + \frac{A_{v_1}}{A}T - \left(1+\frac{1}{\alpha_1}\right)\frac{A}{u_1+v_1} - \frac{T}{u_1+v_1} = -\left(1+\frac{1}{\alpha}\right)\frac{1}{u+v}, \tag{2.4}$$

$$\frac{A_v}{A} = -\frac{1}{u+v}, \tag{2.5}$$

$$M\left(\frac{A_{v_1}}{A} - \frac{1}{u_1+v_1}\right) = 0, \tag{2.6}$$

$$N\left(\frac{A_{v_1}}{A} - \frac{1}{u_1+v_1}\right) = 0. \tag{2.7}$$

It follows from (2.5)-(2.7) that

$$A = \frac{u_1+v_1}{u+v}S(n,u,u_1),$$

where $S(n,u,u_1)$ is a function depending on n, u, u_1 only. Substitute this expression for A into (2.4), we find that

$$(u+v)(u_1+v_1)\frac{S_u}{S} + (u_1+v_1)^2 S_{u_1} + \left(v_1 - \frac{u_1}{\alpha_1} - \frac{(1+\alpha_1)v_1(u_1+v_1)^2}{\alpha_1}\right)S + \frac{u_1+v_1}{\alpha} = 0. \tag{2.8}$$

Differentiating the last equation three times with respect to v_1, we get

$$-6\left(1+\frac{1}{\alpha_1}\right)S = 0, \quad \text{hence,} \quad S = 0.$$

Hence, $A = 0$ is the only solution when $\alpha \neq \alpha_1$.

Now we consider the case when α is a constant, that is α is independent of n. We have:

$$g = \left(1+\frac{1}{\alpha}\right)\left(\frac{Av_1}{u_1+v_1} - \frac{vA^{\alpha}}{u+v}\left(\frac{u+v}{u_1+v_1}\right)^{\alpha}\right)u_x + \left(A\frac{u+v}{u_1+v_1}\right)^{\alpha} v_x. \tag{2.9}$$

We substitute the expressions for f and g into (2.1) and compare the coefficients at v_x, u_x and the free term. This gives:

$$\frac{A_x}{A}=0, \tag{2.10}$$

$$\frac{A_u}{A}+A_{u_1}+\left(1+\frac{1}{\alpha}\right)\left[A_{v_1}\frac{v_1}{u_1+v_1}-\frac{vA^\alpha A_{v_1}}{A(u+v)}\left(\frac{u+v}{u_1+v_1}\right)^\alpha-\frac{A}{u_1+v_1}\right.$$
$$\left.-\frac{Av_1}{(u_1+v_1)^2}+\frac{vA^\alpha}{(u+v)(u_1+v_1)}\left(\frac{u+v}{u_1+v_1}\right)^\alpha+\frac{1}{u+v}\right]=0, \tag{2.11}$$

$$\frac{A_v}{A}+\frac{A_{v_1}A^\alpha}{A}\left(\frac{u+v}{u_1+v_1}\right)^\alpha-\frac{A^\alpha}{u_1+v_1}\left(\frac{u+v}{u_1+v_1}\right)^\alpha+\frac{1}{u+v}=0. \tag{2.12}$$

Let

$$\mathcal{D}_1=\left(\frac{u+v}{u_1+v_1}\right)^\alpha A^\alpha. \tag{2.13}$$

In terms of the function $\mathcal{D}_1$, the equations (2.11) and (2.12) cast into the form

$$(u+v)\mathcal{D}_{1u}+(u_1+v_1)\mathcal{D}_1^{\alpha^{-1}}\mathcal{D}_{1u_1}+\frac{\alpha+1}{\alpha}(v_1\mathcal{D}_1^{\alpha^{-1}}-v\mathcal{D}_1)\mathcal{D}_{1v_1}-\mathcal{D}_1(\mathcal{D}_1^{\alpha^{-1}}-1)=0, \tag{2.14}$$

$$\frac{\mathcal{D}_{1v}}{\mathcal{D}_1}+\mathcal{D}_{1v_1}=0. \tag{2.15}$$

The set of solutions of the above system is not empty. For example, $\mathcal{D}_1=1$ is a singular solution leading to Darboux integrable system (1.17). Let $\mathcal{D}_1\neq 1$. It is convenient to regard $\mathcal{D}_1$ as a function of n, u, v, u_1, v_1 defined implicitly by the equation as follows

$$W(n,u,v,u_1,v_1,\mathcal{D}_1)=0.$$

Then in terms of function $W=W(n,u,v,u_1,v_1,\mathcal{D}_1)$, equations (2.14) and (2.15) can be rewritten as

$$(u+v)W_u+(u_1+v_1)\mathcal{D}_1^{\alpha^{-1}}W_{u_1}+\frac{\alpha+1}{\alpha}(v_1\mathcal{D}_1^{\alpha^{-1}}-v\mathcal{D}_1)W_{v_1}+\mathcal{D}_1(\mathcal{D}_1^{\alpha^{-1}}-1)W_{\mathcal{D}_1}=0, \tag{2.16}$$

$$\frac{W_v}{\mathcal{D}_1}+W_{v_1}=0. \tag{2.17}$$

Under the change of variables

$$\tilde v=v,\qquad \tilde v_1=v_1-v\mathcal{D}_1,\qquad \tilde u=u,\qquad \tilde u_1=u_1,\qquad \tilde{\mathcal{D}}_1=\mathcal{D}_1,$$

the above equations cast into the form

$$(\tilde u+\tilde v)W_{\tilde u}+(\tilde u_1+\tilde v_1+\tilde v\tilde{\mathcal{D}}_1)\tilde{\mathcal{D}}_1^{\alpha^{-1}}W_{\tilde u_1}+\left(\left(1+\frac{1}{\alpha}\right)\tilde v_1\tilde{\mathcal{D}}_1^{\alpha^{-1}}+\frac{1}{\alpha}\tilde v(\tilde{\mathcal{D}}_1^{1+\alpha^{-1}}-\tilde{\mathcal{D}}_1)\right)W_{\tilde v_1}$$
$$+(\tilde{\mathcal{D}}_1^{1+\alpha^{-1}}-\tilde{\mathcal{D}}_1)W_{\tilde{\mathcal{D}}_1}=0,$$

$$W_{\tilde v}=0.$$

We differentiate the first equation with respect to $\tilde v$, then use the identity $W_{\tilde v}=0$, and get two new equations

$$W_{\tilde u}+\tilde{\mathcal{D}}_1^{1+\alpha^{-1}}W_{\tilde u_1}+\frac{1}{\alpha}(\tilde{\mathcal{D}}_1^{1+\alpha^{-1}}-\tilde{\mathcal{D}}_1)W_{\tilde v_1}=0,$$

$$\tilde uW_{\tilde u}+(\tilde u_1+\tilde v_1)\tilde{\mathcal{D}}_1^{\alpha^{-1}}W_{\tilde u_1}+\frac{\alpha+1}{\alpha}\tilde v_1\tilde{\mathcal{D}}_1^{\alpha^{-1}}W_{\tilde v_1}+(\tilde{\mathcal{D}}_1^{1+\alpha^{-1}}-\tilde{\mathcal{D}}_1)W_{\tilde{\mathcal{D}}_1}=0.$$

In the latter system, we make the change of variables

$$u_1^*=\tilde u_1-\tilde{\mathcal{D}}_1^{1+\alpha^{-1}}\tilde u,\qquad v_1^*=\alpha\tilde{\mathcal{D}}_1^{\alpha^{-1}}\tilde v_1+(1-\tilde{\mathcal{D}}_1^{\alpha^{-1}})\tilde u_1,\qquad u^*=\tilde u,\qquad v^*=\tilde v,\qquad \mathcal{D}_1^*=\tilde{\mathcal{D}}_1$$

and we get:

$$W_{u^*} = 0,$$

$$\Big((\mathcal{D}_1^{*\alpha^{-1}} + \alpha^{-1}(1 - \mathcal{D}_1^{*-\alpha^{-1}}))u_1^* + \alpha^{-1}v_1^*\mathcal{D}_1^{*-\alpha^{-1}}\Big)W_{u_1^*} + \frac{\alpha+1}{\alpha}v_1^*\mathcal{D}_1^{*\alpha^{-1}}W_{v^*} + \mathcal{D}_1^*(\mathcal{D}_1^{*\alpha^{-1}} - 1)W_{\mathcal{D}_1^*} = 0.$$

The last equation has a general solution $H(n, K_1, L_1) = 0$, where K_1, L_1 rewritten in old variables are given by (1.15), (1.16) and H is a smooth function for each n. Now, using identities (2.13), (2.2) and (2.9), we obtain system (1.14). This completes the proof.

3. Proof of Theorem 1.2

The identity $DJ_2 = J_2$ implies that

$$\frac{f_x + f_u u_x + f_v v_x + f_{u_1} f + f_{v_1} g + f_{u_x} u_{xx} + f_{v_x} v_{xx}}{f} + \frac{(d_1 - c_1)v_1 f - c_1 u_1 g}{c_1(u_1v_1 + d_1)} = \frac{u_{xx}}{u_x} + \frac{(d-c)vu_x - cuv_x}{c(uv+d)}. \tag{3.1}$$

Comparing the coefficients at u_{xx} and v_{xx} in the above identity, we get

$$f_{v_x} = 0 \quad \text{and} \quad \frac{f_{u_x}}{f} = \frac{1}{u_x}.$$

Hence,

$$f = A(x, n, u, v, u_1, v_1)u_x. \tag{3.2}$$

The identity $DI_2 = I_2$ implies

$$\frac{(d_1 - c_1)v_1^2 A^2 u_x}{2(u_1v_1 + d_1)^2} - \frac{cAg}{u_1v_1 + d_1} = \frac{(d-c)v^2 u_x}{2(uv+d)^2} - \frac{cv_x}{uv+d}. \tag{3.3}$$

It follows from (3.3) that

$$g = \left(\frac{(d_1 - c_1)v_1^2 A}{2c_1(u_1v_1 + d_1)} - \frac{(d-c)v_1^2(u_1v_1 + d_1)}{2c_1 A(uv+d)^2}\right)u_x + \frac{c(u_1v_1 + d_1)}{c_1 A(uv+d)}v_x. \tag{3.4}$$

Substituting the expressions for f and g into (3.1) and comparing the coefficients at u_x, v_x and the free term, we get

$$\frac{A_x}{A} = 0, \tag{3.5}$$

$$\frac{A_u}{A} + A_{u_1} + \left(\frac{A_{v_1}}{A} - \frac{u_1}{u_1v_1 + d_1}\right)\left(\frac{(d_1 - c_1)v_1^2 A}{2c_1(u_1v_1 + d_1)} - \frac{(d-c)v^2(u_1v_1 + d_1)}{2c_1 A(uv+d)^2}\right) + \frac{(d_1 - c_1)v_1 A}{c_1(u_1v_1 + d_1)} - \frac{(d-c)v}{c(uv+d)} = 0, \tag{3.6}$$

$$\frac{A_v}{A} + \frac{c(u_1v_1 + d_1)}{c_1 A(uv+d)}\left(\frac{A_{v_1}}{A} - \frac{u_1}{u_1v_1 + d_1}\right) + \frac{u}{uv+d} = 0. \tag{3.7}$$

One can check that

$$A = \frac{v(u_1v_1 + d)}{v_1(uv+d)}$$

is a particular solution provided $d_1 = d$ and $c_1 = c$.

Now assuming that

$$A \neq \frac{v(u_1v_1 + d)}{v_1(uv+d)},$$

we introduce a new function

$$\mathcal{D}_2 = \frac{v_1(uv+d)}{v(u_1v_1+d_1)}A. \tag{3.8}$$

In terms of $\mathcal{D}_2$, system (3.6) becomes

$$\begin{aligned}
&\mathcal{D}_{2x} = 0,\\
&(uv+d)\mathcal{D}_{2u} + \frac{v(u_1v_1+d_1)\mathcal{D}_2}{v_1}\mathcal{D}_{2u_1} + \frac{vv_1}{2c_1}\left((d_1-c_1)\mathcal{D}_2 - (d-c){\mathcal{D}_2}^{-1}\right)\mathcal{D}_{2v_1}\\
&\qquad - \frac{dv}{c}\mathcal{D}_2 + \frac{(d_1+c_1)v}{2c_1}{\mathcal{D}_2}^2 + \frac{v(d-c)}{2c_1} = 0,\\
&c_1v\mathcal{D}_2\mathcal{D}_{2v} + cv_1\mathcal{D}_{2v_1} + (-c\mathcal{D}_2 + c_1{\mathcal{D}_2}^2) = 0.
\end{aligned}$$

In the same way as in the proof of Theorem 1.1, we introduce a function $W(n,u,v,u_1,v_1,\mathcal{D}_2)$. For the function $W = W(n,u,v,u_1,v_1,\mathcal{D}_2)$ the last two equations become

$$\begin{aligned}
&(uv+d)W_u + \frac{v(u_1v_1+d_1)}{v_1}\mathcal{D}_2W_{u_1} + \frac{vv_1}{2c_1}\left((d_1-c_1)\mathcal{D}_2 - (d-c){\mathcal{D}_2}^{-1}\right)W_{v_1}\\
&\qquad + \left(\frac{dv}{c}\mathcal{D}_2 - \frac{(d_1+c_1)v}{2c_1}{\mathcal{D}_2}^2 - \frac{v(d-c)}{2c_1}\right)W_{\mathcal{D}_2} = 0,\\
&c_1v\mathcal{D}_2W_v + cv_1W_{v_1} + (c\mathcal{D}_2 - c_1{\mathcal{D}_2}^2)W_{\mathcal{D}_2} = 0.
\end{aligned}$$

In new variables

$$\tilde{u} = u, \qquad \tilde{u}_1 = u_1, \qquad \tilde{v} = v(c_1\mathcal{D}_2 - c), \qquad \tilde{v}_1 = v_1(c_1\mathcal{D}_2 - c)\mathcal{D}_2^{-1}, \qquad \tilde{\mathcal{D}_2} = \mathcal{D}_2,$$

the last system can be rewritten as

$$\begin{aligned}
&\left((c_1\tilde{\mathcal{D}_2} - c)\tilde{u}\tilde{v} + d(c_1\tilde{\mathcal{D}_2} - c)^2\right)W_{\tilde{u}} + \frac{\tilde{v}}{\tilde{v}_1}\left(\tilde{u}_1\tilde{v}_1\tilde{\mathcal{D}_2}(c_1\tilde{\mathcal{D}_2} - c) + d_1(c_1\tilde{\mathcal{D}_2} - c)^2\right)W_{\tilde{u}_1}\\
&\qquad + \tilde{v}^2\left(\frac{c_1d\tilde{\mathcal{D}_2}}{c} - \frac{(d_1+c_1){\tilde{\mathcal{D}_2}}^2}{2} + \frac{c-d}{2}\right)W_{\tilde{v}}\\
&\qquad + \tilde{v}\tilde{v}_1\left(\frac{(d_1-c_1){\tilde{\mathcal{D}_2}}^2}{2} - \frac{cd_1\tilde{\mathcal{D}_2}}{c_1} + \frac{c+d}{2}\right)W_{\tilde{v}_1} = 0,\\
&W_{\tilde{\mathcal{D}_2}} = 0.
\end{aligned} \tag{3.9}$$

Special solutions of (3.9) may exist only when $\tilde{\mathcal{D}_2} = c_1/c$. We differentiate equation (3.9) with respect to $\tilde{\mathcal{D}_2}$ three times and get the following system of three equations

$$(dc^2 - c\tilde{u}\tilde{v})W_{\tilde{u}} + c^2d_1\frac{\tilde{v}}{\tilde{v}_1}W_{\tilde{u}_1} + \frac{(c-d)\tilde{v}^2}{2}W_{\tilde{v}} + \frac{(c+d)\tilde{v}\tilde{v}_1}{2}W_{\tilde{v}_1} = 0, \tag{3.10}$$

$$(c_1\tilde{u}\tilde{v} - 2dc_1c)W_{\tilde{u}} - \left(2d_1c_1c\frac{\tilde{v}}{\tilde{v}_1} + c\tilde{u}_1\tilde{v}\right)W_{\tilde{u}_1} + \frac{c_1d\tilde{v}^2}{c}W_{\tilde{v}} - \frac{cd_1\tilde{v}\tilde{v}_1}{c_1}W_{\tilde{v}_1} = 0, \tag{3.11}$$

$$dc_1^2W_{\tilde{u}} + \left(\frac{d_1c_1^2\tilde{v}}{\tilde{v}_1} + c_1\tilde{u}_1\tilde{v}\right)W_{\tilde{u}_1} - \frac{(d_1+c_1)\tilde{v}^2}{2}W_{\tilde{v}} + \frac{(d_1-c_1)\tilde{v}\tilde{v}_1}{2}W_{\tilde{v}_1} = 0, \tag{3.12}$$

that has no solutions if $c_1 \neq c$ or $d_1 \neq d$. In the case $c_1 = c$ and $d_1 = d$ the system becomes

$$W_{\tilde{u}} - \frac{\tilde{v}^2(2c^2d + (c-d)\tilde{u}_1\tilde{v}_1)}{2c(\tilde{u}\tilde{u}_1\tilde{v}\tilde{v}_1 + cd(\tilde{u}\tilde{v} - \tilde{u}_1\tilde{v}_1))}W_{\tilde{v}} - \frac{\tilde{v}\tilde{v}_1(2c^2d + (c+d)\tilde{u}_1\tilde{v}_1)}{2c(\tilde{u}\tilde{u}_1\tilde{v}\tilde{v}_1 + cd(\tilde{u}\tilde{v} - \tilde{u}_1\tilde{v}_1))}W_{\tilde{v}_1} = 0, \tag{3.13}$$

$$W_{\tilde{u}_1} - \frac{\tilde{v}\tilde{v}_1(-2c^2d + (c+d)\tilde{u}\tilde{v})}{2c(\tilde{u}\tilde{u}_1\tilde{v}\tilde{v}_1 + cd(\tilde{u}\tilde{v} - \tilde{u}_1\tilde{v}_1))}W_{\tilde{v}} + \frac{\tilde{v}_1^2(2c^2d + (-c+d)\tilde{u}\tilde{v})}{2c(\tilde{u}\tilde{u}_1\tilde{v}\tilde{v}_1 + cd(\tilde{u}\tilde{v} - \tilde{u}_1\tilde{v}_1))}W_{\tilde{v}_1} = 0. \tag{3.14}$$

Under the change of variables

$$u_1^* = \tilde{u}_1, \qquad v_1^* = \tilde{v}_1, \qquad v^* = \frac{\tilde{v}}{\tilde{v}_1}\left(2c^2d + (c+d)\tilde{u}_1\tilde{v}_1\right)^{\frac{2d}{c+d}},$$
$$u^* = \tilde{u}\tilde{v}_1\left(2c^2d + (c+d)\tilde{u}_1\tilde{v}_1\right)^{\frac{c-d}{c+d}} - 2c^2d\tilde{u}_1\tilde{v}_1^2\tilde{v}^{-1}\left(2c^2d + (c+d)\tilde{u}_1\tilde{v}_1\right)^{-\frac{2d}{c+d}},$$

equations (3.13) and (3.14) become $W_{v_1^*} = 0$ and $W_{u_1^*} = 0$, respectively. We rewrite these first integrals in old variables and get the general solution in an implicit form $H(n, K_2, L_2) = 0$, where, for each n, H is a smooth function and K_2, L_2 are given by (1.24). The form of system (1.23) follows from (3.2), (3.4) and (3.8). The proof is complete.

4. Proof of Theorem 1.3

The identity $DJ_3 = J_3$ implies

$$-\frac{f_x + f_u u_x + f_v v_x + f_{u_1} f + f_{v_1} g + f_{u_x} u_{xx} + f_{v_x} v_{xx}}{f} + \frac{2v_1 f + u_1 g}{u_1 v_1 + d_1} = -\frac{u_{xx}}{u_x} + \frac{2v u_x + u v_x}{uv + d}. \tag{4.1}$$

Comparing the coefficients at u_{xx} and v_{xx} in the above identity, we get

$$f_{v_x} = 0, \qquad \frac{f_{u_x}}{f} = \frac{1}{u_x}.$$

Hence,

$$f = A(x, n, u, v, u_1, v_1)u_x. \tag{4.2}$$

It follows from the identity $DI_3 = I_3$ that

$$\frac{f^{\beta_1} g}{(u_1 v_1 + d_1)^{\beta_1}} + \frac{\beta_1 v_1^2 f^{\beta_1+1}}{(u_1 v_1 + d_1)^{\beta_1+1}} = \frac{u_x^{\beta} v_x}{(uv + d)^{\beta}} + \frac{\beta v^2 u_x^{\beta+1}}{(uv + d)^{\beta+1}}. \tag{4.3}$$

First we consider the case $\beta_1 \neq \beta$. We have:

$$g = T v_x u_x^{\beta-\beta_1} + M u_x^{1+\beta-\beta_1} + N u_x, \tag{4.4}$$

where

$$T = \frac{A^{-\beta_1}(u_1 v_1 + d_1)^{\beta_1}}{(uv + d)}, \quad M = \frac{\beta v^2 A^{-\beta_1}(u_1 v_1 + d_1)^{\beta_1}}{(uv + d)^{\beta+1}}, \quad N = -\frac{\beta_1 v_1^2 A}{(u_1 v_1 + d_1)}.$$

We substitute this expression for g into equation (4.1) and compare the coefficients at u_x^0, u_x, $u_x^{\beta-\beta_1}$, v_x, $u_x^{1+\beta-\beta_1}$ in the resulting equation. This gives:

$$\frac{A_x}{A} = 0, \tag{4.5}$$
$$\frac{A_u}{A} + A_{u_1} + \frac{A_{v_1} N}{A} + \frac{2Av_1}{u_1 v_1 + d_1} + \frac{u_1 N}{u_1 v_1 + d_1} = \frac{2v}{uv + d}, \tag{4.6}$$
$$\frac{A_v}{A} = \frac{u}{uv + d}, \tag{4.7}$$
$$T\left(\frac{A_{v_1}}{A} + \frac{u_1}{u_1 v_1 + d_1}\right) = 0, \tag{4.8}$$
$$M\left(\frac{A_{v_1}}{A} + \frac{u_1}{u_1 v_1 + d_1}\right) = 0. \tag{4.9}$$

If $T = 0$ or $M = 0$, then $A = 0$. Hence, in order to have $A \neq 0$, we assume that $TM \neq 0$. If $TM \neq 0$, then equations (4.7)-(4.9) imply

$$A = \frac{uv + d}{u_1 v_1 + d_a} S,$$

where $S = S(n, u, u_1)$ is a function depending on n, u and u_1 only. We substitute the above expression for A into (4.6) and we find that

$$(u_1v_1 + d_1)^2(uv + d)\frac{S_u}{S} + (u_1v_1 + d_1)(uv + d)^2S_{u_1} + v_1(uv + d)^2S - v(u_1v_1 + d_1)^2 = 0. \quad (4.10)$$

Then we differentiate the last equation twice with respect to v_1 and we obtain:

$$2u_1^2(uv + d)\frac{S_u}{S} - 2u_1^2v = 0,$$

that is,

$$\frac{S_u}{S} = \frac{v}{uv + d}.$$

This contradicts to the fact that S is independent of v, v_1. Hence, $\beta_1 = \beta$.

We proceed to the case when β is a constant, that is, β is independent of n. Let

$$\mathcal{D}_3 = \frac{u_1v_1 + d_1}{A(uv + d)}. \quad (4.11)$$

Then it follows from (4.3) that

$$g = \left(-\frac{\beta v_1^2}{\mathcal{D}_3(uv + d)} + \frac{\beta v^2\mathcal{D}_3^\beta}{(uv + d)}\right)u_x + \mathcal{D}_3^\beta v_x. \quad (4.12)$$

Being rewritten in terms of $\mathcal{D}_3$, identity (4.1) casts into the form

$$\frac{\mathcal{D}_{3x}}{\mathcal{D}_3} + \left(\frac{\mathcal{D}_{3u}}{\mathcal{D}_3} + \frac{u_1v_1 + d_1}{{\mathcal{D}_3}^2(uv + d)}\mathcal{D}_{3u_1} + \frac{\beta(v^2\mathcal{D}_3^\beta - v_1^2\mathcal{D}_3^{-1})}{\mathcal{D}_3(uv + d)}\mathcal{D}_{3v_1} + \frac{v_1}{\mathcal{D}_3(uv + d)} - \frac{v}{(uv + d)}\right)u_x$$
$$+ \left(\frac{\mathcal{D}_{3v}}{\mathcal{D}_3} + \mathcal{D}_3^{\beta-1}\mathcal{D}_{3v_1}\right)v_x = 0.$$

We compare the coefficients at u_x, v_x and the free term and we get:

$$\mathcal{D}_{3x} = 0,$$
$$\frac{uv + d}{\mathcal{D}_3}\mathcal{D}_{3u} + \frac{u_1v_1 + d_1}{{\mathcal{D}_3}^2}\mathcal{D}_{3u_1} + \frac{\beta v^2\mathcal{D}_3^\beta - \beta v_1^2\mathcal{D}_3^{-1}}{\mathcal{D}_3}\mathcal{D}_{3v_1} + \frac{v_1}{\mathcal{D}_3} - v = 0, \quad (4.13)$$
$$\mathcal{D}_{3v} + \mathcal{D}_3^\beta\mathcal{D}_{3v_1} = 0. \quad (4.14)$$

We introduce a function W in the same way as in the proof of Theorem 1.1 and in new variables

$$\tilde{v}_1 = v_1 - v\mathcal{D}_3^\beta, \qquad \tilde{v} = v, \qquad \tilde{u} = u, \qquad \tilde{u}_1 = u_1, \qquad \tilde{\mathcal{D}}_3 = \mathcal{D}_3,$$

equations (4.14) and (4.13) for the function $W = W(n, \tilde{u}, \tilde{v}, \tilde{u}_1, \tilde{v}_1, \tilde{\mathcal{D}}_3)$ can be rewritten as follows

$$W_{\tilde{v}} = 0,$$
$$\tilde{\mathcal{D}}_3(\tilde{u}\tilde{v} + d)W_{\tilde{u}} + (\tilde{u}_1(\tilde{v}_1 + \tilde{v}\tilde{\mathcal{D}}_3^\beta) + d_1)W_{\tilde{u}_1}$$
$$+ \tilde{\mathcal{D}}_3(\tilde{v}(\tilde{\mathcal{D}}_3 - \tilde{\mathcal{D}}_3^\beta) - \tilde{v}_1)W_{\tilde{\mathcal{D}}_3} - \beta\tilde{v}_1(\tilde{v}_1 + \tilde{v}\tilde{\mathcal{D}}_3^\beta)W_{\tilde{v}_1} = 0.$$

We differentiate the latter equation with respect to $\tilde{v}$, employ the identity $W_{\tilde{v}} = 0$, and get a new system of equations:

$$\tilde{u}\tilde{\mathcal{D}}_3W_{\tilde{u}} + \tilde{u}_1\tilde{\mathcal{D}}_3^\beta W_{\tilde{u}_1} + (\tilde{\mathcal{D}}_3^2 - \tilde{\mathcal{D}}_3^{\beta+1})W_{\tilde{\mathcal{D}}_3} - \beta\tilde{v}_1\tilde{\mathcal{D}}_3^\beta W_{\tilde{v}_1} = 0,$$
$$d\tilde{\mathcal{D}}_3W_{\tilde{u}} + (\tilde{u}_1\tilde{v}_1 + d_1)W_{\tilde{u}_1} - \tilde{\mathcal{D}}_3\tilde{v}_1W_{\tilde{\mathcal{D}}_3} - \beta\tilde{v}_1^2W_{\tilde{v}_1} = 0,$$

which can be rewritten as

$$W_{\tilde{u}} + \frac{d_1\tilde{\mathcal{D}}_3 - d_1\tilde{\mathcal{D}}_3^{\beta} + \tilde{\mathcal{D}}_3\tilde{u}_1\tilde{v}_1}{d_1\tilde{u} - d\tilde{\mathcal{D}}_3^{\beta}\tilde{u}_1 + \tilde{u}\tilde{u}_1\tilde{v}_1}W_{\tilde{\mathcal{D}}_3} - \frac{\beta d_1\tilde{v}_1\tilde{\mathcal{D}}_3^{\beta-1}}{d_1\tilde{u} - d\tilde{\mathcal{D}}_3^{\beta}\tilde{u}_1 + \tilde{u}\tilde{u}_1\tilde{v}_1}W_{\tilde{v}_1} = 0,$$

$$W_{\tilde{u}_1} - \frac{\tilde{\mathcal{D}}_3(d\tilde{\mathcal{D}}_3 - d\tilde{\mathcal{D}}_3^{\beta} + \tilde{u}\tilde{v}_1)}{d_1\tilde{u} - d\tilde{\mathcal{D}}_3^{\beta}\tilde{u}_1 + \tilde{u}\tilde{u}_1\tilde{v}_1}W_{\tilde{\mathcal{D}}_3} + \frac{\beta\tilde{v}_1(d\tilde{\mathcal{D}}_3^{\beta} - \tilde{u}\tilde{v}_1)}{d_1\tilde{u} - d\tilde{\mathcal{D}}_3^{\beta}\tilde{u}_1 + \tilde{u}\tilde{u}_1\tilde{v}_1}W_{\tilde{v}_1} = 0.$$

In these equations, we make the change of variables

$$u^* = \tilde{u}\tilde{v}_1^{1/\beta}d_1^{1/(1-\beta)}\tilde{\mathcal{D}}_3^{-1} - dd_1^{\beta/(1-\beta)}\tilde{u}_1\tilde{v}_1^{1/\beta},$$
$$\mathcal{D}_3^* = \tilde{v}_1^{(1-\beta)/\beta}\tilde{\mathcal{D}}_3^{\beta-1} - \tilde{v}_1^{(1-\beta)/\beta} + (\beta-1)d_1^{-1}\tilde{u}_1\tilde{v}_1^{1/\beta},$$
$$u_1^* = \tilde{u}_1, \qquad v^* = \tilde{v}, \qquad v_1^* = \tilde{v}_1,$$

and these equations become $W_{v_1^*} = 0$ and $W_{u_1^*} = 0$, respectively. We rewrite these first integrals in old variables and get that the general solution is given implicitly by $H(n, K_3, L_3) = 0$, where, for each n, the symbol H denotes an arbitrary smooth function and K_3, L_3 are given by (1.27), (1.28). The form of system (1.26) follows from (4.2), (4.12) and (4.11). The proof is complete.

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